Representations of positive projections

W. A. J. Luxemburg
B. de Pagter
Department of Mathematics, California Institute of Technology, Pasadena, CA, USA
Department of Mathematics, Delft University of Technology, Delft, The Netherlands

to Dorothy Maharam in admiration

Abstract

In this paper a number of Maharam-type slice integral representations, with respect to scalar measures, is obtained for positive projections in Dedekind complete vector lattices. In the approach to these results the theory of f-algebras plays a crucial role.

1 Introduction

About half a century ago Dorothy Maharam published an important series of papers dealing with the characterization of measure algebras and their decomposition properties and applied them to prove a number of deep representation theorems for positive linear transformations between spaces of measurable functions as kernel operators by means of what is now called Maharam's slice integral method. For a discussion of these decomposition theorems for measure algebras and their subalgebras we refer the reader also to D. H. Fremlin's contribution in [12].

In [5], the first author presented a brief summary and discussion of some of Maharam's groundbreaking work. By using the language of the theory of Riesz spaces (vector lattices) it was suggested that an algebraization of her work would be a worthwhile undertaking. This led to the paper [7] dealing with Maharam extensions of positive linear operators in the setting of the

AMS Classification: 47B65, 46A40, 06F25

theory of f-modules (i.e., vector lattices which are modules over f-algebras). For an alternative exposition of results on Maharam extensions we refer the reader also to the book [4].

In the present paper we present a general theory of Maharam-type representation theorems mentioned above for positive projections on unital f-algebras as well as on Dedekind complete vector lattices which are what we call locally order separable. The main results of the paper are collected in the theorems in Section 10.

Section 2 is devoted to some preliminary results and constructions in the theory of Riesz spaces which will be used frequently throughout the paper. In addition, we introduce the notion of local order separability, referred to above, a somewhat weaker concept than that of order separability. A Riesz space is called locally order separable if there exists a maximal disjoint system consisting of order separable elements, i.e., the principal ideals generated by these elements are order separable. In essence, such (Archimedean) Riesz spaces are those who have an order dense and order separable ideal.

A representation theory for Dedekind (σ-) complete Riesz spaces as special quotient spaces is presented in Section 3. A measurable space (X, Σ) is called a representation space of a Dedekind σ-complete Riesz space L with a weak order unit $0 < w \in L$ if there exists an ideal $\widehat{L}$ in $M(X, \Sigma)$, the Riesz space of all Σ-measurable real functions on X, such that $\mathbf{1}_X \in \widehat{L}$ and there exists a σ-order continuous Riesz homomorphism Φ from $\widehat{L}$ onto L with $\Phi(\mathbf{1}_X) = w$. Consequently, L is Riesz isomorphic to the quotient space $\widehat{L} / \mathrm{Ker}(\Phi)$. The existence of such a representation depends on the classical Sikorski-Loomis theorem for σ-complete Boolean algebras.

Using this concept we introduce the notion of a Maharam-type representation space for a positive projection P in a Dedekind σ-complete Riesz space L with weak order unit w (denoted from here by (L, w)) onto a Riesz subspace $K \subseteq L$ with $w \in K$. Such a representation space is a product space $(\Omega, \mathcal{F}) = (X \times Y, \Sigma \otimes \Lambda)$ of a measurable space (X, Σ) and a σ-finite measure space (Y, Λ, μ), such that $(\Omega, \mathcal{F})$ is a representation space for (L, w) with representation homomorphism Φ. Furthermore, there exists an $\mathcal{F}$-measurable function $m \geq 0$ on Ω satisfying $\int_Y m(x, y)\, d\mu(y) = 1$ and $\int_Y m(x, y) |f(x, y)|\, d\mu(y) < \infty$ for all $f \in \widehat{L}$, $x \in X$ and moreover the function Rf, defined by

$$(Rf)(x, y) = \int_Y m(x, z) f(x, z)\, d\mu(z) \tag{1}$$

for all $(x, y) \in \Omega$ satisfies $Rf \in \widehat{L}$ and

$$P(\Phi f) = \Phi(Pf)$$

for all $f \in \widehat{L}$. We stress the fact that in such a Maharam-type integral representation (1) the measure μ is a scalar valued measure and not a vector valued measure (as is the case in e.g. the integral representations discussed in Section 6.3 of [4]). Actually one of the main efforts in the present paper will be the construction of this measure μ (see Sections 7 and 8).

As will be shown, if $(X \times Y, \Sigma \otimes \Lambda)$ is a representation space for the restriction of the projection P to the principal ideal L_w generated by the weak order unit w, then it is already a representation space for P. Since Dedekind complete Riesz spaces with a strong order unit have a unital f-algebra structure, we switch our attention to the study of representations of positive projections in unital f-algebras. For this reason Section 4 is devoted to a discussion of the theory of f-algebras that plays a role in our representation theory. First we show that the averaging property of positive projections, as stated in Lemma 4.1, in combination with the Radon-Nikodym theorem for Maharam operators, implies the following interesting result that plays a crucial role later in Section 8. Suppose that A is a Dedekind complete unital f-algebra in which the unit element $\mathbf{1}$ is a strong order unit as well and that B is an f-subalgebra with $\mathbf{1} \in B$, which is the range of a strictly positive order continuous projection P in A. If C is a complete f-subalgebra of A such that $B \subseteq C$, then there exists a unique positive projection Q in A onto C such that $P = P \circ Q$; moreover, Q is order continuous and strictly positive. By means of an example it is shown that the existence of a strong order unit in A is essential.

In the theory of measure algebras Maharam ([10], Section 11) introduced the following important concept. Suppose that A is a Dedekind complete unital f-algebra and that B is a complete unital f-subalgebra of A. The complete Boolean algebras of components of the unit element in A and B are denoted by $\mathcal{C}_A$ and $\mathcal{C}_B$ respectively. Then $p \in \mathcal{C}_A$ is called B-full (or, of order zero over $\mathcal{C}_B$) if $\{e \in \mathcal{C}_A : e \leq p\} = \{pq : q \in \mathcal{C}_B\}$. Furthermore, A is called nowhere full with respect to B if the zero element is the only B-full element of $\mathcal{C}_A$. If for every $0 < p \in \mathcal{C}_A$ there exists a B-full component $0 < q \in \mathcal{C}_A$ such that $q \leq p$, then A is called everywhere full with respect to B. From these definitions it is easily seen that A splits in a part that is everywhere full and a part that is nowhere full over B. Section 4 finishes with a technical result showing how families of representations of restrictions of a positive projection can be glued together to obtain a global representation of the projection, a technique that will yield the general representation theorems in Section 10.

In [11] Maharam proved the following important and non-trivial version of a Radon-Nikodym theorem (see Theorem 5.8 and its proof). Assume that (X, Σ, μ) is a finite measure space with measure algebra (Σ_μ, μ) and assume that Λ is a σ-subalgebra of Σ and let Λ_μ be the corresponding complete Boolean subalgebra of Σ_μ. If Σ_μ is nowhere full with respect to Λ_μ and if ν is a measure on Λ such that $\nu \leq \mu$ on Λ, then there exists $F_0 \in \Sigma$ such that $\nu(D) = \mu(D \cap F_0)$ for all $D \in \Lambda$. In Section 5 we present a vector-valued version of Maharam's theorem of the following form. Assume that B is an f-subalgebra of the Dedekind complete unital f-algebra A with $\mathbf{1} \in B$ and that P is a positive order continuous projection of A onto B. If A is nowhere full with respect to B and if $0 < e \in \mathcal{C}_A$ and $g \in B$ satisfy $0 \leq g \leq P(e)$, then there exists $p \in \mathcal{C}_A$ such that $p \leq e$ and $P(p) = g$. In Theorem 5.7 an application of this result is presented, which is of independent interest.

If P is a strictly positive order continuous projection in the unital f-algebra A onto the unital f-subalgebra B and if A is nowhere full with respect to B, then it follows from the results in Section 5 that there exist for any $\lambda \in [0,1]$ components $p \in \mathcal{C}_A$ with the property that $P(p) = \lambda\mathbf{1}$. The collection of all such components is denoted by $\mathcal{S}$. In Section 6 the structure of this set $\mathcal{S}$ and its complete Boolean subalgebras is discussed. The construction of such special Boolean subalgebras is contained in Sections 7 and 8. We would like to point out that these special subalgebras are the essential ingredient in the construction of the measure space (X, Σ, μ) occurring in the representation (1) of the projection P. For more details we have to refer the reader to these sections. In particular the results in Section 8 use Maharam's decomposition of Boolean algebras in homogeneous parts.

Finally, in Section 10, the results of the previous sections are combined to obtain the desired Maharam-type representations of order continuous strictly positive projections in Dedekind complete Riesz spaces and f-algebras.

2 Preliminaries

In this section we will discuss some results concerning Riesz spaces (vector lattices) which will be used in the sequel. For unexplained terminology and the general theory of Riesz spaces and positive linear operators we refer the reader to the books [1], [9], [13] and [16].

Suppose that L is an Archimedean Riesz space and that K is a Riesz subspace of L. We will say that K is *(σ-) regularly embedded* in L if for any net (sequence) $\{f_\tau\}$ in K such that $f_\tau \downarrow 0$ in K, it follows that $f_\tau \downarrow 0$ in L. Furthermore, K will be called a *(σ-) complete Riesz subspace of* L if for any net (sequence) $\{f_\tau\}$ in K and $f \in L$ such that $f_\tau \uparrow f$ in L it follows that $f \in K$. The latter terminology is inspired by the corresponding notions in the theory of Boolean algebras (see Section 23 in [15]). If L is Dedekind (σ-) complete and K is a (σ-) complete Riesz subspace of L, then it is easy to see that K itself is Dedekind (σ-) complete and K is (σ-) regularly embedded in L.

If L is an Archimedean Riesz space and $0 \leq w \in L$ then we will denote by $\mathcal{C}_L(w) = \mathcal{C}(w)$ the set of all *components* of w, i.e.,

$$\mathcal{C}(w) = \{p \in L : p \wedge (w - p) = 0\}.$$

With respect to the ordering induced by L, the set $\mathcal{C}(w)$ is a Boolean algebra. Now we assume that L is Dedekind σ-complete and that w is a weak order unit in L. Then $\mathcal{C}(w)$ is a Boolean σ-algebra, which is isomorphic to the Boolean algebra $\mathcal{P}(L)$ of all band projections in L. The Boolean isomorphism from $\mathcal{P}(L)$ onto $\mathcal{C}(w)$ is given by the mapping $P \mapsto Pw$. Consequently, the Boolean algebra $\mathcal{C}(w)$ is complete if and only if L is Dedekind complete (see [9], Theorem 30.6).

Next we will discuss a construction that will be used frequently in this paper. Let L be a Dedekind σ-complete Riesz space with a fixed weak order unit $0 \leq w \in L$. For $f \in L$ we will denote by $\left\{p_\alpha^f : \alpha \in \mathbb{R}\right\}$ the *left-continuous spectral system of f with respect to w*, i.e., p_α^f is the component of w in the band $\left\{(\alpha w - f)^+\right\}^{dd}$. The *right-continuous spectral system of f with respect to w* is denoted by $\left\{q_\alpha^f : \alpha \in \mathbb{R}\right\}$, i.e., q_α^f is the component of w in the band $\left\{(f - \alpha w)^+\right\}^d$. For the properties of these spectral systems we refer to [9], Chapter 6.

Definition 2.1 *Given a Boolean σ-subalgebra $\mathcal{E}$ of $\mathcal{C}(w)$ we define*

$$L(\mathcal{E}) = \left\{f \in L : p_\alpha^f \in \mathcal{E} \text{ for all } \alpha \in \mathbb{R}\right\}.$$

We will show that $L(\mathcal{E})$ is a σ-complete Riesz subspace of L. For this purpose we need a result that goes back to H. Nakano ([14], Theorem 14.4) and which is of interest in its own right. For the convenience of the reader we will indicate the proof of this result. We need the following lemma.

Lemma 2.2 *For $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$ the following statements hold;*

(i). $p_\alpha^f \wedge p_\beta^g \leq p_{\alpha+\beta}^{f+g}$;

(ii). $p_{\alpha+\beta}^{f+g} \wedge \left(w - p_\alpha^f\right) \leq p_\beta^g$.

Proof.

(i). For all $a, b \in L$ we have $a + b \geq 2(a \wedge b)$ and so $(a+b)^{+} \geq 2(a^{+} \wedge b^{+})$, which implies that

$$\left\{a^{+}\right\}^{dd} \cap \left\{b^{+}\right\}^{dd} \subseteq \left\{(a+b)^{+}\right\}^{dd}.$$

Applying this to $a = \alpha w - f$ and $b = \beta w - g$ we find that

$$\left\{(\alpha w - f)^{+}\right\}^{dd} \cap \left\{(\beta w - g)^{+}\right\}^{dd} \subseteq \left\{((\alpha + \beta) w - (f+g))^{+}\right\}^{dd},$$

from which (i) follows.

(ii). It follows from

$$((\alpha + \beta) w - (f+g))^{+} \leq (\alpha w - f)^{+} + (\beta w - g)^{+}$$

that

$$\left\{((\alpha + \beta) w - (f+g))^{+}\right\}^{dd} \subseteq \left\{(\alpha w - f)^{+}\right\}^{dd} + \left\{(\beta w - g)^{+}\right\}^{dd}$$

and this implies that

$$\left\{((\alpha + \beta) w - (f+g))^{+}\right\}^{dd} \cap \left\{(\alpha w - f)^{+}\right\}^{d} \subseteq \left\{(\beta w - g)^{+}\right\}^{dd}.$$

Since $p_{\alpha+\beta}^{f+g} \wedge \left(w - p_{\alpha}^{f}\right)$ is the component of w in the band

$$\left\{((\alpha + \beta) w - (f+g))^{+}\right\}^{dd} \cap \left\{(\alpha w - f)^{+}\right\}^{d},$$

it is now clear that $p_{\alpha+\beta}^{f+g} \wedge \left(w - p_{\alpha}^{f}\right) \leq p_{\beta}^{g}$.

■

Proposition 2.3 (H. Nakano, [14]) *If $f, g \in L$ and $\lambda \in \mathbb{R}$, then*

$$p_{\lambda}^{f+g} = \sup\left\{p_{\alpha}^{f} \wedge p_{\beta}^{g} : \alpha, \beta \in \mathbb{Q}, \alpha + \beta \leq \lambda\right\}.$$

Proof. Since the set $\mathbb{Q}$ of rationals is countable, the supremum on the right hand side, which we denote by p, exists in L. From (ii) of Lemma 2.2 it is clear that $p \leq p_{\lambda}^{f+g}$. To prove the reverse inequality, take $\mu \in \mathbb{Q}$ such that $\mu \leq \lambda$ and take $k \in \mathbb{N}$. Define $\alpha_n = 2^{-k} n$ for all $n \in \mathbb{Z}$. Since $p_{\alpha_n}^{f} \downarrow 0$ as $n \to -\infty$ and $p_{\alpha_n}^{f} \uparrow w$ as $n \to \infty$, we have

$$w = \bigvee_{n \in \mathbb{Z}} \left(p_{\alpha_{n+1}}^{f} - p_{\alpha_n}^{f}\right) = \bigvee_{n \in \mathbb{Z}} \left(w - p_{\alpha_n}^{f}\right) \wedge p_{\alpha_{n+1}}^{f}$$

in $\mathcal{C}(w)$ and so it follows that

$$p^{f+g}_{\mu-2^{-k}} = p^{f+g}_{\mu-2^{-k}} \wedge w = \bigvee_{n\in\mathbb{Z}} p^{f+g}_{\mu-2^{-k}} \wedge \left(w - p^f_{\alpha_n}\right) \wedge p^f_{\alpha_{n+1}}.$$

Since $\left(\mu - 2^{-k}\right) - \alpha_n = \mu - \alpha_{n+1}$, Lemma 2.2 (ii) implies that

$$p^{f+g}_{\mu-2^{-k}} \wedge \left(w - p^f_{\alpha_n}\right) \leq p^g_{\mu-\alpha_{n+1}}$$

for all $n \in \mathbb{Z}$. Hence,

$$p^{f+g}_{\mu-2^{-k}} \leq \bigvee_{n\in\mathbb{Z}} p^f_{\alpha_{n+1}} \wedge p^g_{\mu-\alpha_{n+1}} \leq p,$$

as $\alpha_{n+1}, \mu - \alpha_{n+1} \in \mathbb{Q}$ and $\alpha_{n+1} + (\mu - \alpha_{n+1}) = \mu \leq \lambda$ for all $n \in \mathbb{Z}$. Since $p^{f+g}_{\mu-2^{-k}} \uparrow p^{f+g}_{\mu}$ as $k \to \infty$, it follows that $p^{f+g}_{\mu} \leq p$. Finally, take a sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathbb{Q}$ such that $\mu_n \uparrow \lambda$. Then $p^{f+g}_{\mu_n} \leq p$ for all n and $p^{f+g}_{\mu_n} \uparrow p^{f+g}_{\lambda}$, so $p^{f+g}_{\lambda} \leq p$. We may conclude that $p^{f+g}_{\lambda} = p$, by which the proof is complete. ■

Proposition 2.4 *Let L be a Dedekind σ-complete Riesz space with weak order unit $0 \leq w \in L$ and suppose that $\mathcal{E}$ is a Boolean σ-subalgebra of $\mathcal{C}(w)$. Then $L(\mathcal{E})$, as introduced in Definition 2.1, is a σ-complete Riesz subspace of L and $\mathcal{C}_{L(\mathcal{E})}(w) = \mathcal{E}$. In particular, $L(\mathcal{E})$ is Dedekind σ-complete and σ-regularly embedded in L.*

Proof. First we observe that $L(\mathcal{E})$ is also given by

$$L(\mathcal{E}) = \left\{f \in L : q^f_\alpha \in \mathcal{E} \text{ for all } \alpha \in \mathbb{R}\right\}.$$

Indeed, this follows immediately from the fact that if $\alpha_n < \alpha$, $\alpha_n \uparrow \alpha$, then $q^f_{\alpha_n} \uparrow p^f_\alpha$ and if $\alpha_n > \alpha$, $\alpha_n \downarrow \alpha$, then $p^f_{\alpha_n} \downarrow q^f_\alpha$. Since $p^{-f}_\alpha = w - q^f_{-\alpha}$ for all $\alpha \in \mathbb{R}$, it is now clear that $f \in L(\mathcal{E})$ if and only if $-f \in L(\mathcal{E})$. Moreover, if $0 < \lambda \in \mathbb{R}$ then $p^{\lambda f}_\alpha = p^f_{\alpha/\lambda}$ and so $f \in L(\mathcal{E})$ implies that $\lambda f \in L(\mathcal{E})$. This shows that $\lambda f \in L(\mathcal{E})$ for all $f \in L(\mathcal{E})$ and all $\lambda \in \mathbb{R}$. Furthermore, it follows from Proposition 2.3 that $f + g \in L(\mathcal{E})$ for all $f, g \in L(\mathcal{E})$, so $L(\mathcal{E})$ is a linear subspace of L. Since $p^{f\vee g}_\alpha = p^f_\alpha \wedge p^g_\alpha$ for all $\alpha \in \mathbb{R}$, we see that $f, g \in L(\mathcal{E})$ implies that $f \vee g$, hence $L(\mathcal{E})$ is a Riesz subspace of L. Now assume that $f_n \in L(\mathcal{E})$ for $n = 1, 2, \ldots$ and that $f \in L$ such that $f_n \downarrow f$ in L. Then $p^{f_n}_\alpha \uparrow p^f_\alpha$ for all $\alpha \in \mathbb{R}$, which implies that $f \in L(\mathcal{E})$. Therefore, $L(\mathcal{E})$ is a σ-complete Riesz subspace of L. Finally, it follows immediately from the definition of $L(\mathcal{E})$ that a component $p \in \mathcal{C}(w)$ belongs to $L(\mathcal{E})$ if and only if $p \in \mathcal{E}$. The proof is complete. ■

We will need some further properties of the space $L(\mathcal{E})$. In particular, we will give an alternative description of the space $L(\mathcal{E})$. For this purpose we give the following definition.

Definition 2.5

$$S(\mathcal{E}) = \left\{ \sum_{j=1}^{n} \alpha_j e_j : \alpha_j \in \mathbb{R}, e_j \in \mathcal{E}, j = 1, ..., n, n \in \mathbb{N} \right\}. \tag{2}$$

It is clear that $S(\mathcal{E})$ is a linear subspace of L. Furthermore, every $s \in S(\mathcal{E})$ can be written as

$$s = \sum_{j=1}^{n} \alpha_j e_j, \text{ with } e_i \wedge e_j = 0 \ (i \neq j), \ \sum_{j=1}^{n} e_j = w. \tag{3}$$

From this observation it follows easily that $S(\mathcal{E})$ is actually a Riesz subspace of L, in which w is a strong order unit. Moreover, since $\mathcal{E} \subseteq L(\mathcal{E})$, it follows that $S(\mathcal{E}) \subseteq L(\mathcal{E})$. For $0 \leq u \in L$ we will denote by P_u the band projection in L onto $\{u\}^{dd}$.

Proposition 2.6 *If L is a Dedekind σ-complete Riesz space with weak order unit $0 \leq w \in L$ and $\mathcal{E}$ is a Boolean σ-subalgebra of $\mathcal{C}(w)$, then the following statements hold.*

(i). If $0 \leq f \in L$, then $f \in L(\mathcal{E})$ if and only if there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in $S(\mathcal{E})$ such that $0 \leq s_n \uparrow f$.

(ii). All the band projections in $L(\mathcal{E})$ are all of the form $(P_e)_{|L(\mathcal{E})}$ and all the bands in $L(\mathcal{E})$ are given by $\{e\}^{dd} \cap L(\mathcal{E})$ with $e \in \mathcal{E}$.

(iii). $L(\mathcal{E})$ is a complete Riesz subspace of L if and only if $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}(w)$.

In particular, if L is Dedekind complete and if $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}(w)$, then $L(\mathcal{E})$ is a complete Riesz subspace of L and hence, $L(\mathcal{E})$ is Dedekind complete and regularly embedded in L.

Proof.

(i). This follows immediately from the Freudenthal spectral theorem ([9], Theorem 40.3) and from the fact that $L(\mathcal{E})$ is a σ-complete Riesz subspace of L.

(ii). If $e \in \mathcal{E}$ and $0 \leq f \in L(\mathcal{E})$ then $f \wedge (ne) \in L(\mathcal{E})$ for all $n = 1, 2, \ldots$ and $f \wedge (ne) \uparrow P_e f$ in L. Since $L(\mathcal{E})$ is a σ-complete Riesz subspace of L, it follows that $P_e f \in L(\mathcal{E})$. This shows that $P_e[L(\mathcal{E})] \subseteq L(\mathcal{E})$ and now it is clear that $(P_e)_{|L(\mathcal{E})}$ is a band projection in $L(\mathcal{E})$. Since

$P_e(e) = e$ and $P_e(f) = 0$ whenever $f \in L(\mathcal{E})$ such that $f \perp e$, it follows that $(P_e)_{|L(\mathcal{E})}$ is the band projection in $L(\mathcal{E})$ onto the band generated by e in $L(\mathcal{E})$. Note that this also shows that the band generated by e in $L(\mathcal{E})$ is equal to $\{e\}^{dd} \cap L(\mathcal{E})$.

Now suppose that Q is a band projection in $L(\mathcal{E})$. Then $e = Qw \in L(\mathcal{E})$ is a component of w in $L(\mathcal{E})$ and so $e \in \mathcal{E}$. From the above we know that $(P_e)_{|L(\mathcal{E})}$ is a band projection in $L(\mathcal{E})$ and that $(P_e)_{|L(\mathcal{E})}(w) = e = Qw$. Since w is a weak order unit in $L(\mathcal{E})$, it follows that $Q = (P_e)_{|L(\mathcal{E})}$. Since all bands in $L(\mathcal{E})$ are principal projection bands, the final statement of (ii) is now obvious.

(iii). First assume that $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}(w)$. Take $f_\tau \in L(\mathcal{E})$ and $f \in L$ such that $f_\tau \downarrow f$ in L. Then $p_\alpha^{f_\tau} \uparrow p_\alpha^f$ in $\mathcal{C}(w)$ for all $\alpha \in \mathbb{R}$. Since $p_\alpha^{f_\tau} \in \mathcal{E}$ and $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}(w)$, it follows that $p_\alpha^f \in \mathcal{E}$ for all $\alpha \in \mathbb{R}$. Hence $f_\tau \in L(\mathcal{E})$. This shows that $L(\mathcal{E})$ is a complete Riesz subspace of L.

Now assume that $L(\mathcal{E})$ is a complete Riesz subspace of L. Let $p_\tau \in \mathcal{E}$ and $p \in \mathcal{C}(w)$ be such that $p_\tau \uparrow p$ in $\mathcal{C}(w)$. Since L is Dedekind σ-complete, this implies that $p_\tau \uparrow p$ in L and so $p \in L(\mathcal{E})$, which implies that $p \in \mathcal{E}$. Hence, $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}(w)$.

The final statement of the proposition is an immediate consequence of (iii). ■

Now we will discuss a number of results related to order separability of a Riesz space. Recall that the Archimedean Riesz space L is called *order separable* if every non-empty subset $D \subseteq L$ for which $\sup D = g_0$ exists has an at most countable subset $\{f_n\} \subseteq D$ such that $\sup_n f_n = g_0$ (see [9], Definition 23.1). As is well known, L is order separable if and only if every order bounded disjoint system in L is at most countable ([9], Theorem 29.3). The following result will be useful. Recall that a positive operator T from a Riesz space L into a Riesz space M is called *strictly positive* whenever $0 \leq u \in L$ and $Tu = 0$ imply that $u = 0$.

Lemma 2.7 *Let L and M be Archimedean Riesz spaces and suppose that M is order separable. Let $0 \leq T : L \to M$ be a strictly positive linear operator. Then L is order separable.*

Proof. Since the Dedekind completion of M is order separable as well (see [9], Theorem 32.9), we may assume without loss of generality that M is Dedekind complete. Take $0 < u \in L$ and let $\{u_\tau : \tau \in \mathbb{T}\}$ be a disjoint system in $[0,u]$. Let $\mathbb{S}$ be the collection of all finite subsets of $\mathbb{T}$, directed by inclusion. For $\sigma \in \mathbb{S}$ we define

$$v_\sigma = \sum_{\tau \in \sigma} u_\tau .$$

Then $0 \leq v_\sigma \uparrow \leq u$ and so $0 \leq Tv_\sigma \uparrow \leq Tu$ in M. Let $w = \sup_\sigma Tv_\sigma$. Since M is order separable, there exists an at most countable collection $\{\sigma_k\}$ in $\mathbb{S}$ such that $w = \sup_k Tv_{\sigma_k}$. Define $\mathbb{T}_0 = \bigcup_k \sigma_k$ and suppose that there exists $\tau \in \mathbb{T}$ such that $\tau \notin \mathbb{T}_0$. Defining $\sigma_k' = \sigma_k \cup \{\tau\}$ for all $k = 1, 2, \ldots$ we find that

$$Tv_{\sigma_k} + Tu_\tau = Tv_{\sigma_k'} \leq w$$

for all k and hence $w + Tu_\tau \leq w$. This implies that $Tu_\tau = 0$, and so $u_\tau = 0$, which is a contradiction. Consequently, $\mathbb{T} = \mathbb{T}_0$ and this shows that the disjoint system $\{u_\tau\}$ is at most countable. We may conclude that L is order separable. ■

Next we will discuss a property of Riesz spaces which is weaker than order separability.

Definition 2.8 *Let L be an Archimedean Riesz space.*

(i). An element $0 < u \in L$ will be called ***order separable*** *if the principal ideal generated by u is order separable.*

(ii). The space L is called ***locally order separable*** *if there exists a maximal disjoint system $\{u_\tau\}$ in L consisting of order separable elements.*

In the following lemmas we will present several characterizations of locally order separable spaces. The proof of the following lemma is straightforward and therefore omitted.

Lemma 2.9 *If L is an Archimedean Riesz space and $0 < u \in L$, then the following statements are equivalent:*

(i). u is order separable;

(ii). the band generated by u is order separable;

(iii). every disjoint system in $[0,u]$ is at most countable.

Lemma 2.10 *For an Archimedean Riesz space L the following statements are equivalent:*

(i). L is locally order separable;

(ii). there exists an order separable ideal I in L such that $I^d = \{0\}$;

(iii). for every $0 < v \in L$ there exists $0 < u \leq v$ such that u is order separable.

Proof. First assume that L is locally order separable and let $\{u_\tau\}$ be a maximal disjoint system in L consisting of order separable elements. Let I be the ideal generated by $\{u_\tau\}$. It is clear that $I^d = \{0\}$. We claim that I is order separable. Indeed, suppose that $\{v_\sigma\}$ is an order bounded disjoint system in I, so there exists $0 < w \in I$ such that $0 \leq v_\sigma \leq w$ for all σ. Now $0 < w \in I$ implies that there exist $\tau_1, \dots, \tau_n$ such that $0 < w \leq \alpha_1 u_{\tau_1} + \cdots + \alpha_2 u_{\tau_2}$ for some $0 \leq \alpha_1, \dots, \alpha_n \in \mathbb{R}$. For each j we have $v_\sigma \wedge u_{\tau_j} \neq 0$ for at most countably many σ and consequently $\{v_\sigma\}$ is at most countable.

To proof that (ii) implies (iii), assume that I is an order separable ideal in L such that $I^d = \{0\}$. If $0 < v \in L$, then there exists $u \in I$ such that $0 < u \leq v$ and since I is order separable, it is clear that u is order separable.

Finally we show that (iii) implies (i). By Zorn's lemma, there exists a maximal system $\{u_\tau\}$ of mutually disjoint order separable elements. Assuming that $\{u_\tau\}$ is not a maximal disjoint system in L, there exists $0 < v \in L$ such that $v \wedge u_\tau = 0$ for all τ. By hypothesis, there exists an order separable element u such that $0 < u \leq v$ and this contradicts the maximality of $\{u_\tau\}$. Hence $\{u_\tau\}$ is a maximal disjoint system in L consisting of order separable elements, i.e., L is locally order separable. ■

The following simple observation will be useful.

Lemma 2.11 *Let L be a Dedekind σ-complete Riesz space with weak order unit $0 < w \in L$ and let $\mathcal{C}(w)$ be the Boolean algebra of components of w. Then the following two statements are equivalent.*

(i). L is locally order separable.

(ii). For every $0 < e \in \mathcal{C}(w)$ there exists $p \in \mathcal{C}(w)$ such that $0 < p \leq e$ and $\{p\}^{dd}$ is order separable.

Proof. First we show that (i) implies (ii). Given $0 < e \in \mathcal{C}(w)$ it follows from the above lemma that there exists an order separable element u such that $0 < u \leq e$. Let $p \in \mathcal{C}(w)$ be the component of w in $\{u\}^{dd}$. Then $\{p\}^{dd} = \{u\}^{dd}$ and by Lemma 2.9 the band $\{u\}^{dd}$ is order separable.

Now assume that (ii) holds and let $0 < v \in L$ be given. Let e be the component of w in $\{v\}^{dd}$. By hypothesis, there exists $p \in \mathcal{C}(w)$, $0 < p \leq e$

such that $\{p\}^{dd}$ is order separable. Let u be the component of v in $\{p\}^{dd}$. Then u is order separable and $0 < u \leq v$. By Lemma 2.10 we may conclude that L is locally order separable. ■

Example 2.12 *(a). Let L be an Archimedean Riesz space such that ${}^{\perp}(L_n^{\sim}) = \{0\}$. We claim that L is locally order separable. Indeed, let $\{\varphi_\tau : \tau \in \mathbb{T}\}$ be a maximal disjoint system in $L_n^{\sim}$. Since φ_τ is strictly positive on its carrier C_τ, it is easy to see that C_τ is order separable (this is actually a special case of Lemma 2.7). Let I be the ideal generated by $\{C_\tau : \tau \in \mathbb{T}\}$. Then $I^d = \{0\}$ and I is order separable, hence L is locally order separable. As is the case in this example, if an Archimedean Riesz space has an order dense ideal on which a strictly positive functional exists, then L is locally order separable. In particular, the condition that ${}^{\perp}(L_n^{\sim}) = \{0\}$ may be weakened to the requirement that the extended order dual separates the points of L (see [6]).*

(b). For an example of a locally order separable space which is not order separable, let (X, Σ, μ) be a localizable measure space with the finite subset property which is not σ-finite and take $L = L_\infty(\mu)$. Then the ideal $I = L_\infty(\mu) \cap L_1(\mu)$ is order separable and satisfies $I^d = \{0\}$, so L is locally order separable. But L is not order separable since (X, Σ, μ) is not σ-finite.

(c). There exist Riesz spaces which are not locally order separable. Indeed, consider the quotient space $L = l_\infty / c_0$; for $u \in l_\infty$ we denote the corresponding equivalence class in l_∞ / c_0 by $[u]$. We will show that for every $0 < [u] \in L$ there exists an uncountable disjoint system $\{[v_\tau]\}$ such that $0 < [v_\tau] \leq [u]$ for all τ. We may assume that $0 < u \in l_\infty$. Since $u \notin c_0$, there exists $\varepsilon > 0$ such that the set $F = \{n \in \mathbb{N} : u(n) \geq \varepsilon\}$ is infinite. Then there exists an uncountable collection $\{F_\tau\}$ of infinite subsets of F such that $F_\tau \cap F_{\tau'}$ is finite whenever $\tau \neq \tau'$. Defining $v_\tau = \varepsilon \mathbf{1}_{F_\tau}$, it is easy to verify that $\{[v_\tau]\}$ has the desired properties. Observe that this actually shows that the only order separable ideal in L is $\{0\}$. In particular, the carrier of every positive functional on L is $\{0\}$. In this connection we mention that l_∞ / c_0 is not Dedekind σ-complete and has σ-order continuous norm (reflecting the fact that every non-empty G_δ-set in $\beta\mathbb{N} \diagdown \mathbb{N}$ has non-empty interior). This example also illustrates that, in contrast to order separability, local order separability is in general not inherited by Riesz subspaces. Indeed, identifying L with the space $C(X)$, where $X = \beta\mathbb{N} \diagdown \mathbb{N}$, we have $L \subseteq \mathbb{R}^X$ as a (σ-regular) Riesz subspace and $\mathbb{R}^X$ is clearly locally order separable. Observe that

this example also shows that the analogue of Lemma 2.7 fails for locally order separable spaces (take for T the embedding of L into $M = \mathbb{R}^X$). However, it is not difficult to show that an Archimedean Riesz space is locally order separable if and only if its Dedekind completion is locally order separable.

We end this section with recalling some facts concerning order convergence. A net $\{f_\alpha\}_{\alpha\in\mathbb{A}}$ in the Archimedean Riesz space L is called *order convergent* to the element $f \in L$ if there exists a net $\{u_\beta\}_{\beta\in\mathbb{B}}$ in L satisfying $u_\beta \downarrow 0$ such that for every $\beta \in \mathbb{B}$ there exists $\alpha_\beta \in \mathbb{A}$ such that $|f - f_\alpha| \leq u_\alpha$ for all $\alpha \geq \alpha_\beta$. This will be denoted by $f_\alpha \xrightarrow{(o)} f$. A subset $G \subseteq L$ is called *order closed* if $\{f_\alpha\} \subseteq G$ and $f_\alpha \xrightarrow{(o)} f \in L$ imply that $f \in G$. The order closed subsets of L are the closed subsets of a topology τ_o in L, which is called the *order topology*. The τ_o-closure of a subset $D \subseteq L$ is denoted by $\overline{D}^{(o)}$ and is called the *order closure* of D. Although τ_o is in general not a vector space topology, it is not difficult to see that the order closure of any Riesz subspace of L is a Riesz subspace. If K is an order closed Riesz subspace of L, then it is clear that K is a complete Riesz subspace of L. Moreover, if L is Dedekind complete, then a Riesz subspace K is order closed if and only if K is a complete Riesz subspace of L. In particular, if K is an order closed Riesz subspace of the Dedekind complete space L, then K itself is Dedekind complete and K is regularly embedded in L.

Now we assume that L is a Dedekind complete Riesz space and let $\{f_\alpha : \alpha \in \mathbb{A}\}$ be a net in L. As is easily verified, $f_\alpha \xrightarrow{(o)} f$ if and only if there exists $\alpha_0 \in \mathbb{A}$ such that $\{f_\alpha : \alpha \geq \alpha_0\}$ is order bounded and

$$\bigvee_{\gamma\geq\alpha_0} \bigwedge_{\alpha\geq\gamma} f_\alpha = \bigwedge_{\gamma\geq\alpha_0} \bigvee_{\alpha\geq\gamma} f_\alpha = f. \tag{4}$$

Note that it follows in particular from this characterization of order convergence that $f_\alpha \xrightarrow{(o)} f$ if and only if there exist $\alpha_0 \in \mathbb{A}$ and a net $\{u_\alpha : \alpha \geq \alpha_0\}$ in L such that $u_\alpha \downarrow 0$ and $|f_\alpha - f| \leq u_\alpha$ for all $\alpha \geq \alpha_0$. The following characterization of order convergence will be useful. For a net $\{f_\alpha : \alpha \in \mathbb{A}\}$ in L we will also consider the net $\{f_\alpha - f_{\alpha'} : (\alpha, \alpha') \in \mathbb{A} \times \mathbb{A}\}$, where $\mathbb{A} \times \mathbb{A}$ is given the product order.

Lemma 2.13 *A net $\{f_\alpha\}$ in the Dedekind complete Riesz space L is order convergent if and only if $f_\alpha - f_{\alpha'} \xrightarrow{(o)} 0$.*

Proof. If $f_\alpha \xrightarrow{(o)} f$, then it is clear that $f_\alpha - f_{\alpha'} \xrightarrow{(o)} 0$. Now suppose that $f_\alpha - f_{\alpha'} \xrightarrow{(o)} 0$, i.e., there exist a net $\{u_\beta\}_{\beta\in\mathbb{B}}$ in L satisfying $u_\beta \downarrow 0$ such that for every $\beta \in \mathbb{B}$ there exists $(\alpha_\beta, \alpha'_\beta) \in \mathbb{A}\times\mathbb{A}$ such that $|f_\alpha - f_{\alpha'}| \le u_\beta$ whenever $(\alpha,\alpha') \ge (\alpha_\beta,\alpha'_\beta)$. Replacing α_β and α'_β by some $\alpha''_\beta \ge \alpha_\beta, \alpha'_\beta$, we may assume without loss of generality that $\alpha_\beta = \alpha'_\beta$. Fix some $\beta_0 \in \mathbb{B}$. It is clear that $\{f_\alpha : \alpha \ge \alpha_{\beta_0}\}$ is order bounded. Moreover,

$$0 \le \bigvee_{\alpha\ge\alpha_\beta} f_\alpha - \bigwedge_{\alpha'\ge\alpha_\beta} f_{\alpha'} = \bigvee_{\alpha,\alpha'\ge\alpha_\beta} |f_\alpha - f_{\alpha'}| \le u_\beta$$

for all $\beta \ge \beta_0$. Since $u_\beta \downarrow 0$, it follows that

$$\bigvee_{\beta\ge\beta_0}\bigwedge_{\alpha\ge\alpha_\beta} f_\alpha = \bigwedge_{\beta\ge\beta_0}\bigvee_{\alpha\ge\alpha_\beta} f_\alpha,$$

and this implies that (4) holds with $\alpha_0 = \alpha_{\beta_0}$, so $\{f_\alpha\}$ is order convergent. ∎

Next we will prove a simple extension result for positive operators which will be used in a later section of the paper. Given a Riesz subspace K of the Riesz space L we will denote by K' the set of all $f \in L$ for which there exists a net $\{f_\alpha\}$ in K such that $f_\alpha \xrightarrow{(o)} f$ in L. It is easy to see that K' is a Riesz subspace of L.

Lemma 2.14 *Let K be a Riesz subspace of the Archimedean Riesz space L and let M be a Dedekind complete Riesz space. Suppose that $0 \le T : K \to M$ is a positive linear operator with the property that $Tf_\alpha \xrightarrow{(o)} 0$ in M whenever $\{f_\alpha\}$ is a net in K satisfying $f_\alpha \xrightarrow{(o)} 0$ in L. Then there exists a unique positive linear operator $0 \le T' : K' \to M$ such that for every $f \in K'$ and any net $\{f_\alpha\} \subseteq K$ with $f_\alpha \xrightarrow{(o)} f$ we have $Tf_\alpha \xrightarrow{(o)} T'f$ (in particular, $T'_{|K} = T$).*

Proof. Take $f \in K'$ and let $\{f_\alpha\}$ be a net in K such that $f_\alpha \xrightarrow{(o)} f$ in L. Then $f_\alpha - f_{\alpha'} \xrightarrow{(o)} 0$ in L and so, by hypothesis, $Tf_\alpha - Tf_{\alpha'} \xrightarrow{(o)} 0$ in M. Now it follows from Lemma 2.13 that $\{Tf_\alpha\}$ is order convergent in M, i.e., $Tf_\alpha \xrightarrow{(o)} g$ for some $g \in M$. We claim that g does not depend on the choice of the particular net $\{f_\alpha\}$. Indeed, suppose that $\{h_\beta\}$ is any other net in K such that $h_\beta \xrightarrow{(o)} f$ in L. Then $f_\alpha - h_\beta \xrightarrow{(o)} 0$ in L and by the assumption on T it follows that $Tf_\alpha - Th_\beta \xrightarrow{(o)} 0$ in M. This implies that $Th_\beta \xrightarrow{(o)} g$. Defining $T'f = g$, it is easy to verify that $T' : K' \to M$ is a positive linear operator which has, by definition, the desired property. The uniqueness is obvious. ∎

3 Representations

Many of the familiar Dedekind (σ-) complete Riesz spaces arise as quotient spaces of ideals of measurable functions by a σ-ideal (usually, the null functions with respect to some measure). In this section we discuss the representation of Dedekind σ-complete Riesz spaces as such quotient spaces. First we introduce some notation. Given a measurable space (X, Σ) (i.e., X is a non-empty set and Σ is a σ-algebra of subsets of X), we will denote by $M(X, \Sigma)$ the space of all real-valued Σ-measurable functions on X and $M_b(X, \Sigma)$ denotes the ideal in $M(X, \Sigma)$ consisting of all bounded functions. The characteristic function of a set $F \subseteq X$ will be denoted by $\mathbf{1}_F$.

Definition 3.1 *Let L be a Dedekind σ-complete Riesz space with weak order unit $0 < w \in L$. The measurable space (X, Σ) is called **a representation space for** (L, w) if there exist an ideal $\widehat{L}$ in $M(X, \Sigma)$ such that $M_b(X, \Sigma) \subseteq \widehat{L}$ and a surjective σ-order continuous Riesz homomorphism $\Phi : \widehat{L} \to L$ with $\Phi(\mathbf{1}_X) = w$.*

Given such a homomorphism $\Phi : \widehat{L} \to L$ as in the above definition, we will sometimes call Φ *the representation* (corresponding to (X, Σ)). It is clear that the kernel $\mathrm{Ker}(\Phi)$ is a σ-ideal in $\widehat{L}$. Hence Φ induces a Riesz isomorphism from $\widehat{L} / \mathrm{Ker}\,(\Phi)$ onto L.

The proof of the first result in this section will be divided into two lemmas. The *principal ideal generated by an element* $0 \leq w \in L$ will be denoted by L_w.

Lemma 3.2 *Let L be a Dedekind σ-complete Riesz space with weak order unit $0 < w \in L$. If (X, Σ) is a representation space for (L_w, w), then (X, Σ) is also a representation space for (L, w).*

Proof. Since (X, Σ) is a representation space for (L_w, w), it follows that there exists a surjective σ-order continuous Riesz homomorphism $\Phi_b : M_b(X, \Sigma) \to L_w$ with $\Phi_b\,(\mathbf{1}_X) = w$. Define

$$\widehat{L} = \left\{ g \in M(X, \Sigma) : \{\Phi_b\,(|g| \wedge n\mathbf{1}_X)\}_{n=1}^{\infty} \text{ is order bounded in } L \right\}. \tag{5}$$

It is easy to see that $\widehat{L}$ is an ideal in $M(X, \Sigma)$ and $M_b(X, \Sigma) \subseteq \widehat{L}$. For $0 \leq g \in \widehat{L}$ we define

$$\Phi(g) = \sup_{n \geq 1} \Phi_b\,(g \wedge n\mathbf{1}_X)\,.$$

The following properties of the mapping $\Phi : \widehat{L}^+ \to L^+$ are readily verified:

(a). $\Phi(g) = \Phi_b(g)$ for all $0 \le g \in M_b(X, \Sigma)$.

(b). $\Phi(g_1 + g_2) = \Phi(g_1) + \Phi(g_2)$ for all $0 \le g_1, g_2 \in \widehat{L}$.

(c). $\Phi(\lambda g) = \lambda \Phi(g)$ for all $0 \le g \in \widehat{L}$ and all $0 \le \lambda \in \mathbb{R}$.

(d). If $0 \le g_1, g_2 \in \widehat{L}$ are such that $g_1 \wedge g_2 = 0$, then $\Phi(g_1) \wedge \Phi(g_2) = 0$ in L.

Consequently, Φ has a unique extension to a positive linear operator from $\widehat{L}$ into L, which we will denote by Φ again. From (d) it follows that Φ is a Riesz homomorphism. Furthermore, Φ is σ-order continuous. Indeed, suppose that $0 \le g_n \uparrow g$ in $\widehat{L}$. For every $k \in \mathbb{N}$ we have $g_n \wedge k\mathbf{1}_X \uparrow_n g \wedge k\mathbf{1}_X$ in $M_b(X, \Sigma)$ and since Φ_b is σ-order continuous this implies that $\Phi_b\left(g_n \wedge k\mathbf{1}_X\right) \uparrow_n \Phi_b\left(g \wedge k\mathbf{1}_X\right)$ in L. Hence,

$$\Phi(g) = \sup_k \Phi_b\left(g \wedge k\mathbf{1}_X\right) = \sup_{k,n} \Phi_b\left(g_n \wedge k\mathbf{1}_X\right) = \sup_n \Phi(g_n).$$

It remains to show that Φ is surjective. To this end, let $0 \le f \in L$ be given. For every $1 \le n \in \mathbb{N}$ there exists $0 \le g_n \in M_b(X, \Sigma)$ such that $\Phi_b(g_n) = f \wedge nw$. Since Φ_b is a Riesz homomorphism we may assume that $0 \le g_n \uparrow$. Let

$$F = \left\{x \in X : g_n(x) \uparrow_n \infty\right\}.$$

For every $m \in \mathbb{N}$ we have $(m\mathbf{1}_F) \wedge g_n \uparrow_n m\mathbf{1}_F$ and since Φ_b is σ-order continuous it follows that

$$m\Phi_b(\mathbf{1}_F) \wedge f \wedge nw \uparrow_n m\Phi_b(\mathbf{1}_F).$$

This implies that $0 \le m\Phi_b(\mathbf{1}_F) \le f$ for all $m \in \mathbb{N}$ and so $\Phi_b(\mathbf{1}_F) = 0$. Now define $h_n = g_n \mathbf{1}_{X \setminus F}$. Then $\Phi_b(h_n) = \Phi_b(g_n) = f \wedge nw$ for all n and $0 \le h_n \uparrow h \in M(X, \Sigma)$. Since

$$\Phi_b(h \wedge k\mathbf{1}_X) = \sup_n \Phi_b(h_n \wedge k\mathbf{1}_X) \le f$$

for all k, it is clear that $h \in \widehat{L}$. Using that Φ is σ-order continuous and that w is a weak order unit in L, we find that

$$\Phi(h) = \sup_n \Phi(h_n) = \sup_n f \wedge nw = f,$$

which completes the proof of the lemma. ■

Lemma 3.3 *Given a Dedekind σ-complete Riesz space L with weak order unit $0 < w \in L$, there exists a representation space (X, Σ) for (L_w, w).*

Proof. As before, we denote by $\mathcal{C}(w)$ the Boolean algebra of all components of w. It follows from the Loomis-Sikorski theorem (see [15], Theorem 29.1) that there exist a measurable space (X, Σ) and a surjective Boolean σ-homomorphism $\theta : \Sigma \to \mathcal{C}(w)$. We denote by $\mathrm{sim}(X, \Sigma)$ the space of all simple functions on (X, Σ), i.e.,

$$\mathrm{sim}(X, \Sigma) = \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{1}_{D_i} : \alpha_i \in \mathbb{R}, D_i \in \Sigma, i = 1, ..., n, n \in \mathbb{N} \right\}.$$

By $S(\mathcal{C}(w))$ we denote the Riesz subspace of L defined by (2). Define the mapping $\Phi_0 : \mathrm{sim}(X, \Sigma) \to S(\mathcal{C}(w))$ by

$$\Phi_0 \left(\sum_{i=1}^{n} \alpha_i \mathbf{1}_{D_i} \right) = \sum_{i=1}^{n} \alpha_i \theta(D_i).$$

It follows via standard arguments that Φ_0 is a well-defined σ-order continuous surjective Riesz homomorphism. Now Φ_0 extends uniquely to a σ-order continuous Riesz homomorphism $\Phi_b : M_b(X, \Sigma) \to L_w$. Indeed, given $0 \leq g \in M_b(X, \Sigma)$, take a sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathrm{sim}(X, \Sigma)$ such that $0 \leq s_n \uparrow g$ in $M_b(X, \Sigma)$. Since $0 \leq g \leq k\mathbf{1}_X$ for some $k \in \mathbb{N}$, we have $0 \leq \Phi_0(s_n) \uparrow \leq kw$ in L_w and so $\Phi_b(g) = \sup_n \Phi_0(s_n)$ exists in L_w. It follows easily from the σ-order continuity of Φ_0 that this definition is independent of the choice of the particular sequence $\{s_n\}$ and that Φ_b is a σ-order continuous Riesz homomorphism. Moreover, Φ_b is surjective. Indeed, let $0 \leq f \leq w$ be given in L_w. By the Freudenthal spectral theorem there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $S(\mathcal{C}(w))$ such that $0 \leq f_n \uparrow f$. For each n there exists $s_n \in \mathrm{sim}(X, \Sigma)$ such that $\Phi_0(s_n) = f_n$. Since Φ_0 is a Riesz homomorphism, we may assume that $0 \leq s_n \uparrow \leq \mathbf{1}_X$, and so $0 \leq s_n \uparrow g \in M_b(X, \Sigma)$. From the definition of Φ_b it is clear that $\Phi_b(g) = f$, showing that Φ_b is surjective. Consequently, (X, Σ) is a representation space for L_w. ■

Combining the results of Lemmas 3.2 and 3.3 we get the following proposition.

Proposition 3.4 *Given a Dedekind σ-complete Riesz space L with weak order unit $0 < w \in L$, there exists a representation space (X, Σ) for (L, w).*

In the following corollary we will denote by L^u the *universal completion* of a Riesz space L (see e.g. [9], Definition 50.4).

Corollary 3.5 *Let (X, Σ) be a representation space for the Dedekind complete Riesz space L with weak order unit $0 \le w \in L$. Then (X, Σ) is a representation space for (L^u, w) as well and we may actually take $\widehat{L^u} = M(X, \Sigma)$.*

Proof. Let $\widehat{L}$ be an ideal such that $M_b(X, \Sigma) \subseteq L \subseteq M(X, \Sigma)$ and $\Phi : \widehat{L} \to L$ a surjective σ-order continuous Riesz homomorphism. Let $\Phi_b = \Phi_{|M_b(X,\Sigma)}$, then $\Phi_b : M_b(X, \Sigma) \to L_w$ is a surjective σ-order continuous Riesz homomorphism as well. Since L is Dedekind complete, $(L^u)_w = L_w$ and so it follows immediately from Lemma 3.2 that (X, Σ) is a representation space for (L^u, w). Taking $\widehat{L^u}$ as defined by (5) we have $\widehat{L^u} = M(X, \Sigma)$. Indeed, let $0 \le g \in M(X, \Sigma)$ be given and let $\{g_k\}_{k=1}^{\infty}$ be a disjoint sequence in $M_b(X, \Sigma)$ such that $\sup_k g_k = g$. Then $\{\Phi_b(g_k)\}_{k=1}^{\infty}$ is a disjoint system in L and so $\sup_k \Phi_b(g_k) = f \in L^u$ exists. Since Φ_b is a σ-order continuous Riesz homomorphism it follows that

$$\Phi_b(g \wedge n\mathbf{1}_X) = \Phi_b\left(\sup_k (g_k \wedge n\mathbf{1}_X)\right) = \sup_k \Phi_b(g_k \wedge n\mathbf{1}_X) \le f$$

for all $n \in \mathbb{N}$ and so $g \in \widehat{L^u}$. Note that, by σ-order continuity, the corresponding Riesz homomorphism $\Phi^u : M(X, \Sigma) \to L^u$ coincides with Φ on $\widehat{L}$. ■

Next we make some remarks concerning the relation between σ-ideal in ideals in $M(X, \Sigma)$ and Boolean σ-ideals in Σ. We introduce some further notation. Given a σ-ideal $\mathfrak{N} \subseteq \Sigma$ we write

$$M(\mathfrak{N}) = \{f \in M(X, \Sigma) : \{x \in X : f(x) \ne 0\} \in \mathfrak{N}\}$$

and $M_b(\mathfrak{N}) = M(\mathfrak{N}) \cap M_b(X, \Sigma)$. Then $M(\mathfrak{N})$ and $M_b(\mathfrak{N})$ are σ-ideals in $M(X, \Sigma)$ and $M_b(X, \Sigma)$ respectively. Now assume that E is an ideal in $M(X, \Sigma)$ such that $M_b(X, \Sigma) \subseteq E$. It is clear that $E \cap M(\mathfrak{N})$ is a σ-ideal in E whenever $\mathfrak{N}$ is a σ-ideal in Σ. We claim that all σ-ideals in E are of this form. Indeed, given a σ-ideal N in E we define $\mathfrak{N} = \{F \in \Sigma : \mathbf{1}_F \in N\}$ and we will show that $N = E \cap M(\mathfrak{N})$. Take $f \in N$ and let $F = \{x \in X : f(x) \ne 0\}$. Then $n|f| \wedge \mathbf{1}_X \in N$ for all $n \in \mathbb{N}$ and $n|f| \wedge \mathbf{1}_X \uparrow \mathbf{1}_F$, so $\mathbf{1}_F \in N$, i.e., $F \in \mathfrak{N}$ and hence $f \in E \cap M(\mathfrak{N})$. Conversely, if $f \in E$ and $F = \{x \in X : f(x) \ne 0\} \in \mathfrak{N}$, then $|f| \wedge n\mathbf{1}_F \in N$ for all $n \in \mathbb{N}$ and $|f| \wedge n\mathbf{1}_F \uparrow |f|$, so $|f| \in N$ and hence $f \in N$.

Assume that L is a Dedekind σ-complete Riesz space with weak order unit $0 \le w \in L$ and that (X, Σ) is a representation space for (L, w) with representation $\Phi : \widehat{L} \to L$, where $M_b(X, \Sigma) \subseteq \widehat{L} \subseteq M(X, \Sigma)$ is an ideal. Then $\mathrm{Ker}\,(\Phi)$ is a σ-ideal in $\widehat{L}$ and it follows from the above observations that $\mathrm{Ker}(\Phi) = \widehat{L} \cap M(\mathfrak{N}_\Phi)$, where the σ-ideal $\mathfrak{N}_\Phi$ is given by $\mathfrak{N}_\Phi = \{F \in \Sigma : \Phi(\mathbf{1}_F)\} = 0$. Consequently, L is Riesz isomorphic to $\widehat{L} \big/ \widehat{L} \cap M(\mathfrak{N}_\Phi)$. Furthermore it is easy to see that the following three conditions are equivalent:

(a). if $f_n \in \widehat{L}$ $(n = 1, 2, \dots)$ and $f \in M(X, \Sigma)$ such that $0 \leq f_n \uparrow f$ and if $\Phi(f_n) \leq h$ for all n and some $h \in L$, then $f \in \widehat{L}$;

(b). $M(\mathfrak{N}_\Phi) \subseteq \widehat{L}$;

(c). if $f \in \widehat{L}$ and $g \in M(X, \Sigma)$ such that $f = g$ $\mathfrak{N}_\Phi$-a.e. (i.e., f and g are equal except on a set belonging to $\mathfrak{N}_\Phi$), then $g \in \widehat{L}$.

Note that the ideal $\widehat{L}$ constructed in Lemma 3.2 has these properties.

Now we introduce some terminology concerning representations of positive projections. We assume that L is an Archimedean Riesz space and that $P : L \to L$ is a *positive projection* (i.e., P is a positive linear operator such that $P^2 = P$). Furthermore we assume that $K = \mathrm{Ran}(P)$ is a Riesz subspace of L. First a simple observation.

Lemma 3.6 *If in addition P is (σ-) order continuous, then K is a (σ-) complete Riesz subspace of L. In particular, if L is Dedekind (σ-) complete, then K is Dedekind (σ-) complete and (σ-) regularly embedded in L.*

Proof. We assume that P is order continuous and that $\{f_\tau\} \subseteq K$ and $f \in L$ are such that $f_\tau \uparrow f$ in L. This implies that $f_\tau = Pf_\tau \uparrow Pf$ in L and hence $f = Pf \in K$, which shows that K is a complete Riesz subspace of L. ■

Given two measurable spaces (X, Σ) and (Y, Λ) we denote by $\mathcal{F} = \Sigma \otimes \Lambda$ the product σ-algebra of Σ and Λ in the product space $\Omega = X \times Y$, i.e., $\mathcal{F}$ is the σ-algebra generated by all rectangles $F \times G$ with $F \in \Sigma$ and $G \in \Lambda$. If $f \in M(X, \Sigma)$ then we define $f \otimes \mathbf{1}_Y \in M(\Omega, \mathcal{F})$ by $(f \otimes \mathbf{1}_Y)(x, y) = f(x)$ for all $(x, y) \in \Omega$. The mapping $f \longmapsto f \otimes \mathbf{1}_Y$ is an f-algebra isomorphism from $M(X, \Sigma)$ onto the σ-complete f-subalgebra $M(X, \Sigma) \otimes \mathbf{1}_Y$ of $M(\Omega, \mathcal{F})$, where

$$M(X, \Sigma) \otimes \mathbf{1}_Y = \{f \otimes \mathbf{1}_Y : f \in M(X, \Sigma)\}.$$

It will be convenient in the sequel to identify $M(X, \Sigma)$ with this f-subalgebra $M(X, \Sigma) \otimes \mathbf{1}_Y$. Similarly, $M(Y, \Lambda)$ will be identified with the σ-complete f-subalgebra $\mathbf{1}_X \otimes M(Y, \Lambda)$ of $M(\Omega, \mathcal{F})$. Note that it is implicit in these identifications that the σ-algebra Σ is identified with the σ-subalgebra Σ_1

of $\mathcal{F}$ given by $\Sigma_1 = \{F \times Y : F \in \Sigma\}$, and similarly Λ is identified with the corresponding σ-subalgebra Λ_1 of $\mathcal{F}$. Assuming in addition that μ is a σ-finite measure on (Y, Λ) and that $0 \leq m \in M(\Omega, \mathcal{F})$ satisfies

$$\int_Y m(x,y)\, d\mu(y) = 1 \ \ \forall x \in X, \tag{6}$$

we define the linear operator $R_b : M_b(\Omega, \mathcal{F}) \to M_b(\Omega, \mathcal{F})$ by

$$R_b f(x,y) = \int_Y m(x,z)\, f(x,z)\, d\mu(z), \quad (x,y) \in \Omega, \tag{7}$$

for all $f \in M_b(\Omega, \mathcal{F})$. Then R_b is a σ-order continuous positive projection onto $M_b(X, \Sigma)$.

Definition 3.7 *Let L be a Dedekind σ-complete Riesz space with weak order unit $0 < w \in L$ and suppose that $P : L \to L$ is a positive projection onto the Riesz subspace $K \subseteq L$ with $w \in K$. Give a measurable space (X, Σ) and a σ-finite measure space (Y, Λ, μ), we will say that the product space $(\Omega, \mathcal{F}) = (X \times Y, \Sigma \otimes \Lambda)$ is a* ***representation space*** *for P if:*

(i). $(\Omega, \mathcal{F})$ is a representation space for (L, w) with representation homomorphism $\Phi : \widehat{L} \to L$;

(ii). there exists $0 \leq m \in M(\Omega, \mathcal{F})$ satisfying $\int_Y m(x,y)\, d\mu(y) = 1$ for all $x \in X$;

(iii). if $f \in \widehat{L}$, then

$$\int_Y m(x,y)\, |f(x,y)|\, d\mu(y) < \infty$$

for all $x \in X$ and the function Rf on Ω, defined by

$$Rf(x,y) = \int_Y m(x,z)\, f(x,z)\, d\mu(z)$$

for all $(x,y) \in \Omega$, satisfies $Rf \in \widehat{L}$;

(iv). $P(\Phi f) = \Phi(Rf)$ for all $f \in \widehat{L}$.

If, in addition, μ is a probability measure and $m(x,y) = 1$ for all $(x,y) \in \Omega$, then $(\Omega, \mathcal{F})$ will be called a ***proper representation space*** *for P.*

It is readily verified that the existence of a representation space for a projection P implies that P is σ-order continuous and so we will restrict our attention to such projections only.

Remark 3.8 *Suppose that* $(\Omega, \mathcal{F}) = (X \times Y, \Sigma \otimes \Lambda)$ *is a representation space for the positive projection* $P : L \to L$ *and* $K = \operatorname{Ran}(P)$ *as in the above definition. Define* $\widehat{L}_X = \widehat{L} \cap M(X, \Sigma)$*. Then* (X, Σ) *is a representation space for* K*, where the representation is given by* $\Phi_{|\widehat{L}_X} : \widehat{L}_X \to K$*. If there is already given some representation* $\Phi_K : \widehat{K} \to K$ *with* $M_b(X, \Sigma) \subseteq \widehat{K} \subseteq M(X, \Sigma)$ *and if* $\Phi_K = \Phi_{|\widehat{L}_X}$*, then we will say that the representation* Φ *is* ***compatible*** *with* Φ_K*.*

The following lemma will be very useful.

Lemma 3.9 *Let* L *be a Dedekind* σ*-complete Riesz space with weak order unit* $0 < w \in L$ *and suppose that* $P : L \to L$ *is a positive projection onto the Riesz subspace* $K \subseteq L$ *with* $w \in K$*. Denote by* $P_w = P_{|L_w}$*, which is a positive projection in* P_w *onto* K_w*. If* $(X \times Y, \Sigma \otimes \Lambda)$ *is a representation space for* P_w *and* (L_w, w)*, then* $(X \times Y, \Sigma \otimes \Lambda)$ *is a representation space for* P *and* (L, w)*.*

Proof. Denote $(\Omega, \mathcal{F}) = (X \times Y, \Sigma \otimes \Lambda)$ and let $\Phi_b : M_b(\Omega, \mathcal{F}) \to L_w$ be the representation of P_w on (L_w, w) and let $R_b : M_b(\Omega, \mathcal{F}) \to M_b(\Omega, \mathcal{F})$ be given by (7) with $0 \le m \in M(\Omega, \mathcal{F})$ satisfying (6) such that $P_w \circ \Phi_b = \Phi_b \circ R_b$. From Lemma 3.2 we know that there exists an ideal $\widehat{L}$ such that $M_b(\Omega, \mathcal{F}) \subseteq \widehat{L} \subseteq M(\Omega, \mathcal{F})$ and a representation homomorphism $\Phi : \widehat{L} \to L$ such that $\Phi_{|M_b(\Omega, \Sigma)} = \Phi_b$ and that we may assume that $M(\mathfrak{N}_\Phi) \subseteq \widehat{L}$. Define

$$E = \left\{ f \in \widehat{L} : \int_Y m(x, y) \left| f(x, y) \right| d\mu(y) < \infty \ \ \forall x \in X \right\}.$$

It is clear that E is an ideal satisfying $M_b(\Omega, \mathcal{F}) \subseteq E \subseteq M(\Omega, \mathcal{F})$. For $f \in E$ we define

$$Rf(x, y) = \int_Y m(x, z) f(x, z) \, d\mu(z)$$

for all $(x, y) \in \Omega$. It is clear that $Rf \in M(\Omega, \mathcal{F})$. First we show that $Rf \in E$ whenever $f \in E$. Take $0 \le f \in E$ and put $f_n = f \wedge (n\mathbf{1})$. Then $0 \le Rf_n \uparrow Rf$ and

$$\Phi(Rf_n) = \Phi_b(R_b f_n) = P_w(\Phi_b f_n) \le P(\Phi f)$$

for all n. As we have observed before, since we assume that $M(\mathfrak{N}_\Phi) \subseteq \widehat{L}$, this implies that $Rf \in \widehat{L}$. Hence $Rf \in \widehat{L} \cap M(X, \Sigma)$ and it is clear that $\widehat{L} \cap M(X, \Sigma) \subseteq E$. Consequently, we have a positive linear operator $R : E \to E$ with $\operatorname{Ran}(R) = E \cap M(X, \Sigma)$. It is easy to see that $P(\Phi f) = \Phi(Rf)$ for all $f \in E$.

Defining $\Phi_1 = \Phi_{|E}$, it is clear that $\Phi_1 : E \to L$ is a σ-order continuous Riesz homomorphism. We will show now that Φ_1 is surjective. Let $0 \leq g \in L$ be given and take $0 \leq f, h \in \widehat{L}$ such that $\Phi f = g$ and $\Phi h = Pg$. Define

$$F = \left\{ x \in X : \int_Y m(x, y) f(x, y) \, d\mu(y) = \infty \right\}.$$

Letting $f_n = f \wedge (n\mathbf{1})$ we have

$$\Phi(Rf_n) = P(\Phi f_n) \leq Pg = \Phi h,$$

which implies that $\Phi(Rf_n - h)^+ = 0$. Therefore, the sets

$$F_n = \{(x, y) \in \Omega : Rf_n(x, y) > h(x, y)\}$$

satisfy $F_n \in \mathfrak{N}_\Phi$ for all n. Furthermore, it follows from $0 \leq f_n \uparrow f$ that

$$Rf_n(x, y) \uparrow \int_Y m(x, z) f(x, z) \, d\mu(z)$$

for all $(x, y) \in \Omega$ and so $F \times Y \in \bigcup_{n=1}^\infty F_n$. This shows that $F \times Y \in \mathfrak{N}_\Phi$. Defining $\tilde{f} = f \mathbf{1}_{(F \times Y)^c}$, it is clear that $\tilde{f} \in E$ and $\Phi\left(\tilde{f}\right) = g$, so we may conclude that Φ_1 is surjective. Therefore, $\Phi_1 : E \to L$ is the desired representation for P on L. ■

The above lemma shows that for the construction of representations of positive projections we may restrict our attention to spaces with a strong order unit. All such Dedekind complete Riesz spaces have the structure of an f-algebra and complete Riesz subspaces containing the unit element are f-subalgebras. Therefore, in the next sections we will study in detail the properties and structure of positive projections in f-algebras onto f-subalgebras.

4 Some properties of f-subalgebras

For the basic theory of f-algebras we refer the reader to the books [1], [13] and [16]. Let A be an Archimedean f-algebra with a unit element $\mathbf{1}$, which is weak order unit in A. We denote the Boolean algebra of all components of $\mathbf{1}$ by $\mathcal{C}_A = \mathcal{C}_A(\mathbf{1})$. Note that $\mathcal{C}_A$ is equal to the set of all idempotents in A and that for all $e_1, e_2 \in \mathcal{C}_A$ in A we have

$$\begin{array}{l} e_1 \wedge e_2 = e_1 e_2 \\ e_1 \vee e_2 = e_1 + e_2 - e_1 e_2 \end{array} .$$

Denoting by $\mathcal{P}(A)$ the set if all band projections in A, the mapping $P \mapsto P\mathbf{1}$ is a Boolean isomorphism from $\mathcal{P}(A)$ onto $\mathcal{C}_A$. For every $e \in \mathcal{C}_A$ the corresponding band projection P_e onto $\{e\}^{dd}$ is given by $P_e f = ef$ for all $f \in A$. Assuming in addition that A is Dedekind complete, $\mathcal{P}(A)$ is complete and hence $\mathcal{C}_A$ is a complete Boolean algebra. Now suppose that B is an f-*subalgebra* of A (i.e., B is a Riesz subspace as well as a subalgebra of A) with $\mathbf{1} \in B$ and let $\mathcal{C}_B$ be the Boolean algebra of all components of $\mathbf{1}$ in B. Then $\mathcal{C}_B$ is a Boolean subalgebra of $\mathcal{C}_A$. If we assume that B is Dedekind complete as well, then $\mathcal{C}_B$ is a complete Boolean algebra in its own right, however in general the infinite joins and meets in $\mathcal{C}_B$ and $\mathcal{C}_A$ need not coincide. However, if we assume in addition that B is regularly embedded in A, then $\mathcal{C}_B$ is a complete Boolean subalgebra of $\mathcal{C}_A$ (in the sense of [15], Section 23). This is in particular the case if B is a complete f-subalgebra of A (i.e., B is a complete Riesz subspace of A, as defined at the beginning of Section 2).

If A is a Dedekind σ-complete f-algebra with unit element $\mathbf{1}$ and if $\mathcal{E}$ is a Boolean σ-algebra of $\mathcal{C}_A$, then $A(\mathcal{E})$ is defined as in Definition 2.1 and by Proposition 2.4, $A(\mathcal{E})$ is a σ-complete Riesz subspace of A. We claim that $A(\mathcal{E})$ is an f-subalgebra of A. Indeed, it is easy to see that $S(\mathcal{E})$ as defined by (2) is a subalgebra of A and now it follows from Proposition 2.6 (i), in combination with the order continuity of the multiplication in A, that $A(\mathcal{E})$ is a subalgebra as well. In particular, if A is Dedekind complete and $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$, then $A(\mathcal{E})$ is a complete f-subalgebra of A with $\mathcal{C}_{A(\mathcal{E})} = \mathcal{E}$ (see Proposition 2.6).

Next we make some comments about the results in Section 3. Given a Dedekind σ-complete f-algebra with unit element $\mathbf{1}$, it follows from Proposition 3.4 that there exists a representation space (X, Σ) for $(A, \mathbf{1})$. Let $\widehat{A}$ be an ideal such that $M_b(X, \Sigma) \subseteq \widehat{A} \subseteq M(X, \Sigma)$ with corresponding surjective σ-order continuous Riesz homomorphism $\Phi : \widehat{A} \to A$. Defining

$$A_b = \{f \in A : |f| \leq n\mathbf{1} \text{ for some } n \in \mathbb{N}\} . \tag{8}$$

let $\Phi_b : M_b(X, \Sigma) \to A_b$ be the restriction of Φ to $M_b(X, \Sigma)$, which is surjective as well. Since Φ_b is a Riesz homomorphism satisfying $\Phi_b(\mathbf{1}_X) = \mathbf{1}$, it is well known that Φ_b is actually an f-*algebra homomorphism* (i.e., a lattice and algebra homomorphism; see e.g. [2]). Replacing, if necessary, the ideal $\widehat{A}$ by the corresponding ideal defined by (5), which is easily seen to be a subalgebra of $M(X,\Sigma)$, it follows that we may assume in this situation, that $\widehat{A}$ is an f-subalgebra of $M(X,\Sigma)$ and that Φ is an f-algebra homomorphism. If (X,Σ) is a representation space for $(A,\mathbf{1})$, then we will simply say that (X,Σ) is a representation space for A.

Although the result of the following lemma is known (the result is even true under some weaker assumptions; see [3], Theorem 6.1), we indicate the simple proof for the convenience of the reader.

Lemma 4.1 *Let A be an Archimedean f-algebra with unit element $\mathbf{1}$ and suppose that B is an f-subalgebra of A with $\mathbf{1}\in B$. If P is a positive projection in A onto B (i.e.,* $\mathrm{Ran}(P)=B$*), then $P(ba)=bP(a)$ for all $b\in B$ and $a\in A$ (i.e., the projection P is so-called* ***averaging****).*

Proof. Fix some $0\leq a\in A$. First we assume in addition that $0\leq a\leq n\mathbf{1}$ for some $n\in\mathbb{N}$. Define the linear operator $T_a:B\to B$ by $T_a(b)=P(ba)$ for all $b\in B$. Then $0\leq T_a(b)\leq nb$ for all $0\leq b\in B$. Hence $T_a\in Z(B)$, the center of B. Since B is an f-algebra with unit element every operator in $Z(B)$ is given by multiplication by some element of B, so there exists $b_1\in B$ such that $T_a(b)=bb_1$ for all $b\in B$. Taking $b=\mathbf{1}$ we see that $b_1=P(a)$. It follows that $P(ba)=T_a(b)=bP(a)$. If $0\leq a\in A$ is arbitrary, then we can use that $a\wedge n\mathbf{1}\uparrow_n a$ (a^2-uniformly) and the result now follows via approximation (see e.g. [16], Theorem 142.7). ■

The following proposition will play an important role in the sequel. As usual, positive projection $P:A\to A$ will be called *strictly positive* if $Pa>0$ whenever $0<a\in A$. The result of the following proposition, which is of interest in its own right, will play a crucial role in the proof of Lemma 8.2.

Proposition 4.2 *Let A be a Dedekind complete f-algebra with unit element $\mathbf{1}$, which is a strong unit as well. Let $P:A\to A$ be a strictly positive order continuous projection onto the f-subalgebra B, with $\mathbf{1}\in B$. Suppose that C is a complete f-subalgebra of A such that $B\subseteq C$. Then there exists a unique positive projection $Q:A\to A$ onto C such that $P=P\circ Q$. Moreover, this projection Q is strictly positive and order continuous.*

Proof. Given $0\leq a\in A$ define the positive linear operator $T_a:C\to B$ by

$$T_a(c)=P(ac)$$

for all $c\in C$. Since C is a regular Riesz subspace of A and P is order continuous, it follows that T_a is order continuous as well. Furthermore, if $b\in B$ then

$$T_a(bc)=P(bac)=bP(ac)=bT_a(c)$$

for all $c \in C$, by the averaging property of P (see Lemma 4.1). Considering A and C as f-modules over B via multiplication (see e.g. [7], Section 4 for the definition of an f-module), we see that T_a is B-linear and so $T_a \in \mathcal{L}_n^B(C, B)$. Since $\mathbf{1}$ is assumed to be a strong unit in A, there exists $n \in \mathbb{N}$ such that $0 \leq a \leq n\mathbf{1}$, hence

$$0 \leq T_a(c) = P(ac) \leq nP(c) = nT_{\mathbf{1}}(c)$$

for all $0 \leq c \in C$, i.e., $0 \leq T_a \leq nT_{\mathbf{1}}$ in $\mathcal{L}_n^B(C, B)$. From the Radon-Nikodym theorem in $\mathcal{L}_n^B(C, B)$ (see [7], Proposition 3.10 and Examples 4.5 (3)) it follows that there exists $\pi_a \in Z(C)$ such that $0 \leq \pi_a \leq nI_C$ and $T_a = T_{\mathbf{1}}\pi_a$. Since C is a unital f-algebra there exists $Q(a) \in C$ such that $0 \leq Q(a) \leq n\mathbf{1}$ and $\pi_a(c) = Q(a)c$ for all $c \in C$. Hence $T_a(c) = T_{\mathbf{1}}(Q(a)c)$ for all $c \in C$, i.e.,

$$P(ac) = P(Q(a)c) \quad \text{for all } c \in C. \tag{9}$$

The element $Q(a) \in C$ is uniquely determined by (9). Indeed, if $f \in C$ is such that $P(fc) = 0$ for all $c \in C$, then in particular $P(f^2) = 0$. Since $f^2 \geq 0$ and P is strictly positive, it follows that $f^2 = 0$ and hence $f = 0$.

Now it is clear that the mapping $a \longmapsto Q(a)$ is additive on A^+ and that $Q(a) = a$ whenever $0 \leq a \in C$. Consequently Q extends uniquely to a linear positive projection $Q : A \to A$ with $\mathrm{Ran}(Q) = C$, satisfying (9) for all $a \in A$. Taking $c = \mathbf{1}$ in (9) it follows that $P = P \circ Q$. Since P is strictly positive, this implies that Q is strictly positive as well. To show that Q is order continuous, suppose that $a_\tau \downarrow 0$ in A and assume that $Q(a_\tau) \geq a$ for all τ and some $0 \leq a \in A$. This implies that $P(a_\tau) = P(Q(a_\tau)) \geq P(a) \geq 0$ for all τ. Since $P(a_\tau) \downarrow 0$ it follows that $P(a) = 0$, hence $a = 0$. This shows that $Q(a_\tau) \downarrow 0$ in A, so Q is order continuous.

It remains to prove that Q is unique. To this end suppose that $Q_1 : A \to A$ is a positive projection such that $Q_1(A) = C$ and $P = P \circ Q_1$. By the averaging property of Q_1it follows that

$$P(ac) = P(Q_1(ac)) = P(Q_1(a)c)$$

for all $a \in A$ and all $c \in C$. As observed above, $Q(a) \in C$ is uniquely determined by (9), hence $Q_1(a) = Q(a)$ for all $a \in A$, i.e., $Q_1 = Q$. ■

Remark 4.3 *If we drop the assumption that* $\mathbf{1}$ *is a strong order unit in* A, *then the result of the above theorem does not hold in general, as can be seen by the following example. We consider the positive real axis* $(0, \infty)$ *equipped with the probability measure* $d\mu = e^{-x}dx$ *defined on the* σ*-algebra* Σ *of all*

Lebesgue measurable subsets. Define the ideal A in $L_0(\mu)$ by

$$A = \left\{ f \in \bigcap_{1 \leq p < \infty} L_p(\mu) : |f| \mathbf{1}_{(0,1)} \leq c\mathbf{1}_{(0,1)} \text{ for some } 0 \leq c \in \mathbb{R} \right\}.$$

Since $f \in A$ implies that $f^2 \in A$, it follows that A is a Dedekind complete f-algebra with unit element $\mathbf{1} = \mathbf{1}_{(0,\infty)}$. Note already that $\mathbf{1}$ is not a strong order unit in A, since for example the function f_0, given by

$$f_0(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \text{ or } x > 2 \\ |\log(x-1)| & \text{if } 1 < x \leq 2 \end{cases}, \tag{10}$$

belongs to A and is unbounded. Let the f-subalgebra B be given by $B = \{\lambda \mathbf{1} : \lambda \in \mathbb{R}\}$. For $f \in A$ we define $\varphi_0(f) = \int_0^\infty f(x)\, d\mu(x)$ and $P(f) = \varphi_0(f)\mathbf{1}$. Then $P : A \to A$ is an order continuous strictly positive projection onto B. Define

$$C = \{f \in A : f(x+1) = f(x) \ \ \mu\text{-a.e. on } (0,\infty)\},$$

which is a complete f-subalgebra of A with $B \subseteq C$. Let Σ_1 be the sub-σ-algebra of Σ defined by

$$\Sigma_1 = \{E \in \Sigma : \mathbf{1}_E \in C\}.$$

If $f \in C$, then, by the definition of A, there exists $0 \leq c \in \mathbb{R}$ such that $|f| \mathbf{1}_{(0,1)} \leq c\mathbf{1}_{(0,1)}$, consequently $|f| \leq c\mathbf{1}$. Therefore, C consists of bounded functions. Now suppose that there exists a positive projection $Q : A \to A$ such that $Q(A) = C$ and $P = P \circ Q$. First observe that Q is the restriction to A of the conditional expectation operator with respect to Σ_1. Indeed, for all $f \in A$ we have

$$\varphi_0(f)\mathbf{1} = P(f) = P(Qf) = \varphi_0(Qf)\mathbf{1}$$

and so $\varphi_0(f) = \varphi_0(Qf)$. If $E \in \Sigma_1$, then $\mathbf{1}_E \in C$ and so it follows from the averaging property of Q (see Lemma 4.1) that

$$\int_E f\, d\mu = \varphi_0(\mathbf{1}_E f) = \varphi_0(Q(\mathbf{1}_E f)) = \varphi_0(\mathbf{1}_E Q(f)) = \int_E Q(f)\, d\mu, \tag{11}$$

which shows that $Q(f)$ is the conditional expectation of f with respect to Σ_1. Take $0 \leq f \in A$ and $E_1 \in \Sigma$ such that $E_1 \subseteq (0,1)$. Define $E \in \Sigma_1$ by $E = \bigcup_{k=0}^\infty (E_1 + k)$. Then

$$\begin{aligned}\int_E f(x)\, d\mu(x) = \sum_{k=0}^\infty \int_{E_1+k} f(x) e^{-x} dx = \sum_{k=0}^\infty \int_{E_1} f(x+k) e^{-(x+k)} dx \\ = \int_{E_1} \left(\sum_{k=0}^\infty f(x+k) e^{-k} \right) e^{-x} dx\end{aligned}$$

and, using that Qf is 1-periodic,

$$\int_E Qf(x)\,d\mu(x) = \sum_{k=0}^{\infty}\int_{E_1+k} Qf(x)\,e^{-x}dx = \sum_{k=0}^{\infty}\int_{E_1} Qf(x)\,e^{-(x+k)}dx$$
$$= \int_{E_1} \frac{e}{e-1} Qf(x)\,e^{-x}dx.$$

In combination with (11) this shows that

$$\int_{E_1} \frac{e}{e-1} Qf(x)\,e^{-x}dx = \int_{E_1}\left(\sum_{k=0}^{\infty} f(x+k)\,e^{-k}\right)e^{-x}dx$$

for all measurable subsets $E_1 \subseteq (0,1)$. This implies that

$$\sum_{k=0}^{\infty} f(x+k)\,e^{-k} = \frac{e}{e-1} Qf(x) \tag{12}$$

μ-a.e. on $(0,1)$. Now we take in particular the function $f_0 \in A$ defined by (10). Then

$$\sum_{k=0}^{\infty} f_0(x+k)\,e^{-k} = f_0(x+1)\,e^{-1} = |\log x|\,e^{-1}$$

and so it follows from (12) that

$$Qf_0(x) = \frac{e-1}{e^2}|\log x|$$

μ-a.e. on $(0,1)$. Hence Qf_0 is not bounded. However, as observed above, all functions in C are bounded, which is a contradiction. This shows that such a projection Q does not exist.

Next we will discuss some further notation and properties concerning subalgebras. As above we assume that A is a Dedekind complete f-algebra with unit element $\mathbf{1}$ and we assume that B is a Dedekind complete regularly embedded f-subalgebra of A with $\mathbf{1} \in B$. Hence $\mathcal{C}_B$ is a complete Boolean subalgebra of $\mathcal{C}_A$. For $p \in \mathcal{C}_A$ we will denote

$$A(p) = \{pf : f \in A\}, \tag{13}$$

which is the band in A generated by p. Note that $A(p)$ is an f-algebra in its own right with unit element p. The Boolean algebra of components of p in $A(p)$ is given by

$$\mathcal{C}_A(p) = \{q \in \mathcal{C}_A : q \leq p\}.$$

Furthermore we define

$$B_{\restriction p} = \{pf : f \in B\} . \tag{14}$$

Then $B_{\restriction p}$ is a Dedekind complete regularly embedded f-subalgebra of $A(p)$ and it is easy to see that the Boolean algebra of components of p in $B_{\restriction p}$ is given by

$$p\mathcal{C}_B = \{pq : q \in \mathcal{C}_B\} ,$$

which is a complete Boolean subalgebra of $\mathcal{C}_A(p)$. If $p \in \mathcal{C}_B$, then $B_{\restriction p}$ will also be denoted by $B(p)$.

For $p \in \mathcal{C}_A$ we define

$$\overline{p} = \inf \{q \in \mathcal{C}_B : p \leq q\} , \tag{15}$$

where the infimum is taken in $\mathcal{C}_A$. Since $\mathcal{C}_B$ is a complete Boolean subalgebra of $\mathcal{C}_A$, it follows that $\overline{p} \in \mathcal{C}_B$. The element $\overline{p}$ is sometimes called *the closure of p with respect to $\mathcal{C}_B$* ([10]). It is clear that $p \leq \overline{p}$.

Lemma 4.4 *The mapping $\psi : B(\overline{p}) \to B_{\restriction p}$, defined by $\psi(f) = pf$ for all $f \in B(\overline{p})$, is a surjective f-algebra isomorphism.*

Proof. It is clear that ψ is an f-algebra homomorphism. Now take $g \in B_{\restriction p}$. Then $g = pf$ for some $f \in B$ and hence $\psi(\overline{p}f) = p\overline{p}f = pf = g$, which shows that ψ is surjective. It remains to proof that ψ is injective. To this end suppose that $f \in B(\overline{p})$ such that $pf = 0$. From (15) it follows that

$$\mathbf{1} - \overline{p} = \sup \{q \in \mathcal{C}_B : p \wedge q = 0\} .$$

Since $f = f\overline{p}$, i.e., $f(\mathbf{1} - \overline{p}) = 0$, this implies that $fq = 0$ for all $q \in \mathcal{C}_B \cap \{p\}^d$ and hence $fs = 0$ for all $s \in S(\mathcal{C}_B) \cap \{p\}^d$. Therefore, by Freudenthal's spectral theorem we have $fg = 0$ for all $g \in B \cap \{p\}^d$. Since $pf = 0$ implies that $f \in B \cap \{p\}^d$, it follows that $f^2 = 0$, hence $f = 0$, which shows that ψ is injective. ■

Remark 4.5 *Note that the last argument in the above proof actually shows that*

$$B \cap \{p\}^{d(A)} = B(\overline{p})^{d(B)},$$

where $d(A)$ and $d(B)$ indicate that the disjoint complements are taken in A and B respectively.

As we have seen in Example 2.12, a Riesz subspace of a locally order separable Riesz space need not be locally order separable. However, using Lemma 4.4 we can prove the following result, which will be used later.

Lemma 4.6 *Let A be a locally order separable Dedekind complete f-algebra with unit element $\mathbf{1}$ and let B be a Dedekind complete regular f-subalgebra of A with $\mathbf{1} \in B$. Then B is locally order separable as well.*

Proof. Take $0 < q \in \mathcal{C}_B$. It follows from Lemma 2.11 that there exists $p \in \mathcal{C}_A$ such that $0 < p \leq q$ and $A(p)$ is order separable. Since $B_{\restriction p}$ is a Riesz subspace of $A(p)$, it follows that $B_{\restriction p}$ is order separable. By Lemma 4.4, $B(\overline{p})$ is Riesz isomorphic to $B_{\restriction p}$, hence $B(\overline{p})$ is order separable as well. Since $0 < \overline{p} \leq q$ in $\mathcal{C}_B$, it follows from Lemma 2.11 that B is locally order separable. ■

Definition 4.7 *Let A be a Dedekind complete f-algebra with unit element $\mathbf{1}$ and let B be a Dedekind complete regular f-subalgebra of A with $\mathbf{1} \in B$. An element $p \in \mathcal{C}_A$ will be called B-**full** if $\mathcal{C}_A(p) = p\mathcal{C}_B$, i.e., if*

$$\{e \in \mathcal{C}_A : e \leq p\} = \{pq : q \in \mathcal{C}_B\}. \tag{16}$$

We note that the B-full elements in $\mathcal{C}_A$ correspond to the *elements of order zero over* $\mathcal{C}_B$, as defined in [10], Section 11. In the next lemma we will use the notation introduced in (8).

Remark 4.8 *(i). If $p \in \mathcal{C}_A$ and $q \in \mathcal{C}_A(p)$, then q is B-full if and only if q is $B_{\restriction p}$-full in $\mathcal{C}_{A(p)} = \mathcal{C}_A(p)$.*

(ii). If $p \in \mathcal{C}_A$ is B-full and $q \in \mathcal{C}_A(p)$, then q is B-full.

Lemma 4.9 *For an element $p \in \mathcal{C}_A$ the following statements are equivalent:*

(i). p is B-full;

(ii). $B_{\restriction p}$ is an order ideal in A;

(iii). $A_b(p) = (B_b)_{\restriction p}$;

(iv). the mapping $f \longmapsto pf$ is an f-algebra isomorphism from $B_b(\overline{p})$ onto $A_b(p)$;

(v). the mapping $q \longmapsto pq$ is a Boolean isomorphism from $\mathcal{C}_B(\overline{p})$ onto $\mathcal{C}_A(p)$.

Proof. Suppose that p is B-full and take $f \in A$ satisfying $0 \leq f \leq pg$ for some $0 \leq g \in B$. Since $f \in A(p)$, there exists a sequence $\{s_k\}_{k=1}^{\infty}$ in $S\left(\mathcal{C}_A(p)\right)$ such that $0 \leq s_k \uparrow f$. It follows from (16) that every s_k can be written as $s_k = pt_k$ for some $0 \leq t_k \in S\left(\mathcal{C}_B\right)$ and we may assume that $0 \leq t_k \uparrow$. Hence $0 \leq t_k \wedge g \uparrow \leq g$ in B and so $0 \leq t_k \wedge g \uparrow h \in B$. Now

$$p\left(t_k \wedge g\right) = s_k \wedge pg \uparrow f \wedge pg = f,$$

so $f = ph \in B_{\restriction p}$. This proves that (i) implies (ii). Since

$$A_b(p) = \{f \in A : |f| \leq np \text{ for some } n \in \mathbb{N}\}$$

and $p \in \left(B_b\right)_{\restriction p} = \left(B_{\restriction p}\right)_b$it is clear that (ii) implies (iii). Lemma 4.4 shows that (iii) implies (iv) and it is also clear that (iv) implies (v), which obviously implies (i). ■

It will be convenient to introduce the following definition.

Definition 4.10 *Let A be a Dedekind complete f-algebra with unit element $\mathbf{1}$ and let B be a Dedekind complete regular f-subalgebra of A with $\mathbf{1} \in B$. Then:*

*(a). A is called **nowhere full with respect to** B (or, **nowhere** B-**full**) if 0 is the only B-full element in $\mathcal{C}_A$. In this case we will also say that $\mathcal{C}_A$ is nowhere full with respect to $\mathcal{C}_B$;*

*(b). A is called **everywhere full with respect to** B (or, **everywhere** B-**full**) if for every $0 < p \in \mathcal{C}_A$ there exists $0 < q \in \mathcal{C}_A(p)$ which is B-full.*

The following lemma is straightforward.

Lemma 4.11 *The following three statements are equivalent:*

(i). A is everywhere B-full;

(ii). there exists a disjoint system $\{p_\tau\}$ in $\mathcal{C}_A$ consisting of B-full elements such that $\bigvee_\tau p_\tau = \mathbf{1}$;

(iii). $\sup\{p \in \mathcal{C}_A : p$ *is B-full*$\} = \mathbf{1}$.

With these observations, the next result follows immediately.

Proposition 4.12 *Let A be a Dedekind complete f-algebra with unit element* $\mathbf{1}$ *and let B be a Dedekind complete regular f-subalgebra of A with $\mathbf{1} \in B$. Then there exist $p_1, p_2 \in \mathcal{C}_A$ with $p_1 + p_2 = \mathbf{1}$ such that:*

(i). $A(p_1)$ is everywhere $B_{\restriction p_1}$-full;

(ii). $A(p_2)$ is nowhere $B_{\restriction p_2}$-full.

Proof. Defining $p_1 = \sup\{p \in \mathcal{C}_A : p \text{ is } B\text{-full}\}$ and $p_2 = \mathbf{1} - p_1$, the result follows from Remark 4.8 and Lemma 4.11. ∎

Next we will discuss restrictions of positive projections to bands in A. We assume that A is a Dedekind complete f-algebra, that B is an f-subalgebra of A with $\mathbf{1} \in B$ and that $P : A \to A$ is a strictly positive order continuous projection onto B. Let $0 < p_0 \in \mathcal{C}_A$ be fixed. We will need the following observation.

Lemma 4.13 *The element $P(p_0)$ is a weak order unit in $B(\overline{p_0})$.*

Proof. Since $0 \leq p_0 \leq \overline{p_0}$, it follows that $0 \leq P(p_0) \leq P(\overline{p_0}) = \overline{p_0}$, so $0 \leq P(p_0) \in B(\overline{p_0})$. Now suppose that $0 \leq g \in B(\overline{p_0})$ is such that $gP(p_0) = 0$. Then $P(gp_0) = 0$ and so $gp_0 = 0$, as P is strictly positive. Hence $\psi(g) = 0$ and by Lemma 4.4 this implies that $g = 0$. Consequently, $P(p_0)$ is a weak order unit in $B(\overline{p_0})$. ∎

Denoting by $B(\overline{p_0})^u$ the universal completion of the f-algebra $B(\overline{p_0})$, it is well known that the f-algebra structure of $B(\overline{p_0})$ extends uniquely to $B(\overline{p_0})^u$ and that every weak order unit in $B(\overline{p_0})^u$ has an inverse. Hence, it follows from the above lemma that in particular the element $P(p_0)$ is invertible in $B(\overline{p_0})^u$, i.e., there exists $0 < w_0 \in B(\overline{p_0})^u$ such that $w_0 P(p_0) = \overline{p_0}$. We will denote this element w_0 by $P(p_0)^{-1}$.

Lemma 4.14 *We assume in addition that the unit element* $\mathbf{1}$ *in A is a strong order unit. The mapping $P_{\restriction p_0} : A(p_0) \to A(p_0)$, defined by*

$$P_{\restriction p_0}(f) = p_0 P(p_0)^{-1} P(f)$$

for all $f \in A(p_0)$, is a strictly positive order continuous projection onto $B_{\restriction p_0}$.

Proof. First we show that $P_{\restriction p_0}$ is well defined. Take $0 \leq f \in A(p_0)$. Then $0 \leq f \leq np_0$ for some $n \in \mathbb{N}$ and so $0 \leq P(f) \leq nP(p_0) \leq n\overline{p_0}$. Therefore, in $B(\overline{p_0})^u$ we have

$$0 \leq P(p_0)^{-1} P(f) \leq nP(p_0)^{-1} P(p_0) = n\overline{p_0}.$$

Since $B(\overline{p_0})$ is Dedekind complete, $B(\overline{p_0})$ is an ideal in $B(\overline{p_0})^u$ and so $0 \leq P(p_0)^{-1}P(f) \in B(\overline{p_0})$. Hence $0 \leq P_{\upharpoonright p_0}(f) \in B_{\upharpoonright p_0} \subseteq A(p_0)$. This shows that $P_{\upharpoonright p_0}$ is a well defined positive linear mapping with its range contained in $B_{\upharpoonright p_0}$.

If $f \in B_{\upharpoonright p_0}$ then, by definition, $f = p_0 g$ for some $g \in B$. Hence

$$\begin{aligned} P_{\upharpoonright p_0}(f) = p_0 P(p_0)^{-1} P(p_0 g) &= p_0 P(p_0)^{-1} P(p_0) g \\ &= p_0 \overline{p_0} g = p_0 g = f, \end{aligned}$$

which shows that $P_{\upharpoonright p_0}$ is a projection onto $B_{\upharpoonright p_0}$. It is clear that $P_{\upharpoonright p_0}$ is order continuous, so it remains to prove that $P_{\upharpoonright p_0}$ is strictly positive. Suppose that $0 \leq f \in A(p_0)$ such that $P_{\upharpoonright p_0}(f) = 0$. Since $0 \leq P(p_0)^{-1}P(f) \in B(\overline{p_0})$, it follows from Lemma 4.4 that $P(p_0)^{-1}P(f) = 0$ and so $\overline{p_0}P(f) = P(p_0)P(p_0)^{-1}P(f) = 0$. Since $0 \leq P(f) \leq n\overline{p_0}$ for some $n \in \mathbb{N}$, this implies that $P(f) = 0$ and so $f = 0$. ∎

In the above situation we will say that $P_{\upharpoonright p_0}$ is *the restriction of* P *to* $A(p_0)$ (or, *the restriction to* p_0). We end this section with a technical result, which shows how representations of restrictions of a positive projection P can be glued together to obtain a representation of P (as defined in Definition 3.7). We assume that A is a Dedekind complete f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and that $P : A \to A$ is a strictly positive order continuous projection onto the f-subalgebra B with $\mathbf{1} \in B$. Suppose that (X, Σ) is a representation space for B with representation homomorphism $\Phi_B : M_b(X, \Sigma) \to B$. If $0 < p_0 \in \mathcal{C}_A$, then (X, Σ) is also a representation space for $B_{\upharpoonright p_0}$, the representation given by the homomorphism

$$\Phi_B^{p_0} : M_b(X, \Sigma) \to B_{\upharpoonright p_0}, \tag{17}$$

defined by $\Phi_B^{p_0}(f) = p_0 \Phi_B(f)$ for all $f \in M_b(X, \Sigma)$. In this situation we will call $\Phi_B^{p_0}$ the *induced representation* of $B_{\upharpoonright p_0}$.

Lemma 4.15 *Let* $\Phi_B : M_b(X, \Sigma) \to B$ *be a representation of* B*. We assume furthermore that:*

(i). $\{p_j\}$ *is an at most countable disjoint system in* $\mathcal{C}_A$ *such that* $\sum_j p_j = \mathbf{1}$*;*

(ii). *For every* j *there exists a representation space* $(X \times Y_j, \Sigma \otimes \Lambda_j)$ *for* $P_{\upharpoonright p_j}$ *and* $A(p_j)$ *such that the corresponding representation*

$$\Phi_j : M_b(X \times Y_j, \Sigma \otimes \Lambda_j) \to A(p_j)$$

is compatible with the induced representation $\Phi_B^{p_j}$ *of* $B_{\upharpoonright p_j}$*.*

Then there exists a representation space $(X \times Y, \Sigma \otimes \Lambda)$ *for* P *and* A *such that the corresponding representation* $\Phi : M_b(X \times Y, \Sigma \otimes \Lambda) \to A$ *is compatible with* Φ_B.

Proof. We will denote $(\Omega_j, \mathcal{F}_j) = (X \times Y_j, \Sigma \otimes \Lambda_j)$. By assumption, there exists a σ-finite measure μ_j on Λ_j and for each $j = 1, 2, \ldots$ there is an operator $R_j : M_b(\Omega_j, \mathcal{F}_j) \to M_b(\Omega_j, \mathcal{F}_j)$, given by

$$R_j f(x, y) = \int_{Y_j} m_j(x, z) f(x, z)\, d\mu_j(z)$$

for all $(x, y) \in \Omega_j$ and all $f \in M_b(\Omega_j, \mathcal{F}_j)$, with $0 \leq m_j \in M(\Omega_j, \mathcal{F}_j)$ and $\int_{Y_j} m(x, z)\, d\mu_j(z) = 1$ for all $x \in X$, such that $\Phi_j \circ R_j = P_{\restriction p_j} \circ \Phi_j$.

We define $Y = \bigsqcup_j Y_j$ (i.e., Y is the disjoint union of the Y_j, $j = 1, 2, \ldots$), $\Lambda = \bigoplus_j \Lambda_j$ and $\mu = \bigoplus_j \mu_j$. Then (Y, Λ, μ) is a σ-finite measure space. Denoting $(\Omega, \mathcal{F}) = (X \times Y, \Sigma \otimes \Lambda)$, we identify $M_b(\Omega_j, \mathcal{F}_j)$ with the ideal in $M_b(\Omega, \mathcal{F})$ consisting of all functions of the form $f\mathbf{1}_{\Omega_j}$ with $f \in M_b(\Omega, \mathcal{F})$. If $0 \leq f \in M_b(\Omega, \mathcal{F})$, then $\left\{\Phi_j\left(f\mathbf{1}_{\Omega_j}\right)\right\}$ is a disjoint order bounded sequence in A, so we can define

$$\Phi(f) = \sum_j \Phi_j\left(f\mathbf{1}_{\Omega_j}\right).$$

It is easily verified that this defines a σ-order continuous surjective f-algebra homomorphism $\Phi : M_b(\Omega, \mathcal{F}) \to A$, which is compatible with Φ_B, as all Φ_j are compatible with $\Phi_B^{p_j}$. Note that $\Phi\left(f\mathbf{1}_{\Omega_j}\right) = p_j \Phi(f)$ for all $f \in M_b(\Omega, \mathcal{F})$. For $f \in M_b(\Omega, \mathcal{F})$ we will from now on consider the function $R_j\left(f\mathbf{1}_{\Omega_j}\right)$ as an element of $M_b(\Omega, \mathcal{F})$.The commutation relation $\Phi_j \circ R_j = P_{\restriction p_j} \circ \Phi_j$ then corresponds to

$$\Phi\left[\mathbf{1}_{\Omega_j} R_j\left(f\mathbf{1}_{\Omega_j}\right)\right] = P_{\restriction p_j}\left[\Phi\left(f\mathbf{1}_{\Omega_j}\right)\right] = P_{\restriction p_j}\left[p_j \Phi(f)\right] \tag{18}$$

for all $f \in M_b(\Omega, \mathcal{F})$.

For $j = 1, 2, \ldots$ we choose $0 \leq u_j \in M_b(X, \Sigma)$ such that $\Phi_B(u_j) = P(p_j)$. Take $f \in M_b(\Omega, \mathcal{F})$ and put $g = \Phi(f)$. Define $h_j \in M_b(X, \Sigma)$ by

$$h_j = \Phi\left[u_j R_j\left(f\mathbf{1}_{\Omega_j}\right)\right].$$

Since $0 \leq P(p_j) \leq \bar{p}_j$, $\Phi\left[R_j\left(f\mathbf{1}_{\Omega_j}\right)\right] \in B$ and $h_j = P(p_j)\Phi\left[R_j\left(f\mathbf{1}_{\Omega_j}\right)\right]$, it follows that $h_j \in B(\bar{p}_j)$. Furthermore, using (18) and the definition of $P_{\restriction p_j}$ (see Lemma 4.14), we find

$$\begin{aligned} p_j h_j = \Phi\left(\mathbf{1}_{\Omega_j}\right) P(p_j) \Phi\left[R_j\left(f\mathbf{1}_{\Omega_j}\right)\right] &= P(p_j)\Phi\left[\mathbf{1}_{\Omega_j} R_j\left(f\mathbf{1}_{\Omega_j}\right)\right] \\ = P(p_j) P_{\restriction p_j}(p_j g) &= P(p_j) p_j P(p_j)^{-1} P(p_j g) \\ &= p_j \bar{p}_j P(p_j g) = p_j P(p_j g). \end{aligned}$$

Since $P(p_j g) \in B(\bar{p}_j)$, it follows from Lemma 4.4 that $h_j = P(p_j g)$. We thus have shown that

$$\Phi\left[u_j R_j\left(f\mathbf{1}_{\Omega_j}\right)\right] = P\left[p_j \Phi(f)\right] \tag{19}$$

for all $f \in M_b(\Omega, \mathcal{F})$ and all j.

Define the $\mathcal{F}$-measurable function $m : \Omega \to [0, \infty]$ by

$$m(x, y) = \sum_j u_j(x)\, m_j(x, y)\, \mathbf{1}_{\Omega_j}(x, y) \tag{20}$$

for all $(x, y) \in \Omega$ and let $k_n \in M(\Omega, \mathcal{F})$ be the nth partial sum of (20). For $n = 1, 2, \dots$ we define the positive linear operators $R^{(n)} : M_b(\Omega, \mathcal{F}) \to M_b(\Omega, \mathcal{F})$ by

$$R^{(n)} f(x, y) = \int_Y k_n(x, z)\, f(x, z)\, d\mu(z)$$

for all $(x, y) \in \Omega$ and all $f \in M_b(\Omega, \mathcal{F})$. Since

$$R^{(n)} f = \sum_{j=1}^{n} u_j R_j\left(f\mathbf{1}_{\Omega_j}\right),$$

it follows from (19) that

$$\Phi\left(R^{(n)} f\right) = P\left[\left(\sum_{j=1}^{n} p_j\right)\Phi(f)\right] \tag{21}$$

for all $f \in M_b(\Omega, \mathcal{F})$. This implies in particular that

$$\Phi\left(R^{(n)} f\right) \uparrow P(\Phi f) \tag{22}$$

in A for all $0 \leq f \in M_b(\Omega, \mathcal{F})$.

Our next objective is to show that we can actually choose the functions u_j such that the function m, defined by (20), satisfies $\int_Y m(x, y)\, d\mu(y) = 1$ for all $x \in X$. To this define $h : \Omega \to [0, \infty]$ by $h(x, y) = \int_Y m(x, z)\, d\mu(z)$ for all $(x, y) \in \Omega$ and put $h_n = R^{(n)}\mathbf{1}_\Omega$ for all n. From (22) it follows that $\Phi(h_n) \uparrow \mathbf{1}$ in A. Defining $h_n' = h_n \wedge \mathbf{1}_\Omega$, we have $\Phi(h_n') = \Phi(h_n)$ for all n and $0 \leq h_n' \uparrow h'$ for some $h' \in M_b(\Omega, \mathcal{F})$. From the σ-order continuity of Φ it follows that $\Phi(h') = \mathbf{1} = \Phi(\mathbf{1}_\Omega)$. Hence, $h' = \mathbf{1}_\Omega$ $\mathfrak{N}_\Phi$-a.e. and $h_n = h_n'$ $\mathfrak{N}_\Phi$-a.e. Since $h_n(x, y) \uparrow h(x, y)$ for all $(x, y) \in \Omega$, we may conclude that $h = \mathbf{1}_\Omega$ $\mathfrak{N}_\Phi$-a.e., which implies that there exists a set $F \in \Sigma$ such that $F \times Y \in \mathfrak{N}_\Phi$ and $\int_Y m(x, y)\, d\mu(y) = 1$ for all $x \in X \setminus F$. Now it is easy to see that we can adjust the values of the functions u_j on the set $F \times Y$ in such a way that $\int_Y m(x,y)\, d\mu(y) = 1$ for all $x \in X$. With this assumption, we define the positive projection $R : M_b(\Omega, \mathcal{F}) \to M_b(\Omega, \mathcal{F})$ onto $M_b(X, \Sigma)$ by

$$Rf(x,y) = \int_Y m(x,z)\, f(x,z)\, d\mu(z)$$

for all $(x,y) \in \Omega$ and all $f \in M_b(\Omega, \mathcal{F})$. It is clear that $R^{(n)} f \uparrow Rf$ for all $0 \leq f \in M_b(\Omega, \mathcal{F})$, and so (22) implies that $\Phi(Rf) = P(\Phi f)$ for all $f \in M_b(\Omega, \mathcal{F})$. Therefore, we may conclude that Φ is a representation of P on A and the proof is complete. ■

We end this section with an example which illustrates the use of the above lemma and which will also be used later.

Example 4.16 *(i). Let A be a Dedekind complete f-algebra in which the unit element* $\mathbf{1}$ *is a strong order unit. The most trivial projection $P : A \to A$ satisfying the conditions above is the projection $P = I$ onto $B = A$. Let $\Phi_B : M_b(X, \Sigma) \to B$ be a representation for B. Let (Y, Λ, μ) be the one point measure space with $Y = \{y\}$ and $\mu(\{y\}) = 1$. Then $M_b(X \times Y, \Sigma \otimes \Lambda) = M_b(X, \Sigma)$. Define the representation Φ_A of A by $\Phi_A f = \Phi_B f$ for all $f \in M_b(X \times Y, \Sigma \otimes \Lambda)$. This defines a representation for P and A, where the corresponding representing projection R_b in $M_b(X \times Y, \Sigma \otimes \Lambda)$ is the identity operator. This will be called* ***the trivial representation****.*

(ii). Now we assume that A is a Dedekind complete order separable f-algebra in which the unit element $\mathbf{1}$ *is a strong order unit and that $P : A \to A$ is a strictly positive order continuous projection onto the f-subalgebra B with* $\mathbf{1} \in B$. *Moreover, we suppose that A is everywhere B-full. Let $\Phi_B : M_b(X, \Sigma) \to B$ be a representation of B. By Lemma 4.11, there exists a disjoint system $\{p_\tau\}$ in $\mathcal{C}_A$ consisting of B-full elements such that $\sup_\tau p_\tau = \mathbf{1}$. Since A is assumed to be order separable, this system is at most countable, $\{p_n : n = 1, 2, \ldots\}$ say. For each n, let $\Phi_B^{p_n} : M_b(X, \Sigma) \to B_{\restriction p_n}$ be the induced representation defined by (17). As p_n is B-full, it follows from Lemma 4.9 that $A(p_n) = B_{\restriction p_n}$ and so the restriction $P_{\restriction p_n} : A(p_n) \to B_{\restriction p_n}$ is the identity operator. Hence, by (i), $P_{\restriction p_n}$ has the trivial representation $\Phi_n : M_b(X \times Y_n, \Sigma \otimes \Lambda_n) \to A(p_n)$, where $Y_n = \{y_n\}$ and $\mu_n(\{y_n\}) = 1$. Defining $Y = \{y_1, y_2, \ldots\}$ and $\Lambda = \mathcal{P}(Y)$ with counting measure μ, it follows from Lemma 4.15 (and its proof) that $(X \times Y, \Sigma \otimes \Lambda)$ is a representation space for P and A, with a representation $\Phi_A : M_b(X \times Y, \Sigma \otimes \Lambda) \to A$ which is compatible with Φ_B.*

Given a Dedekind complete order separable f-algebra A in which the unit element $\mathbf{1}$ is a strong order unit and a strictly positive order continuous projection $P : A \to A$ onto the f-subalgebra B with $\mathbf{1} \in B$, let $A = A(p_1) \oplus A(p_2)$ be the decomposition of Proposition 4.12. As shown in the above example, it is easy to construct a representation for $P_{\restriction p_1}$ on $A(p_1)$. The main effort in the remainder of the present paper is to construct a representation for the projection $P_{\restriction p_2}$ on the nowhere B-full part $A(p_2)$.

5 Maharam's theorem for positive projections

In this section we will discuss a generalization of a result of D. Maharam ([11], Lemma 3) to positive projections, which will play an important role in the sequel. Actually, the main theorem in the present section can be considered as a vector valued version of Maharam's lemma. As before, A will denote a Dedekind complete f-algebra with unit element $\mathbf{1}$ and B will be an f-subalgebra with $\mathbf{1} \in B$. Furthermore we will assume that $P : A \to A$ is a positive order continuous projection onto B. As observed in Lemma 3.6, this implies that B is a complete f-subalgebra of A. The proof of the main result in the present section is divided into a number of lemmas.

Lemma 5.1 *Assume that A is nowhere full with respect to B. If $0 \neq e \in \mathcal{C}_A$, then there exists $p_1 \in \mathcal{C}_A$ such that $0 \neq p_1 \leq e$ and $P(p_1) \leq \frac{1}{2}P(e)$.*

Proof. Since $e \neq 0$, the element e is not B-full and so there exists $p \in \mathcal{C}_A(e)$ such that $p \notin e\mathcal{C}_B$. Let $q = e - p$ and put $u = \overline{p}.\overline{q}$. First observe that $u \neq 0$. Indeed, suppose that $u = 0$, then $\overline{p}q = 0$, which implies that

$$e\overline{p} = (p + q)\,\overline{p} = p\overline{p} = p,$$

so $p \in e\mathcal{C}_B$, a contradiction.

Let $e_1 \in \mathcal{C}_B$ be the component of $\overline{p}$ in the band $\left\{\left[P\left(p\right) - \frac{1}{2}P\left(e\right)\right]^+\right\}^d$ in B. Then $e_1\left[P\left(p\right) - \frac{1}{2}P\left(e\right)\right]^+ = 0$ and so from Lemma 4.1 it follows that

$$P\left(e_1 p\right) = e_1 P\left(p\right) \leq \frac{1}{2}P\left(e\right).$$

If $e_1 \neq 0$, then, by Lemma 4.4, $e_1 p \neq 0$ and so we can take $p_1 = e_1 p$ in this case.

Now assume that $e_1 = 0$. Then $\overline{p} \in \left\{\left[P\left(p\right) - \frac{1}{2}P\left(e\right)\right]^+\right\}^{dd}$ and since $0 < u \leq \overline{p}$, it follows that $u\left[P\left(p\right) - \frac{1}{2}P\left(e\right)\right]^+ > 0$ and $u\left[P\left(p\right) - \frac{1}{2}P\left(e\right)\right]^- = 0$, so

$$u\left[P\left(p\right)-\frac{1}{2}P\left(e\right)\right]=u\left[P\left(p\right)-\frac{1}{2}P\left(e\right)\right]^{+}>0,$$

which implies that

$$uP(p)>\frac{1}{2}uP\left(e\right). \tag{23}$$

Let $e_2 \in \mathcal{C}_B$ be the component of $\overline{q}$ in the band $\left\{\left[P\left(q\right)-\frac{1}{2}P\left(e\right)\right]^{+}\right\}^{d}$ in B. If $e_2 = 0$, then a similar argument as above shows that

$$uP(q)>\frac{1}{2}uP\left(e\right). \tag{24}$$

If (23) and (24) hold simultaneously we find that $uP(e) = uP(p) + uP(q) > uP(e)$, which is impossible. Consequently, $e_2 \neq 0$. As above it now follows that $e_2 q \neq 0$ and $P(e_2 q) \leq \frac{1}{2}P(e)$ and so in this case we can take $p_1 = e_2 q$. ■

Lemma 5.2 *Assume that A is nowhere full with respect to B. If $0 \neq e \in \mathcal{C}_A$, then for every $\varepsilon > 0$ there exists $p \in \mathcal{C}_A$ such that $0 \neq p \leq e$ and $P(p) \leq \varepsilon P(e)$.*

Proof. Repeated application of Lemma 5.1 immediately shows that for every $n \in \mathbb{N}$ there exists $p_n \in \mathcal{C}_A$ such that $0 \neq p_n \leq e$ and $P(p_n) \leq 2^{-n}P(e)$. ■

Lemma 5.3 *Assume that A is nowhere full with respect to B. If $0 \neq e \in \mathcal{C}_A$ and $g \in B$ such that $0 < g \leq P(e)$, then there exists $p \in \mathcal{C}_A$ such that $0 \neq p \leq e$ and $P(p) \leq g$.*

Proof. Since $0 < g \leq P(e)$, there exists $\varepsilon > 0$ such that $[g - \varepsilon P(e)]^+ > 0$. Note that $e \leq \overline{e} \in \mathcal{C}_B$, so $P(e) \leq \overline{e}$ and hence

$$0<[g-\varepsilon P(e)]^{+}\leq g\leq P(e)\leq\overline{e}.$$

Let $q \in \mathcal{C}_B$ be the component of $\overline{e}$ in the band generated by $[g - \varepsilon P(e)]^+$. From the above observation it follows that $0 < q \leq \overline{e}$. Now $q\left[g - \varepsilon P(e)\right]^- = 0$ implies that

$$\varepsilon P(qe)=\varepsilon qP(e)\leq qg\leq g.$$

Furthermore, it follows from Lemma 4.4 that $qe \neq 0$. Now Lemma 5.2 implies that there exists $p \in \mathcal{C}_A$ such that $0 \neq p \leq qe$ and $P(p) \leq \varepsilon P(qe)$. Hence, $0 \neq p \leq e$ and $P(p) \leq \varepsilon P(qe) \leq g$. ■

Theorem 5.4 *Let A be a Dedekind complete f-algebra with unit element $\mathbf{1} \in A$ and let $P : A \to A$ be a positive order continuous projection onto the f-subalgebra B with $\mathbf{1} \in B$. Assume that A is nowhere full with respect to B. If $0 \neq e \in \mathcal{C}_A$ and $g \in B$ such that $0 \leq g \leq P(e)$, then there exists $p \in \mathcal{C}_A$ such that $p \leq e$ and $P(p) = g$.*

Proof. Define

$$\mathcal{K} = \{u \in \mathcal{C}_A(e) : P(u) \leq g\}.$$

It is clear that $\mathcal{K} \neq \emptyset$ and by the order continuity of P, any upward directed system in $\mathcal{K}$ has a supremum in $\mathcal{K}$. Hence, $\mathcal{K}$ has a maximal element $p \in \mathcal{K}$. Suppose that $P(p) < g$. Then

$$0 < g - P(p) \leq P(e) - P(p) = P(e - p).$$

By Lemma 5.3 there exists $q \in \mathcal{C}_A$ such that $0 < q \leq e - p$ and $P(q) \leq g - P(p)$. This implies that $p + q \in \mathcal{K}$, which contradicts the maximality of p. Consequently $P(p) = g$ and the proof is complete. ■

Note that the above theorem actually shows that the restriction $P : \mathcal{C}_A \to B$ is a full-valued (i.e., interval preserving) mapping.

5.1 An application

Now we shall discuss a consequence of the above theorem which may be of interest in its own right. In what follows L will be a Dedekind complete Riesz space and K will be a regular Riesz subspace of L which is itself Dedekind complete. Furthermore we will assume that K is order dense in L in the sense that $K^d = \{0\}$. Recall that $\mathcal{P}(L)$ and $\mathcal{P}(K)$ denote the Boolean algebras of band projections in L and K respectively. As is well known, every $\pi \in Z(K)$ has an extension $\widehat{\pi} \in Z(L)$ (see e.g. [8] Theorem 1.5) and this extension is unique since $K^{dd} = L$. It is easily verified that the mapping $\pi \longmapsto \widehat{\pi}$ is an f-algebra isomorphism from $Z(K)$ into $Z(L)$. Consequently we can identify $Z(K)$ with an f-subalgebra of $Z(L)$, i.e.,

$$Z(K) = \{\pi \in Z(L) : \pi(K) \subseteq K\}.$$

Then $Z(K)$ is actually a complete f-subalgebra of $Z(L)$.

Proposition 5.5 *Assume in addition that $0 < \varphi_0 \in L_n^{\sim}$ is a fixed strictly positive normal linear functional on L. Then there exists a unique strictly positive order continuous projection $P : Z(L) \to Z(L)$ onto $Z(K)$ such that*

$$\langle \pi f, \varphi_0 \rangle = \langle P(\pi) f, \varphi_0 \rangle \tag{25}$$

for all $f \in K$ and all $\pi \in Z(L)$.

Proof. First observe that $\varphi_{|K} \in K_n^\sim$ whenever $\varphi \in L_n^\sim$, as K is regularly embedded in L. Take $0 \leq \pi \in Z(L)$. Then $0 \leq \pi \leq nI_L$ for some $n \in \mathbb{N}$ and so $0 \leq (\varphi_0 \circ \pi)_{|K} \leq n\,(\varphi_0)_{|K}$ in $K_n^\sim$. Hence, by the Radon-Nikodym theorem for normal functionals (see [8], Theorem 3.4 or [16], Lemma 145.1), there exists $0 \leq \sigma \leq nI_K$ in $Z(K)$ such that

$$(\varphi_0 \circ \pi)_{|K} = (\varphi_0)_{|K} \circ \sigma. \tag{26}$$

We claim that this $\sigma \in Z(K)$ is uniquely determined by (26). Indeed, suppose that $\sigma_1, \sigma_2 \in Z(K)$ both satisfy (26) and put $\sigma_3 = \sigma_1 - \sigma_2$. This implies that $(\varphi_0)_{|K} \circ \sigma_3 = 0$ and so

$$(\varphi_0)_{|K} \circ |\sigma_3| = \left|(\varphi_0)_{|K} \circ \sigma_3\right| = 0$$

in $K_n^\sim$. Therefore $\langle |\sigma_3|\, u, \varphi_0 \rangle = 0$ and hence $|\sigma_3|\, u = 0$ for all $0 \leq u \in K$, as φ_0 is strictly positive. This shows that $\sigma_3 = 0$, by which the claim is proved. Therefore we may define $\sigma = P(\pi)$. It is clear that P is additive on $Z(L)^+$ and it follows that P extends to a positive projection $P : Z(L) \to Z(L)$ onto $Z(K)$. Note that for every $\pi \in Z(L)$ the element $P(\pi)$ is uniquely determined by (25). Since φ_0 is strictly positive and K order dense in L, it is clear that P is strictly positive. Finally we show that P is order continuous. To this end assume that $\pi_\alpha \downarrow 0$ in $Z(L)$ and that $\sigma \in Z(L)$ such that $P(\pi_\alpha) \geq \sigma \geq 0$ for all α. It follows from (25) that

$$0 \leq \langle \sigma u, \varphi_0 \rangle \leq \langle P(\pi_\alpha) u, \varphi_0 \rangle = \langle \pi_\alpha u, \varphi_0 \rangle \downarrow 0$$

and hence $\sigma u = 0$ for all $0 \leq u \in K$. Since K is order dense in L, we may conclude that $\sigma = 0$, and so $P(\pi_\alpha) \downarrow 0$ in $Z(L)$. This completes the proof of the proposition. ■

We will call the projection P in the above proposition *the conditional expectation projection associated with* φ_0. Note that $\mathcal{P}(L)$ and $\mathcal{P}(K)$ are the Boolean algebras of components of the unit element I in $Z\,(L)$ and $Z\,(K)$ respectively. Recall from Definition 4.7 that $Q_0 \in \mathcal{P}(L)$ is called $Z(K)$-full if

$$\{Q \in \mathcal{P}(L) : Q \leq Q_0\} = \{Q_0 R : R \in \mathcal{P}(K)\}\,. \tag{27}$$

In the next lemma we present some characterizations of $Z(K)$-full band projections in L.

Lemma 5.6 *Let $Q_0 \in \mathcal{P}(L)$ be the projection onto the band B_0 in L. The following statements are equivalent:*

(i). Q_0 is $Z(K)$-full;

(ii). for every $\sigma \in Z(B_0)$ there exists $\pi \in Z(K)$ such that $\pi_{|B_0} = \sigma$;

(iii). $[0, Q_0 u]_L = Q_0 [0, u]_K$ for all $0 \leq u \in K$ (where the subscripts L and K indicate that the order intervals are taken in L and K respectively);

(iv). $Q_0(K)$ is an ideal in L.

Proof. We will only show that (iv) implies (i), as the remaining implications follow by standard arguments. So we assume that $Q_0(K)$ is an ideal in L. Define $\overline{Q_0} \in \mathcal{P}(K)$ by

$$\overline{Q_0} = \inf \{R \in \mathcal{P}(K) : Q_0 \leq R\}$$

and define $\psi : \overline{Q_0}(K) \to Q_0(K)$ by $\psi(f) = Q_0 f$ for all $f \in \overline{Q_0}(K)$. It is clear that ψ is a surjective Riesz homomorphism. We claim that ψ is injective. Suppose that $0 \leq f \in \overline{Q_0}(K)$ and $Q_0 f = 0$. Since

$$I - \overline{Q_0} = \sup \{R \in \mathcal{P}(K) : RQ_0 = 0\}$$

and $\left(I - \overline{Q_0}\right) f = 0$, it follows that

$$\sup \{Rf : R \in \mathcal{P}(K), RQ_0 = 0\} = 0,$$

i.e., $Rf = 0$ for all $R \in \mathcal{P}(K)$ with $RQ_0 = 0$. Let R_f be the band projection in L onto $\{f\}^{dd}$. Since $f \in K$, it is easy to see that $R_f \in \mathcal{P}(K)$, and $Q_0 f = 0$ implies that $R_f Q_0 = 0$. Hence $f = 0$, which proves the claim.

Now let $Q \in \mathcal{P}(L)$ be given such that $Q \leq Q_0$. Let Q_1 be the restriction of Q to $Q_0(K)$. Then Q_1 is a band projection in $Q_0(K)$, as $Q_0(K)$ is an ideal in L. By the first part of the proof, there exists a band projection R in $\overline{Q_0}(K)$ such that $\psi \circ R = Q_1 \circ \psi$, i.e., $Q_0 R f = Q_1 Q_0 f = Qf$ for all $f \in \overline{Q_0}(K)$. Considering R as an element of $\mathcal{P}(K)$, this shows that $Q_0 R = Q$ and we may conclude that Q_0 satisfies (27), i.e., Q_0 is $Z(K)$-full. ∎

We will say that L is nowhere full with respect to K if 0 is the only band projection in L which is full with respect to $Z(K)$. By definition, this is equivalent to saying that $Z(L)$ is nowhere full with respect to $Z(K)$. Therefore the following result is now an immediate consequence of Theorem 5.4.

Theorem 5.7 *Let L be a Dedekind complete Riesz space and K an order dense Dedekind complete regular Riesz subspace of L, such that L is nowhere full with respect to K. Suppose that $0 < \varphi_0 \in L_n^\sim$ is strictly positive. If $\psi \in K^\sim$ is such that $0 \leq \psi \leq (\varphi_0)_{|K}$, then there exists a band projection $Q \in \mathcal{P}(L)$ such that $\psi = (\varphi_0 \circ Q)_{|K}$, i.e., ψ is the restriction of a component of φ_0.*

Proof. Let $P : Z(L) \to Z(L)$ be the conditional expectation projection onto $Z(K)$ associated with φ_0. By the Radon-Nikodym theorem in $K_n^\sim$, it follows from $0 \leq \psi \leq (\varphi_0)_{|K}$ that $\psi = (\varphi_0)_{|K} \circ \sigma$ for some $0 \leq \sigma \leq I$ in $Z(K)$. By Theorem 5.4 there exists $Q \in \mathcal{P}(L)$ in such that $\sigma = P(Q)$. Now it follows from (25) that

$$\langle Qf, \varphi_0 \rangle = \langle P(Q)f, \varphi_0 \rangle = \langle \sigma f, \varphi_0 \rangle = \langle f, \psi \rangle$$

for all $f \in K$, i.e., $(\varphi_0 \circ Q)_{|K} = \psi$. ■

5.2 Maharam's theorem

Next we will discuss the result of Maharam by which Theorem 5.4 was inspired. Let (X, Σ, μ) be a measure space. For the sake of simplicity we will assume that μ is finite. We denote the corresponding measure algebra by (Σ_μ, μ). Suppose that Λ is a σ-subalgebra of Σ and let Λ_μ denote the corresponding complete Boolean subalgebra of Σ_μ. As before we shall say that $p \in \Sigma_\mu$ is Λ_μ-full if

$$\{e \in \Sigma_\mu : e \leq p\} = \{eq : q \in \Lambda_\mu\},$$

and Σ_μ is called nowhere full with respect to Λ_μ if $0 \in \Sigma_\mu$ is the only Λ_μ-full element.

Theorem 5.8 (D. Maharam, [11]) *Assume that Σ_μ is nowhere full with respect to Λ_μ. If ν is a measure on Λ such that $0 \leq \nu(D) \leq \mu(D)$ for all $D \in \Lambda$, then there exists $F_0 \in \Sigma$ such that $\nu(D) = \mu(D \cap F_0)$ for all $D \in \Lambda$.*

Proof. Let $A = L_\infty(X, \Sigma, \mu)$ and $B = L_\infty(X, \Lambda, \mu)$. Then B is an order closed f-subalgebra of A. Let $P : A \to A$ be the conditional expectation projection onto B, i.e., $P = \mathbb{E}(\cdot \mid \Lambda)$, which is an order continuous positive projection. Since $\mathcal{C}_A \cong \Sigma_\mu$ and $\mathcal{C}_B \cong \Lambda_\mu$, it is clear that A is nowhere full with respect to B.

By the classical Radon-Nikodym theorem there exists $g \in B$ such that $0 \leq g \leq \mathbf{1}_X$ and

$$\nu(D) = \int_D g d\mu$$

for all $D \in \Lambda$. It follows from Theorem 5.4 that there exists $p \in \mathcal{C}_A = \Sigma_\mu$ such that $P(p) = g$. Let $F_0 \in \Sigma$ be such that $\mathbf{1}_{F_0}$ is a representative of p. From the defining property of $P = \mathbb{E}(\cdot \mid \Lambda)$ it follows that

$$\nu(D) = \int_D P(p) d\mu = \int_D \mathbf{1}_{F_0} d\mu = \mu(D \cap F_0)$$

for all $D \in \Lambda$, by which the theorem is proved. ■

6 Special subalgebras

As before we assume that A is a Dedekind complete f-algebra with unit element $\mathbf{1}$ and that B is an f-subalgebra with $\mathbf{1} \in B$. Furthermore we assume that $P : A \to A$ is an order continuous strictly positive projection onto B. In Section 7 we will construct complete Boolean subalgebras $\mathcal{E}$ of $\mathcal{C}_A$ with the property that for every $e \in \mathcal{E}$ there exists $\lambda \in [0,1]$ such that $P(e) = \lambda\mathbf{1}$. In the present section we will collect some properties of such special Boolean subalgebras.

First we introduce some notation. We define the subset $\mathcal{S} = \mathcal{S}_P$ of $\mathcal{C}_A$ by

$$\mathcal{S} = \{p \in \mathcal{C}_A : P(p) = \lambda\mathbf{1} \text{ for some } \lambda \in [0,1]\}. \tag{28}$$

The following simple observation will be useful.

Lemma 6.1 *The subset $\mathcal{S}$ is closed for monotone convergence in $\mathcal{C}_A$.*

Proof. Since $p \in \mathcal{S}$ implies that $\mathbf{1} - p \in \mathcal{S}$, it suffices to show that $\mathcal{S}$ is closed for upwards convergence in $\mathcal{C}_A$. To this end suppose that $p_\alpha \in \mathcal{S}$ and $p \in \mathcal{C}_A$ such that $p_\alpha \uparrow p$ in $\mathcal{C}_A$. Then $P(p_\alpha) = \lambda_\alpha \mathbf{1}$ for all α and $0 \leq \lambda_\alpha \uparrow \leq 1$. Let $\lambda = \sup_\alpha \lambda_\alpha$. By the order continuity of P we have $P(p_\alpha) \uparrow P(p)$ and so $P(p) = \lambda\mathbf{1}$. Hence $p \in \mathcal{S}$. ■

For any non-empty subset $\mathcal{D}$ of $\mathcal{C}_A$ we denote by $\mathfrak{S}(\mathcal{D})$ the complete Boolean subalgebra of $\mathcal{C}_A$ generated by $\mathcal{D}$.

Lemma 6.2 *Let $\mathcal{S} \subseteq \mathcal{C}_A$ be defined by (28).*

(i). If $\mathcal{F}$ is a Boolean subalgebra of $\mathcal{C}_A$ and $\mathcal{F} \subseteq \mathcal{S}$, then $\mathfrak{S}(\mathcal{F}) \subseteq \mathcal{S}$.

(ii). If $\mathcal{F}$ is a Boolean subalgebra of $\mathcal{C}_A$ such that $\mathcal{F} \subseteq \mathcal{S}$, and if we write $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{F}$, then $\lambda : \mathcal{F} \to [0,1]$ is a strictly positive (in general, finitely additive) measure on $\mathcal{F}$.

(iii). If $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$ such that $\mathcal{E} \subseteq \mathcal{S}$, and if we write $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$, then $\lambda : \mathcal{E} \to [0,1]$ is a completely additive and strictly positive measure.

Proof.

(i). This follows via a standard argument for Boolean algebras from Lemma 6.1.

(ii). If $e_1, e_2 \in \mathcal{F}$ such that $e_1 \wedge e_2 = 0$, then

$$P(e_1 \vee e_2) = P(e_1 + e_2) = \{\lambda(e_1) + \lambda(e_2)\}\, \mathbf{1},$$

hence $\lambda(e_1 \vee e_2) = \lambda(e_1) + \lambda(e_2)$, which shows that λ is a finitely additive measure on $\mathcal{F}$. The measure λ is strictly positive as P is strictly positive.

(iii). Since P is order continuous and $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$ it follows immediately that λ is completely additive.

■

We assume that $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$, such that $\mathcal{E} \subseteq \mathcal{S}$, so there exists a completely additive strictly positive measure $\lambda : \mathcal{E} \to [0,1]$ such that $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$. Recall from Section 4 the definitions of the complete f-subalgebra $A(\mathcal{E})$ generated by $\mathcal{E}$ and of the f-subalgebra $S(\mathcal{E})$ given by (2). If $s \in S(\mathcal{E})$ is given by $s = \sum_{i=1}^{n} \alpha_i e_i$ with $\alpha_i \in \mathbb{R}$ and $e_i \in \mathcal{E}$, then $P(s) = \varphi_P(s)\,\mathbf{1}$, where

$$\varphi_P(s) = \sum_{i=1}^{n} \alpha_i \lambda(e_i)\,.$$

Clearly, φ_P is a positive functional on $S(\mathcal{E})$. If $0 \leq A(\mathcal{E})$ then, by Proposition 2.6, there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in $S(\mathcal{E})$ such that $0 \leq s_n \uparrow f$. By the order continuity of P we then have $\varphi_P(s_n)\,\mathbf{1} \uparrow P(f)$, which implies that there exists $\lambda \in \mathbb{R}$ such that $P(f) = \lambda\mathbf{1}$. Consequently, φ_P extends to a positive functional on $A(\mathcal{E})$, denoted by φ_P as well, such that

$$P(f) = \varphi_P(f)\,\mathbf{1} \tag{29}$$

holds for all $f \in A(\mathcal{E})$. Since P is order continuous and $A(\mathcal{E})$ regularly embedded in A, it follows that φ_P is order continuous and strictly positive on $A(\mathcal{E})$.

We assume that $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$ such that $\mathcal{C}_A \subseteq \mathcal{S}$ and $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$. We will denote by $K(\mathcal{C}_B, \mathcal{E})$ the collection of all $f \in A$ which can be written as

$$f = \sum_{i=1}^{n} q_i f_i, \tag{30}$$

where $q_i \in \mathcal{C}_B$, $f_i \in A(\mathcal{E})$ for all $i = 1, \ldots n$ and $n \in \mathbb{N}$. It is clear that $K(\mathcal{C}_B, \mathcal{E})$ is a subalgebra of A with $\mathcal{C}_B \subseteq K(\mathcal{C}_B, \mathcal{E})$ and $\mathcal{E} \subseteq A(\mathcal{E}) \subseteq$

$K(\mathcal{C}_B, \mathcal{E})$. If $f \in K(\mathcal{C}_B, \mathcal{E})$, then it is easy to see that f can be written as is (30) where $\{q_i\}_{i=1}^n$ a disjoint system in $\mathcal{C}_B$ with $\bigvee_{i=1}^n q_i = \mathbf{1}$ and $f_i \neq f_j$ whenever $i \neq j$. This will be called the *standard form* of f. We claim that this standard form of an element $f \in K(\mathcal{C}_B, \mathcal{E})$ is unique. First we observe that if $p, q \in \mathcal{C}_B$ and $g, h \in A(\mathcal{E})$ such that $pg = qh \neq 0$, then $p = q$ and $g = h$. Indeed, using the averaging property of P and (29) we find that

$$\varphi_P(g)\, p = pP(g) = P(pg) = P(qh) = qP(h) = \varphi_P(h)\, q.$$

Since $\varphi_P(g)\, p \neq 0$, it follows that $p = q$. Hence, $p(g-h) = 0$, so $p|g-h| = 0$, which implies that

$$\varphi_P(|g-h|)\, p = pP(|g-h|) = P(p|g-h|) = 0.$$

Therefore $\varphi_P(|g-h|) = 0$, as $p \neq 0$, and since φ_P is strictly positive we may conclude that $g = h$. From this observation the uniqueness of the standard form of elements in $K(\mathcal{C}_B, \mathcal{E})$ follows by a standard argument. For future reference we collect some properties of $K(\mathcal{C}_B, \mathcal{E})$ in the next lemma.

Lemma 6.3 *With the notation as introduced above, the following hold.*

(i). $K(\mathcal{C}_B, \mathcal{E})$ is an f-subalgebra of A.

(ii). The order closure of $K(\mathcal{C}_B, \mathcal{E})$ in A is equal to A.

Proof.

(i). As observed already, $K(\mathcal{C}_B, \mathcal{E})$ is a subalgebra of A. Writing $f \in K(\mathcal{C}_B, \mathcal{E})$ in standard form $f = \sum_{i=1}^n q_i f_i$, it follows that $|f| = \sum_{i=1}^n q_i |f_i|$ and so $|f| \in K(\mathcal{C}_B, \mathcal{E})$.

(ii). Let $L = \overline{K(\mathcal{C}_B, \mathcal{E})}^{(o)}$ be the order closure of $K(\mathcal{C}_B, \mathcal{E})$ in A. Then L is regularly embedded in A and L is Dedekind complete. Hence the Boolean algebra $\mathcal{C}_L$ of components of $\mathbf{1}$ in L is a complete Boolean subalgebra of $\mathcal{C}_A$. Since $\mathcal{C}_B \subseteq \mathcal{C}_L$ and $\mathcal{E} \subseteq \mathcal{C}_L$, this implies that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E}) = \mathcal{C}_L$. Now it follows from the Freudenthal spectral theorem that $L = A$, as L is a complete Riesz subspace of A.

■

Now suppose that A_1 and A_2 are two Dedekind complete f-algebras with unit elements $\mathbf{1}_{A_1}$ and $\mathbf{1}_{A_2}$ respectively. For $i = 1, 2$, let $\mathbf{1}_{A_i} \in B_i \subseteq A_i$ be an f-subalgebra and let $P_i : A_i \to A_i$ be a strictly positive order continuous projection onto B_i. Furthermore we assume that $\mathcal{E}_i \subseteq \mathcal{C}_{A_i}$ is a complete

Boolean subalgebra such that $\mathcal{C}_{A_i} = \mathfrak{S}(\mathcal{C}_{B_i} \cup \mathcal{E}_i)$ and that $\lambda_i : \mathcal{E}_i \to [0,1]$ is a completely additive measure satisfying $P_i(e) = \lambda_i(e)\,\mathbf{1}_{A_i}$ for all $e \in \mathcal{E}_i$. In addition to this, we assume in the following proposition that the unit elements $\mathbf{1}_{A_i}$ are actually *strong order units* in A_i $(i = 1, 2)$.

Proposition 6.4 *In the above situation, suppose that:*

(i). $\alpha : A_1(\mathcal{E}_1) \to A_2(\mathcal{E}_2)$ *is an f-algebra homomorphisms with* $\alpha(\mathbf{1}_{A_1}) = \mathbf{1}_{A_2}$ *and* $\lambda_2(\alpha(e)) = \lambda_1(e)$ *for all* $e \in \mathcal{E}_1$*;*

(ii). $\beta : B_1 \to B_2$ *is an order continuous f-algebra homomorphisms with* $\beta(\mathbf{1}_{A_1}) = \mathbf{1}_{A_2}$.

Then there exists an f-algebra homomorphism $\Psi : A_1 \to A_2$ *such that:*

(a). $\Psi_{|A_1(\mathcal{E}_1)} = \alpha$ *and* $\Psi_{|B_1} = \beta$*;*

(b). $\Psi \circ P_1 = P_2 \circ \Psi$.

Moreover, Ψ is order continuous and unique. If α and β are surjective isomorphisms, then Ψ is a surjective isomorphism as well.

Proof. First we show that an f-algebra homomorphism $\Psi : A_1 \to A_2$ satisfying (a) and (b) is necessarily order continuous. To this end suppose that $f_\tau \downarrow 0$ in A_1 and that $\Psi(f_\tau) \geq g \geq 0$ for all τ and some $g \in A_2$. Then $P_2\Psi(f_\tau) \geq P_2 g \geq 0$ for all τ and

$$P_2\Psi(f_\tau) = \Psi P_1(f_\tau) = \beta P_1(f_\tau) \downarrow 0,$$

since P_1 and β are order continuous. Therefore $P_2 g = 0$, which implies that $g = 0$, as P_2 is strictly positive. This shows that Ψ is order continuous. To prove the uniqueness, assume that $\Psi' : A_1 \to A_2$ is an f-algebra homomorphism satisfying (a) and (b) as well. Define

$$L = \{f \in A_1 : \Psi(f) = \Psi'(f)\}.$$

Then L is an f-subalgebra of A_1 and the order continuity of Ψ and Ψ' implies that L is order closed. Furthermore, it is clear that B_1 and $A_1(\mathcal{E}_1)$ are contained in L, hence $K(\mathcal{C}_{B_1}, \mathcal{E}_1) \subseteq L$. It follows from Lemma 6.3 that $L = A_1$, consequently $\Psi = \Psi'$.

Now we turn to the existence proof of the homomorphism Ψ. For $i = 1, 2$, let $0 \leq \varphi_{P_i} \in A_i(\mathcal{E}_i)_n^\sim$ be the strictly positive functional satisfying $P_i(f) =$

$\varphi_{P_i}(f)\,\mathbf{1}_{A_i}$ for all $f \in A_i(\mathcal{E}_i)$ (see (29)). Since $\lambda_2(\alpha(e)) = \lambda_1(e)$ for all $e \in \mathcal{E}_1$ it follows that $\varphi_{P_2}(\alpha(f)) = \varphi_{P_1}(f)$ for all $f \in A_1(\mathcal{E}_1)$. We define

$$\Psi_0 : K(\mathcal{C}_{B_1}, \mathcal{E}_1) \to K(\mathcal{C}_{B_2}, \mathcal{E}_2)$$

by

$$\Psi_0\left(\sum_{i=1}^{n} q_i f_i\right) = \sum_{i=1}^{n} \beta(q_i)\,\alpha(f_i) \tag{31}$$

for all $\{q_i\}_{i=1}^n$ in $\mathcal{C}_{B_1}$ and all $\{f_i\}_{i=1}^n$ in $A_1(\mathcal{E}_1)$. First we observe that Ψ_0 is well defined. Indeed, if $f = \sum_{i=1}^n q_i f_i$ has standard form $f = \sum_{j=1}^m p_j g_j$, then it is easy to see that

$$\sum_{i=1}^{n} \beta(q_i)\,\alpha(f_i) = \sum_{j=1}^{m} \beta(p_j)\,\alpha(g_j).$$

Since the standard form of an element in $K(\mathcal{C}_{B_1}, \mathcal{E}_1)$ is unique, it follows that Ψ_0 is well defined by (31). Since α and β are f-algebra homomorphisms it is now also clear that Ψ_0 is an f-algebra homomorphism.

Take $f \in K(\mathcal{C}_{B_1}, \mathcal{E}_1)$ and write $f = \sum_{i=1}^n q_i f_i$ with $\{q_i\}_{i=1}^n$ in $\mathcal{C}_{B_1}$ and $\{f_i\}_{i=1}^n$ in $A_1(\mathcal{E}_1)$. Then

$$P_1 f = \sum_{i=1}^{n} P_1(q_i f_i) = \sum_{i=1}^{n} q_i P_1(f_i) = \sum_{i=1}^{n} \varphi_{P_1}(f_i)\, q_i$$

and so $P_1[K(\mathcal{C}_{B_1}, \mathcal{E}_1)] \subseteq K(\mathcal{C}_{B_1}, \mathcal{E}_1)$. Moreover, using that $\varphi_{P_2}(\alpha(f_i)) = \varphi_{P_1}(f_i)$, we find that

$$\begin{aligned}
\Psi_0(P_1 f) &= \sum_{i=1}^{n} \varphi_{P_1}(f_i)\,\beta(q_i) = \sum_{i=1}^{n} \varphi_{P_2}(\alpha(f_i))\,\beta(q_i) \\
&= \sum_{i=1}^{n} \beta(q_i)\, P_2(\alpha(f_i)) = \sum_{i=1}^{n} P_2(\beta(q_i)\,\alpha(f_i)) \\
&= P_2\left(\sum_{i=1}^{n} \beta(q_i)\,\alpha(f_i)\right) = P_2(\Psi_0 f).
\end{aligned}$$

This shows that $\Psi_0 \circ P_1 = P_2 \circ \Psi_0$.

Let K_1 be the $\mathbf{1}_{A_1}$-uniform closure of $K(\mathcal{C}_{B_1}, \mathcal{E}_1)$ in A_1. Then Ψ_0 extends uniquely to an f-algebra homomorphism $\Psi_1 : K_1 \to A_2$. Since $\mathcal{C}_{B_1} \subseteq K_1$ and $\Psi_1(q) = \Psi_0(q) = \beta(q)$ for all $q \in \mathcal{C}_{B_1}$, it follows from Freudenthal's spectral theorem that $B_1 \subseteq K_1$ and $\Psi_1(g) = \beta(g)$ for all $g \in B_1$. Note that at this point we use that $\mathbf{1}_{A_1}$ is a strong order unit in A_1, which guarantees that every $f \in B_1$ is a $\mathbf{1}_{A_1}$-uniform limit if a sequence in $S(\mathcal{C}_{B_1})$. Moreover, it is easy to see that $\Psi_1 \circ P_1 = P_2 \circ \Psi_1$.

Now let $\mathcal{K}$ be the collection of all pairs (K, Ψ_K), where:

(i). K is an f-subalgebra of A_1 such that $K_1 \subseteq K$;

(ii). $\Psi_K : K \to A_2$ is an f-algebra homomorphism such that

$$(\Psi_K)_{|K_1} = \Psi_1;$$

(iii). $\Psi_K \circ P_1 = P_2 \circ \Psi_K$.

We note that the reason for introducing the uniformly closed f-subalgebra K_1 is that $B_1 \subseteq K_1$ and $K_1 \subseteq K$ imply that $P_1(K) \subseteq K$ and so (iii) makes sense. From the above observations it is clear that $(K_1, \Psi_1) \in \mathcal{K}$, so $\mathcal{K}$ is non-empty. If $(K, \Psi_K), (K', \Psi_{K'}) \in \mathcal{K}$, then we will say that $(K, \Psi_K) \preceq (K', \Psi_{K'})$ whenever $K \subseteq K'$ and $(\Psi_{K'})_{|K} = \Psi_K$. It is easy to see that every chain in the partially ordered set $(\mathcal{K}, \preceq)$ has an upper bound. Hence, by Zorn's lemma, $(\mathcal{K}, \preceq)$ contains a maximal element (K_m, Ψ_m). It is our aim to show that $K_m = A_1$.

Suppose that $\{g_\tau : \tau \in \mathbb{T}\}$ is a net in K_m such that $g_\tau \xrightarrow{(o)} 0$ in A_1. We claim that $\Psi_m(g_\tau) \xrightarrow{(o)} 0$ in A_2. Without loss of generality we may assume that $\{g_\tau\}$ is order bounded in A_1. Since $\mathbf{1}_{A_1}$ is a strong order unit in A_1 and $\Psi_m(\mathbf{1}_{A_1}) = \mathbf{1}_{A_2}$, it is clear that $\{\Psi_m(g_\tau)\}$ is order bounded in A_2. For $\tau \in \mathbb{T}$ define

$$u_\tau = \bigvee_{\sigma \geq \tau} |g_\sigma|, \qquad h_\tau = \bigvee_{\sigma \geq \tau} |\Psi_m(g_\sigma)|,$$

and assume that $h_\tau \geq h \geq 0$ for all $\tau \in \mathbb{T}$ and some $h \in A_2$. Fix τ and define for any finite subset F of $\{\sigma : \sigma \geq \tau\}$ the element $h_F = \bigvee_{\sigma \in F} |\Psi_m(g_\sigma)|$. Then

$$P_2 h_F = P_2 \Psi_m \left(\bigvee_{\sigma \in F} |g_\sigma| \right) = \Psi_m P_1 \left(\bigvee_{\sigma \in F} |g_\sigma| \right) = \beta P_1 \left(\bigvee_{\sigma \in F} |g_\sigma| \right).$$

Since $h_F \uparrow_F h_\tau$ and $\bigvee_{\sigma \in F} |g_\sigma| \uparrow_F u_\tau$, and since β, P_1 and P_2 are order continuous, it follows that $P_2 h_\tau = \beta P_1 u_\tau$. Hence $\beta P_1 u_\tau \geq P_2 h \geq 0$ for all τ. By hypothesis we have $u_\tau \downarrow 0$ in A_1 and so $\beta P_1 u_\tau \downarrow 0$ in A_2. This implies that $P_2 h = 0$ and since P_2 is strictly positive we may conclude that $h = 0$, by which our claim is proved.

Now we are in a position to apply Lemma 2.14, from which it follows that Ψ_m has a unique extension

$$\Psi'_m : K'_m \to A_2$$

with the property that $\Psi_m(f_\tau) \xrightarrow{(o)} \Psi'(f)$ in A_2 whenever $f \in K'_m$ and $\{f_\tau\}$ is a net in K_m such that $f_\tau \xrightarrow{(o)} f$ in A_1. It is straightforward to verify that K'_m is an f-subalgebra of A_1 and that Ψ'_m is an f-algebra homomorphism. Hence the pair (K'_m, Ψ'_m) satisfies (i) and (ii) above. To show that condition (iii) is satisfied as well, take any $f \in K'_m$ and let $\{f_\tau\}$ be a net in K_m such that $f_\tau \xrightarrow{(o)} f$ in A_1. By the order continuity of P_1 of follows that $P_1 f_\tau \xrightarrow{(o)} P_1 f$ and so $\Psi_m(P_1 f_\tau) \xrightarrow{(o)} \Psi'_m(P_1 f)$. On the other hand, since $\Psi_m f_\tau \xrightarrow{(o)} \Psi'_m f_\tau$ and P_2 is order continuous we find that

$$\Psi_m(P_1 f_\tau) = P_2(\Psi_m f_\tau) \xrightarrow{(o)} P_2(\Psi'_m f),$$

consequently $\Psi'_m(P_1 f) = P_2(\Psi'_m f)$. This shows that $\Psi'_m \circ P_1 = P_2 \circ \Psi'_m$and so $(K'_m, \Psi'_m) \in \mathcal{K}$.

Since (K'_m, Ψ'_m) is a maximal element of $\mathcal{K}$, we may conclude that $K'_m = K_m$. Therefore, K_m is order closed in A_1. By Lemma 6.3, the order closure of $K(\mathcal{C}_{B_1}, \mathcal{E}_1)$ is equal to A_1, so $K_m = A_1$. Hence $\Psi = \Psi_m$ is the desired f-algebra homomorphism.

Finally, suppose that α and β are surjective isomorphisms. Then we can apply the above construction to α^{-1} and β^{-1} to obtain a corresponding f-algebra homomorphism $\Psi_1 : A_2 \to A_1$. Then it is clear that $\Psi_1(\Psi f) = f$ for all $f \in K(\mathcal{C}_{B_1}, \mathcal{E}_1)$. Defining

$$M = \{f \in A_1 : \Psi_1(\Psi f) = f\},$$

it follows that M is an order closed f-subalgebra of A_1 with $K(\mathcal{C}_{B_1}, \mathcal{E}_1) \subseteq M$. Again using Lemma 6.3 we conclude that $M = A_1$, so $\Psi_1(\Psi f) = f$ for all $f \in A_1$. Similarly we see that $\Psi(\Psi_1 g) = g$ for all $g \in A_2$, hence Ψ is a surjective isomorphism. By this the proof of the proposition is complete. ■

7 The construction of special subalgebras

In this section we will discuss in some more detail the structure of positive projections onto f-subalgebras. The construction presented in this section are inspired by the results in [10], Section 14. We start by listing the hypotheses which will be assumed throughout this section:

- A is a Dedekind complete f-algebra with unit element $\mathbf{1} \in A$;
- B is an f-subalgebra of A with $\mathbf{1} \in B$;
- $P : A \to A$ is a strictly positive order continuous projection onto B;
- A is nowhere full with respect to B.

As before, for any non-empty subset $\mathcal{D} \subseteq \mathcal{C}_A$ we will denote by $\mathfrak{S}(\mathcal{D})$ the complete Boolean subalgebra of $\mathcal{C}_A$ generated by $\mathcal{D}$, i.e., $\mathfrak{S}(\mathcal{D})$ is the smallest complete Boolean subalgebra of $\mathcal{C}_A$ containing the set $\mathcal{D}$. The main objective in the present section is to show that given any sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{C}_A$, there exists a complete Boolean subalgebra $\mathcal{E}$ of $\mathcal{C}_A$ such that $a_n \in \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$ for all n and $\mathcal{E} \subseteq \mathcal{S}$. Here $\mathcal{S} = \mathcal{S}_P$ is defined by (28). It follows from Lemma 6.2 that the inclusion $\mathcal{E} \subseteq \mathcal{S}$ is equivalent to saying that there exists a completely additive strictly positive measure $\lambda : \mathcal{E} \to [0,1]$ such that $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$. The proof will be divided in a number of lemmas.

Lemma 7.1 *Let $a, p \in \mathcal{C}_A$ be given. Then there exist elements $H_k(p,a) \in \mathcal{C}_A$ ($k = 0,1,2,3$) such that*

(i). $H_0(p,a), ..., H_3(p,a)$ are mutually disjoint;

(ii). $H_0(p,a) + H_1(p,a) = ap$;

(iii). $H_2(p,a) + H_3(p,a) = (\mathbf{1}-a)\,p$;

(iv). $\sum_{k=0}^{3} H_k(p,a) = p$;

(v). $P(H_k(p,a)) \leq \frac{1}{2}P(p)$ for $k = 0,1,2,3$.

Proof. Since $0 \leq \frac{1}{2}P(ap) \leq P(ap)$ in B, it follows from Theorem 5.4 that there exists $H_0(p,a) \in \mathcal{C}_A$ such that

$$H_0(p,a) \leq ap \quad \text{and} \quad P(H_0(p,a)) = \frac{1}{2}P(ap).$$

Define $H_1(p,a) = ap - H_0(p,a)$. Similarly there exists $H_2(p,a) \in \mathcal{C}_A$ such that

$$H_2(p,a) \leq (\mathbf{1}-a)\,p \quad \text{and} \quad P(H_2(p,a)) = \frac{1}{2}P((\mathbf{1}-a)\,p).$$

Define $H_3(p,a) = (\mathbf{1}-a)\,p - H_2(p,a)$. It is clear that the elements $H_k(p,a)$ satisfy all the requirements. ■

Observe that the elements $H_k(p,a)$ as constructed in the above lemma are in general not uniquely determined by a and p. Before stating the next lemma we introduce some notation. For $n \in \mathbb{N}$ and $0 \leq j \leq 4^n - 1$ we define the intervals

$$I_{n,j} = \left[j4^{-n}, (j+1)\,4^{-n}\right).$$

For each n the intervals $I_{n,0}, ..., I_{n,4^n-1}$ are a partition of $I_{0,0} = [0,1)$. Furthermore, for all $n \in \mathbb{N}$ and $0 \leq j \leq 4^n - 1$ we have

$$I_{n,j} = \bigcup_{k=0}^{3} I_{n+1,4j+k}. \tag{32}$$

We denote by $\mathfrak{A}_n$ the algebra of subsets of $[0,1)$ generated by the intervals $\{I_{n,j} : 0 \leq j \leq 4^n - 1\}$. From (32) it is clear that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ for all n. Define $\mathfrak{A}_\infty = \cup_{n=0}^{\infty} \mathfrak{A}_n$, which is an algebra of subsets of $[0,1)$. Let $\mathbb{D}$ denote the set of all dyadic numbers in $[0,1]$. It is clear that $\mathfrak{A}_\infty$ is equal to the algebra generated by all intervals $[\alpha, \beta)$ with $\alpha, \beta \in \mathbb{D}$.

Let $\{a_n\}_{n=1}^{\infty}$ be a fixed sequence in $\mathcal{C}_A$.

Lemma 7.2 *There exists a Boolean homomorphism $h : \mathfrak{A}_\infty \to \mathcal{C}_A$ such that:*

(a). $h(I_{0,0}) = \mathbf{1}$;

(b). $P\left[h\left(I_{n,j}\right)\right] \leq 2^{-n}\mathbf{1}$ for all $n \in \mathbb{N}$ and $0 \leq j \leq 4^n - 1$;

(c). $a_n \in h(\mathfrak{A}_\infty)$ for all $n = 1, 2, ...$

Proof. We define recursively Boolean homomorphisms $h_n : \mathfrak{A}_n \to \mathcal{C}_A$ with $(h_{n+1})_{|\mathfrak{A}_n} = h_n$ for all $n \in \mathbb{N}$, as follows. The homomorphism $h_0 : \mathfrak{A}_0 \to \mathcal{C}_A$ is simply given by $h(I_{0,0}) = \mathbf{1}$ and $h(\emptyset) = 0$. Now assume that $h_n : \mathfrak{A}_n \to \mathcal{C}_A$ has been defined for some $n \in \mathbb{N}$. For $0 \leq j \leq 4^n - 1$ and $0 \leq k \leq 3$ we put

$$h_{n+1}\left(I_{n+1,4j+k}\right) = H_k\left(h_n\left(I_{n,j}\right), a_{n+1}\right). \tag{33}$$

Using (i) and (iv) of Lemma 7.1 it is easy to see that this defines a Boolean homomorphism $h_{n+1} : \mathfrak{A}_{n+1} \to \mathcal{C}_A$ such that $(h_{n+1})_{|\mathfrak{A}_n} = h_n$. We claim that

$$P\left[h_n\left(I_{n,j}\right)\right] \leq 2^{-n}\mathbf{1} \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq j \leq 4^n - 1. \tag{34}$$

Indeed, for $n = 0$ this is trivial. Furthermore, by (v) in Lemma 7.1 and (33) we have

$$P\left[h_{n+1}\left(I_{n+1,4j+k}\right)\right] \leq \frac{1}{2} P\left[h_n\left(I_{n,j}\right)\right],$$

and (34) now follows by induction on n. Next we observe that it follows from (33) and Lemma 7.1 (ii) that

$$\begin{aligned} h_{n+1}&\left(\bigcup_{j=0}^{4^n-1} I_{n+1,4j}\cup I_{n+1,4j+1}\right)\\ &=\sum_{j=0}^{4^n-1}\left\{h_{n+1}\left(I_{n+1,4j}\right)+h_{n+1}\left(I_{n+1,4j+1}\right)\right\}\\ &=\sum_{j=0}^{4^n-1}\left\{H_0\left(h_n\left(I_{n,j}\right),a_{n+1}\right)+H_1\left(h_n\left(I_{n,j}\right),a_{n+1}\right)\right\}\\ &=\sum_{j=0}^{4^n-1} h_n\left(I_{n,j}\right)a_{n+1}=h(I_{0,0})a_{n+1}=a_{n+1}, \end{aligned}$$

and so $a_{n+1}\in h_{n+1}\left(\mathfrak{A}_{n+1}\right)$ for all $n\in\mathbb{N}$.

Defining the Boolean homomorphism $h:\mathfrak{A}_\infty\to\mathcal{C}_A$ by $h_{|\mathfrak{A}_n}=h_n$ for all $n\in\mathbb{N}$ it is clear that h has all the desired properties. ■

Lemma 7.3 *There exists a function $p_\bullet:\mathbb{D}\to\mathcal{C}_A$ such that:*

(i). $p_0=0$ and $p_1=\mathbf{1}$;

(ii). if $\tau_1\le\tau_2$ in $\mathbb{D}$, then $p_{\tau_1}\le p_{\tau_2}$ in $\mathcal{C}_A$;

(iii). if $\tau_1,\tau_2\in\mathbb{D}$ such that $0\le\tau_2-\tau_1<4^{-n}$ for some $n\in\mathbb{N}$, then

$$0\le P\left(p_{\tau_2}\right)-P\left(p_{\tau_1}\right)\le 2^{-n+1}\mathbf{1}$$

in $\mathcal{C}_A$;

(iv). the sequence $\{a_n\}_{n=1}^\infty$ belongs to the algebra generated by $\{p_\tau:\tau\in\mathbb{D}\}$ in $\mathcal{C}_A$.

Proof. Let $h:\mathfrak{A}_\infty\to\mathcal{C}_A$ be the Boolean homomorphism constructed in the previous lemma. For $\tau\in\mathbb{D}$ we define $p_\tau=h([0,\tau))$. It is clear that $p_\bullet$ has properties (i) and (ii). Since $h([\tau_1,\tau_2))=p_{\tau_2}-p_{\tau_1}$ for all $\tau_1\le\tau_2$ in $\mathbb{D}$, the algebra generated by $\{p_\tau:\tau\in\mathbb{D}\}$ is equal to $h(\mathfrak{A}_\infty)$ and so (iv) follows from (c) in the above lemma. It remains to proof (iii). If $\tau_1,\tau_2\in\mathbb{D}$ such that $0\le\tau_2-\tau_1<4^{-n}$, then $[\tau_1,\tau_2)\subseteq I_{n,j}\cup I_{n,j+1}$ for some $0\le j\le 4^n-1$. Hence,

$$0\le p_{\tau_2}-p_{\tau_1}=h([\tau_1,\tau_2))\le h(I_{n,j})+h(I_{n,j+1}),$$

and so by (b) in Lemma 7.2,

$$0 \le P(p_{\tau_2}) - P(p_{\tau_1}) \le P\left[h(I_{n,j})\right] + P\left[h(I_{n,j+1})\right] \le 2^{-n+1}\mathbf{1}.$$

■

Given the system $\{p_\tau : \tau \in \mathbb{D}\}$ as in Lemma 7.3 we denote by $q(\tau,\lambda) \in \mathcal{C}_B$ the component of $\mathbf{1}$ in the band $\left\{[P(p_\tau) - \lambda\mathbf{1}]^+\right\}^d$ for all $\lambda \in [0,1]$. In the following lemma we collect some of the relevant properties of the system $\{q(\tau,\lambda) : \tau \in \mathbb{D}, 0 \le \lambda \le 1\}$.

Lemma 7.4 *(a). If $q \in \mathcal{C}_B$ such that $0 \le q \le q(\tau,\lambda)$, then $qP(p_\tau) \le \lambda q$.*

(b). If $q \in \mathcal{C}_B$ such that $0 < q \le \mathbf{1} - q(\tau,\lambda)$, then $qP(p_\tau) > \lambda q$.

(c). If $\tau_1 \le \tau_2$ in $\mathbb{D}$, then $q(\tau_1,\lambda) \ge q(\tau_2,\lambda)$ for all $0 \le \lambda \le 1$.

(d). If $0 \le \lambda_1 \le \lambda_2 \le 1$, then $q(\tau,\lambda_1) \le q(\tau,\lambda_2)$ for all $\tau \in \mathbb{D}$.

(e). $q(0,\lambda) = \mathbf{1}$ for all $0 \le \lambda \le 1$, $q(1,\lambda) = 0$ for all $0 \le \lambda < 1$, and $q(1,1) = \mathbf{1}$.

(f). $q(\tau,1) = \mathbf{1}$ for all $\tau \in \mathbb{D}$, and $q(\tau,0)$ is the component of $\mathbf{1}$ in $\{P(p_\tau)\}^d$ for all $\tau \in \mathbb{D}$.

(g). $q(\tau,0)p_\tau = 0$ for all $\tau \in \mathbb{D}$.

Proof. Properties (c), (d), (e) and (f) are obvious. To prove (a), take $q \in \mathcal{C}_B$ such that $0 \le q \le q(\tau,\lambda)$. Then $q \wedge [P(p_\tau) - \lambda\mathbf{1}]^+ = 0$, so

$$[qP(p_\tau) - \lambda q]^+ = q\left[P(p_\tau) - \lambda\mathbf{1}\right]^+ = 0,$$

which shows that $qP(p_\tau) \le \lambda q$.

Now take $q \in \mathcal{C}_B$ such that $0 < q \le \mathbf{1} - q(\tau,\lambda)$. Since $\mathbf{1} - q(\tau,\lambda)$ is the component of $\mathbf{1}$ in the band $\left\{[P(p_\tau) - \lambda\mathbf{1}]^+\right\}^{dd}$, it is clear that $q\left[P(p_\tau) - \lambda\mathbf{1}\right]^+ > 0$. Moreover $q\left[P(p_\tau) - \lambda\mathbf{1}\right]^- = 0$, so

$$q\left[P(p_\tau) - \lambda\mathbf{1}\right] = q\left[P(p_\tau) - \lambda\mathbf{1}\right]^+ > 0$$

and this proves (b).

Finally, for the proof of (g), observe that $q(\tau,0)P(p_\tau) = 0$, as $q(\tau,0)$ is the component of $\mathbf{1}$ in $\{P(p_\tau)\}^d$. Hence,

$$P\left(q(\tau,0)p_\tau\right) = q(\tau,0)P\left(p_\tau\right) = 0$$

and since P is strictly positive it follows that $q(\tau,0)p_\tau$. ■

Now we define the system $\{e_\lambda : 0 \leq \lambda \leq 1\}$ in $\mathcal{C}_A$ by

$$e_\lambda = \sup\left\{q(\tau,\lambda)p_\tau : \tau \in \mathbb{D}\right\}. \tag{35}$$

From (d) in Lemma 7.4 it is clear that $e_{\lambda_1} \leq e_{\lambda_2}$ whenever $0 \leq \lambda_1 \leq \lambda_2 \leq 1$. Since $q(\tau,0)p_\tau = 0$ for all $\tau \in \mathbb{D}$ and since $q(1,1)p_1 = \mathbf{1}$, it follows that $e_0 = 0$ and $e_1 = \mathbf{1}$. Our next objective is to show that $P(e_\lambda) = \lambda\mathbf{1}$ for all $\lambda \in [0,1]$. The proof is divided in two lemmas.

Lemma 7.5 *For all* $0 \leq \lambda \leq 1$ *we have* $P(e_\lambda) \leq \lambda\mathbf{1}$.

Proof. For $\lambda = 1$ the inequality is trivial, so we may assume that $0 \leq \lambda < 1$. Since P is order continuous, it is sufficient to show that

$$P\left(\bigvee_{k=0}^{N} q\left(\tau_k,\lambda\right)p_{\tau_k}\right) \leq \lambda\mathbf{1}$$

for any partition $0 = \tau_0 < \tau_1 < ... < \tau_N = 1$ in $\mathbb{D}$. Given such a partition we have

$$0 = q\left(\tau_N,\lambda\right) \leq q\left(\tau_{N-1},\lambda\right) \leq ... \leq q\left(\tau_0,\lambda\right) = \mathbf{1}$$

(note that $q\left(1,\lambda\right) = 0$, as $\lambda < 1$). For $k = 1,2,...,N$ define

$$q_k = q\left(\tau_{k-1},\lambda\right) - q\left(\tau_k,\lambda\right). \tag{36}$$

Then $\{q_k\}_{k=1}^N$ is a disjoint system in $\mathcal{C}_B$ and $\sum_{j=k+1}^N q_j = q\left(\tau_k,\lambda\right)$ for all $k = 0,1,...,N-1$. Now

$$\begin{aligned}\bigvee_{k=0}^{N} q\left(\tau_k,\lambda\right)p_{\tau_k} &= \bigvee_{k=0}^{N-1}\bigvee_{j=k+1}^{N} q_j p_{\tau_k} = \bigvee_{j=1}^{N} q_j \bigvee_{k=0}^{j-1} p_{\tau_k} \\ &= \bigvee_{j=1}^{N} q_j p_{\tau_{j-1}} = \sum_{j=1}^{N} q_j p_{\tau_{j-1}}.\end{aligned} \tag{37}$$

Since $0 \leq q_j \leq q\left(\tau_{j-1},\lambda\right)$ in $\mathcal{C}_B$, it follows from (a) in Lemma 7.4 that

$$P\left(q_j p_{\tau_{j-1}}\right) = q_j P\left(p_{\tau_{j-1}}\right) \leq \lambda q_j$$

for all $j = 1,...,N$. Hence it follows from (37) that

$$P\left(\bigvee_{k=0}^{N} q\left(\tau_k,\lambda\right)p_{\tau_k}\right) = \sum_{j=1}^{N} P\left(q_j p_{\tau_{j-1}}\right) \leq \lambda\sum_{j=1}^{N} q_j = \lambda\mathbf{1},$$

which completes the proof of the lemma. ■

Lemma 7.6 *For all* $0 \le \lambda \le 1$ *we have* $P(e_\lambda) = \lambda \mathbf{1}$.

Proof. For $\lambda = 0$ and $\lambda = 1$ the inequality is trivial, so we may assume that $0 < \lambda < 1$. Take a partition $0 = \tau_0 < \tau_1 < ... < \tau_N = 1$ in $\mathbb{D}$ and define $\{q_k\}_{k=1}^N$ in $\mathcal{C}_B$ by (36). Since $q_k \le \mathbf{1} - q(\tau_k, \lambda)$, it follows from (b) in Lemma 7.4 that

$$P(q_k p_{\tau_k}) = q_k P(p_{\tau_k}) \ge \lambda q_k$$

for all $k = 1, ..., N$. Using (37) we find that

$$\begin{aligned} P\left(\bigvee_{k=0}^{N} q(\tau_k, \lambda) p_{\tau_k}\right) &= \sum_{k=1}^{N} q_k P\left(p_{\tau_{k-1}}\right) \\ = \sum_{k=1}^{N} q_k P(p_{\tau_k}) &- \sum_{k=1}^{N} q_k \left\{P(p_{\tau_k}) - P\left(p_{\tau_{k-1}}\right)\right\} \\ &\ge \lambda \sum_{k=1}^{N} q_k - \sum_{k=1}^{N} q_k \left\{P(p_{\tau_k}) - P\left(p_{\tau_{k-1}}\right)\right\} \\ &= \lambda \mathbf{1} - \sum_{k=1}^{N} q_k \left\{P(p_{\tau_k}) - P\left(p_{\tau_{k-1}}\right)\right\}. \end{aligned}$$

Take $n \in \mathbb{N}$ and choose the partition $0 = \tau_0 < \tau_1 < ... < \tau_N = 1$ such that $0 < \tau_j - \tau_{j-1} < 4^{-n}$ for all $j = 1, ..., N$. From (iii) in Lemma 7.3 it follows that

$$0 \le P\left(p_{\tau_j}\right) - P\left(p_{\tau_{j-1}}\right) \le 2^{-n+1}$$

for all j, hence

$$P\left(\bigvee_{k=0}^{N} q(\tau_k, \lambda) p_{\tau_k}\right) \ge \lambda \mathbf{1} - 2^{-n+1} \sum_{k=1}^{N} q_k = \left(\lambda - 2^{-n+1}\right) \mathbf{1}.$$

This shows that $P(e_\lambda) \ge (\lambda - 2^{-n+1}) \mathbf{1}$ for all $n \in \mathbb{N}$ and so $P(e_\lambda) \ge \lambda \mathbf{1}$. In combination with the previous lemma we may conclude that $P(e_\lambda) = \lambda \mathbf{1}$ for all $0 \le \lambda \le 1$. ■

Now we shall show that

$$p_\tau \in \mathfrak{S}\left(\mathcal{C}_B \cup \{e_\lambda : 0 \le \lambda \le 1\}\right)$$

for all $\tau \in \mathbb{D}$. The proof is divided in three lemmas.

Lemma 7.7 *If $\tau \in \mathbb{D}$ and $0 \leq \alpha < \beta \leq 1$, then*

$$[q(\tau,\beta) - q(\tau,\alpha)]\, e_\alpha \leq p_\tau.$$

Proof. Let $\tau \in \mathbb{D}$ be fixed. It follows from the definition (35) that

$$[q(\tau,\beta) - q(\tau,\alpha)]\, e_\alpha = \sup_{\sigma\in\mathbb{D}} [q(\tau,\beta) - q(\tau,\alpha)]\, q(\sigma,\alpha)\, p_\sigma.$$

Since $q(\sigma,\alpha) \leq q(\tau,\alpha)$ if $\sigma \in \mathbb{D}$ with $\tau \leq \sigma$, it follows that

$$[q(\tau,\beta) - q(\tau,\alpha)]\, q(\sigma,\alpha) = q(\tau,\beta)\,[\mathbf{1} - q(\tau,\alpha)]\, q(\sigma,\alpha) = 0$$

for all such $\sigma \in \mathbb{D}$. This implies that

$$[q(\tau,\beta) - q(\tau,\alpha)]\, e_\alpha = \sup\{[q(\tau,\beta) - q(\tau,\alpha)]\, q(\sigma,\alpha)\, p_\sigma : \sigma \in \mathbb{D}, \sigma \leq \tau\}$$

and from this the lemma follows. ■

Given $\tau \in \mathbb{D}$ and a partition $\pi : 0 = \lambda_0 < \lambda_1 < ... < \lambda_n = 1$ of the interval $[0,1]$ we define $s_\pi \in \mathcal{C}_A$ by

$$s_\pi = \sum_{k=1}^{n} [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})]\, e_{\lambda_{k-1}}. \tag{38}$$

Furthermore we denote $|\pi| = \max_k (\lambda_k - \lambda_{k-1})$.

Lemma 7.8 *With the notation introduced above, we have*

$$0 \leq P(p_\tau - s_\pi) \leq |\pi|\, \mathbf{1}$$

for all $\tau \in \mathbb{D}$ and for every partition π of $[0,1]$.

Proof. Since

$$s_\pi = \bigvee_{k=1}^{n} [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})]\, e_{\lambda_{k-1}},$$

it follows immediately from the Lemma 7.7 that $0 \leq s_\pi \leq p_\tau$. As observed in Lemma 7.4, $q(\tau,\lambda_n) = q(\tau,1) = \mathbf{1}$ and $q(\tau,\lambda_0) = q(\tau,0)$ is the component of $\mathbf{1}$ in $\{P(p_\tau)\}^d$, hence

$$P(p_\tau) = [\mathbf{1} - q(\tau,0)]\, P(p_\tau) = \sum_{k=1}^{n} [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})]\, P(p_\tau).$$

Moreover, since $q(\tau,\lambda_k) - q(\tau,\lambda_{k-1}) \leq q(\tau,\lambda_k)$ in $\mathcal{C}_B$, it follows from Lemma 7.4 (a) that

$$[q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})]\, P(p_\tau) \leq \lambda_k [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})]$$

for all $k = 1, ..., n$. Therefore,

$$P(p_\tau) \leq \sum_{k=1}^{n} \lambda_k [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})].$$

By Lemmas 4.1 and 7.6 we have

$$P(s_\pi) = \sum_{k=1}^{n} \lambda_{k-1} [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})].$$

Consequently,

$$\begin{aligned} 0 \leq P(p_\tau - s_\pi) &= P(p_\tau) - P(s_\pi) \\ \leq \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) &[q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})] \\ &\leq |\pi| \sum_{k=1}^{n} [q(\tau,\lambda_k) - q(\tau,\lambda_{k-1})] \\ &= |\pi| [\mathbf{1} - q(\tau,0)] \leq |\pi|\, \mathbf{1}, \end{aligned}$$

by which the lemma is proved. ■

Lemma 7.9 *For all $\tau \in \mathbb{D}$ we have $p_\tau \in \mathfrak{S}(\mathcal{C}_B \cup \{e_\lambda : 0 \leq \lambda \leq 1\})$.*

Proof. Let $\tau \in \mathbb{D}$ be fixed. Take a sequence $\{\pi_n\}_{n=1}^{\infty}$ of partitions of $[0,1]$ such that $|\pi_n| \to 0$ as $n \to \infty$ and define $s \in \mathcal{C}_A$ by

$$s = \sup\{s_{\pi_n} : n = 1, 2, ...\}.$$

From Lemma 7.7 and (38) it is clear that $s_{\pi_n} \leq p_\tau$ for all n, so $s \leq p_\tau$. Now it follows from the above lemma that

$$0 \leq P(p_\tau - s) \leq P(p_\tau - s_{\pi_n}) \leq |\pi_n|\, \mathbf{1},$$

which implies that $P(p_\tau - s) = 0$. Since P is assumed to be strictly positive, it follows that $p_\tau = s$. This shows that

$$p_\tau = \sup\{s_{\pi_n} : n = 1, 2, ...\}. \tag{39}$$

Since $q(\tau, \lambda) \in \mathcal{C}_B$ for all $0 \leq \lambda \leq 1$, it is clear from (38) that $s_{\pi_n} \in \mathfrak{S}\left(\mathcal{C}_B \cup \{e_\lambda : 0 \leq \lambda \leq 1\}\right)$ for all n. The result of the lemma now follows from (39). ■

From the above lemma in combination with (iv) in Lemma 7.3 it follows that

$$\{a_n\}_{n=1}^{\infty} \subseteq \mathfrak{S}\left(\mathcal{C}_B \cup \{e_\lambda : 0 \leq \lambda \leq 1\}\right).$$

For later reference we summarize the above results in the following proposition.

Proposition 7.10 *Let A, B and P satisfy the hypotheses stated at the beginning of the present section. Given any sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{C}_A$, there exists a system $\{e_\lambda : 0 \leq \lambda \leq 1\}$ in $\mathcal{C}_A$ such that:*

(i). $e_0 = 0$, $e_1 = \mathbf{1}$ *and* $e_{\lambda_1} \leq e_{\lambda_2}$ *whenever* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$*;*

(ii). $P(e_\lambda) = \lambda \mathbf{1}$ *for all* $0 \leq \lambda \leq 1$*;*

(iii). $\{a_n\}_{n=1}^{\infty} \subseteq \mathfrak{S}\left(\mathcal{C}_B \cup \{e_\lambda : 0 \leq \lambda \leq 1\}\right)$.

Let the system $\{e_\lambda : 0 \leq \lambda \leq 1\}$ be as in the Proposition 7.10 and let $\mathcal{F}$ denote the Boolean subalgebra of $\mathcal{C}_A$ generated by this system. It is easy to see that $\mathcal{F}$ consists precisely of those elements in $p \in \mathcal{C}_A$ which can be written as

$$p = \bigvee_{k=1}^{n} \left(e_{\lambda_k} - e_{\mu_k}\right) \tag{40}$$

with $0 \leq \mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq ... \leq \mu_n \leq \lambda_n \leq 1$. Hence

$$P(p) = \sum_{k=1}^{n} \left\{P\left(e_{\lambda_k}\right) - P\left(e_{\mu_k}\right)\right\} = \sum_{k=1}^{n} \left(\lambda_k - \mu_k\right) \mathbf{1}$$

for all such $p \in \mathcal{F}$. This shows that $\mathcal{F} \subseteq \mathcal{S}$. Since $\mathcal{F}$ is a Boolean subalgebra of $\mathcal{C}_A$ it follows from Lemma 6.2 (i) that $\mathfrak{S}(\mathcal{F}) \subseteq \mathcal{S}$. We denote $\mathcal{E} = \mathfrak{S}(\mathcal{F})$, and we will write $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$. If $p \in \mathcal{F}$ is given by (40), then it is clear that

$$\lambda(p) = \sum_{k=1}^{n} \left(\lambda_k - \mu_k\right) \tag{41}$$

Note that, by definition, $\mathcal{E}$ is the complete Boolean subalgebra of $\mathcal{C}_A$ generated by $\{e_\lambda : 0 \leq \lambda \leq 1\}$.

We denote by m the Lebesgue measure on $[0, 1]$ and let $(\mathcal{I}, m)$ denote the corresponding measure algebra.

Lemma 7.11 *There exists a measure preserving Boolean isomorphism h from $\mathcal{I}$ onto $\mathcal{E}$.*

Proof. Let $\mathcal{R}$ denote the subalgebra of $\mathcal{I}$ generated by all (elements corresponding to) intervals $[\alpha, \beta)$ with $0 \le \alpha \le \beta \le 1$. Define the Boolean isomorphism $h_0 : \mathcal{R} \to \mathcal{F}$ by $h_0([\alpha, \beta)) = e_\beta - e_\alpha$. It follows immediately from (40) and (41) that $\lambda(h_0(R)) = m(R)$ for all $R \in \mathcal{R}$, i.e., h_0 is measure preserving. Since $\mathcal{R}$ is dense in $\mathcal{I}$ with respect to the metric induced by m and $\mathcal{F}$ is dense in $\mathcal{E}$ with respect to the metric induced by λ, and since h_0 is an isometry for with respect to these metrics, it now follows that h_0 extends uniquely to a surjective measure preserving isomorphism $h : \mathcal{I} \to \mathcal{E}$. ∎

We collect the above results in the following proposition.

Proposition 7.12 *Let A, B and P satisfy the hypotheses stated at the beginning of the present section and let the sequence $\{a_n\}_{n=1}^\infty$ in $\mathcal{C}_A$ be given. Then there exists a complete Boolean subalgebra $\mathcal{E}$ of $\mathcal{C}_A$ such that:*

(i). $\{a_n\}_{n=1}^\infty \subseteq \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$;

(ii). there exists a strictly positive completely additive measure $\lambda : \mathcal{E} \to [0, 1]$ such that $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$;

(iii). the measure algebra $(\mathcal{E}, \lambda)$ is isomorphic with the measure algebra $(\mathcal{I}, m)$ of the Lebesgue measure on $[0, 1]$.

Remark 7.13 *Observe that $\mathcal{C}_B$ and $\mathcal{E}$ in the above proposition are independent Boolean subalgebras (in the sense of [15], Section 28) of $\mathcal{C}_A$. Indeed, if $q \in \mathcal{C}_B$ and $e \in \mathcal{E}$, then*

$$P(qe) = qP(e) = \lambda(e)q.$$

Hence, if $qe = 0$, then $P(qe) = 0$ and so $q = 0$ or $e = 0$.

If there exists a sequence $\{a_n\}_{n=1}^\infty$ in $\mathcal{C}_A$ such that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \{a_n\}_{n=1}^\infty)$ then we will say that $\mathcal{C}_A$ *is separable over* $\mathcal{C}_B$ (or, A *is separable over* B). The following result is now an immediate consequence of Proposition 7.12.

Corollary 7.14 *Let A, B and P satisfy the hypotheses stated at the beginning of the present section and suppose that $\mathcal{C}_A$ is separable over $\mathcal{C}_B$. Then there exists a complete Boolean subalgebra $\mathcal{E}$ of $\mathcal{C}_A$ such that:*

(i). $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$;

(ii). there exists a strictly positive completely additive measure $\lambda : \mathcal{E} \to [0,1]$ such that $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$;

(iii). the measure algebra $(\mathcal{E}, \lambda)$ is isomorphic with the measure algebra $(\mathcal{I}, m)$ of the Lebesgue measure on $[0,1]$.

Hence, in the situation of the above corollary, $\mathcal{C}_A$ is a Boolean product of the independent Boolean subalgebras $\mathcal{C}_B$ and $\mathcal{E}$.

8 The structure of positive projections

In this section we assume again that A, B and P satisfy the hypotheses stated at the beginning of Section 7. The main objective in this section is to show that there exists a disjoint system $\{p_\tau\}$ in $\mathcal{C}_A$ with $\sup_\tau p_\tau = \mathbf{1}$, such that for each τ there exists a complete Boolean subalgebra $\mathcal{E}_\tau$ of $\mathcal{C}_A(p_\tau)$ such that:

(a). $\mathcal{C}_A(p_\tau) = \mathfrak{S}(p_\tau \mathcal{C}_B \cup \mathcal{E}_\tau)$ in $\mathcal{C}_A(p_\tau)$;

(b). there exists a strictly positive completely additive non-atomic measure $\lambda_\tau : \mathcal{E}_\tau \to [0,1]$ such that $P_{\upharpoonright p_\tau}(e) = \lambda_\tau(e)\mathbf{1}$ for all $e \in \mathcal{E}_\tau$,

where $P_{\upharpoonright p_\tau}$ is the restriction of P to $A(p_\tau)$ as defined in Lemma 4.14. We follow the ideas of Maharam ([10]). First we introduce some terminology. For any subset $\mathcal{D}$ of $\mathcal{C}_A$ we denote by $|\mathcal{D}|$ the cardinal number corresponding to $\mathcal{D}$. The *order of* $p \in \mathcal{C}_A$ *over* $\mathcal{C}_B$ is defined by

$$\min\{|\mathcal{D}| : \mathcal{D} \subseteq \mathcal{C}_A \text{ such that } \mathcal{C}_A(p) = p\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D})\}. \tag{42}$$

Note that $p \in \mathcal{C}_A$ is of order 0 over $\mathcal{C}_B$ if and only if $\mathcal{C}_A(p) = p\mathcal{C}_B$, i.e., if and only if p is B-full (see Definition 4.7). If there exists a cardinal $\mathfrak{m}$ such that every $0 < p \in \mathcal{C}_A$ is of order $\mathfrak{m}$ over $\mathcal{C}_B$, then $\mathcal{C}_A$ is called *homogeneous of order* $\mathfrak{m}$ *over* $\mathcal{C}_B$ (and we will also say that A is *homogeneous of order* $\mathfrak{m}$ *over* B).

In the first step of the construction we will assume in addition that A is homogeneous of order $\mathfrak{m}$ over B for some cardinal $\mathfrak{m} > 0$. Under this assumption we will show that there exists a complete non-atomic Boolean subalgebra $\mathcal{E}$ of $\mathcal{C}_A$ such that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$ and $\mathcal{E} \subseteq \mathcal{S}$, where $\mathcal{S}$ is given by (28).

Remark 8.1 *If $0 < p \in \mathcal{C}_A$ has finite order over $\mathcal{C}_B$, then there exists $0 < q \leq p \in \mathcal{C}_A$ such that q is B-full. Indeed, let $\{p_k\}_{k=1}^n \subseteq \mathcal{C}_A$ be such that*

$$\mathcal{C}_A(p) = p\mathfrak{S}\left(\mathcal{C}_B \cup \{p_k\}_{k=1}^n\right). \tag{43}$$

Replacing, if necessary, $\{p_k\}_{k=1}^n$ by a basis for the Boolean subalgebra generated by $\{p_k\}_{k=1}^n$ we may assume that $\{p_k\}_{k=1}^n$ is a disjoint system with $\sum_{k=1}^n p_k = \mathbf{1}$. Then it is easily verified that

$$\mathfrak{S}\left(\mathcal{C}_B \cup \{p_k\}_{k=1}^n\right) = \left\{\sum_{k=1}^n p_k q_k : q_k \in \mathcal{C}_B, k = 1, ..., n\right\}.$$

Let $1 \leq m \leq n$ be such that $q = pp_m > 0$. Take $a \in \mathcal{C}_A(q) \subseteq \mathcal{C}_A(p)$. It follows from (43) that there exist $q_k \in \mathcal{C}_B$ $(k = 1, \ldots, n)$ such that $a = p\sum_{k=1}^n p_k q_k$. Since $a \leq q \leq p_m$, this implies that $a = pp_m q_m = qq_m \in q\mathcal{C}_B$, which shows that $\mathcal{C}_A(q) \subseteq q\mathcal{C}_B$. Hence q is B-full. From this observation it follows that, if A is nowhere full with respect to B, then every $0 < p \in \mathcal{C}_A$ has order at least $\aleph_0$ over $\mathcal{C}_B$.

By the above remark, we may assume that A is homogeneous of order $\mathfrak{m}$ over B for some cardinal $\mathfrak{m} \geq \aleph_0$. In case that $\mathfrak{m} = \aleph_0$, the desired result has already been obtained in Corollary 7.14. Therefore in the proof of Proposition 8.3 below we may assume that $\mathfrak{m} > \aleph_0$. But first we prove the following lemma.

Lemma 8.2 *Let $\mathcal{R}_0$ be a complete Boolean subalgebra of $\mathcal{C}_A$ such that $\mathcal{R}_0 \subseteq \mathcal{S}$. Suppose that $\mathcal{C}_A$ is nowhere full with respect to $\mathfrak{S}\left(\mathcal{C}_B \cup \mathcal{R}_0\right)$ and let $a \in \mathcal{C}_A$ be given. Then there exists a complete Boolean subalgebra $\mathcal{R}_1$ of $\mathcal{C}_A$ such that:*

(i). $\mathcal{R}_0 \subseteq \mathcal{R}_1$ and $a \in \mathfrak{S}\left(\mathcal{C}_B \cup \mathcal{R}_1\right)$;

(ii). $\mathcal{R}_1 \subseteq \mathcal{S}$ and $\mathcal{R}_1$ is non-atomic;

(iii). $\mathcal{R}_1$ is separable over $\mathcal{R}_0$, i.e.,there exists a sequence $\{e_n\}_{n=1}^\infty \subseteq \mathcal{R}_1$ such that $\mathcal{R}_1 = \mathfrak{S}\left(\mathcal{R}_0 \cup \{e_n\}_{n=1}^\infty\right)$.

Proof. Considering the restriction of P to A_b, we may assume without loss of generality that $A = A_b$, i.e., we may assume that the unit element $\mathbf{1}$ is a strong order unit in A. Now let

$$C = A\left(\mathfrak{S}\left(\mathcal{C}_B \cup \mathcal{R}_0\right)\right),$$

as defined in Section 2. It follows from Proposition 2.6 (and the remarks made at the beginning of Section 4) that C is complete f-subalgebra of A. Hence, by Proposition 4.2 there exists a strictly positive order continuous projection $Q : A \to A$ onto C such that $P = P \circ Q$. Now we apply Proposition 7.12 to A, C, Q and the constant sequence $a_n = a$ for all n. Consequently there exists a complete Boolean subalgebra $\mathcal{E}$ of $\mathcal{C}_A$ such that $a \in \mathfrak{S}(\mathcal{C}_C \cup \mathcal{E})$, and there exists a strictly positive completely additive measure $\lambda : \mathcal{E} \to [0,1]$ such that $Q(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$, and $(\mathcal{E}, \lambda)$ is isomorphic with the measure algebra $(\mathcal{I}, m)$ of the Lebesgue measure on $[0,1]$.

Defining $\mathcal{R}_1 = \mathfrak{S}(\mathcal{R}_0 \cup \mathcal{E})$, it is clear the $\mathcal{R}_1$ satisfies (iii), as $(\mathcal{I}, m)$ is a separable measure algebra. Furthermore,

$$\mathfrak{S}(\mathcal{C}_C \cup \mathcal{E}) = \mathfrak{S}(\mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_0) \cup \mathcal{E}) = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_0 \cup \mathcal{E}) = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_1),$$

so $\mathcal{R}_1$ satisfies (i) as well. For the proof of (ii) we denote by $\lambda_0 : \mathcal{R}_0 \to [0,1]$ the measure defined by $P(q) = \lambda_0(q)\mathbf{1}$ for all $q \in \mathcal{R}_0$. Let $\mathcal{F}$ be the Boolean subalgebra of $\mathcal{C}_A$ generated by $\mathcal{R}_0 \cup \mathcal{E}$. Every $p \in \mathcal{F}$ is of the form

$$p = \bigvee_{k=1}^{n} q_k e_k, \tag{44}$$

with $q_k \in \mathcal{R}_0$, $e_k \in \mathcal{E}$, $k = 1, ..., n$ and $n \in \mathbb{N}$. It is easy to see that we can take the elements q_k in (44) mutually disjoint, so $p = \sum_{k=1}^{n} q_k e_k$. Hence, using Lemma 4.1 and that $P = P \circ Q$, we find

$$\begin{aligned} P(p) &= P\left(\sum_{k=1}^{n} Q(q_k e_k)\right) = P\left(\sum_{k=1}^{n} q_k Q(e_k)\right) \\ &= P\left(\sum_{k=1}^{n} q_k Q(e_k)\right) = P\left(\sum_{k=1}^{n} \lambda(e_k) q_k\right) \\ &= \sum_{k=1}^{n} \lambda(e_k) P(q_k) = \sum_{k=1}^{n} \lambda(e_k) \lambda_0(q_k) \mathbf{1}. \end{aligned} \tag{45}$$

This shows that for every $p \in \mathcal{F}$ we have $P(p) = \alpha\mathbf{1}$ for some $\alpha \in [0,1]$ and so $\mathcal{F} \subseteq \mathcal{S}$. Since $\mathcal{F}$ is a Boolean subalgebra of $\mathcal{C}_A$, it follows from Lemma 6.2 that $\mathcal{R}_1 = \mathfrak{S}(\mathcal{F}) \subseteq \mathcal{S}$. Hence, for every $p \in \mathcal{R}_1$ we have $P(p) = \lambda_1(p)\mathbf{1}$ for some $\lambda_1(p) \in [0,1]$ and this defines a strictly positive completely additive measure $\lambda_1 : \mathcal{R}_1 \to [0,1]$. Finally, using that λ is non-atomic on $\mathcal{E}$, it follows easily from (45) that for every $p \in \mathcal{F}$ there exist $p_1, p_2 \in \mathcal{F}$ such that $p_1 + p_2 = p$, $p_1 \wedge p_2 = 0$ and $\lambda_1(p_1) = \lambda_1(p_2) = \frac{1}{2}\lambda_1(p)$. This implies that λ_1 is a non-atomic measure and hence, $\mathcal{R}_1$ is a non-atomic Boolean subalgebra of $\mathcal{C}_A$. By this the proof of the lemma is complete. ∎

Now we are in a position to prove one of the main results of the present section.

Proposition 8.3 *Assume that A, B and P satisfy the hypotheses stated at the beginning of Section 7. Furthermore we assume that A is homogeneous over B of order $\mathfrak{m}$ for some cardinal $\mathfrak{m} > 0$. Then there exists a complete Boolean subalgebra $\mathcal{E} \subseteq \mathcal{C}_A$ such that:*

(i). $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$;

(ii). there exists a strictly positive completely additive non-atomic measure $\lambda : \mathcal{E} \to [0,1]$ such that $P(e) = \lambda(e)\mathbf{1}$ for all $e \in \mathcal{E}$.

Proof. As observed after Remark 8.1, we may assume that $\mathfrak{m} > \aleph_0$. Let $\mathcal{D} = \{d_\alpha : \alpha \in \mathbb{A}\}$ be a subset of $\mathcal{C}_A$ such that $|\mathbb{A}| = \mathfrak{m}$ and $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{D})$. Identifying $\mathfrak{m}$ with the initial ordinal associated with $\mathfrak{m}$, we will write $\mathbb{A} = \{\alpha : 1 \leq \alpha < \mathfrak{m}\}$. We will show that there exists a collection $\{\mathcal{E}_\alpha : \alpha \in \mathbb{A}\}$ of complete non-atomic Boolean subalgebras of $\mathcal{C}_A$ such that:

(a). $\mathcal{E}_\beta \subseteq \mathcal{E}_\alpha$ whenever $\beta < \alpha$ in $\mathbb{A}$;

(b). $\{d_\beta : \beta \in \mathbb{A}, \beta \leq \alpha\} \subseteq \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E}_\alpha)$ for all $\alpha \in \mathbb{A}$;

(c). $\mathcal{E}_\alpha \subseteq \mathcal{S}$ for all $\alpha \in \mathbb{A}$ (where $\mathcal{S}$ is defined by (28));

(d). for each $\alpha \in \mathbb{A}$ there exists a subset $\mathcal{G}_\alpha \subseteq \mathcal{E}_\alpha$ such that $\mathcal{E}_\alpha = \mathfrak{S}(\mathcal{G}_\alpha)$ and $|\mathcal{G}_\alpha| \leq \max\{|\alpha|, \aleph_0\}$.

We proceed by induction. If $\alpha = 1$, then we apply Lemma 8.2 to $\mathcal{R}_0 = \{0, \mathbf{1}\}$ and $a = d_1$. It is clear that $\mathcal{E}_1 = \mathcal{R}_1$ satisfies (b), (c) and (d) above. Now assume that $\alpha \in \mathbb{A}$ is such that $\{\mathcal{E}_\beta : \beta \in \mathbb{A}, \beta < \alpha\}$ have been constructed with the above properties. By (a) and (c), $\bigcup_{\beta<\alpha} \mathcal{E}_\beta$ is a Boolean subalgebra of $\mathcal{C}_A$ that is contained in $\mathcal{S}$. Defining

$$\mathcal{R}_\alpha = \mathfrak{S}\left(\bigcup_{\beta<\alpha} \mathcal{E}_\beta\right),$$

we know from Lemma 6.2 that $\mathcal{R}_\alpha \subseteq \mathcal{S}$. We claim that $\mathcal{C}_A$ is nowhere full with respect to $\mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_\alpha)$. Indeed, defining $\mathcal{D}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\beta$, it is clear that $\mathcal{R}_\alpha = \mathfrak{S}(\mathcal{D}_\alpha)$ and hence $\mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_\alpha) = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}_\alpha)$. By assumption $|\mathcal{G}_\beta| \leq \max\{|\beta|, \aleph_0\}$ for all $\beta < \alpha$, and so $|\mathcal{D}_\alpha| \leq |\alpha| \max\{|\alpha|, \aleph_0\} = \max\{|\alpha|, \aleph_0\}$. Since $\mathfrak{m} > \aleph_0$, it follows in particular that $|\mathcal{D}_\alpha| < \mathfrak{m}$. If $0 < p \in \mathcal{C}_A$, then by hypothesis p is of order $\mathfrak{m}$ over $\mathcal{C}_B$. Hence, $\mathcal{C}_A(p) \neq p\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}_\alpha)$ and this shows that p is not full with respect to $\mathfrak{S}(\mathcal{C}_B \cup \mathcal{R}_\alpha)$, which proves our claim.

Lemma 8.2, applied to $\mathcal{R}_\alpha$ and d_α, implies that there exists a complete non-atomic Boolean subalgebra $\mathcal{E}_\alpha$ of $\mathcal{C}_A$ such that $\mathcal{R}_\alpha \subseteq \mathcal{E}_\alpha \subseteq \mathcal{S}$ and $d_\alpha \in$

$\mathfrak{S}(\mathcal{C}_B \cup \mathcal{E}_\alpha)$. Moreover, there exists a sequence $\{e_n\}_{n=1}^\infty \subseteq \mathcal{E}_\alpha$ such that $\mathcal{E}_\alpha = \mathfrak{S}(\mathcal{R}_\alpha \cup \{e_n\}_{n=1}^\infty)$. It is clear that $\mathcal{E}_\alpha$ satisfies (a), (b) and (c). Define $\mathcal{G}_\alpha = \mathcal{D}_\alpha \cup \{e_n\}_{n=1}^\infty$. Since $|\mathcal{D}_\alpha| \leq \max\{|\alpha|, \aleph_0\}$, it is obviously that $|\mathcal{G}_\alpha| \leq \max\{|\alpha|, \aleph_0\}$. Furthermore,

$$\begin{aligned}\mathcal{E}_\alpha = \mathfrak{S}(\mathcal{R}_\alpha \cup \{e_n\}_{n=1}^\infty) &= \mathfrak{S}(\mathfrak{S}(\mathcal{D}_\alpha) \cup \{e_n\}_{n=1}^\infty) \\ &= \mathfrak{S}(\mathcal{D}_\alpha \cup \{e_n\}_{n=1}^\infty) = \mathfrak{S}(\mathcal{G}_\alpha),\end{aligned}$$

which shows that (d) is satisfied as well. The construction of the collection $\{\mathcal{E}_\alpha : \alpha \in \mathbb{A}\}$ is complete.

Defining

$$\mathcal{F} = \bigcup_{\alpha \in \mathbb{A}} \mathcal{E}_\alpha,$$

it follows from (a) that $\mathcal{F}$ is a Boolean subalgebra of $\mathcal{C}_A$ and by (c) we have $\mathcal{F} \subseteq \mathcal{S}$. Lemma 6.2 implies that $\mathcal{E} = \mathfrak{S}(\mathcal{F}) \subseteq \mathcal{S}$. Hence, for every $e \in \mathcal{E}$ there exists $\lambda(e) \in [0,1]$ such that $P(e) = \lambda(e)\mathbf{1}$, and by Lemma 6.2, $\lambda : \mathcal{E} \to [0,1]$ is a completely additive strictly positive measure. Since λ is non-atomic on $\mathcal{F}$, a similar argument as used in the proof of Lemma 8.2 shows that λ is non-atomic on $\mathcal{E}$ as well. Finally, it follows from (b) that $\mathcal{D} \subseteq \mathfrak{S}(\mathcal{C}_B \cup \mathcal{F})$, so

$$\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}) \subseteq \mathfrak{S}(\mathcal{C}_B \cup \mathcal{F}) = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E}),$$

which shows that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$ and the proof is complete. ■

Now we will drop the assumption that A is homogeneous over B. In the proof of the next lemma the following simple observation will be used.

Lemma 8.4 *Let $0 < p_0 \in \mathcal{C}_A$ be fixed. If $p \in \mathcal{C}_A(p_0)$, then the order of p over $\mathcal{C}_B$ is equal to the order of p over $p_0\mathcal{C}_B$ in $\mathcal{C}_A(p_0)$. In particular, given any cardinal $\mathfrak{m}$, the following two statements are equivalent:*

- *(i). every $0 < p \in \mathcal{C}_A(p_0)$ has order $\mathfrak{m}$ over $\mathcal{C}_B$;*
- *(ii). $\mathcal{C}_A(p_0)$ is homogeneous of order $\mathfrak{m}$ over $p_0\mathcal{C}_B$.*

Proof. For a non-empty subset $\mathcal{H}_0$ of $\mathcal{C}_A(p_0)$ we will denote by $\mathfrak{S}_0(\mathcal{H}_0)$ the complete Boolean subalgebra of $\mathcal{C}_A(p_0)$ generated by $\mathcal{H}_0$. It is readily verified that

$$p_0\mathfrak{S}(\mathcal{H}) = \mathfrak{S}_0(p_0\mathcal{H}). \tag{46}$$

for any non-empty subset $\mathcal{H} \subseteq \mathcal{C}_A$. Given $p \in \mathcal{C}_A(p_0)$, let $\mathfrak{m}$ be the order of p over $\mathcal{C}_B$ and let $\mathfrak{m}_0$ denote the order of p over $p_0\mathcal{C}_B$ in $\mathcal{C}_A(p_0)$. Take $\mathcal{D} \subseteq \mathcal{C}_A$ such that $|\mathcal{D}| = \mathfrak{m}$ and $\mathcal{C}_A(p) = p\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D})$. Using (46) it follows that

$$\mathcal{C}_A(p) = pp_0\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}) = p\mathfrak{S}_0(p_0\mathcal{C}_B \cup p_0\mathcal{D}),$$

which shows that $\mathfrak{m}_0 \leq |p_0\mathcal{D}| \leq \mathfrak{m}$. Now take $\mathcal{D}_0 \subseteq \mathcal{C}_A(p_0)$ such that $|\mathcal{D}_0| = \mathfrak{m}_0$ and $\mathcal{C}_A(p) = p\mathfrak{S}_0(p_0\mathcal{C}_B \cup \mathcal{D}_0)$. Using (46) once more, we find that

$$\mathcal{C}_A(p) = p\mathfrak{S}_0(p_0\mathcal{C}_B \cup \mathcal{D}_0) = pp_0\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}_0) = p\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}_0),$$

and so $\mathfrak{m} \leq |\mathcal{D}_0| = \mathfrak{m}_0$. We may conclude that $\mathfrak{m} = \mathfrak{m}_0$ and this suffices to prove the lemma. ■

Lemma 8.5 *Assume that A, B and P satisfy the hypotheses stated at the beginning of Section 7. Then there exists a disjoint system $\{p_\tau\}$ in $\mathcal{C}_A$ such that:*

(i). *each $A(p_\tau)$ is homogeneous of order $\mathfrak{m}_\tau > 0$ over $B_{\upharpoonright p_\tau}$ (equivalently, $\mathcal{C}_A(p_\tau)$ is homogeneous of order $\mathfrak{m}_\tau > 0$ over $p_\tau\mathcal{C}_B$);*

(ii). $\sum_\tau p_\tau = \mathbf{1}$.

Proof. By Zorn's Lemma there exists a maximal disjoint system $\{p_\tau\}$ with property (i). We will show that $\sum_\tau p_\tau = \mathbf{1}$. Put $p_0 = \mathbf{1} - \sum_\tau p_\tau$ and suppose that $p_0 > 0$. Define

$$\mathfrak{m}_0 = \min\{\mathfrak{m} : \exists 0 < e \leq p_0, e \text{ is of order } \mathfrak{m} \text{ over } \mathcal{C}_B\}.$$

Since $\mathcal{C}_A$ is nowhere full with respect to $\mathcal{C}_B$, it follows that $\mathfrak{m}_0 > 0$ and actually $\mathfrak{m}_0 \geq \aleph_0$, by Remark 8.1. Take $0 < e_0 \leq p_0$ such that e_0 is of order $\mathfrak{m}_0$ over $\mathcal{C}_B$, and let $\mathcal{D} \subseteq \mathcal{C}_A$ be such that $|\mathcal{D}| = \mathfrak{m}_0$ and $\mathcal{C}_A(e_0) = e_0\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D})$. If $0 < e \in \mathcal{C}_A(e_0)$, then

$$\mathcal{C}_A(e) = e\mathcal{C}_A(e_0) = e\mathfrak{S}(\mathcal{C}_B \cup \mathcal{D}),$$

so the order of e over $\mathcal{C}_B$ is less or equal to $\mathfrak{m}_0$. By the definition of $\mathfrak{m}_0$ it is clear that e is of order $\mathfrak{m}_0$ over $\mathcal{C}_B$. This shows that every $0 < e \in \mathcal{C}_A(e_0)$ has order $\mathfrak{m}_0$ over $\mathcal{C}_B$. It follows from Lemma 8.4 that $\mathcal{C}_A(e_0)$ is homogeneous of order $\mathfrak{m}_0$ over $e_0\mathcal{C}_B$. Since e_0 is disjoint to $\{p_\tau\}$, this contradicts the maximality of the system $\{p_\tau\}$. Consequently $p_0 = 0$, by which the lemma is proved. ■

The following proposition is the main result in the present section.

Proposition 8.6 *Assume that A, B and P satisfy the hypotheses stated at the beginning of Section 7. Then there exists a disjoint system $\{p_\tau\}$ in $\mathcal{C}_A$ with* $\sup_\tau p_\tau = \mathbf{1}$, *such that for each τ there exists a complete Boolean subalgebra $\mathcal{E}_\tau$ of $\mathcal{C}_A(p_\tau)$ such that:*

(a). $\mathcal{C}_A(p_\tau) = \mathfrak{S}(p_\tau \mathcal{C}_B \cup \mathcal{E}_\tau)$ *in* $\mathcal{C}_A(p_\tau)$*;*

(b). there exists a strictly positive completely additive non-atomic measure $\lambda_\tau : \mathcal{E}_\tau \to [0,1]$ *such that* $P_{\restriction p_\tau}(e) = \lambda_\tau(e) p_\tau$ *for all* $e \in \mathcal{E}_\tau$.

Proof. Let the disjoint system $\{p_\tau\}$ be as in Lemma 8.5. For each τ we can now apply Proposition 8.3 to $A(p_\tau)$, $B_{\restriction p_\tau}$ and $P_{\restriction p_\tau}$, which immediately yields the result of the proposition. ∎

9 Representations of positive projections: special case

In this section we will obtain a special case of the representation of positive projections, which will be one of the essential building blocks for the general case, treated in the next section. Before we start, recall some of the notation introduced in Section 3. Given two measurable spaces (X, Σ) and (Y, Λ) we denote $\Omega = X \times Y$ and $\mathcal{F} = \Sigma \otimes \Lambda$. As before, we denote by $M(\Omega, \mathcal{F})$ the f-algebra of all real-valued $\mathcal{F}$-measurable functions on Ω and $M_b(\Omega, \mathcal{F})$ is the f-subalgebra of all bounded functions in $M(\Omega, \mathcal{F})$. Recall furthermore that $M(X, \Sigma)$ is identified with an f-subalgebra of $M(\Omega, \mathcal{F})$.

Now we assume in addition that μ is a *probability measure* on (Y, Λ) and we define the linear operator $R_b : M_b(\Omega, \mathcal{F}) \to M_b(\Omega, \mathcal{F})$ by

$$R_b f(x, y) = \int_Y f(x, z)\, d\mu(z) \tag{47}$$

for all $(x, y) \in \Omega$ and all $f \in M_b(\Omega, \mathcal{F})$, which is a positive σ-order continuous projection onto $M_b(X, \Sigma)$. Furthermore suppose that B is a Dedekind complete f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and that (X, Σ) is a representation space for B with corresponding representation homomorphism $\Phi_B : M_b(X, \Sigma) \to B$. Define

$$S : M_b(\Omega, \mathcal{F}) \xrightarrow{R_b} M_b(X, \Sigma) \xrightarrow{\Phi_B} B,$$

i.e., $S = \Phi_B \circ R_b$, where R_b is the projection given by (47). Clearly, S is a linear σ-order continuous positive operator onto B. Denote by N_S the *null* *ideal* of S, i.e.,

$$N_S = \{f \in M_b(\Omega, \mathcal{F}) : S|f| = 0\},$$

which is a σ-ideal as well as an algebra ideal in $M_b(\Omega, \mathcal{F})$. Let E be the corresponding quotient space,

$$E = M_b(\Omega, \mathcal{F}) \diagup N_S$$

and let $Q : M_b(\Omega, \mathcal{F}) \to E$ be the quotient f-algebra homomorphism. Then E is Dedekind σ-complete and Q is σ-order continuous. The unit element in E will be denoted by $\mathbf{1}_E$, which is also a strong order unit in E. Furthermore we define the f-subalgebra F of E by

$$F = Q[M_b(X, \Sigma)].$$

Since $N_S \subseteq \mathrm{Ker}(S)$ we can define the linear mapping

$$\overline{S} : E \to B \tag{48}$$

by $\overline{S}(Qf) = Sf$ for all $f \in M_b(\Omega, \mathcal{F})$. In the next lemma we collect some properties of the operator $\overline{S}$.

Lemma 9.1 *The operator $\overline{S}$ in (48) has the following properties.*

(i). $\overline{S}$ is a strictly positive, σ-order continuous and surjective linear operator.

(ii). The restriction $\overline{S}_{|F} : F \to B$ is an f-algebra isomorphism onto B.

Proof. The proof of (i) is easy and therefore omitted. For the proof of (ii), we first show that $\overline{S}_{|F}$ is injective. Suppose that $h \in F$ is such that $\overline{S}h = 0$. Take $g \in M_b(X, \Sigma)$ such that $Qg = h$. Then

$$S|g| = \Phi_B R_b |g| = \Phi_B |g| = |\Phi_B g| = |\Phi_B R_b g| = |Sg| = \left|\overline{S}h\right| = 0,$$

so $g \in N_S$. This implies that $h = Qg = 0$. Next we show that $\overline{S}_{|F}$ is surjective with a positive inverse. If $0 \leq b \in B$, then there exists $0 \leq g \in M_b(X, \Sigma)$ such that $\Phi_B g = b$. Hence, $0 \leq Qg \in F$ and

$$\overline{S}(Qg) = Sg = \Phi_b(R_b g) = \Phi_b g = b.$$

Now it is clear that $\overline{S}_{|F}$ is an f-algebra isomorphism from F onto B. ■

Since B is Dedekind complete, it is clear that F is Dedekind complete as well. We claim that F is regularly embedded in E. Indeed, suppose that $\{f_\tau\}$ is a net in F such that $f_\tau \downarrow 0$ in F and let $g \in E$ be such that $f_\tau \geq g \geq 0$ for all τ. Since $\overline{S}_{|F}$ is a Riesz isomorphism from F onto B, we have $\overline{S} f_\tau \downarrow 0$ in B and $\overline{S} f_\tau \geq \overline{S} g \geq 0$ for all τ, so $\overline{S} g = 0$. Hence, $g = 0$, as $\overline{S}$ is strictly positive, which shows that $f_\tau \downarrow 0$ in E.

Next we introduce a second f-subalgebra G of E, defined by

$$G = Q\left[M_b\left(Y, \Lambda\right)\right],$$

where, as before, $M_b\left(Y, \Lambda\right)$ is identified with the f-subalgebra $\mathbf{1}_X \otimes M_b\left(Y, \Lambda\right)$ of $M_b\left(\Omega, \mathcal{F}\right)$. For any $f \in M_b\left(Y, \Lambda\right)$ we have $R_b f = \left(\int_Y f \, d\mu\right) \mathbf{1}_\Omega$ and so

$$Sf = \left(\int_Y f \, d\mu\right) \mathbf{1} \in B. \tag{49}$$

In particular, $f \in N_S$ if and only if $\int_Y |f| \, d\mu = 0$, i.e., if and only if $f \in M_b\left(\mathfrak{N}_\mu\right)$, where $\mathfrak{N}_\mu$ denotes the σ-ideal in Λ of all μ-null sets. Consequently, G is precisely the quotient space $M_b\left(Y, \Lambda\right) \diagup M_b\left(\mathfrak{N}_\mu\right)$, as usual denoted by $L_\infty\left(\mu\right)$. The integral with respect to μ on $G = L_\infty\left(\mu\right)$ will be denoted by φ_μ, i.e., $\varphi_\mu\left(Qf\right) = \int_Y f \, d\mu$ for all $f \in M_b\left(Y, \Lambda\right)$, and the corresponding measure on $\mathcal{C}_G$ will be denoted by $\bar{\mu}$, i.e., $\bar{\mu}\left(Q\mathbf{1}_V\right) = \mu\left(V\right)$ for all $V \in \Lambda$. It follows in particular that G is Dedekind complete and order separable. Moreover, it is now easy to see that G is regularly embedded in E.

We denote, as before, by $\mathcal{C}_E$, $\mathcal{C}_F$ and $\mathcal{C}_G$ the Boolean algebras of components of $\mathbf{1}_E$ in E, F and G respectively. Since F and G are Dedekind complete and regularly embedded in E, it follows that $\mathcal{C}_F$ and $\mathcal{C}_G$ are both complete Boolean subalgebras of $\mathcal{C}_E$. Observing that $\mathcal{C}_E = \{Q\left(\mathbf{1}_W\right) : W \in \mathcal{F}\}$, $\mathcal{C}_F = \{Q\left(\mathbf{1}_U\right) : U \in \Sigma\}$ and $\mathcal{C}_G = \{Q\left(\mathbf{1}_V\right) : V \in \Lambda\}$ the next lemma follows immediately.

Lemma 9.2 *The Boolean σ-algebra generated by $\mathcal{C}_F$ and $\mathcal{C}_G$ is equal to $\mathcal{C}_E$ and hence $\mathcal{C}_E = \mathfrak{S}\left(\mathcal{C}_F \cup \mathcal{C}_G\right)$.*

Now we define the positive linear operator $P_E : E \to E$ by

$$P_E : E \xrightarrow{\overline{S}} B \overset{\left(\overline{S}_{|F}\right)^{-1}}{\longrightarrow} F \hookrightarrow E,$$

so $P_E = \left(\overline{S}_{|F}\right)^{-1} \circ \overline{S}$.

Lemma 9.3 *The operator P_E is a strictly positive σ-order continuous projection in E onto the f-subalgebra F. Moreover, $P_E\left(Qf\right) = Q\left(R_b f\right)$ for all $f \in M_b\left(\Omega, \mathcal{F}\right)$.*

Proof. The first statement of the lemma follows immediately from Lemma 9.1 and from the fact that F is regularly embedded in E. To prove the second statement, let $f \in M_b(\Omega, \mathcal{F})$ be given. Then $\overline{S}(Qf) = Sf = \Phi_B(R_b f)$ and $\overline{S}(QR_b f) = S(R_b f) = \Phi_B(R_b f)$. Since $Q(R_b f) \in F$, this implies that $P_E(Qf) = \left(\overline{S}_{|F}\right)^{-1} \overline{S}(Qf) = Q(R_b f)$. ∎

Next we observe that, if $g \in G$ and we write $g = Qf$ with $f \in M_b(Y, \Lambda)$, then it follows from the above lemma that

$$P_E g = Q(R_b f) = Q\left(\left(\int_Y f d\mu\right) \mathbf{1}_\Omega\right) = \varphi_\mu(g) \mathbf{1}_E,$$

where $0 \leq \varphi_\mu \in G_n^\sim$ is the integral on $G = L_\infty(\mu)$. In particular, $P_E(p) = \bar{\mu}(p)$ for all $p \in \mathcal{C}_G$.

Now we assume that A is a Dedekind complete f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and that $P : A \to A$ is a strictly positive order continuous projection onto the f-subalgebra $B \subseteq A$ with $\mathbf{1} \in B$. Furthermore, let $\mathcal{E} \subseteq \mathcal{C}_A$ be a complete Boolean subalgebra such that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$ and suppose that $\lambda : \mathcal{E} \to [0,1]$ a completely additive measure such that $P(e) = \lambda(e) \mathbf{1}$ for all $e \in \mathcal{E}$. Let (Y, Λ) be a representation space for $A(\mathcal{E})$ with representation $\Phi_{A(\mathcal{E})} : M_b(Y, \Lambda) \to A(\mathcal{E})$. Defining $\mu(V) = \lambda\left(\Phi_{A(\mathcal{E})} \mathbf{1}_V\right)$ for all $V \in \Lambda$, it is clear that μ is a probability measure on (Y, Λ) and that $\mathfrak{N}_\mu = \mathfrak{N}_{\Phi_{A(\mathcal{E})}}$. Consequently, Ker $\left(\Phi_{A(\mathcal{E})}\right) = M_b(\mathfrak{N}_\mu)$ and so $\Phi_{A(\mathcal{E})}$ induces a surjective f-algebra isomorphism $\alpha : L_\infty(\mu) \to A(\mathcal{E})$ satisfying $\lambda(\alpha(p)) = \bar{\mu}(p)$ for all $p \in \mathcal{C}_{L_\infty(\mu)}$, where we denote by $\bar{\mu}$ the measure induced by μ on $\mathcal{C}_{L_\infty(\mu)}$.

Suppose that (X, Σ) is a representation space for B with corresponding homomorphism $\Phi_B : M_b(X, \Sigma) \to B$. Now we can apply the construction discussed in the first part of the present section to (Y, Λ, μ), (X, Σ) and B. We will use the same notation as introduced before. Moreover, *we assume in addition that A is order separable.* This implies that B is order separable as well. By Lemma 9.1, the operator $\overline{S} : E \to B$ is strictly positive and so it follows from Lemma 2.7 that E is order separable. Since E is Dedekind σ-complete, it follows that E is actually Dedekind complete and that the positive projection $P_E : E \to E$ is order continuous.

Collecting the above, we are now in the following situation:

- A is a Dedekind complete order separable f-algebra in which the unit element $\mathbf{1}$ is a strong order unit, $P : A \to A$ is an order continuous strictly positive projection onto the f-subalgebra $B \subseteq A$ with $\mathbf{1} \in B$;

- $\mathcal{E}$ is a complete Boolean subalgebra of $\mathcal{C}_A$ such that $\mathcal{C}_A = \mathfrak{S}(\mathcal{C}_B \cup \mathcal{E})$ and there is a completely additive measure $\lambda : \mathcal{E} \to [0,1]$ such that $P(e) = \lambda(e) \mathbf{1}$ for all $e \in \mathcal{E}$;

- E is a Dedekind complete f-algebra in which the unit element $\mathbf{1}_E$ is a strong order unit, $P_E : E \to E$ is an order continuous strictly positive projection onto the f-subalgebra $F \subseteq E$ with $\mathbf{1}_E \in F$;
- $\mathcal{C}_G$ is a complete Boolean subalgebra of $\mathcal{C}_E$ such that $\mathcal{C}_E = \mathfrak{S}(\mathcal{C}_F \cup \mathcal{C}_G)$ and there is a completely additive measure $\bar{\mu} : \mathcal{C}_G \to [0,1]$ such that $P_E(p) = \bar{\mu}(p)\mathbf{1}_E$ for all $p \in \mathcal{C}_G$, where $G = L_\infty(\mu)$;
- $\alpha : G \to A(\mathcal{E})$ is a surjective f-algebra isomorphism with $\alpha(\mathbf{1}_E) = \mathbf{1}$ such that $\lambda(\alpha(g)) = \bar{\mu}(g)$ for all $p \in \mathcal{C}_G$ and $\beta = \overline{S}_{|F} : F \to B$ is a surjective f-algebra isomorphism with $\beta(\mathbf{1}_E) = \mathbf{1}$.

Consequently, we are in a position to apply Proposition 6.4. Hence, there exists a unique surjective f-algebra isomorphism $\Psi : E \to A$ such that $\Psi_{|G} = \alpha$, $\Psi_{|F} = \overline{S}_{|F}$ and $\Psi \circ P_E = P \circ \Psi$. Recalling that E is the quotient space $M_b(\Omega, \mathcal{F}) / N_S$ with corresponding σ-order continuous quotient homomorphism $Q : M_b(\Omega, \mathcal{F}) \to E$, we define

$$\Phi_A : M_b(\Omega, \mathcal{F}) \to A$$

by $\Phi_A = \Psi \circ Q$. Then Φ_A is a σ-order continuous f-algebra homomorphism from $M_b(\Omega, \mathcal{F})$ onto A with $\Phi_A(\mathbf{1}_\Omega) = \mathbf{1}$. Therefore, $(\Omega, \mathcal{F})$ is a representation space for A with representation homomorphism Φ_A. In the next lemma we collect some properties of this representation Φ_A. Recall that we identify $M_b(X, \Sigma)$ and $M_b(Y, \Lambda)$ with f-subalgebras of $M_b(\Omega, \mathcal{F})$.

Lemma 9.4 *In the above situation the following hold:*

(i). the representation Φ_A is compatible with Φ_B, i.e., $(\Phi_A)_{|M_b(X,\Sigma)} = \Phi_B$;

(ii). $\Phi_A[M_b(Y, \Lambda)] = A(\mathcal{E})$;

(iii). $P \circ \Phi_A = \Phi_A \circ R_b$.

Proof. If $f \in M_b(X, \Sigma)$, then $Qf \in F$ and so $\Phi_A(f) = \Psi(Qf) = \overline{S}(Qf) = Sf = \Phi_B(R_b f) = R_b f$, which proves (i). Since $\Psi_{|G} = \alpha$, $Q[M_b(Y, \Lambda)] = G$ and $\alpha(G) = A(\mathcal{E})$ it is clear that (ii) holds. For the proof of (iii), recall from Lemma 9.3 that $P_E \circ Q = Q \circ R_b$, which implies that

$$P \circ \Phi_A = P \circ \Psi \circ Q = \Psi \circ P_E \circ Q = \Psi \circ Q \circ R = \Phi_A \circ R_b,$$

and we are done. ■

A combination of the above results with Proposition 8.3 immediately yields the following theorem.

Theorem 9.5 *Let A be a Dedekind complete order separable f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and let $P : A \to A$ be a strictly positive order continuous projection onto the f-subalgebra $B \subseteq A$ with $\mathbf{1} \in B$. Suppose that A is homogeneous over B and nowhere B-full. Let (X, Σ) be a representation space for B with representation homomorphism $\Phi_B : M_b(X, \Sigma) \to B$. Then there exists a non-atomic probability space (Y, Λ, μ) such that $(X \times Y, \Sigma \otimes \Lambda)$ is a pure representation space for P and A with representation $\Phi_A : M_b(X \times Y, \Sigma \otimes \Lambda) \to A$, which is compatible with Φ_B.*

10 Representations of positive projections: general case

We start this section with an extension of Theorem 9.5.

Proposition 10.1 *Let A be a Dedekind complete order separable f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and let $P : A \to A$ be a strictly positive order continuous projection onto the f-subalgebra $B \subseteq A$ with $\mathbf{1} \in B$. We assume furthermore that A is nowhere B-full. Let $\Phi_B : M_b(X, \Sigma) \to B$ be a representation of B.*
Then there exists a non-atomic σ-finite measure space (Y, Σ, μ) such that $(X \times Y, \Sigma \otimes \Lambda)$ is a representation space for P and A such that the corresponding representation $\Phi_A : M_b(X \times Y, \Sigma \otimes \Lambda) \to A$ is compatible with Φ_B.

Proof. Applying Proposition 8.6, let $\{p_\tau\}$ be a disjoint system in $\mathcal{C}_A$ with the stated properties. Since A is order separable, $\{p_\tau\}$ is at most countable and we will denote this system by $\{p_n : n = 1, 2, \dots\}$ with corresponding Boolean algebras $\{\mathcal{E}_n\}$ and measures $\{\lambda_n\}$. Now we apply Theorem 9.5 to the restrictions $P_{\restriction p_n} : A(p_n) \to A(p_n)$ and the induced representations $\Phi_B^{p_n} : M_b(X, \Sigma) \to B_{\restriction p_n}$. Consequently, for each n there exists a non-atomic σ-finite measure space (Y_n, Λ_n, μ_n) such that $(X \times Y_n, \Sigma \otimes \Lambda_n)$ is a representation space for $P_{\restriction p_n}$ and $A(p_n)$, such that the corresponding representation $\Phi_n : M_b(X \times Y_n, \Sigma \otimes \Lambda_n) \to A(p_n)$ is compatible with $\Phi_B^{p_n}$. Finally, an application of Lemma 4.15 finishes the proof. ■

A combination of Proposition 4.12, Lemma 4.15, Example 4.16 and Proposition 10.1 yields the following theorem.

Theorem 10.2 *Let A be a Dedekind complete order separable f-algebra in which the unit element $\mathbf{1}$ is a strong order unit and let $P : A \to A$ be a strictly positive order continuous projection onto the f-subalgebra $B \subseteq A$ with $\mathbf{1} \in B$. Let $\Phi_B : M_b(X, \Sigma) \to B$ be a representation of B.*

Then there exists a σ-finite measure space (Y, Σ, μ) such that $(X \times Y, \Sigma \otimes \Lambda)$ is a representation space for P and A, where the corresponding representation $\Phi_A : M_b(X \times Y, \Sigma \otimes \Lambda) \to A$ is compatible with Φ_B.

As already observed at the end of Section3, the next result is now an immediate consequence of Lemma 3.9.

Theorem 10.3 *Let L be a Dedekind complete order separable Riesz space with weak order unit $0 \leq w \in L$ and let $P : L \to L$ be a strictly positive order continuous projection onto the Riesz subspace $K \subseteq L$ with $w \in K$. Suppose that (X, Σ) is a representation space for K with corresponding representation $\Phi_K : \widehat{K} \to K$.*
Then there exists a σ-finite measure space (Y, Λ, μ) such that $(X \times Y, \Sigma \otimes \Lambda)$ is a representation space for P and L, where the corresponding representation $\Phi_L : \widehat{L} \to A$ is compatible with Φ_K.

Proof. Equip L_w with the f-algebra structure for which w is the unit element. Then K_w is an f-subalgebra of L_w and so we may apply the above theorem to the restriction $P_w = P_{|L_w} : L_w \to L_w$. We only need to refer to Lemma 3.9 to finish the proof. ∎

Next we will consider the extension of the above result to the situation in which L is locally order separable. Moreover, we will drop the assumption that L has a weak order unit.

Theorem 10.4 *Let L be a Dedekind complete locally order separable Riesz space and suppose that $P : L \to L$ is a strictly positive order continuous projection onto the Riesz subspace $K \subseteq L$ with $K^d = \{0\}$. Then there exists a disjoint system $\{Q_\tau\}$ of band projections in L such that $\bigvee_\tau Q_\tau = I$ and*

(i). $PQ_\tau = Q_\tau P$ for all τ;

(ii). defining $L_\tau = \operatorname{Ran}(Q_\tau)$ and $P_\tau = P_{|L_\tau} : L_\tau \to L_\tau$, there exists a representation space $(X_\tau \times Y_\tau, \Sigma_\tau \otimes \Lambda_\tau)$ for P_τ and L_τ.

Proof. For $0 \leq u \in L$ we will denote by $L(u)$ the band generated by u and if $0 \leq u \in K$, then $K(u)$ will denote the band generated by u in K. As observed in Lemma 3.6, K is a complete Riesz subspace of L and this implies that $K(u) = L(u) \cap K$ for all $0 \leq u \in K$. We claim that K is locally order separable. Indeed, take $0 < u \in K$ and equip the principal ideal L_u with the f-algebra structure for which u is the unit element. Then L_u is locally order separable and K_u is a complete f-subalgebra of L_u, so our claim follows from Lemma 4.6.

Let $\{w_\tau\}$ be a maximal disjoint system in K consisting of order separable elements, so $K(w_\tau)$ is order separable for all τ. Since K is a complete Riesz subspace of L and $K^d = \{0\}$ it follows that $\{w_\tau\}$ be a maximal disjoint system in L as well. Denoting by Q_τ the band projection in L onto $L(w_\tau)$, we have $\sup_\tau Q_\tau = I$. Next we will show that $Q_\tau P = PQ_\tau$ for all τ. To this end take $0 \leq f \in L(w_\tau)$. Then $f = \sup_n f \wedge nw_\tau$, so $Pf = \sup_n P(f \wedge nw_\tau)$ and this shows that $Pf \in L(w_\tau)$. Hence, P leaves $L(w_\tau)$ invariant. Using that $\sup_\tau Q_\tau = I$ and that P is order continuous, it follows easily that P leaves $L(w_\tau)^d$ invariant as well. Consequently, $Q_\tau P = PQ_\tau$. Let $P_\tau : L(w_\tau) \to L(w_\tau)$ be the restriction of P. Then P_τ is a strictly positive order continuous projection in $L(w_\tau)$ onto $K(w_\tau)$. By Lemma 2.7, $L(w_\tau)$ is order separable. Application of Theorem 10.3 to P_τ finishes the proof. ∎

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators,* Academic Press, Orlando, 1985.

[2] C. B. Huijsmans and B. de Pagter, Subalgebras and Riesz subspaces of an f-algebra, Proc. London Math. Soc., 48 (1984), 161-174.

[3] C. B. Huijsmans and B. de Pagter, Averaging Operators and Positive Contractive Projections, J. Math. Analysis and Applications, 113 (1986), 163-184.

[4] A. G. Kusraev, *Dominated Operators,* Math. and Its Appl., Vol. 519, Kluwer Academic Publishers, Dordrecht, 2000.

[5] W. A. J. Luxemburg, The work of Dorothy Maharam on kernel representations of linear operators, in: *Measure and measurable dynamics (Rochester, NY, 1987),* 177-183, Am. Math. Soc., Providence, RI, 1989.

[6] W. A. J. Luxemburg and J. J. Masterson, An extension of the concept of the order dual of a Riesz space, Can. J. Math., 19 (1967), 488-498.

[7] W. A. J. Luxemburg and B. de Pagter, Maharam extensions of positive operators and f-modules, Positivity 6 (2002), 147-190.

[8] W. A. J. Luxemburg and A.R. Schep, A Radon-Nikodym type theorem for positive operators and a dual, Indag. Math., 40 (1978), 357-375.

[9] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I,* North-Holland, Amsterdam, 1971.

[10] Dorothy Maharam, The representation of abstract measure functions, Trans. Amer. Math. Soc., 65 (1949), 279-330.

[11] Dorothy Maharam, Decompositions of measure algebras and spaces, Trans. Amer. Math. Soc., 69 (1950), 142-160.

[12] J. D. Monk and R. Bonnet (eds.), *Handbook of Boolean Algebras, Volume 3,* North-Holland, Amsterdam, 1989.

[13] P. Meyer-Nieberg, *Banach Lattices,* Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[14] H. Nakano, *Modern Spectral Theory,* Maruzen, Tokyo, 1950.

[15] Roman Sikorski, *Boolean Algebras,* Third Edition, Springer-Verlag, New York, 1969.

[16] A. C. Zaanen, *Riesz Spaces II,* North-Holland, Amsterdam-London, 1983.