GLOBAL EXISTENCE FOR CRITICAL POWER YANG-MILLS-HIGGS EQUATIONS IN $\mathbb{R}^{3+1}$

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ABSTRACT. We prove global existence for both smooth and finite energy solutions of the Yang-Mills-Higgs equations with a critical-power Higgs self-interaction in $\mathbb{R}^{3+1}$. The phrase "critical-power" denotes here a Higgs self-interaction behaving as $|\phi|^6$ for large ϕ.

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1. Introduction

This paper extends the global existence results of [4], [15] on the Yang-Mills-Higgs equations to include a wider class of Higgs self-interactions. Before stating these results, we set notation.

Let $\mathcal{G}$ denote a compact, semisimple Lie group and $\mathfrak{g}$ its Lie algebra. $\mathcal{G}$ and $\mathfrak{g}$ can be represented as matrices. We denote the positive definite inner product (Killing form) on $\mathfrak{g}$ by $< \cdot, \cdot >$. This inner product is compatible with commutators in that for all $A, B, C \in \mathfrak{g}$,

$$< [A, B], C > \quad = \quad < A, [B, C] > . \tag{1.1}$$

Here $[A, B] = AB - BA$ is the usual commutator of matrices. In the case of matrix groups, the inner product can be written as a suitably normalized trace.

For example, set $\mathcal{G} = \mathrm{SO}(n)$, the group of orthogonal matrices of determinant one (respectively $\mathcal{G} = \mathrm{SU}(n)$ the group of unitary matrices of determinant one). Then $\mathfrak{g} = \mathrm{so}(n)$, the algebra of skew symmetric matrices (resp. $\mathfrak{g} = \mathrm{su}(n)$ the algebra of trace-free skew hermetian matrices) with Killing form

$$\begin{aligned} < X, Y > \quad &= \quad -\mathrm{Tr}\left(X \cdot Y^{\mathrm{T}}\right) \\ (\text{resp.} \quad < X, Y > \quad &= \quad -\mathrm{Tr}\left(X \cdot Y^{*}\right)\Big) . \end{aligned}$$

Here Tr denotes the usual trace of matrices.

The variables for the Yang-Mills-Higgs field theory are a $\mathfrak{g}$-valued one form

$$A_\alpha : \mathbb{R}^{3+1} \longrightarrow \mathfrak{g}$$

and a *scalar field* [1]

$$\phi : \mathbb{R}^{3+1} \longrightarrow \mathfrak{g}.$$

The one form A_α is called the *gauge potential.* The space $\mathbb{R}^{3+1}$ is the four dimensional Euclidean space equipped with the flat Minkowski metric

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1).$$

We follow the convention that Greek indices take values in $\{0, 1, 2, 3\}$ while Latin indices are valued in $\{1, 2, 3\}$. Repeated indices are summed. The x^0 coordinate is also labelled t and is thought of as representing time.

Indices are raised and lowered using the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. For example, $A^\alpha \equiv g^{\alpha\beta} A_\beta$. We'll denote derivatives by $\frac{\partial}{\partial x^\alpha} \equiv \partial_\alpha$. The symbol ∂ represents a space-time derivative, ∂_0 or ∂_i.

Suppose given a $\mathfrak{g}$-valued function ψ on $\mathbb{R}^{3+1}$. The *covariant derivative* operator associated to A is defined by

$$D_\alpha^{(A)} \psi \quad \equiv \quad \partial_\alpha \psi + [A_\alpha, \psi].$$

One easily checks that (1.1) implies the following compatibility between the covariant derivative and the Killing form,

$$\partial_\alpha < X, Y > \quad = \quad < D_\alpha X, Y > + < X, D_\alpha Y > . \tag{1.2}$$

[1] Also called the *Higgs field*

Associated to a gauge potential A one has a two form $F_{\alpha\beta}$ called the *curvature* of A which measures the commutator of covariant derivatives in the following sense,

$$D_\alpha D_\beta \psi - D_\beta D_\alpha \psi \quad = \quad [F_{\alpha\beta}, \psi].$$

A straightforward calculation gives the following formula for the components of the curvature:

$$F_{\alpha\beta} \quad = \quad \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]. \tag{1.3}$$

Using (1.3), one easily proves the Bianchi identity,

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} \quad = \quad 0. \tag{1.4}$$

Recall that $O \in \mathcal{G}$ acts on $\mathfrak{g}$ via the adjoint action, which in the case of matrices is simply the mapping $X \rightarrow OXO^{-1}$ for all $X \in \mathfrak{g}$. Using this, one defines the following *gauge transformations* induced by a $\mathcal{G}$-valued function O(t,x),

$$A_\alpha \quad \longrightarrow \quad A_\alpha^{(O)} = OA_\alpha O^{-1} + O\partial_\alpha O^{-1} \tag{1.5}$$

$$\phi \quad \longrightarrow \quad \phi^{(O)} = O\phi O^{-1}. \tag{1.6}$$

A simple computation shows that the curvatures transform according to

$$\begin{aligned} F^{(A^{(O)})} \quad &= \quad OF^{(A)}O^{-1} \\ D^{(A^{(O)})}\phi^{(O)} \quad &= \quad OD^{(A)}(\phi)O^{-1}. \end{aligned}$$

Recall that the adjoint action is unitary on $\mathfrak{g}$. That is, if ψ_1, ψ_2 are two $\mathfrak{g}$-valued functions, then

$$< \psi_1, \psi_2 > = \quad < O\psi_1 O^{-1}, O\psi_2 O^{-1} > \quad = < \psi_1^{(O)}, \psi_2^{(O)} > . \tag{1.7}$$

For example, the quantity $|\phi|^2 \equiv < \phi, \phi >$ is invariant under gauge transformations.

We come finally to the equations: the Yang-Mills-Higgs equations arise as the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L} \quad = \quad -\frac{1}{4} < F_{\mu\nu}, F^{\mu\nu} > -\frac{1}{2} < D_\mu\phi, D^\mu\phi > -V(\phi) \tag{1.8}$$

where F is the curvature associated to the potential A and V is a positive gauge-invariant function called the *Higgs self-interaction*. We assume that V behaves like $|\phi|^6$ at infinity, in the sense that $|V(\phi)| = O(|\phi|^6), |V'(\phi)| = O(|\phi|^5)$. To simplify the exposition we will assume in fact that V is the pure power self-interaction

$$V(\phi) \quad = \quad \frac{|\phi|^6}{6}. \tag{1.9}$$

For various reasons (see below), we call this the *critical power* Higgs self-interaction.

The Yang-Mills-Higgs equations arise from (1.8) via the variational principle. Using (1.1) and (1.2) one easily finds,

$$D_\mu F^{\nu\mu} \quad = \quad -[\phi, D^\nu\phi] \tag{1.10}$$

$$D_\mu D^\mu\phi \quad = \quad V'(\phi) \; = \; |\phi|^4\,\phi. \tag{1.11}$$

In light of the gauge-invariance of (1.8), solutions of (1.10) - (1.11) are defined only up to gauge transformations.

Definition 1.1. *A* classical solution *to the Yang-Mills-Higgs equations is an equivalence class*[2] $\widetilde{(A,\phi)}$ *of smooth* $\mathfrak{g}$*-valued pairs* (A,ϕ) *such that* $F^{(A)},\phi$ *satisfy* (1.10) - (1.11).

Definition 1.2. *A* generalized solution *to the Yang-Mills-Higgs equations is an equivalence class* $\widetilde{(A,\phi)}$ *which contains a representative*[3] *pair* (A,ϕ) *for which* $F^{(A)},\phi$ *satisfy* (1.10) - (1.11) *in the sense of distributions.*

One can fix an equivalence class representative by appending an additional equation to the system (1.10) - (1.11). This process is called *fixing the gauge*, and the additional equation is called a *gauge condition.* We will at various points in this paper consider two gauge conditions: $\partial^i A_i = 0$ and $A_0 = 0$. These denote the Coulomb gauge and Temporal gauge, respectively.

As in all Lagrangian field theories, the conservation laws are encoded in the *energy-momentum tensor* which is defined in terms of the Lagrangian by

$$T_{\alpha\beta} = -\left(\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}\mathcal{L}\right).$$

For Yang-Mills-Higgs, this is

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{4}\left(< F_{\alpha\mu}, F_\beta{}^\mu > + < *F_{\alpha\mu}, *F_\beta{}^\mu >\right) + \frac{1}{2} < D_\alpha\phi, D_\beta\phi > \\ &\quad -\frac{1}{4}g_{\alpha\beta} < D_\mu\phi, D^\mu\phi > -\frac{1}{2}g_{\alpha\beta}V(\phi) \end{aligned} \tag{1.12}$$

where

$$*F_{\alpha\mu} = \frac{1}{2}\epsilon_{\alpha\mu\beta\gamma}F^{\beta\gamma}.$$

Here, $\epsilon_{\alpha\mu\beta\gamma}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Using Noether's theorem, one obtains conservation laws by contracting (1.12) with the vector fields generating the conformal group of Minkowski space-time. This is explained in greater detail in Section 3. For example, contracting with the vectorfield $T_0 = \partial_t$ gives that the following energy

$$(\mathcal{E}(t))^2 = \int_{\mathbb{R}^3} |E(t,\cdot)|^2 + |H(t,\cdot)|^2 + |D\phi|^2 + V(\phi)dx \tag{1.13}$$

is constant in time. Here $E_i = F_{0i}$ and $H_i = *F_{0i} = \frac{1}{2}\epsilon_{ijk}F^{jk}$ represent the electric and magnetic components of the curvature.

One way to parametrize the solutions of (1.10)-(1.11) is to assign data

$$(\bar{A}_i, \bar{E}_i, \bar{\phi}, \overline{D_0\phi})$$

on the hypersurface $t = 0$ in $\mathbb{R}^{3+1}$ and consider the evolution of this data under the Yang-Mills-Higgs equations. The statement

$$(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi}) \in H^s(\mathbb{R}^3)$$

[2] Where two pairs are equivalent if they are related by a gauge transformation.
[3] In our work, this representative will be chosen in the temporal gauge.

means $\bar{A}, \bar{\phi} \in H^s(\mathbb{R}^3)$; $\bar{E}, \overline{D_0\phi} \in H^{s-1}(\mathbb{R}^3)$. An initial data set must satisfy the compatibility condition

$$\partial^i \bar{E}_i + [\bar{A}^i, \bar{E}_i] \quad = \quad -[\phi, \overline{D_0\phi}] \tag{1.14}$$

The following notation is used to describe various sets in $\mathbb{R}^{3+1}$:

Denote the light cone with vertex $(t_0, x_0) \in \mathbb{R}^{3+1}$ by

$$\bigwedge^{(t_0,x_0)} = \left\{(t,x) \;\middle|\; 0 \le t \le t_0; |x - x_0| = t_0 - t\right\}.$$

The restriction of this set to some smaller time interval $[t_1, t_2]$ will be written $/\ \ \backslash_{t_1}^{t_2}$. Filling in these sets, we write

$$\begin{aligned} \blacktriangle^{(t_0,x_0)} &= \left\{(t,x) \;\middle|\; t \le t_0; |x - x_0| \le t_0 - t\right\} \\ \blacksquare_{t_1}^{t_2} &= \left\{(t,x) \;\middle|\; t_1 \le t \le t_2; |x - x_0| \le t_0 - t\right\}. \end{aligned}$$

Finally, write $\bigcirc_t$ and $\bullet_t$ for the time$=$ t slices of $\bigwedge$ and $\blacktriangle$, respectively. Hence $\bigcirc_t$ is a sphere, while $\bullet_t$ is a solid ball in $\mathbb{R}^3$.

The main results of this paper are contained in the following two theorems.

Theorem 1.1 (Global Smooth Solutions). *Consider Yang-Mills-Higgs with a critical power Higgs self-interaction. Then any finite energy H^s_{loc} initial data set $(s \ge 2)$ admits a unique, global solution in the temporal gauge.*[4]

$$A, \phi \in C([0,\infty); H^s_{loc}(\mathbb{R}^3)) \cap C^1[0,\infty); H^{s-1}_{loc}(\mathbb{R}^3))$$

Theorem 1.2 (Global Energy Class Solution). *Consider Yang-Mills-Higgs with a critical power Higgs self-interaction. Any finite energy data set has a unique, global energy class development in the temporal gauge* [5]

$$A, \phi \in C([0,\infty); H^1_{loc}(\mathbb{R}^3)) \cap C^1([0,\infty); L^2_{loc}(\mathbb{R}^3)) \tag{1.15}$$

Remark that uniqueness here holds in the class[6] *of all such functions which can be approximated, in the norm evident in* (1.15)*, by smooth solutions.*

In section two, we briefly recall what a *critical power* involves in the simpler case of the nonlinear wave equation

$$\Box\, \phi - |\phi|^{p-1}\, \phi = 0 \qquad (t,x) \in \mathbb{R}^{1+3} \tag{1.16}$$

which highlights the difference between *subcritical* (when $p < 5$), and *critical* ($p = 5$) problems. Roughly speaking, the classification is based on how the energy transforms under the natural scaling[7] of (1.16). For the critical power problem, two basic

[4] The proof will also show that data in $H^s(\mathbb{R}^3)$ implies evolution in $H^s(\mathbb{R}^3)$. That is, one can remove the "loc" from the statements assumption and conclusion.

[5] The proof will show in addition the existence of an H^1 solution for H^1 data.

[6] This is called the class of *admissible* solutions.

[7] If ϕ solves (1.16), then so does $\tilde{\phi}(t,x) = \lambda^{-\frac{2}{p-1}} \phi(\frac{t}{\lambda}, \frac{x}{\lambda})$. For subcritical problems, the initial energy scales to zero as $\lambda \nearrow \infty$. For critical powers, energy is scale invariant.

ideas make for a simple proof of regularity [10]. First, there is nonconcentration of energy shown by Struwe [28]. To take advantage of this, Grillakis [10] noticed that certain space-time estimates can be employed. In Section two, we prove a simple extension of the nonconcentration result and a simplification in $\mathbb{R}^{3+1}$ of the argument using space-time estimates[8]

Section three proves the key energy nonconcentration result for the Higgs field in (1.10)-(1.11).

Section four proves Temporal and Coulomb gauge local existence theorems for (1.10)-(1.11).

Section five provides the proofs of Theorems 1.1 and 1.2 modulo some basic estimates which are found in sections six and seven. Section seven also includes an important derivation of an integral-differential system of equations giving Coulomb gauge solutions of (1.10)-(1.11).

The breakthrough paper of Eardley and Moncrief [4] proves Theorem 1.1 for *subcritical* Higgs nonlinearities. The main step in their work uses the fundamental solution for the wave equation and a judicious change of gauge to obtain the following pointwise bound,

$$||\phi||_{L^\infty([0,T]\times\mathbb{R}^3)} + ||F||_{L^\infty([0,T]\times\mathbb{R}^3)} \quad \leq C.$$

This argument's dependence on L^∞ bounds makes it very difficult to extend to the case of a critical Higgs self-interaction for the simple fact that the L^∞-based argument of Grillakis in [9] is quite a bit more involved than his local approach in [10]. In addition, the argument in [4] doesn't apply to relatively rough initial data. (Theorem 1.2).

The motivation for our results comes rather from the recent work of Klainerman and Machedon ([15],[14]) establishing global energy-class solutions to the Yang-Mills equations. Their techniques are basic to the proofs of both our theorems, as they allow us to localize the analysis and take advantage of energy nonconcentration in the Higgs field.

2. A Model Problem: The Critical Power Semilinear Wave Equation

The bootstrap argument given in Section 6 for the Higgs field ϕ is complicated by the presence of the gauge potential and a scaling factor. Nevertheless, there are two very simple new ideas which make the argument possible. In Lemmas 2.3 and 2.4 of this section, we'll encounter these ideas in the uncluttered context of the critical power scalar wave equation. [9]

The critical power wave equation in $\mathbb{R}^{3+1}$ is

$$\Box\phi \quad = \quad |\phi|^4\,\phi. \tag{2.1}$$

Global existence was shown for small energy data by Rauch [22], for spherically symmetric data by Struwe [28], and for general data by Grilllakis [9]. In [10], Grillakis simplifies his earlier argument by using two ideas. First, he generalizes the following local nonconcentration result of Struwe.

[8] This simpler approach is found also in [24].

[9] For a careful exposition of Jörgen's regularity argument for *subcritical* powers, see [22].

Theorem 2.1 (Struwe). *Given a solution of* (2.1) *with*

$$\phi(t,x) \quad \in \quad C([0,T^*);H^2(\mathbb{R}^3)) \cap C^1([0,T^*);H^1(\mathbb{R}^3))$$

and a point $p = (T^*, x_p)$.

Then for all $\epsilon > 0$, *there exists* δ_1 *such that for* $t \in [T^* - \delta_1, T^*)$,

$$\int_{\bullet_t} |\phi|^6 \, dx < \epsilon$$

Here $\bullet_t$ *is the time* $= t$ *slice of* $\blacktriangle^p$. *See figure* (2.1).)

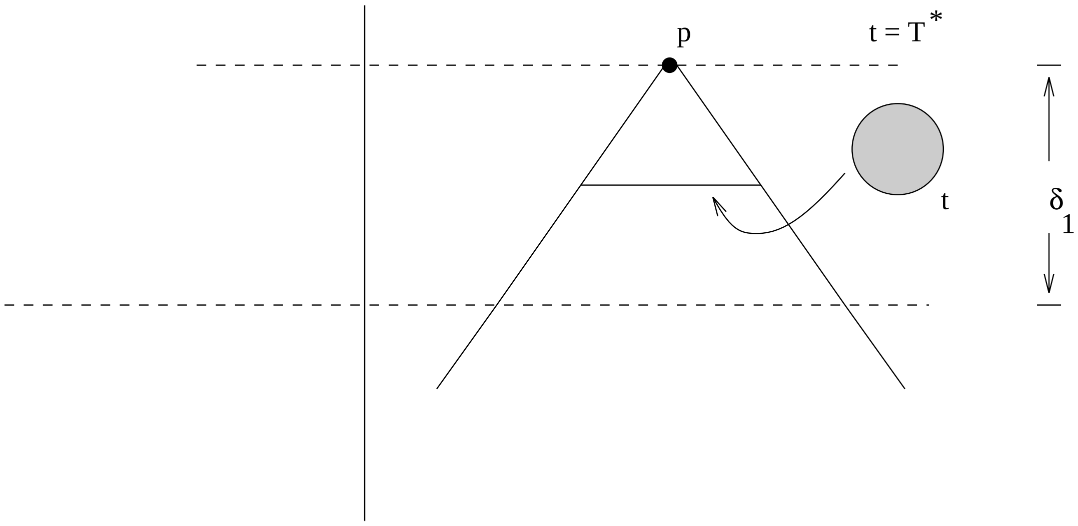

FIGURE 2.1. Decay inside a cone

In [10], it's shown that $||\phi||_{L^6}$ actually decays on constant time slices under a space-like hypersurface, see Figure 2.2.

Along with this decay, Grillakis uses a space-time estimate for the linear wave equation which follows quickly from the work of Marshall [18] and which belongs to a family of results known as "Strichartz estimates" (See [27]).

Theorem 2.2 (Grillakis). [10] *The solution of*

$$\begin{aligned} \Box\phi &= F \\ \phi(0,\cdot) &= \phi_0(\cdot) \\ \partial_t\phi(0,\cdot) &= \phi_1(\cdot) \end{aligned}$$

[10] *This is the* $3+1$ *dimensional case of a more general result in* [10].

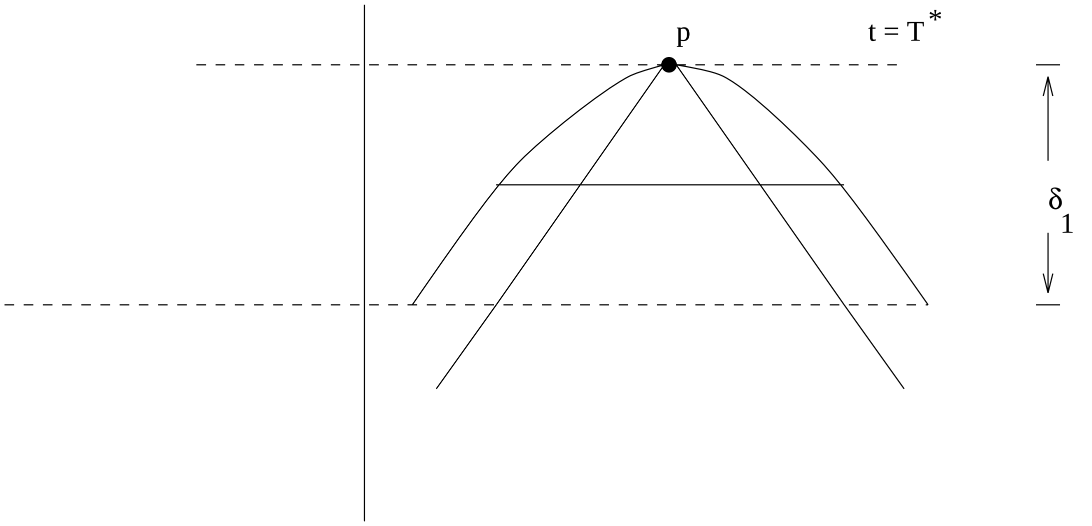

FIGURE 2.2. Decay under space-like hypersurface

enjoys the following estimate:

$$\left(\int_0^T \|\phi(s,\cdot)\|^r_{L^q(\bigcirc_s)}\,dt\right)^{\frac{1}{r}} \quad\leq\quad C(\epsilon)\left(\|\phi_0\|_{H^1(\bigcirc_0)} + \|\phi_1\|_{L^2(\bigcirc_0)} + \int_0^T \left\|\phi^5\right\|_{L^2(\bigcirc_s)}\,ds\right)$$

where

$$q(\epsilon) = \frac{6}{\epsilon} \qquad r(\epsilon) = \frac{2}{1-\epsilon} \qquad \epsilon \in (0,1]$$

and $C(\epsilon)$ stays bounded as long as $\epsilon > 0$.

We will handle the critical power Higgs nonlinearity in Section 6 using Grillakis' basic approach. However, two modifications are necessary.

First of all, we need to show that the L^6 norm of the Higgs field can be made small in a region which is slightly bigger than those regions provided in the results of Struwe and Grillakis.

Given a point $p = (T^*, x_p)$, consider the region

$$\bigtriangleup^p_{(\delta_1,\delta_2)} \quad=\quad \{(t,x)|T^* - \delta_1 \leq t < T^*;\ |x - x_p| \leq T^* - t + \delta_2\} \tag{2.2}$$

as in fig. (2.3). We present the decay result first in the context of the critical wave equation. The corresponding lemma for Yang-Mills-Higgs is in Section 3.

Lemma 2.3. *Given*

- $p = (T^*, x_p)$,
- *$\phi(t,x)$ solving the critical wave equation on $[0,T^*)$,*
- $\epsilon > 0$.

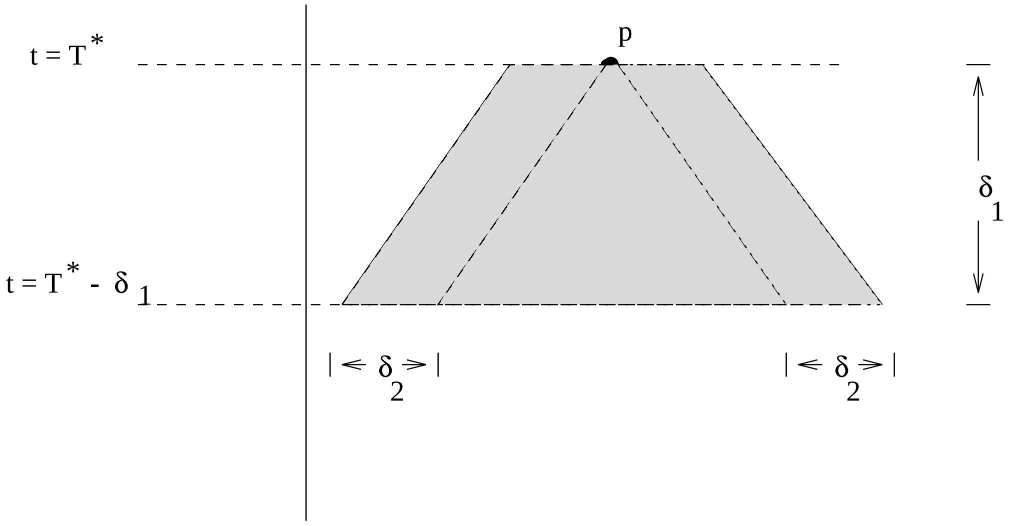

FIGURE 2.3. The trapezoidal region of decay

there exist δ_1, δ_2 such that for $t \in [T^* - \delta_1, T^*)$

$$\int_{\bullet^{trap}_t} |\phi|^6 \, dx \quad < \epsilon$$

where $\bullet^{trap}_t$ *is the time $= t$ slice of* $\blacktriangle^{p}_{(\delta_1, \delta_2)}$

Proof The proof follows easily from the local energy estimate, Theorem 2.1, and the decay of the flux.

Using Theorem 2.1 and decay of flux[11], choose δ_1 so that

$$\int_{\bullet_t} |\phi|^6 \, dx \quad \leq \quad \frac{\epsilon}{2} \qquad t \in [T^* - \delta_1, T^*) \tag{2.3}$$

$$\int_{\bigwedge_{T^*-\delta_1}} |\partial_- \phi|^2 + |\not\nabla_i \phi|^2 + \frac{|\phi|^6}{6} d\sigma dt \quad \leq \quad \frac{\epsilon}{4}. \tag{2.4}$$

where ∂_-, $\not\nabla_i$ span the cone at each point. ($\partial_- = \partial_t - \partial_r$, $\not\nabla_i$ span the sphere at constant times.)

Since the total energy is finite for each time $t < T^*$, one can find a δ_2 small so that

$$\int_{\Omega_{T^*-\delta_1}} \frac{1}{2} |\partial \phi|^2 + \frac{|\phi|^6}{6} \quad \leq \quad \frac{\epsilon}{4} \tag{2.5}$$

where $\Omega_{T^*-\delta_1}$ is the region

$$\Omega_{T^*-\delta_1} = \{(t, x) | t = T^* - \delta_1; \, \delta_1 < |x - x_p| < \delta_1 + \delta_2\}$$

[11]See Section 3.

between the spheres of radius δ_1 and $\delta_1+\delta_2$ with centers x_p. The energy estimate yields that for all $T^*-\delta_1 \le t < T^*$,

$$\begin{aligned}\int_{\Omega_t} \frac{|\phi|^6}{6}dx &\le C\left(\int_{\Omega_{T^*-\delta_1}} \frac{1}{2}|\partial\phi|^2 + \frac{|\phi|^6}{6}dx + \int_{/\ \ \setminus^{t}} |\partial_-\phi|^2 + |\nabla_i\phi|^2 + \frac{|\phi|^6}{6}d\sigma dt\right)\\ &\le \frac{\epsilon}{2}\end{aligned}$$

by (2.4) and (2.5). Combining this with (2.3) completes the proof. ■

In [9], Grillakis uses (2.2) to prove the L^8 space-time norm of ϕ stays bounded beneath space-like hypersurfaces, from which he deduces higher Sobolev bounds. In fact, in $\mathbb{R}^{3+1}$ we can use (2.2) to directly bound the H^2 norm of ϕ on $\triangle_{(\delta_1,\delta_2)}$.

For the remainder of this section, denote by $\triangle$ a truncated light cone of base radius 1 and height $\frac{1}{2}$. Also, define the norms

$$\begin{aligned}|||\phi|||_{L^6} &\equiv \sup_{0\le s\le\frac{1}{2}} \|\phi\|_{L^6(\bigcirc_s^{\mathrm{trap}})}\\ \|(\phi,\partial_t\phi)\|_{H^s(\bigcirc_t^{\mathrm{trap}})} &\equiv \|\phi\|_{H^s(\bigcirc_t^{\mathrm{trap}})} + \|\partial_t\phi\|_{H^{s-1}(\bigcirc_t^{\mathrm{trap}})}\end{aligned}$$

where $\bigcirc_t^{\mathrm{trap}}$ is the time $=t$ slice of $\triangle$.

We first see record sufficient conditions to control $\|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_t^{\mathrm{trap}})}$ for all $0\le t\le\frac{1}{2}$.

Lemma 2.4. *Given ϕ a smooth solution of*

$$\Box\phi = |\phi|^4\phi$$

on $\triangle$. Assume either

$$(*) \qquad |||\phi|||_{L^6}\cdot\left(\|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_0^{trap})}\right)^3 \le \tilde{C}$$

or

$$(**) \qquad \|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_0^{trap})} \le \tilde{C}$$

where $\tilde{C}$ is some universal constant.
Conclude for all $0\le t\le\frac{1}{2}$,

$$\int_0^t \left\|\phi^5\right\|_{L^2(\bigcirc_s^{trap})} ds \le C\left(\|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_0^{trap})}\right) \tag{2.6}$$

and

$$\|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_t^{trap})} \le C\|(\phi,\partial_t\phi)\|_{H^1(\bigcirc_0^{trap})} \tag{2.7}$$

for some universal constant C.

Proof The bound in (2.7) follows from (2.6) and the following expression of the energy inequality,

$$\|(\phi, \partial_t\phi)\|_{H^1(\bullet_t^{\mathrm{trap}})} \leq C\left(\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})} + \int_0^t \left\|\phi^5\right\|_{L^2(\bullet_s^{\mathrm{trap}})} ds\right).$$

It remains to prove (2.6).

Applying Theorem 2.2 with $\epsilon = \frac{1}{2}$,

$$\begin{aligned}
X(t) &\equiv \int_0^t \left\|\phi^5\right\|_{L^2(\bullet_s^{\mathrm{trap}})} ds &\quad (2.8)\\
&\leq \||\phi\||_{L^6} \cdot \int_0^t \|\phi\|^4_{L^{12}(\bullet_s^{\mathrm{trap}})} ds &\\
&\leq C \||\phi\||_{L^6} \cdot \left(\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})} + X(t)\right)^4 &\quad (2.9)
\end{aligned}$$

Hence at each time $t \in [0, \frac{1}{2}]$,

$$\begin{aligned}
X(t) &\leq C \||\phi\||_{L^6} \cdot (\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})})^4\\
&\text{or}\\
X(t) &\leq C \||\phi\||_{L^6} \cdot X(t)^4.
\end{aligned}$$

That is

$$\begin{aligned}
X(t) &\leq C \||\phi\||_{L^6} \cdot (\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})})^4\\
&\text{or} &\quad (2.10)\\
\frac{1}{(C\||\phi\||_{L^6})^{\frac{1}{3}}} &\leq X(t).
\end{aligned}$$

Assumption $(*)$ implies $C \||\phi\||_{L^6} \cdot (\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})})^4 \leq \frac{1}{(C\||\phi\||_{L^6})^{\frac{1}{3}}}$. Hence (2.10), together with the fact that $X(t)$ is continuous with $X(0) = 0$, implies

$$X(t) \leq C\left(\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})}\right) \quad (2.11)$$

as desired.

If instead one assumes $(**)$, apply the energy estimate and Sobolev's theorem to the factor $\||\phi\||_{L^6}$ on the right side of (2.9) to conclude

$$X(t) \leq C(\|(\phi, \partial_t\phi)\|_{H^1(\bullet_0^{\mathrm{trap}})} + X(t))^5. \quad (2.12)$$

Under assumption $(**)$, a simple bootstrap argument similar to (2.10) leads From (2.12) to the bound (2.11). ∎

In order to control the H^2 norm it is enough to bound the H^1 norm of differences between two solutions ϕ, ϕ', where one regards ϕ' as a small space-translation of ϕ.

Lemma 2.5. *Given two solutions ϕ, ϕ' to the critical power wave equation on* .
(A). *Assuming condition (*) for both ϕ, ϕ',*

$$\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_t^{trap})} \leq K\left(\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_0^{trap})}\right) \tag{2.13}$$

for a constant K depending only on $\|(\phi, \partial_t\phi)\|_{H^1({}_0^{trap})}, \|(\phi', \partial_t\phi')\|_{H^1({}_0^{trap})}$.
(B). *Assuming condition (**) for ϕ, ϕ',*

$$\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_t^{trap})} \leq C\left(\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_0^{trap})}\right) \tag{2.14}$$

for some universal constant C.

Proof The argument is, modulo some algebra, the same as in Lemma 2.4 above. The energy estimate applied to the equation

$$\Box(\phi-\phi') \;=\; |\phi|^4\phi - |\phi'|^4\phi'$$

gives

$$\begin{aligned}\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_t^{\mathrm{trap}})} \;&\leq\; \|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_t^{\mathrm{trap}})} \\ &\quad + \int_0^t \left\| |\phi|^4\phi(s) - |\phi'|^4\phi'(s)\right\|_{L^2({}_s^{\mathrm{trap}})} ds.\end{aligned}$$

Define the function $H(s) \equiv |s|^4 s$.

$$\begin{aligned}|H(a)-H(b)| \;&=\; \left|\int_0^1 \frac{d}{d\lambda}H(\lambda a + (1-\lambda)b)d\lambda\right| \\ &\leq\; |a-b| \sup_{0\leq\lambda\leq 1} |H'(\lambda a + (1-\lambda)b)| \\ &\leq\; C\,|a-b|\left||a|^4+|b|^4\right|.\end{aligned}$$

Hence

$$\begin{aligned}Y(t) &\equiv \int_0^t \left\| |\phi|^4\phi(s) - |\phi'|^4\phi'(s)\right\|_{L^2({}_s^{\mathrm{trap}})} ds \\ &\leq C\int_0^t \left\|(\phi-\phi')(|\phi|^4+|\phi'|^4)\right\|_{L^2({}_s^{\mathrm{trap}})} ds \\ &\leq \|\phi-\phi'\|_{L^6}\left\{\int_0^t \|\phi\|^4_{L^{12}({}_s^{\mathrm{trap}})} ds + \int_0^t \|\phi'\|^4_{L^{12}({}_s^{\mathrm{trap}})}\right\} \\ &= \|\phi-\phi'\|_{L^6}\,Q(t)\end{aligned}$$

where $Q(t)$ is the quantity inside braces. Assuming either $(*)$ or $(**)$, Lemma 2.4 and Theorem 2.2 give $Q(t)$ is bounded. Clearly $Q(0)=0$. Choose $\tilde{t}>0$ so that $Q(\tilde{t})<\epsilon$ for some small ϵ to be chosen shortly. By the Sobolev and energy estimates,

$$Y(\tilde{t}) \leq C\left(\|((\phi-\phi'), \partial_t(\phi-\phi'))\|_{H^1({}_0^{\mathrm{trap}})} + Y(\tilde{t})\right)\cdot\epsilon$$

hence if $C \cdot \epsilon < \frac{1}{2}$,

$$Y(\tilde{t}) \leq 2C\epsilon \left(\|((\phi - \phi'), \partial_t(\phi - \phi'))\|_{H^1(\bullet_0^{\mathrm{trap}})} \right).$$

Now[12] repeat the above argument, starting at time $t = \tilde{t}$, until the entire height of the truncated cone is traversed. The number of iterations is roughly $\frac{Q(t)}{\epsilon}$ so one obtains

$$Y(t) \leq C(Q(t)) \left(\|((\phi - \phi'), \partial_t(\phi - \phi'))\|_{H^1(\bullet_0^{\mathrm{trap}})} \right)$$

where $C(Q(t))$ is a constant depending on $Q(t)$. The argument in Lemma 2.4 shows that either $(*)$ or $(**)$ is sufficient to give

$$\begin{aligned} Q(t) \quad &\leq \quad C \Big(\|(\phi, \partial_t \phi)\|_{H^1(\bullet_0^{\mathrm{trap}})} \\ &\qquad + \|(\phi', \partial_t \phi')\|_{H^1(\bullet_0^{\mathrm{trap}})} \Big). \end{aligned}$$

This completes the proof of (1) and (2). ■

Global existence for the critical power wave equation follows quickly from Lemmas 2.3, 2.4, and 2.5:

Theorem 2.6 (Grillakis [9]). *Given initial data $(\phi_0, \phi_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$, $s \geq 2$, there exists a solution ϕ of (2.1) with*

$$\phi \quad \in \quad C([0, \infty); H^s(\mathbb{R}^3)) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}^3)).$$

Proof The classical local existence theorem gives a ϕ with

$$\phi(t, x) \quad \in \quad C\left([0, T^*); H^s(\mathbb{R}^3)\right) \cap C^1([0, T^*); H^{s-1}(\mathbb{R}^3))$$

where

$$T^* \quad = \quad \sup \left\{ t > 0 \middle| \, \|\phi(t, \cdot)\|_{H^2(\mathbb{R}^3)} + \|\partial_t \phi(t, \cdot)\|_{H^1(\mathbb{R}^3)} < \infty \right\}.$$

Assume for contradiction $T^* < \infty$. Take $B \subset \mathbb{R}^3$ with complement B^c so that

$$\|\phi_0\|_{H^1(B^c)} + \|\phi_1\|_{L^2(B^c)} \quad < \quad \tilde{C}$$

where $\tilde{C}$ is the universal constant appearing in condition $(**)$. Denote the domain of influence of B by $I(B)$, and the complement of $I(B)$ in $[0, T^*) \times \mathbb{R}^3$ by $J(B)$. See figure 2.4.

Lemmas 2.4 and 2.5 are easily modified to give

$$\sup_{0 < \tau < T^*} \|\phi\|_{H^2(J(B) \cap \{t = \tau\})} \quad \leq \quad C \, \|(\phi_0, \phi_1)\|_{H^2(B^c)}.$$

Hence any singularity must occur in $I(B)$. To reach a contradiction it suffices to show that every point $p = (T^*, x_p)$ in $I(B)$ has a neighborhood on which $\|\phi\|_{H^2} + \|\partial_t \phi\|_{H^1}$ is finite.

[12] The following reasoning is taken from [17]

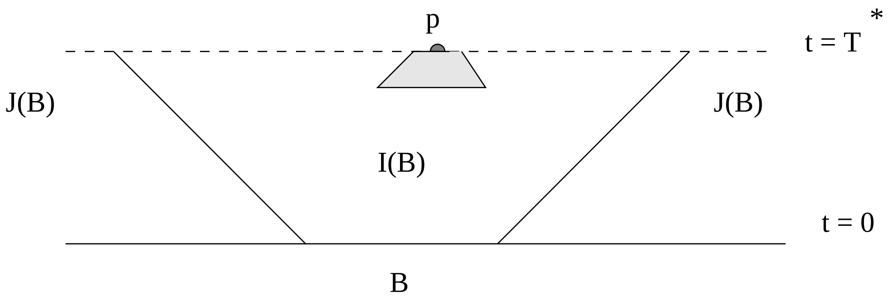

FIGURE 2.4. We aim to show p is not a singularity

Lemma 2.3 gives a region $^{p}_{(\delta_1,\delta_2)}$ on which

$$\sup_{T^*-\delta_1\le t<T^*} \|\phi\|_{L^6(\ _t^{\mathrm{trap}})}\cdot\left(\|(\phi,\partial_t\phi)(0)\|_{H^1(\mathbb{R}^3)}\right)^3 \quad\le\quad \tilde{C}. \tag{2.15}$$

which implies $(*)$ for the truncated cone $^{p}_{\delta_1,\delta_2)}$. Replace δ_2 with $\tilde{\delta}_2=\frac{\delta_2}{2}$. Define

$$\phi_\omega(t,x)\equiv\phi(t,x-\omega).$$

Then for $|\omega|<\frac{\delta_2}{2}$, equation (2.15) holds for ϕ_ω as well. Apply Lemmas 2.4 2.5 to ϕ and $\phi'=\phi_\omega$ to conclude

$$\|\phi\|_{H^2(\ _{T^*}^{\mathrm{trap}})}\quad\le\quad C\,\|(\phi,\partial_t\phi)\|_{H^2(\mathbb{R}^3)}$$

The higher regularity follows easily since the L^∞ norm is bounded by the H^2 norm. ∎

Remarks

The argument used in Lemma 2.4 appears also in the work of Smith and Sogge [24] on the critical problem in the presence of an obstacle. (For related work, see [23].)

For more detailed investigations of the evolution of smooth and finite energy initial data for the critical problem see [10], [30], [12],[31]; also [11],[5], [6], [28], [22] and [29].

3. ENERGY NONCONCENTRATION

The previous section suggests that a global existence proof for the critical power Yang-Mills-Higgs equation might take advantage of some sort of energy nonconcentration. In section 3.2 We will apply some beautiful and rather general techniques to prove such a result for Yang-Mills-Higgs. When compared to the behavior of covariant field theories, this result is not unexpected.

Energy decay properties of wave equations have enjoyed considerable attention since at least Wilcox [33] where the solution to the wave equation in the presence of a spherical obstacle is shown to decay exponentially in time at each point. Morawetz [19], [20] proved similar decay for the energy, and in [21] we finds the identity which drives much of the subsequent work in this area. (See also Lax-Phillips [16].) Indeed for the nonlinear wave equation (for example, see Strauss [25]) and other field theories such as wave maps (Christodoulou-Zadeh [2]), Maxwell and Yang-Mills equations (Costa-Strauss [3], Glassey-Strauss [8],[7], and Strauss [26]), there has been a lot of interesting work using the so-called *Morawetz identities.* The arguments in this section follows most closely the approach of Christodoulou-Klainerman [1].

As was pointed out in the previous section: in the context of the critical power wave equation, the key energy nonconcentration result first appeared in Struwe [28]. The proof of this nonconcentration in Theorem 3.3 below is a modification of a similar argument in Struwe-Shatah [30]. For more general nonconcentration results, see Grillakis [10].

3.1. Basic Energy Estimates. We first recall the basics of the energy method in the context of an arbitrary Lagrangian field theory with field ψ. Let X be an arbitrary vectorfield and P the one-form obtained by contracting the energy-momentum tensor of the field ψ with X.

$$P_\alpha = T_{\alpha\beta} X^\beta$$

Since T is symmetric and divergence-free

$$\begin{aligned} \partial^\alpha P_\alpha &= \left(\partial^\alpha T_{\alpha\beta}\right) X^\beta + T_{\alpha\beta}\left(\partial^\alpha X^\beta\right) \\ &= \frac{1}{2} T^{\alpha\beta}\ {}^{(X)}\pi_{\alpha\beta} \end{aligned} \tag{3.1}$$

where ${}^{(X)}\pi_{\alpha\beta}$ is the deformation tensor of X

$$\begin{aligned} {}^{(X)}\pi_{\alpha\beta} &= \left(\mathcal{L}_X g\right)_{\alpha\beta} \\ &= \partial_\alpha X_\beta + \partial_\beta X_\alpha. \end{aligned}$$

Here, $\mathcal{L}_X$ denotes the Lie derivative.

Conservation laws arise from integrating (3.1) over a region of space-time and using Stoke's theorem. The result is,

Theorem 3.1. *Let P_μ be a one-form satisfying*

$$\partial^\mu P_\mu = F$$

on $\blacktriangle_0^{\tilde{t}}$. *Then for all* $0 \le t_0 \le t_1 \le \tilde{t}$,

$$\int_{\bullet_{t_1}} \langle P, \partial_t\rangle + \int_{/\ \backslash_{t_0}^{t_1}} \langle P, E_-\rangle = \int_{\bullet_{t_0}} \langle P, \partial_t\rangle - \int_{\blacktriangle_{t_0}^{t_1}} F\, dt\, dx. \tag{3.2}$$

Notation: $\int_{/\ \backslash_{t_0}^{t_1}} \langle P, E_-\rangle = \int_{t_0}^{t_1} dt \int_{\bigcirc_t} \langle P, E_-\rangle\, d\sigma_t$, $d\sigma_t$ the area element on the sphere $\bigcirc_t$, and $\langle P, X\rangle$ denotes contraction of the one-form P with the vector field X. Also, $E_- = \partial_t - \partial_r$ is the normal vectorfield to $/\ \backslash$.

In the particular case when X is Killing, its deformation tensor π vanishes identically and (3.1) gives the differential conservation law

$$\partial^\alpha P_\alpha = 0 \tag{3.3}$$

which implies

$$\int_{\bullet_{t_1}} T(\partial_t, X) + \int_{/\ \ \backslash_{t_0}^{t_1}} T(E_-, X) = \int_{\bullet_{t_0}} T(\partial_t, X). \tag{3.4}$$

Equation (3.4) is called the local energy equality. If we apply these arguments to space-time domains of ever wider base, and assumes that the intial data decay at infinity so that the boundary integral in (3.4) goes to zero, we get the global energy equality

$$\int_{\mathbb{R}^n} T(\partial_t, X)(t)dx \quad = \quad \int_{\mathbb{R}^n} T(\partial_t, X)(0)dx \tag{3.5}$$

where X is again assumed Killing. It is important to recognize that (3.3) and (3.4) remain true for X a *conformal* Killing vector field if we assume in addition that T is trace free. Indeed for X conformal killing, ${}^{(X)}\pi_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ and hence $\pi_{\alpha\beta}T^{\alpha\beta} = \Omega^2 \mathrm{tr} T = 0$. When T is trace free, the field theory is called *conformal.* (Alternatively conformally invariant, or covariant.)

3.2. Nonconcentration of the Higgs Field. The energy-momentum tensor for the Yang-Mills-Higgs equations is

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{4}\left(< F_{\alpha\mu}, F_{\beta}^{\ \mu} > + < *F_{\alpha\mu}, *F_{\beta}^{\ \mu} > \right) \\ &+ \frac{1}{2} < D_\alpha\phi, D_\beta\phi > -\frac{1}{4} g_{\alpha\beta} < D_\mu\phi, D^\mu\phi > -\frac{1}{2} g_{\alpha\beta} V(\phi) \end{aligned} \tag{3.6}$$

This can be viewed as

$$T_{\alpha\beta} = T^{\mathrm{YM}}_{\alpha\beta} + T^{\mathrm{Higgs}}_{\alpha\beta},$$

where $T^{\mathrm{YM}}_{\alpha\beta} = \frac{1}{4}\left(< F_{\alpha\mu}, F_{\beta}^{\ \mu} > + < *F_{\alpha\mu}, *F_{\beta}^{\ \mu} > \right)$ is the energy-momentum tensor for the Yang-Mills equations. It is important that T satisfy the positive energy condition, which states that

$$T(X, Y) \geq 0$$

for any future directed time-like vectors X, Y. This positivity property is well known (see for example [1]), but we'll include the proof here for completeness.

Proposition 3.2. *The energy-momentum tensor for Yang-Mills satisfies the positive energy condition. If V is positive, the same conclusion holds for Yang-Mills-Higgs.*

Proof For simplicity, we work in $\mathbb{R}^{3+1}$. Consider two vectors X, Y, at some point $p \in \mathbb{R}^{3+1}$, which are both causal[13] future oriented[14]. The plane spanned by X, Y intersects the null cone at p along two null directions[15]. Let E_+, E_- be the two future directed null vectors corresponding to the two complementary null directions and normalized by the condition

$$<E_+, E_-> = -2$$

[13] $g_{\alpha\beta}X^\alpha X^\beta \leq 0$, similarly for Y.
[14] $X^0, Y^0 > 0$.
[15] If X, Y are linearly dependent any plane passing through their common direction will do.

E_+, E_- are said to form a *null pair*. Since the vectorfields X, Y are linear combinations with positive coefficients of E_+, E_-, the proposition will follow from showing that $T(E_+, E_+) \geq 0$, $T(E_-, E_-) \geq 0$ and $T(E_+, E_-) \geq 0$. To show this consider a frame at p formed by the vectorfields $E_{(4)} = E_+$, $E_{(3)} = E_-$ and $E_{(1)}, E_{(2)}$ with the properties,

$$<E_{(i)}, E_{(3)}> = <E_{(i)}, E_{(4)}> = 0$$

and

$$<E_{(i)}, E_{(j)}> = \delta_{ij}$$

for all $i, j = 1, 2$. A frame with these properties is called a *null frame*.

We first show the positivity of the expression $T^{\text{Higgs}}_{\alpha\beta}$. It's easy to see that

$$\begin{aligned} T^{\text{Higgs}}(E_+, E_+) &= \frac{1}{2}\left|D_4\phi\right|^2 \\ T^{\text{Higgs}}(E_-, E_-) &= \frac{1}{2}\left|D_3\phi\right|^2 \end{aligned}$$

which are clearly non-negative. Now,

$$T^{\text{Higgs}}(E_+, E_-) = \frac{1}{2}\left(<D_4\phi, D_3\phi> + g^{\mu\nu}<D_\mu\phi, D_\nu\phi> + 2V(\phi)\right).$$

In order to simplify this, note that relative to the null frame the only nonvanishing components of the metric $g_{\alpha\beta}$ are

$$g_{34} = -2 \qquad g_{11} = g_{22} = 1$$

and those of the inverse metric $g^{\alpha\beta}$ are

$$g^{34} = -\frac{1}{2} \qquad g^{11} = g^{22} = 1.$$

Therefore,

$$g^{\mu\nu}<D_\mu\phi, D_\nu\phi> = -<D_4\phi, D_3\phi> + \left|\not\nabla\phi\right|^2$$

where

$$\left|\not\nabla\phi\right|^2 \equiv \left|D_1\phi\right|^2 + \left|D_2\phi\right|^2.$$

Therefore,

$$T^{\text{Higgs}}(E_+, E_-) = \frac{1}{2}\left|\not\nabla\phi\right|^2 + V(\phi).$$

To show the positivity of the Yang-Mills energy momentum tensor, first recall $^*F$ as the Hodge dual of F, i.e. $^*F_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}F^{\mu\nu}$ where $\eta_{\alpha\beta\mu\nu}$ is the volume form corresponding to the standard null frame $E_{(1)}, E_{(2)}, E_{(3)}, E_{(4)}$ defined above[16]. Introduce the following null decomposition of F at every point $p \in \mathbb{R}^{3+1}$,

$$\begin{aligned} \alpha_A = F_{A4} \qquad &, \qquad \underline{\alpha}_A = F_{A3} \\ \rho = \frac{1}{2}F_{34} \qquad &, \qquad \sigma = \frac{1}{2}{}^*F_{34}. \end{aligned}$$

[16] So $\eta_{\alpha\beta\mu\nu} = 2\epsilon_{\alpha\beta\mu\nu}$ where the symbol ϵ is completely antisymmetric in its indices.

which completely determines the tensor F. (Here the index $A = 1, 2$ represents components tangent to the sphere). One then calculates that for ${}^*F$,

$$
\begin{array}{rclcrcl}
{}^*F_{A4} &=& -{}^*\alpha_A & , & {}^*F_{A3} &=& {}^*\underline{\alpha}_A \\
{}^*F_{34} &=& 2\sigma & , & {}^{**}F_{34} &=& -2\rho
\end{array}
$$

where ${}^*\alpha_A = \epsilon_{AB}\alpha_B$. Here ϵ_{AB} is the volume form on the unit sphere, hence $\epsilon_{AB} = \frac{1}{2}\eta_{AB34}$, i.e. $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. With this notation,

$$
\begin{aligned}
T^{\mathrm{YM}}(E_{(4)}, E_{(4)}) &= \frac{1}{4}\sum_{A=1}^{2}\left(<F_{4A}, F_{4A}> + <{}^*F_{4A}, {}^*F_{4A}>\right) \\
&= \frac{1}{4}\sum_{A=1}^{2}\left(<\alpha_A, \alpha_A> + <{}^*\alpha_A, {}^*\alpha_A>\right) \\
&= \frac{1}{2}\sum_{A=1}^{2}<\alpha_A, \alpha_A> \\
&= \frac{1}{2}|\alpha|^2 \\
&\geq 0.
\end{aligned}
$$

using the positivity of the Lie algebra's Killing scalar product $<\cdot, \cdot>_{\mathfrak{g}}$. Similarly,

$$
T^{\mathrm{YM}}(E_{(3)}, E_{(3)}) = \frac{1}{2}\sum_{A=1}^{2}<\underline{\alpha}_A, \underline{\alpha}_A> = \frac{1}{2}|\underline{\alpha}|^2 \geq 0
$$

and in the same way we find

$$
T^{\mathrm{YM}}(E_+, E_-) = \frac{1}{2}(\rho^2 + \sigma^2) \geq 0
$$

which proves the assertion. ∎

Contracting $T_{\alpha\beta}$ with ∂_t and integrating over a region ${}^{t_1}_{t_0}$ gives the local energy estimate. In terms of T this is written

$$
\int_{t_1} T_{00} + \frac{1}{2}\int_{/\ \backslash_{t_0}^{t_1}} T_{03} \quad = \quad \int_{t_0} T_{00}. \tag{3.7}
$$

Here

$$
\begin{aligned}
T_{00} &\equiv T(\partial_t, \partial_t) \\
&= \sum_i \frac{1}{2}\left(\frac{1}{2}|E_i|^2 + \frac{1}{2}|H_i|^2 + \frac{1}{2}\left|D_0\phi^2\right| + \frac{1}{2}|D_i\phi|^2 + V(\phi)\right)
\end{aligned}
$$

is the local energy. The nonconcentration result comes by contracting $T_{\alpha\beta}$ with the scaling vector field

$$
S = t\partial_t + x^i\partial_i
$$

which is conformal.[17] Setting $P^\alpha = T^{\alpha\beta} S_\beta$,

$$\begin{aligned}
\partial_\alpha P^\alpha &= \left(\partial_\alpha T^{\alpha\beta}\right) S_\beta + T^{\alpha\beta} g_{\alpha\beta} \\
&= g_{\alpha\beta} T^{\alpha\beta} \\
&= -\frac{1}{2} < D_\alpha \phi, D^\alpha \phi > -2V(\phi).
\end{aligned}$$

If we had an uncoupled Yang-Mills theory, this argument would have lead directly to a conservation law since Yang-Mills is conformally invariant in $\mathbb{R}^{3+1}$. The complication is introduced by the fact that the $T^{\text{Higgs}}_{\alpha\beta}$ component of the energy-momentum tensor is not trace free. We therefore define the modified momentum vector

$$\tilde{P}^\alpha \equiv P^\alpha + \frac{1}{2} < \phi, D^\alpha \phi >$$

for which

$$\partial_\alpha \tilde{P}^\alpha = \frac{|\phi|^6}{6}. \tag{3.8}$$

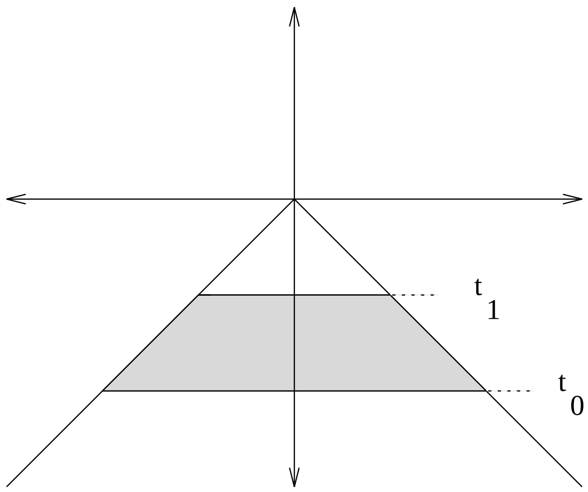

FIGURE 3.1. Region of Integration for Higgs Nonconcentration

Integrating this relation over the region shown in figure (3.1) will lead to a decay result for $\int_{t} |\phi|^6 \, dx$. This decay is 'fattened' just as for the wave equation. (See Lemma 2.3 for notation. The point p here is the origin.)

Lemma 3.3 (Higgs Nonconcentration). *Given a solution (A, ϕ) of the critical power Yang-Mills-Higgs equations on a spacetime region[18] , and a number $\epsilon > 0$.*

[17]The Lie derivative of the metric is $\mathcal{L}_S g_{\alpha\beta} = 2 g_{\alpha\beta}$.
[18]*Assume the top edge of the region lies at time $t = 0$.*

There exists δ_1, δ_2 such that for $t \in [-\delta_1, 0)$,

$$\int_{\bullet_t^{trap}} |\phi|^6 \, dx \quad < \epsilon$$

where $\bullet_t^{trap}$ *is the time $= t$ slice of* $\blacktriangle_{(\delta_1, \delta_2)}$

Remark The statement is gauge invariant.
Proof It suffices to show that

$$\lim_{t \to 0} \int_{\bullet_t} |\phi|^6 \, dx = 0$$

where $\bullet_t$ is the time $= t$ slice of the backward light cone with vertex at the origin. The lemma then follows, line for line, from the proof of Lemma 2.3, since the global energy estimate and local decay of flux hold here as well.

Integrating (3.8) over the region shown in figure (3.1) gives

$$\begin{aligned} \int_{\blacktriangle_{t_0}^{t_1}} \frac{|\phi|^6}{6} dx dt + \int_{\bullet_{t_1}} \tilde{P}^\alpha (\partial_t)_\alpha dx + \int_{/ \quad \backslash_{t_0}^{t_1}} \tilde{P}^\alpha (E_-)_\alpha d\sigma dt & \\ - \int_{\bullet_{t_0}} \tilde{P}^\alpha (\partial_t)_\alpha dx = 0 & \end{aligned} \tag{3.9}$$

Writing $S = \frac{1}{2}\left((t+r)E_+ + (t-r)E_-\right)$ it's easy to see that the second term in (3.9) goes to zero as $t_1 \to 0$,

$$\begin{aligned} \int_{\bullet_{t_1}} \tilde{P}^\alpha (\partial_t)_\alpha dx &= \int_{\bullet_{t_1}} T(S, \partial_t) + \frac{1}{2} < \phi, D_0 \phi > dx \\ &= \int_{\bullet_{t_1}} T(S, \partial_t) + \frac{1}{2} < \phi, D_0 \phi > dx \\ &= \int_{\bullet_{t_1}} \left(\frac{1}{4}(t+r) T_{++} + \frac{1}{4}(t-r) T_{--} \right. \\ & \qquad \left. + \frac{1}{2} t T_{+-} + \frac{1}{2} < \phi, D_0 \phi > \right) dx \end{aligned}$$

The null components T_{++}, T_{+-}, T_{--} of the energy are bounded using the local energy estimate (3.7). Note that their coefficients vanish as $t_1 \nearrow 0$. The contribution of $< \phi, D_0 \phi >$ decays to zero by Holder's inequality and the energy estimate.

We have shown that

$$- \int_{\bullet_{t_0}} \tilde{P}^\alpha (\partial_t)_\alpha dx + \int_{\bigwedge_{t_0}} \tilde{P}^\alpha (E_-)_\alpha d\sigma dt \quad \leq \quad 0. \tag{3.10}$$

It's easy to see that the second term here is $t_0 O_1(t_0)$ where $O_1(t_0)$ denotes a continuous function with $O_1(t_0) \to 0$ as $t_0 \to 0$. This uses the decay of flux, which is the statement

$$\int_{\bigwedge_{t_0}} T(\partial_t, E_-) d\sigma dt \to 0 \quad \text{as} \quad t_0 \to 0.$$

Exactly as in the conformal case, the decay of flux is immediate from the local energy estimate and the dominated convergence theorem. Note that

$$\int_{\bigwedge_{t_0}} T(\partial_t, E_-) d\sigma dt \quad = \quad \frac{1}{2}\int_{\bigwedge_{t_0}} T_{+-} + T_{--} d\sigma dt.$$

The light cone integral in (3.10) is therefore

$$\begin{aligned}
\int_{\bigwedge_{t_0}} \tilde{P}^\alpha (E_-)_\alpha d\sigma dt &= \int_{\bigwedge_{t_0}} T(S, e_-) + \frac{1}{2} < \phi, D_- \phi > d\sigma dt \\
&= \frac{1}{2}\int_{\bigwedge_{t_0}} \underbrace{(t+r)T_{+-}}_{0 \text{ on } \bigwedge} + (t-r)T_{--} + < \phi, D_-\phi > d\sigma dt \\
&= t_0 O_1(t_0)
\end{aligned}$$

since $\int_{\bigwedge_{t_0}} T_{--}$ is part of the flux and

$$\begin{aligned}
\int_{\bigwedge_{t_0}} < \phi, D_-\phi > d\sigma dt \qquad &\leq \qquad \underbrace{\|\phi\|_{L^6(\bigwedge_{t_0})} \cdot \|D_-\phi\|_{L^2(\bigwedge_{t_0})}}_{\text{both are parts of the flux}} \cdot |t_0| \\
&\leq t_0 O_1(t_0).
\end{aligned}$$

As for the first term in (3.10),

$$-\int_{\bullet_{t_0}} \tilde{P}^\alpha (\partial_t)_\alpha dx \quad = \quad -\int_{\bullet_{t_0}} T(t\partial_t + x^i\partial_i, \partial_t) + \frac{1}{2} < \phi, D_0\phi > dx.$$

Proposition 3.2 shows that the Yang-Mills contribution $T^{\mathrm{YM}}(-S, \partial_t)$ here is positive. Throw it away to conclude from (3.10)

$$\begin{aligned}
(3.11) \qquad t_0 O_1(t_0) - \int_{\bullet_{t_0}} \frac{1}{2} t_0 \left(|D_0\phi|^2 + |D_i\phi|^2 + \frac{|\phi|^6}{6} \right) \\
+ \underbrace{\frac{1}{2} < x^i D_i\phi + \phi, D_0\phi >}_{***} dx \quad \leq \quad 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\left| \frac{1}{2} < x^i D_i \phi + \phi, D_0\phi > \right| &= \left| \frac{1}{2} t_0 \frac{x^i}{t_0} < D_0\phi, \frac{x^i}{r^2}\phi + D_i\phi > \right| \\
&\leq \frac{1}{2}|t_0| \cdot \left(|D_0\phi|^2 + \left| D_i\phi + \frac{x^i\phi}{r^2} \right|^2 \right)
\end{aligned}$$

which is *almost* what we have in the positive part of the integral in (3.11). We'll be able to absorb (***) in this positive term, divide the whole of (3.11) by $-t_0$

and conclude the desired decay of the Higgs field, if we can show that the following correction term is $t_0 O_1(t_0)$:

$$\int_{\bullet_{t_0}} t_0 \cdot \left(\frac{1}{2} < \frac{x^i \phi}{r^2}, \frac{x^i \phi}{r^2} > + < D_i \phi, \frac{x^i}{r^2} \phi > \right) dx =$$
$$\int_{\bullet_{t_0}} t_0 \cdot \left(\frac{1}{2} \frac{|\phi|^2}{r^2} + \frac{1}{2} \left(\frac{1}{r} \partial_r < \phi, \phi > \right) \right) dx$$
$$= \int_{\bigcirc_{t_0}} |\phi|^2 \, d\sigma$$

where the last equality is attained by integration by parts. The resulting quantity $\int_{\bigcirc_{t_0}} |\phi|^2 \, d\sigma$ is $t_0 O_1(t_0)$. We show this now:

$$\int_{\bigwedge_{t_0}} < \phi, D_- \phi > d\sigma dt \quad = \quad t_0 O_1(t_0)$$

since the flux decays. On the other hand integration by parts gives

$$\begin{aligned} \int_{\bigwedge_{t_0}} < \phi, D_- \phi > d\sigma dt \quad &= \quad \int_{\bigwedge_{t_0}} \frac{1}{2} \partial_- < \phi, \phi > d\sigma dt \\ &= \quad - \int_{\bigwedge_{t}} \frac{|\phi|^2}{t_0} d\sigma dt + \int_{\bigcirc_{t_0}} \phi^2 d\sigma. \end{aligned}$$

As the first term is $t_0 O_1(t_0)$, (again, due to flux decay), the result follows. ∎

Though we won't need it, let us note here stronger decay in the Higgs field. The inequality to start from is (3.10), which we've shown can be written as:

$$0 \;\geq\; t_0 O_1(t_0) - \int_{\bullet_{t_0}} T(t\partial_t + x^i \partial_i, \partial_t) + \frac{1}{2} < \phi, D_0 \phi > dx \tag{3.13}$$

A straightforward argument shows that

$$\int_{\bullet_{t_0}} \frac{1}{2}(1 + \frac{r}{t_0}) \left| E_+ \phi + \frac{\phi}{r} \right|^2 + \left(|\nabla \phi|^2 + \frac{|\phi|^6}{6} \right) + \frac{1}{2}(1 - \frac{r}{t_0}) \left| E_- \phi - \frac{\phi}{r} \right|^2 dx$$

decays like $O_1(t_0)$.

4. Small time existence

This section contains two simple, classical local existence theorems which will be used later in constructing global-in-time solutions.

4.1. Temporal gauge local existence. Local existence of an $H^s, s \geq 2$ solution to the critical power Yang-Mills-Higgs equations is established in the temporal gauge by a straightforward fixed point argument on the following system of wave equations satisfied by the curvature and Higgs field.

$$\begin{aligned} \Box F_{\alpha\beta} &= [\phi,[\phi,F_{\alpha\beta}]] + 2[D_\alpha\phi, D_\beta\phi] + 2[F_{\alpha\gamma}, F_\beta{}^\gamma] \qquad (4.1)\\ &\quad -2[A^i, \partial_i F_{\alpha\beta}] - [\partial_i A^i, F_{\alpha\beta}] - [A_i,[A^i,F_{\alpha\beta}]] \\ \Box\phi &= [\phi, \partial_i A^i] + 2[\partial_i\phi, A^i] - [A_i,[A^i,\phi]] + |\phi|^4\phi. \qquad (4.2) \end{aligned}$$

Equation (4.1) comes from differentiating the Bianchi identity for $F_{\alpha\beta}$,

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0.$$

The resulting expression is simplified using the field equation

$$D^\nu F_{\mu\nu} = -[\phi, D_\mu\phi] \qquad (4.3)$$

and the Jacobi identity

$$[A_\alpha,[\phi,D_\beta\phi]] + [\phi,[D_\beta\phi,A_\alpha]] + [D_\beta\phi,[A_\alpha,\phi]] = 0 \qquad (4.4)$$

as in [4]. Equation (4.2) follows directly from the field equation for ϕ,

$$D_\mu D^\mu \phi = |\phi|^4\phi.$$

The first result establishes a unique smooth solution for the subset ${}^T_0$ of the spacetime domain

$$\equiv \left\{ (t,x) \in \mathbb{R}^{3+1} \mid 0 \leq t \leq \frac{1}{2}, |x - x_0| \leq 1 - t \right\}.$$

Here $x_0 \in \mathbb{R}^3$ is an arbitrary point. Since truncated light cones like will be used often, we change slightly the notation from Section 3: write

$$_{t_0} = \{t = t_0\} \cap .$$

Previously, this was written ${}^{\text{trap}}_{t_0}$.

Theorem 4.1 (Temporal gauge, H^2 local existence). *Given a data set*

$$(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi}) \in H^s(\,_0) \quad s \geq 2,$$

there exists $0 < T \leq \frac{1}{2}$ depending only on $\left\|(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi})\right\|_{H^2(\,_0)}$ and a unique development (A,ϕ) in the temporal gauge with

$$A, \phi \in C([0,T); H^s(\,_t)) \cap C^1\left([0,T); H^{(s-1)}(\,_t)\right).$$

This solution can be continued[19] *as long as $\|(A,E,\phi,\partial_t\phi)\|_{H^2(\,_t)}$ remains bounded.*

[19] *Until $T = \frac{1}{2}$.*

Remark: A global-in-space version of this theorem[20] follows from essentially the same proof. Recall also the notation

$$\begin{aligned}\|(A,\phi)\|_{H^s(\;_t)} &= \|A\|_{H^s(\;_t)} + \|\phi\|_{H^s(\;_t)}\\ &\quad + \|E\|_{H^{s-1}(\;_t)} + \|\partial_t\phi\|_{H^{s-1}(\;_t)}\end{aligned}$$

Proof The straightforward H^2 fixed-point argument of [15] (Theorem 1) works here as well. One seeks constants T and Δ_0 so that equations (4.1) - (4.2) induce a contraction mapping $\mathcal{L}$ on the closed ball

$$\begin{aligned}(4.5)\quad \mathcal{D} = \big\{A,\phi \in C\big([0,T];H^s(\;_t)\big)\cap C^1\big([0,T];H^{(s-1)}(\;_t)\big)\\ \Big|\ \sup_{0\le s\le T}\|(A,\phi)\|_{H^2(\;_s)} \le \Delta_0\big\}\end{aligned}$$

The bounds for ϕ are not difficult. We'll bound here only the highest order terms in the H^2 norm: by the energy estimate,

$$\|\nabla\partial\phi\|_{L^2(\;_0)} \;\le\; \|\nabla\partial\phi(0,\cdot)\|_{L^2(\;_0)} + \int_0^T \|\nabla\,\Box\,\phi\|_{L^2(\;_s)}\,ds.$$

The integral on the right can be bounded by

$$(4.6)\qquad \int_0^T \|\nabla\,\Box\,\phi\|_{L^2(\;_s)}\,ds \;\le\; K\big(\,|||A,\phi|||_{L^\infty}\big)\int_0^T |||A,\phi|||_{H^2(\;_s)}\,ds$$

where $K(\cdot)$ is an increasing, continuous function[21] of its argument and where we have used the following notation: given any norm $\|\cdot\|_N$, define

$$|||A,\phi|||_N = \sup_{0\le s\le T}\{\,\|A(s,\cdot)\|_N\,,\|\phi(s,\cdot)\|_N\,\}$$

To see (4.6), simply estimate each term from the right hand side of (4.2).

$$\begin{aligned}\int_0^T \big\|\nabla[\phi,\partial_i A^i]\big\|_{L^2(\;_s)}\,ds \;&\le\; \int_0^T \big\|\nabla^2 A\big\|_{L^2(\;_s)}\cdot\|\phi\|_{L^\infty(\;_s)}\,ds\\ &\quad + \int_0^T \|\nabla\phi\cdot\nabla A\|_{L^2(\;_s)}\,ds.\end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\|\nabla\phi\cdot\nabla A\|_{L^2(\;_s)} \;\le\; C\,|||A,\phi|||_{L^\infty}\cdot\big(\|A\|_{H^2(\;_s)} + \|\phi\|_{H^2(\;_s)}\big)$$

Hence

$$\int_0^T \big\|\nabla[\phi,\partial_i A^i]\big\|_{L^2(\;_s)}\,ds \le C\,|||A,\phi|||_{L^\infty}\cdot\int_0^T |||A,\phi|||_{H^2(\;_s)}\,ds.$$

The same argument bounds $\int_0^T \big\|\nabla[A^i,\partial_i\phi]\big\|_{L^2(\;_s)}\,ds$. A typical term in

$$\int_0^T \big\|\nabla[A_i,[A^i,\phi]]\big\|_{L^2(\;_s)}\,ds$$

[20] giving a solution on $[0,T^*]\times\mathbb{R}^3$

[21] In fact, $K(x) = C\cdot(x+x^2+x^4)$ will suffice, where C is some universal constant.

is

$$\begin{aligned}\int_0^T \big\|[\nabla A_i,[A^i,\phi]]\big\|_{L^2(\;_s)}\,ds &\le \int_0^T \big\|[\nabla A_i, A^i]\big\|_{L^2(\;_s)}\,ds\cdot \|\phi\|_{L^\infty}\\ &\le K\big(\,\|A,\phi\|_{L^\infty}\big)\int_0^T \|A,\phi\|_{H^2(\;_s)}\,ds.\end{aligned}$$

Finally,

$$\begin{aligned}\int_0^T \Big\|\nabla\big(\,|\phi|^4\cdot\phi\big)\Big\|_{L^2(\;_s)}\,ds &\le C\int_0^T \|\phi\|^4_{L^\infty(\;_s)}\|\nabla\phi\|_{L^2(\;_s)}\,ds\\ &\le K\left(\|A,\phi\|_{L^\infty}\right)\int_0^T \|A,\phi\|_{H^2(\;_s)}\,ds\end{aligned}$$

which finishes the proof of (4.6).

Klainerman-Machedon [15] have a very simple argument to bound the potential A. The key observation is that if one integrates a solution to the wave equation in time, one gains space regularity. More specifically,

Lemma 4.2 **([15]).** *Let ψ and F be $\mathfrak{g}$-valued fuctions on $\;^T$ with*

$$\begin{aligned}\Box\psi &= F\\ \psi(0,x) &= f_0(x)\\ \partial_t\psi(0,x) &= f_1(x).\end{aligned}$$

Then for $0\le t\le T$,

$$\left\|\nabla^2\int_0^t \psi(s,\cdot)ds\right\|_{L^2(\;_t)} \le C\big(\|f_0\|_{H^1(\;_0)}+\|f_1\|_{L^2(\;_0)}+\int_0^T \|F(s,\cdot)\|_{L^2(\;_s)}\,ds\big).$$

Since $A(t,\cdot)=A(0,\cdot)+\int_0^t E(s,\cdot)ds$, Lemma 4.2 gives,

$$\begin{aligned}\|\nabla^2 A(t,\cdot)\|_{L^2(\;_t)} &\le \|\nabla^2 A(0,\cdot)\|_{L^2(\;_0)}+\left\|\nabla^2\int_0^t E(s,\cdot)ds\right\|_{L^2(\;_t)}\\ &\le \|\nabla^2 A(0,\cdot)\|_{L^2(\;_0)}+C\Big(\|E(0,\cdot)\|_{H^1(\;_0)}\\ &\qquad +\|\partial_t E(0,\cdot)\|_{L^2(\;_0)}+\int_0^t \|\Box E\|_{L^2(\;_s)}\,ds\Big)\end{aligned}\tag{4.7}$$

At the same time, E is bounded in H^1 using the energy inequality

$$\|\partial E(t,\cdot)\|_{L^2(\;_t)} \le \|\partial E(0,\cdot)\|_{L^2(\;_0)}+\int_0^T \|\Box E\|_{L^2(\;_s)}\,ds.\tag{4.8}$$

We bound the quantity $\|\partial_t E(0,\cdot)\|_{L^2(\;_0)}$ on the right of (4.8) using (4.3),

$$\partial_t E_i = -\partial^j F_{ij}-[A_j,F_{ij}]-[\phi,D_i\phi]$$

so $\|\partial_t E\|_{L^2(\bullet_0)} \le K(\|\bar{A}\|_{H^2(\bullet_0)} + \|\bar{\phi}\|_{H^2(\bullet_0)})$, with K as above. Combining (4.7) and (4.8) gives:

$$\begin{aligned} \|\nabla\partial A(t,\cdot)\|_{L^2(\bullet_t)} &\le \left\|\nabla^2 A(t,\cdot)\right\|_{L^2(\bullet_t)} + \|\nabla E(t,\cdot)\|_{L^2(\bullet_t)} \\ &\le K(\|\bar{A}\|_{H^2(\bullet_0)}, \|\bar{\phi}\|_{H^2(\bullet_0)}) + C\int_0^t \|\Box E\|_{L^2(\bullet_s)}\, ds. \end{aligned} \tag{4.9}$$

We claim the quantity $Q(t) \equiv \int_0^t \|\Box E\|_{L^2(\bullet_s)}\, ds$ on the right hand side of (4.9) satisfies

$$Q(t) \le T\cdot K\left(\|\!|A,\phi|\!\|_{H^2}\right)\left(K(\Delta_0) + Q(t)\right). \tag{4.10}$$

Inequality (4.10) is proven by simply considering the right hand side of (4.1) term by term and using the energy estimate. For example,

$$\begin{aligned} \int_0^T \|[\phi,[\phi,E_i]]\|_{L^2(\bullet_s)}\, ds &\le C\int_0^T \|\phi\|^2_{H^1(\bullet_s)} \|E_i\|_{H^1(\bullet_s)}\, ds \\ &\le C\,\|\!|\phi|\!\|^2_{H^2}\, T(\|E\|_{H^1(\bullet_0)} + \|\partial_t E\|_{L^2(\bullet_0)} + Q(t)) \\ &\le TK\left(\|\!|(A,\phi)|\!\|_{H^2}\right)\left(K(\Delta_0) + Q(t)\right). \end{aligned}$$

Similarly for the other terms. For $T < \frac{1}{2\cdot K(\Delta_0)}$, (4.10) gives

$$Q(t) \le 2TK(\Delta_0)^2. \tag{4.11}$$

Using (4.6), (4.9),and (4.11) one chooses Δ_0 large enough to accommodate the initial data, then T small to show the linear mapping induced by (4.1), (4.2) maps $\mathcal{D}$ to $\mathcal{D}$.

To prove the linear mapping $\mathcal{L}$ is a contraction on $\mathcal{D}$, consider it's effect on $(A^1,\phi^1),(A^2,\phi^2) \in \mathcal{D}$. Write $\mathcal{L}(A^i) = \tilde{A}^i$, $i = 1,2$. Similarly for ϕ^i. Using exactly the arguments as above, one shows that

$$\left\|\!\left|(\tilde{A}^1 - \tilde{A}^2, \tilde{\phi}^1 - \tilde{\phi}^2)\right|\!\right\|_{H^2} \le TK(\Delta_0)\cdot \|\!|(A^1 - A^2, \phi^1 - \phi^2)|\!\|_{H^2}.$$

Take T small so that $TK(\Delta_0) < 1$ to complete the proof. ∎

4.2. Coulomb gauge local existence. Section 5 will motivate the following definition: The $\mathfrak{g}$-valued functions (A_0, A_i, ϕ) defined on $\blacktriangle$ represent a *scaled* Yang-Mills-Higgs Coulomb gauge development of the initial data set $(\bar{A}_i, \bar{E}_i, \bar{\phi}, \overline{D_0\phi})$ if they satisfy the following system of equations[22]

$$
\begin{aligned}
&\text{(C.G.a)} & \Box A_i &= B_i + G_i \\
&& B_i &= -\partial_t \partial_i A_0 + [A_j, \partial_i A_j] + [\phi, \partial_i \phi] \\
&& G_i &= -2[A_j, \partial_j A_i] - [\phi, [\phi, A_i]] + [\partial_t A_0, A_i] \\
&& &\quad + 2[A_0, \partial_t A_i] + [\partial_i A_0, A_0] + [A_0, [A_0, A_i]] \\
&& &\quad - [A^j, [A_j, A_i]] \\
&\text{(C.G.b)} & \Box \phi &= [\phi, \partial_t A^0] - 2[A^\mu, \partial_\mu \phi] \\
&& &\quad - [A_\mu, [A^\mu, \phi]] + \lambda^2 |\phi|^4 \phi
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
A_i(0, \cdot) &= \bar{A}_i & \phi(0, \cdot) &= \bar{\phi}(\cdot) \\
\partial_t A_i(0, \cdot) &= \bar{E}_i + \partial_i \bar{A}_0 + [\bar{A}_i, \bar{A}_0] & \partial_t \phi(0, \cdot) &= \overline{D_0 \phi} - [\bar{A}_0, \bar{\phi}]
\end{aligned}
$$

where $\bar{A}_0$ is the solution of A_0 at time zero given below. The number λ appearing in (C.G.b) should be thought of as an arbitrary constant. The data must satisfy the following compatibility condition

$$\text{(C.G.c)} \qquad \partial^i \bar{E}_i = -[\bar{\phi}, \overline{D_0 \phi}] - [\bar{A}_i, \bar{E}_i].$$

The equation for A_0 at time zero is

$$\text{(C.G.d)} \qquad \bar{A}_0(x) = \int_{\bullet_0} G_0(x, y) \Delta \bar{A}_0(y) dy$$

where $\Delta \bar{A}_0 \equiv \bar{M} = \partial^i[\bar{A}_0, \bar{A}_i] + [\bar{A}^i, \bar{E}_i] + [\phi, \overline{D_0 \phi}]$. For $t > 0$, the equation for A_0 is

(C.G.e)

$$
\begin{aligned}
A_0(t, x) = \int_{\bullet_t} G_t(x, y) M(t, y) dy &- \int_{\blacktriangle_t} \partial_s G_s(x, y) M(s, y) ds dy \\
&+ \int_{/\ \backslash_t} G_s(x, y) N(s, y) d\sigma ds
\end{aligned}
$$

where

$$
\begin{aligned}
M &= 2[\partial^i A_0, A_i] + [A_i, \partial_0 A^i] + [A_i, [A_0, A^i]] + [\phi, D_0 \phi] \\
N &= [A_i, (\partial_t - \partial_\eta) A^i] + 2[A_j, \nabla^j A_\eta] + 2[\partial_i A_0, A^i] \\
&\quad -[\partial_t A_0, A_\eta] - 2[A_0, \partial_t A_\eta] + [A_0, \partial_\eta A_0] + [A_j, [A_j, A_\eta]] \\
&\quad -[A_0, [A_0, A_\eta]] + [\phi, (\partial_t - \partial_\eta) \phi] + [\phi, [A_0, \phi]]
\end{aligned}
$$

[22] It is straightforward to prove that the Coulomb condition $\nabla_i A^i = 0$ is preserved by this system.

Also, for positive time $\partial_t A_0$ satisfies

$$\text{(C.G.f)} \qquad \partial_t A_0(t,x) = \int_{\bullet_t} \partial_i G(x,y)\,(J_i - \partial_t[A_0, A_i])\,dy. \\ + \int_{\bigcirc_t} G_t(x,y)\,[A_j\,, \partial_j A_\eta]\,d\sigma_y$$

where $J_i = [F_{i\mu}, A^\mu] - [\phi, D_i\phi]$. In the equations above, $G_t(x,y)$ is the Neumann Greens function on the ball $\bullet_t$ (See Appendix).

Theorem 4.3 (Coulomb gauge local existence). *Given an initial data set*

$$(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi}) \in H^s\left(\mathbb{R}^3\right) \qquad s \geq 2$$

satisfying $\partial_i A^i = 0$ *on* $\bullet_0$*,* $x \cdot A = 0$ *on* $\bigcirc_0$*, with* $\|\bar{A}\|_{H^1(\bullet_0)}, \|\bar{\phi}\|_{H^1(\bullet_0)} < \epsilon$ *for some universal* ϵ*.*

There exists $T = T(\|\bar{A}, \bar{E}, \bar{\phi}, \bar{D_0\phi}\|_{H^s(\bullet_0)}, \lambda) > 0$ *and a unique development* (A, ϕ) *of the data according to the system (C.G.a)-(C.G.f) on* $\blacktriangle^T$ *with*

$$A, \phi \quad \in \quad C\big([0,T], H^s(\bullet_t)\big) \cap C^1\left([0,T], H^{(s-1)}(\bullet)\right).$$

and with the following additional regularity for A_0*,*

$$\partial_t A_0 \quad \in \quad C\left([0,T], H^s(\bullet_t)\right).$$

Proof To solve the integral equation (C.G.d), define a sequence of functions $\{A_0^{(n)}\}$ by

$$\begin{aligned} A_0^{(1)}(x) &= \int_{\bullet_0} G_0(x,y)\big([\bar{A}^i, \bar{E}_i] + [\bar{\phi}, \overline{D_0\phi}]\big)dy \\ A_0^{(n+1)} &= \int_{\bullet_0} G_0(x,y)\big(\partial_i[A_0^{(n)}, \overline{A}_i] + [\bar{A}^i, \bar{E}_i] + [\bar{\phi}, \overline{D_0\phi}]\big)dy \end{aligned}$$

By Lemma 8.4

$$\begin{aligned} \left\|A_0^{(n+1)} - A_0^{(n)}\right\|_{H^2(\bullet_0)} &\leq C\left\|\nabla_i[A_0^{(n)} - A_0^{(n-1)}, \bar{A}_i]\right\|_{L^2(\bullet_0)} \\ &\leq C\left\|\bar{A}\right\|_{H^1(\bullet_0)} \cdot \left\|A_0^{(n)} - A_0^{(n-1)}\right\|_{H^2(\bullet_0)}. \end{aligned}$$

For sufficiently small $\|\bar{A}\|_{H^1(\bullet_0)}$, $\{A_0^{(n)}\}$ converges in H^2 to a solution $\bar{A}_0$.

The local evolution is established by showing the map

$$\text{(4.12)} \qquad \mathcal{L}: \quad (A_0, A, \phi) \quad \rightarrow \quad (A_0^*, A^*, \phi^*)$$

is a contraction on a suitably defined function space, where (A_0^*, A^*, ϕ^*) is defined as the solution of the following integral-hyperbolic system.

$$\partial_t A_0^*(t,x) = \int_{\bullet_t} \partial_i G(x,y)\,(J_i - \partial_t[A_0, A_i])\,dy \tag{4.13}$$
$$+ \int_{\bigcirc_t} G_t(x,y)[A_j, \partial_j A_\eta] d\sigma_y$$
$$\text{with } A_0^*(0,\cdot) = \bar{A}_0(0,\cdot)$$

$$\Box A_i^* = B_i + G_i \tag{4.14}$$
$$A_i^*(0,\cdot) = \bar{A}_i(\cdot)$$
$$\partial_t A_i^*(0,\cdot) = \bar{E}_i(\cdot) + \partial_i \bar{A}_0(\cdot) + [\bar{A}_i, \bar{A}_0]$$

$$\Box\phi^* = \lambda^2 |\phi|^4 \phi + [\phi, \partial_\mu A^\mu] \tag{4.15}$$
$$- 2[A^\mu, \partial_\mu \phi] - [A_\mu, [A^\mu, \phi]]$$
$$\phi^*(0,\cdot) = \bar{\phi}(\cdot)$$
$$\partial_t \phi^*(0,\cdot) = \overline{D_0\phi}(\cdot) - [\bar{A}_0, \bar{\phi}](\cdot).$$

Here B_i and G_i are as in (C.G.a), J_i as in (C.G.f). Define the norm $|||\cdot|||$ by

$$|||(A,\phi)|||\,(s) \equiv \sum_{l=0}^{1}\sum_{k=0}^{1} \left\| \partial^k \nabla^l \phi \right\|_{L^2(\bullet_s)} + \sum_{l=0}^{1}\sum_{k=0}^{1} \left\| \partial^k \nabla^l A \right\|_{L^2(\bullet_s)}$$
$$+ \sum_{l=0}^{2}\sum_{k=0}^{1} \left\| \partial^k \nabla^l A_0 \right\|_{L^2(\bullet_s)}.$$

and write $|||(A,\phi)|||_T \equiv \sup_{0\le s\le T} |||(A,\phi)|||\,(s)$. The aim is to prove $\mathcal{L}$ a contraction on the function space

$$\mathcal{D} = \left\{(A,\phi) \in C([0,T], |||\cdot|||) \mid |||(A,\phi)|||_T \le \Delta_0\right\}$$

for some T, Δ_0.

The necessary estimate for $\left\|\bar{A}_0\right\|_{H^2(\bullet_0)}$ uses Lemma 8.4.

$$\begin{aligned}
\left\|\nabla^2 \bar{A}_0\right\|_{L^2(\bullet_0)} &\le \left\|\bar{M}\right\|_{L^2(\bullet_0)} \\
&\le \left\|\partial^i\,[\bar{A}_0, \bar{A}_i]\,\right\|_{L^2(\bullet_0)} + \left\|\,[\bar{A}_i, \bar{E}_i]\,\right\|_{L^2(\bullet_0)} \\
&\quad + \left\|\,[\bar{\phi}, \overline{D_0\phi}]\,\right\|_{L^2(\bullet_0)} \\
&\le 2\left\|\bar{A}_i\right\|_{H^1(\bullet_0)} \cdot \left\|\bar{A}_0\right\|_{H^2(\bullet_0)} + \left\|\bar{A}_i\right\|_{H^1(\bullet_0)} \cdot \left\|\bar{E}_i\right\|_{H^1(\bullet_0)} \\
&\quad + \left\|\bar{\phi}\right\|_{H^1(\bullet_0)} \cdot \left\|\overline{D_0\phi}\right\|_{H^1(\bullet_0)}
\end{aligned}$$

The assumption $\left\|\bar{A}_i\right\|_{H^1(\bullet_0)} \le \epsilon$ yields

$$\left\|\nabla^2 \bar{A}_0\right\|_{L^2(\bullet_0)} \le K\left(\left\|\,(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi})\,\right\|_{H^2(\bullet_0)}\right)$$

for some K continuous. The lower order terms in $\left\|\bar{A}_0\right\|_{H^2(\bullet_0)}$ are handled as follows: by Lemma 8.4,

$$\left\|\bar{A}_0(0,\cdot)\right\|_{L^2(\bullet_0)} + \left\|\nabla \bar{A}_0(0,\cdot)\right\|_{L^2(\bullet_0)} \le C\left\|\bar{M}(0,\cdot)\right\|_{L^{\frac{6}{5}}(\bullet_0)} \tag{4.16}$$

and straightforward estimates yield

$$\begin{aligned}
\left\|\bar{M}(0,\cdot)\right\|_{L^{\frac{6}{5}}(\bullet_0)} &\leq \left\|\left[A^i,\partial_0 A_i\right]\right\|_{L^{\frac{6}{5}}(\bullet_0)} + 2\left\|\left[\partial^i A_0, A_i\right]\right\|_{L^{\frac{6}{5}}(\bullet_0)} \\
&+ \left\|\left[A^i,[A_0,A_i]\right]\right\|_{L^{\frac{6}{5}}(\bullet_0)} + \|[\phi, D_0\phi]\|_{L^{\frac{6}{5}}(\bullet_0)} \\
&\leq C\cdot\|A_i\|_{L^3(\bullet_0)}\cdot\Big(\|\partial_t A_i\|_{L^2(\bullet_0)} + \|\nabla A_0\|_{L^2(\bullet_0)} \\
&\quad + \|A_i\|_{L^3(\bullet_0)}\|A_0\|_{L^6(\bullet_0)}\Big) + C\cdot\|\phi\|_{L^3(\bullet_0)}\cdot\|D_0\phi\|_{L^2(\bullet_0)} \\
&\leq C\epsilon\Big(\|\nabla A_0\|_{L^2(\bullet_0)} + \|A_0\|_{L^6(\bullet_0)} \\
&\quad + \|\partial_t A_i\|_{L^2(\bullet_0)} + \|D_0\phi\|_{L^2(\bullet_0)}\Big).
\end{aligned}$$

It's easy to see that

$$\|\partial_t A_i\|_{L^2(\bullet_0)} \leq \left\|\bar{E}\right\|_{L^2(\bullet_0)} + \left\|\nabla\bar{A}_0\right\|_{L^2(\bullet_0)} + \left\|\bar{A}_0\right\|_{L^6(\bullet_0)}.$$

Using $\left\|\bar{A}_0\right\|_{L^6(\bullet_0)} \leq C(\left\|\nabla\bar{A}_0\right\|_{L^2(\bullet_0)} + \left\|\bar{A}_0\right\|_{L^2(\bullet_0)})$ gives

$$\begin{aligned}
\left\|\bar{M}\right\|_{L^{\frac{6}{5}}(\bullet_0)} &\leq C\epsilon\Big(\|\nabla A_0\|_{L^2(\bullet_0)} + \|A_0\|_{L^2(\bullet_0)} \\
&\quad + \|E\|_{L^2(\bullet_0)} + \|D_0\phi\|_{L^2(\bullet_0)}\Big).
\end{aligned}$$

Together with (4.16), this yields

$$\left\|\bar{A}_0\right\|_{L^2(\bullet_0)} + \left\|\nabla\bar{A}_0\right\|_{L^2(\bullet_0)} \leq C\cdot\left(\left\|\overline{D_0\phi}\right\|_{L^2(\bullet_0)} + \left\|\bar{E}\right\|_{L^2(\bullet_0)}\right).$$

These estimates on A_0 on time zero are needed to pick Δ_0 large enough to accommodate the initial data. The corresponding estimates for $\|\partial_t\phi\|_{H^1(\bullet_0)}, \|\phi\|_{H^2(\bullet_0)}$ are straightforward.

That $\mathcal{L}$ maps $\mathcal{D}$ to itself follows from

$$\begin{aligned}
|\!|\!|(A^*,\phi^*)|\!|\!|_T &\leq C\left(\left\|\bar{A}\right\|_{H^2(\bullet_0)} + \left\|\bar{E}\right\|_{H^1(\bullet_0)} + \left\|\bar{\phi}\right\|_{H^2(\bullet_0)} + \left\|\overline{D_0\phi}\right\|_{H^1(\bullet_0)}\right) \\
&\quad + CT\left(|\!|\!|A,\phi|\!|\!|_T + |\!|\!|A,\phi|\!|\!|_T^2 + |\!|\!|A,\phi|\!|\!|_T^3 + |\!|\!|A,\phi|\!|\!|_T^5\right).
\end{aligned} \tag{4.17}$$

Indeed, with (4.17), pick $\Delta_0 \geq 2C(\left\|\bar{A},\bar{E},\bar{\phi},\overline{D_0\phi})\right\|_{H^2(\bullet_0)}$ and then $T \leq \frac{\Delta_0}{4C(\Delta_0+\Delta_0^5)}$. The estimate (4.17) is proven below for the highest order terms in $|\!|\!|A^*,\phi^*|\!|\!|$: $\partial\nabla A^*, \partial\nabla\phi^*$ and $\partial_t\nabla^2 A_0^*$. By the energy estimate,

$$\|\partial\nabla A^*\|_{L^2(\bullet_t)} \leq C(\left\|\partial\nabla\bar{A}\right\|_{L^2(\bullet_0)} + \int_0^t \|\nabla B\|_{L^2(\bullet_s)} + \|\nabla G\|_{L^2(\bullet_s)}\,ds).$$

A quick inspection of (C.G.a.) gives,

$$\begin{aligned}
\|\nabla B\|_{L^2(\bullet_s)} &\leq C\left(\left\|\nabla^2\partial_t A_0\right\|_{L^2(\bullet_s)} + |\!|\!|A,\phi|\!|\!|_T^2\right) \\
&\leq C\left(|\!|\!|A,\phi|\!|\!|_T + |\!|\!|A,\phi|\!|\!|_T^2\right)
\end{aligned}$$

and

$$\|\nabla G\|_{L^2(\bullet_s)} \leq C\left(|\!|\!|A,\phi|\!|\!|_T^2 + |\!|\!|A,\phi|\!|\!|_T^3\right).$$

For the next term,

$$\|\partial\nabla\phi^*\|_{L^2(\bigcirc_t)} \quad\leq\quad C\left\|\partial\nabla\bar{\phi}\right\|_{L^2(\bigcirc_o)} + \int_0^t \|\nabla\,\Box\,\phi^*\|_{L^2(\bigcirc_s)}\,ds$$

and by inspecting (C.G.b.),

$$\|\nabla\,\Box\,\phi^*\|_{L^2(\bigcirc_s)} \quad\leq\quad C(1+\lambda^2)\cdot\left(\|A,\phi\|_T^2 + \|A,\phi\|_T^3 + \|A,\phi\|_T^5\right).$$

Finally, apply Lemmas 8.5 and 8.2 to equation (4.13),

$$\begin{aligned}
\left\|\partial_t\nabla^2 A_0^*\right\|_{L^2(\bigcirc_t)} &\leq C\left\|\int_{\bigcirc_t}\nabla^3 G_t(x,y)\,(J_i - \partial_t[A_i,A_0])\,dy\right\|_{L^2(\bigcirc_t)}\\
&\quad + C\left\|\int_{\bigcirc_t}\nabla^2 G_t(x,y)[A,\nabla A]\right\|_{L^2(\bigcirc_t)}\\
&\leq C\,\|J_i - \partial_t[A_i,A_0]\|_{H^1(\bigcirc_t)} + C\,\|[A_j,\partial_j A_j]\|_{H^{\frac{1}{2}}(\bigcirc_t)}\\
&\leq C\left(\|A,\phi\|^2 + \|A,\phi\|^3\right)\\
&\quad + C\,\|[A_j,\partial_j A_j]\|_{H^1(\bigcirc_t)}\\
&\leq C\left(\|A,\phi\|^2 + \|A,\phi\|^3\right)
\end{aligned}$$

The argument proving $\mathcal{L}:\mathcal{D}\to\mathcal{D}$ is a contraction follows the same reasoning and is now completely straightforward. ■

5. Global in time existence

This section contains the proofs of Theorems 1.1 and 1.2. The main estimates for A_0, A_i, and ϕ are merely stated. The proofs of these estimates are found in Sections 6 and 7. Throughout these estimates, the term 'universal constant' will be applied to constants depending only on the dimension in which we work[23] and the constants appearing in the Yang-Mills-Higgs equations (1.10)-(1.11).

5.1. Global smooth solutions. We recall here the statement of Theorem 1.1.

Theorem: Consider Yang-Mills-Higgs with a critical power Higgs self-interaction. Then any initial data set[24]

$$(\bar{A},\bar{E},\bar{\phi},\overline{D_0\phi}) \quad\in\quad H^s_{\mathrm{loc}}(\mathbb{R}^3)\ \ s\geq 2$$

admits a unique global, temporal gauge solution satisfying[25]

$$(A,\phi)\in C([0,\infty);H^s_{\mathrm{loc}}(\mathbb{R}^3))\cap C^1([0,\infty)\,;H^{s-1}_{\mathrm{loc}}(\mathbb{R}^3))$$

Proof: Repeated application of Theorem 4.1 proves that a temporal gauge solution

$$A,\phi \quad\in\quad C\big([0,T^*);H^s_{\mathrm{loc}}\,(\mathbb{R}^3)\,\big)\cap C^1\left([0,T^*);H^{(s-1)}_{\mathrm{loc}}\,(\mathbb{R}^3)\right)$$

---

[23] Namely, three space dimensions.

[24] Recall that the following condition means $\bar{A},\bar{\phi}\in H^s_{\mathrm{loc}}$, $\bar{E},\overline{D_0\phi}\in H^{s-1}_{\mathrm{loc}}$.

[25] In fact, the proof also shows that data in $H^s(\mathbb{R}^3)$ implies evolution in $H^s(\mathbb{R}^3)$.

exists on some maximal interval $[0, T^*)$. Assume for contradiction that $T^* < \infty$.

It suffices to show that every point $p = (T^*, x_p)$ lies on the top of a truncated cone[26] $_{(\delta_1,\delta_2)}$ on which $\|A, \phi\|_{H^2(\ _t)}$ is bounded. One then has data in $H^s_{\mathrm{loc}}(\mathbb{R}^3)$ at time T^* and can apply the local existence theorem once more, contradicting T^* maximal.

The estimates which we get for $\|A, \phi\|_{H^2(\ _t)}$ are accomplished in the Coulomb gauge. In order to change to the Coulomb gauge, we need the following result known as Uhlenbeck's lemma,

Lemma 5.1. *([32]) Suppose $(\bar{A}, \bar{E})$ are the Yang-Mills constituents of an initial data set on a ball of radius 1 and center the origin for which $\left\|H[\bar{A}]\right\|_{L^{\frac{3}{2}}(\ )} \leq \delta$, where δ is some universal constant and $H_i[\bar{A}] \equiv \left(\nabla \times \bar{A}\right)_i + \frac{1}{2}\epsilon_{ijk}[\bar{A}^j, \bar{A}^k]$.*

1). There is a $\mathcal{G}$-valued function O on so that $\tilde{A}_i = O\bar{A}_iO^{-1} + O\partial_iO^{-1}$ satisfies

$$\begin{aligned} \partial_i \tilde{A}^i &= 0 \quad \text{on} \\ x \cdot \tilde{A} &= 0 \quad \text{on} \end{aligned}$$

and

$$\begin{aligned} \left\|\tilde{A}\right\|_{H^1(\ )} &\leq C\left\|H[\bar{A}]\right\|_{L^2(\ )} \\ \|O\|_{H^2(\ )} &\leq C\left\|\bar{A}\right\|_{H^1(\ )} \\ \|O\|_{H^3(\ )} &\leq C\left\|\bar{A}\right\|_{H^2(\ )} \end{aligned}$$

2). There exists $\epsilon > 0$ depending only on $\left\|\bar{A}\right\|_{H^1(\ )}$ so that if $(\bar{A}', \bar{E}')$ are the Yang-Mills constituents of an initial data set with $\left\|\bar{A} - \bar{A}'\right\|_{H^1(\ )} \leq \epsilon$, then there is a gauge transformation O' taking $(\bar{A}', \bar{E}')$ to

$$\begin{aligned} \tilde{A}'_i &= O'\bar{A}'_iO'^{-1} + O'\partial_iO'^{-1} \\ \tilde{E}' &= O'\bar{E}'O'^{-1} \end{aligned}$$

so that $\partial^i\tilde{A}'_i = 0, \tilde{A}' \cdot x = 0$ on . In addition,

$$\left\|\tilde{A}' - \tilde{A}\right\|_{H^1(\ )} + \|O' - O\|_{H^2(\ )} \leq C\left\|\bar{A} - \bar{A}'\right\|_{H^1(\ )}$$

Remark When $\partial_iA^i = 0$ on and $A \cdot x = 0$ on , We will say that A is in a *normalized* Coulomb gauge.

The hypotheses of Uhlenbeck's Lemma are satisfied by small energy initial data. We transform to such data by scaling the equations: if $\left(\underline{A}, \underline{\phi}\right)$ is a solution to the critical power Yang-Mills-Higgs equations, then

$$\begin{aligned} A(t,x) &= \lambda^{-1}\underline{A}(\frac{t}{\lambda}, \frac{x}{\lambda}) \\ \phi(t,x) &= \lambda^{-1}\underline{\phi}(\frac{t}{\lambda}, \frac{x}{\lambda}) \end{aligned}$$

[26]This notation was explained in (2.2) of Section 2.

is a solution of the *scaled* critical power Yang-Mills-Higgs equations,

$$D_\mu F^{\nu\mu} = -[\phi, D^\nu \phi] \tag{5.1}$$

$$D_\mu D^\mu \phi = \lambda^2 |\phi|^4 \phi. \tag{5.2}$$

for which the energy scales as $\mathcal{E}^{\text{scaled}}(t) \equiv \lambda^{-\frac{1}{2}} \underline{\mathcal{E}}(t)$. Here,

$$\mathcal{E}^{\text{scaled}}(t) = \int_{\mathbb{R}^3} |E(t,\cdot)|^2 + |H(t,\cdot)|^2 + |D_t\phi|^2 + |D_i\phi|^2 + \lambda^2 \frac{|\phi|^6}{6} dx$$

is the scaled energy.

Using Lemma 3.3 in Section 3, there is an 'eligible trapezoid' $_{(\delta_1^{\text{elig}}, \delta_2^{\text{elig}})}$ below the point p on which the (unscaled) Higgs field satisfies

$$\|\underline{\phi}\|_{L^6} \cdot \underline{\mathcal{E}}_0^3 \le \epsilon_0. \tag{5.3}$$

Here ϵ_0 is a universal constant, determined in Theorem 6.2.

We choose the scaling constant λ according to the following two criteria:

- Pick λ large enough so that a truncated cone of height $\frac{1}{2\lambda}$ and base radius $\frac{1}{\lambda} + \frac{1}{2\lambda}$ lies strictly within the 'eligible trapezoid' and abutts the top of the eligible trapezoid. (See figure 5.1.) More concisely, we want $_{\frac{1}{2\lambda}, \frac{1}{\lambda}}$ to lie within the eligible trapezoid. This amounts to picking λ large so that

$$\frac{1}{2\lambda} < \delta_1^{\text{elig}} \qquad \frac{1}{\lambda} < \delta_2^{\text{elig}}.$$

- Pick λ large enough so

$$\mathcal{E}_0^{\text{scaled}} \equiv \lambda^{-\frac{1}{2}} \mathcal{E}_0 < \delta, \tag{5.4}$$

where δ is some small universal constant, determined by Uhlenbeck's Lemma and the proof of Theorem 6.2.

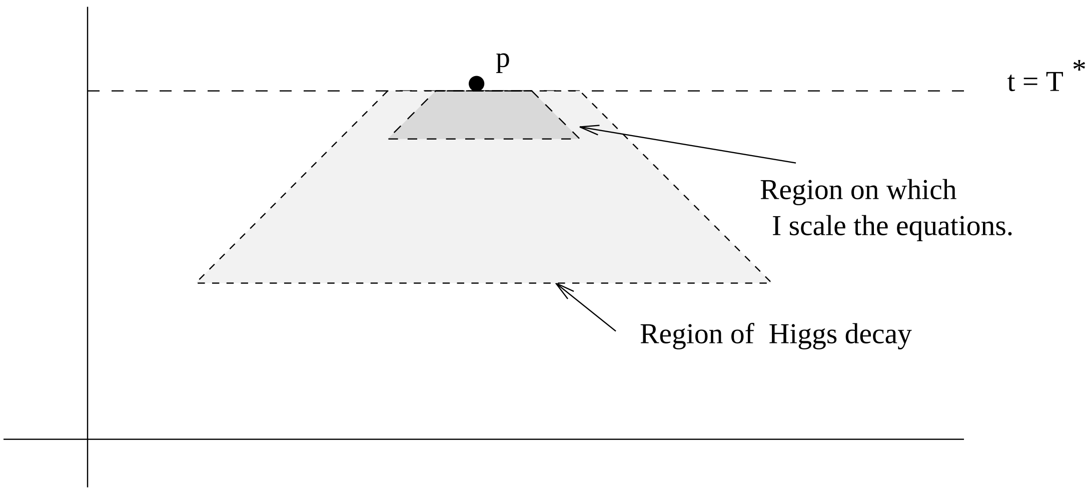

FIGURE 5.1. Pushing past the possible blow-up time T^*.

We then scale the solution and the equations on the truncated cone $^{\text{scaling}} \equiv$ $_{\frac{1}{2\lambda},\frac{1}{2\lambda}}$. Note that all sufficiently small space-translates of $^{\text{scaling}}$ lie entirely within the eligible trapezoid. After scaling, We are left to consider a temporal gauge evolution of equations (5.1) - (5.2) on a truncated cone of base radius 1 and height $\frac{1}{2}$ with small initial energy $\mathcal{E}_0^{\text{scaled}}$. The evolution enjoys the following smallness property[27]

$$\| \phi \|_{L^6} \cdot (\mathcal{E}_0^{\text{scaled}})^3 \quad \leq \quad \lambda^{-2} \epsilon_0. \tag{5.5}$$

In fact, (5.5) also holds for $\phi_\omega(t,x) \equiv \phi(t, x-\omega)$ for all $|\omega| < \frac{1}{2}$[28]. We aim to show that such a development remains bounded in H^2 all the way up to the top of . This clearly implies an H^2 bound for the unscaled solution.

The main steps in the argument are

(1) Change the initial data into the Coulomb gauge and apply the Coulomb gauge local existence theorem 4.3.
(2) Derive good estimates for a Coulomb gauge solution of the scaled Yang-Mills-Higgs equations. These estimates will show that $\|A, \phi\|_{H^2(\ _t)}$ stays finite throughout the evolution. By the local existence theorem 4.3, therefore, the solution extends up to time $\frac{1}{2}$.
(3) Transform this Coulomb gauge solution to the temporal gauge. Show that the gauge transformation doesn't introduce any singularities. By local existence and uniqueness (Theorem 4.1) the transformed solution agrees with the original temporal gauge solution[29] thus proving regularity.

Let us now fill in the details of each step.

STEP ONE: Since $\|H[A]\|_{L^2(\ _0)} < \delta$, Uhlenbeck's Lemma 5.1 applies. See Section 7 for a detailed description of the local Coulomb gauge which we choose.[30] In this gauge, equations (5.1) - (5.2) can be written as the system (C.G.a) - (C.G.f). We apply the local existence and uniqueness result of Theorem 4.3 to conclude

$$A_0, A, \phi \quad \in \quad C\big([0,T]; H^s(\ _t)\big) \cap C^1\left([0,T]; H^{(s-1)}(\ )\right).$$

and the following additional regularity for A_0,

$$\partial_t A_0 \quad \in \quad C\left([0,T]; H^s(\ _t)\right).$$

STEP TWO: The "good estimates" for the gauge potential are very close to those found in [15], the argument here involving only a little extra bookwork. The estimates for the Higgs field are similar in spirit to the discussion of the critical power wave equation in section 2. The result is a rather involved bootstrap argument for the following quantities

$$X(A_i, \phi)(T) \quad = \quad \int_0^T \|\Box A_i\|_{L^2(\ _s)} + \|\Box \phi\|_{L^2(\ _s)}\, ds$$

---

[27] This property follows from (5.3) and scaling.

[28] Here is one of the places where we *need* Higgs nonconcentration on a set larger than a light cone.

[29] That is, the scaled version of the Yang-Mills-Higgs solution.

[30] Such a gauge is not unique, since one needs to impose boundary conditions on A_0.

$$(5.6)\quad Y(A_0)(T) = \Big(\int_0^T \|\partial_t A_0\|^2_{L^3(\bullet_t)}\,dt\Big)^{\frac12} + \Big(\int_0^T \|A_0\|^2_{L^\infty(\bullet_t)}\,dt\Big)^{\frac12} +$$

$$\Big(\int_0^T \|\nabla A_0\|^2_{L^3(\bullet_t)}\,dt\Big)^{\frac12} + \sup_{0\le t\le T}\|\partial_i A_0\|_{L^2(\bullet_t)} + \sup_{0\le t\le T}\|A_0\|_{L^3(\bullet_t)} + \|A_0\|_{L^2(/\ \backslash^T)}$$
$$+\|\partial_t A_0\|_{L^2(/\ \backslash^T)} + \|\partial_i A_0\|_{L^2(/\ \backslash^T)}$$

$$Z(A_0, A_i, \phi)(T) \equiv X(A_i,\phi)(T) + Y(A_0)(T)$$

Recall the following notation,

$$\|(A_i,\phi)\|_{H^1(\bullet_t)} \equiv \|A\|_{H^1(\bullet_t)} + \|\phi\|_{H^1(\bullet_t)}$$
$$+\|\partial_t A_i\|_{L^2(\bullet_t)} + \|\partial_t\phi\|_{L^2(\bullet_t)}$$

Note that Uhlenbeck's lemma gives the bound $\|(A_i,\phi)\|_{H^1(\bullet_0)} \le C\mathcal{E}_0^{\text{scaled}}$. Hence (5.5) can be written

$$(5.7)\qquad \|\phi\|_{L^6}\cdot\|(A_i,\phi)\|^3_{H^1(\bullet_0)} \le \lambda^{-2}\epsilon_0.$$

The bound (5.7) is necessary for the proof of the following proposition which summarizes those parts of Theorems 6.2, 6.4 which we need to construct a global smooth solution.

Proposition 5.2. *1). Suppose* (A_0, A, ϕ) *a solution of (C.G.a) - (C.G.f) on* $\blacktriangle^T$ *satisfying* (5.5), (5.4).

Then there is a universal constant C *so that for all* $0\le t\le T$

$$Z(A_0,A_i,\phi)(t) \le C\,\|(A_i,\phi)\|_{H^1(\bullet_0)}.$$

2). $(A_0,A,\phi), (A_0',A',\phi')$ *two solutions of (C.G.a) - (C.G.f) on* $\blacktriangle^T$ *both satisfying* (5.5), (5.4) *and with* $\|(A_i - A_i', \phi-\phi')\|_{H^1(\bullet_0)} < \epsilon$ *a small universal constant. Then for* $0\le t\le T$,

$$Z(A_0-A_0', A_i - A_i', \phi-\phi')(t) \le \tilde C\,\|(A_i-A_i',\phi-\phi')\|_{H^1(\bullet_0)}.$$

where $\tilde C$ *depends only on* $\|(A_i,\phi)\|_{H^1(\bullet_0)}, \|(A_i',\phi')\|_{H^1(\bullet_0)}$.

Take $A_0'(t,x) = A_0(t,x-\omega)$, similarly A_i', ϕ'. When $|\omega|\to 0$, Proposition 5.2 gives the uniform bound

$$\|(A_0,A,\phi)\|_{H^2(\bullet_t)} \le \tilde C\,\|(A_0,A,\phi)\|_{H^2(\bullet_0)}$$

for all $0\le t\le T$

STEP THREE:

Note that

$$\tilde A_\alpha = \mathcal{U}A_\alpha\mathcal{U}^{-1} + \mathcal{U}\partial_\alpha\mathcal{U}^{-1}$$

satisfies the temporal gauge condition if

$$\partial_t\mathcal{U}^{-1} = -A_0\mathcal{U}^{-1}.$$

The main estimates for Step Three are contained in the following theorem (See [15], Theorem 7). Recall O as the $\mathcal{G}$-valued function on $\;_0$ from Step One. (O is what put the data in the Coulomb gauge to begin with.)

Theorem 5.3. *Let $\mathcal{U}(t,x)$ be the unique $\mathcal{G}$-valued function defined in $\;^{\frac{1}{2}}$ verifying the O.D.E*

$$\partial_t \mathcal{U} = -A_0 \mathcal{U}$$

with data $\mathcal{U}(0) = \mathcal{U}_0 = O^{-1} \in H^2(\;_0)$ and (A_0, A, ϕ) the Coulomb gauge solution of (C.G.a) - (C.G.f) constructed in Step Two above.

1). For all $0 \leq t \leq \frac{1}{2}$,

$$\|\mathcal{U}(t,\cdot)\|_{H^2(\;_t)} + \|\partial_t \mathcal{U}(t,\cdot)\|_{H^1(\;_t)} \quad \leq \quad C\|\mathcal{U}_0\|_{H^2(\;_0)}$$

where the constant C depends only on the initial energy $\mathcal{E}_0^{scaled}$ of the scaled solution.

2). Let (A_0, A, ϕ), (A_0', A', ϕ') be two such Coulomb gauge solutions, yielding $\mathcal{G}$-valued functions $\mathcal{U}$ and $\mathcal{U}'$. Then for all $0 \leq t \leq \frac{1}{2}$,

$$\begin{aligned} &\|(\mathcal{U} - \mathcal{U}')(t,\cdot)\|_{H^2(\;_t)} + \|\partial_t(\mathcal{U} - \mathcal{U}')(t,\cdot)\|_{H^1(\;_t)} \leq \\ &\qquad C\Big(\|(A - A')(t,\cdot)\|_{H^1(\;_t)} + \|\partial_t(A - A')(t,\cdot)\|_{L^2(\;_t)} \\ &\qquad\qquad + \|\mathcal{U}_0 - \mathcal{U}_0'\|_{H^2(\;_0)}\Big) \end{aligned}$$

where the constant C depends only on $\mathcal{E}_0^{scaled}, \mathcal{E}_0^{\prime\, scaled}$.

Proof In Proposition (7.2), Section 7, We show that the quantity $Y(t)$ enjoys the bound $Y(T) \leq C\|(A_i,\phi)\|_{H^1(\;_0)}$ With this, the proof of (5.3) follows line for line the argument supporting Theorem 7 in [15]. ■

Using Theorems 5.3, 5.2, and Theorem 5.3 of [15], the main estimates for a temporal gauge solution are now straightforward.

Theorem 5.4 (Main Temporal Gauge Estimates). *(A). Consider (A_i, ϕ) a temporal gauge solution of (5.1) - (5.2) on $\;^T$, $T \leq \frac{1}{2}$ satisfying condition (5.5) above. Suppose in addition that the initial data satisfies*

$$\left(\bar{A}_i, \bar{E}, \bar{\phi}, \overline{D_0\phi}\right) \in H^s(\;_0)$$

for some $s \geq 2$, with $\left\|(\bar{A}, \bar{\phi})\right\|_{H^1(\;_0)} < \delta$.

Conclusion: The solution (A_i, ϕ) can be extended up to time $T = \frac{1}{2}$. Moreover, the following estimates hold.

$$\|(A,\phi)\|_{H^1(\;_t)} \quad \leq \quad C\left\|(\bar{A}, \bar{\phi}\right\|_{H^1(\;_0)}$$

for all $0 \leq t \leq \frac{1}{2}$, where $C > 0$ depends only on $\left\|(\bar{A}, \bar{\phi})\right\|_{H^1(\;_0)}$.

(B). Suppose $(A,\phi), (A',\phi')$ are two such temporal gauge solutions corresponding to data $(\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi}), (\bar{A}', \bar{E}', \bar{\phi}', \overline{D_0\phi'})$ with

$$\left\|(\bar{A} - \bar{A}', \bar{\phi} - \bar{\phi}'\right\|_{H^1(\;_0)} \leq \epsilon.$$

Then there exists a constant $C > 0$ depending only on $\|(\bar{A},\bar{\phi})\|_{H^1(\bullet_0)}$, $\|(\bar{A}',\bar{\phi}')\|_{H^1(\bullet_0)}$ so that

$$\|(A-A',\phi-\phi')\|_{H^1(\bullet_t)} \;\leq\; C\big(\|(A-A',\phi-\phi')\|_{H^1(\bullet_0)}.$$

(C). Applying (B) to a solution (A,ϕ) and it's translates,

$$\begin{aligned}\|A,\phi\|_{H^2(\bullet_t)} + \|\partial_t A, \partial_t\phi\|_{H^1(\bullet_t)} \;&\leq\; C\big(\|A,\phi\|_{H^2(\bullet_0)} \\ &\quad + \|\partial_t A,\partial_t\phi\|_{H^1(\bullet_0)}\big)\end{aligned}$$

where C here depends only on $\|A,\phi\|_{H^1(\bullet_0)}$.

Proof

Let $(\tilde{A},\tilde{\phi})$ be the Coulomb gauge solution from Step 2, and (A,E_i,ϕ) be

$$\begin{aligned} A_i &= \mathcal{U}\tilde{A}_i\mathcal{U}^{-1} + \mathcal{U}\partial_i\mathcal{U}^{-1} \\ E_i &= \mathcal{U}\tilde{E}_i\mathcal{U}^{-1} \\ \phi &= \mathcal{U}\tilde{\phi}\mathcal{U}^{-1}\end{aligned}$$

be the solution transformed into the temporal gauge. It is easy to see that $(A,E_i,\phi,\partial_t\phi)$ attains the initial data $(\bar{A},\bar{E}_i,\bar{\phi},\overline{D_0\phi})$. The estimates on A_0, A_i are proved exactly as in [15], Theorem 2. As for the Higgs field,

$$\begin{aligned}\|\phi\|_{H^1(\bullet_t)} + \|\partial_t\phi\|_{L^2(\bullet_t)} &= \left\|\mathcal{U}\tilde{\phi}\mathcal{U}^{-1}\right\|_{H^1(\bullet_t)} + \left\|\partial_t\left(\mathcal{U}\tilde{\phi}\mathcal{U}^{-1}\right)\right\|_{L^2(\bullet_t)} \\ &\leq C\big(\left\|\tilde{\phi}\right\|_{H^1(\bullet_t)} + \left\|\partial_t\tilde{\phi}\right\|_{L^2(\bullet_t)} \\ &\qquad + \|\mathcal{U}\|_{H^2(\bullet_t)} + \|\partial_t\mathcal{U}\|_{H^1(\bullet_t)}\big)\end{aligned}$$

where C depends on $\|\mathcal{U}\|_{H^1(\bullet_t)}$, $\|\partial_t\mathcal{U}\|_{H^1(\bullet_t)}$, and $\|A\|_{L^3(\bullet_t)}$. The estimate for the Higgs field in part (i) of the Theorem now follows from

$$\begin{aligned}\left\|\tilde{\phi}\right\|_{H^1(\bullet_t)} + \left\|\tilde{\phi}\right\|_{L^2(\bullet_t)} &\leq C\mathcal{E}_0 \text{ (Prop. 5.2)} \\ \|\mathcal{U}\|_{H^2(\bullet_t)} + \|\partial_t\mathcal{U}\|_{H^1(\bullet_t)} &\leq C\|\mathcal{U}_0\|_{H^2(\bullet_0)} \text{ (Thm. 5.3)} \\ \|\mathcal{U}_0\|_{H^2(\bullet_0)} &\leq C\left\|\bar{A}\right\|_{H^1(\bullet_0)} \text{ (Uhlenbeck's Lemma).}\end{aligned}$$

The proof of the estimates for differences is identical. ■

Statement (C) in Theorem 5.4 concludes the proof of Theorem 1.1.

5.2. Finite Energy Solutions. The existence of strongly continuous energy class solutions for the Yang-Mills equations in $\mathbb{R}^{3+1}$ was established in Klainerman-Machedon [15]. The question of strongly continuous, finite energy solutions to the critical power wave equation was settled for all dimensions in Kapitanski [12], Stuwe-Shatah [31]. we prove here the corresponding result for the critical power Yang-Mills-Higgs equations in $\mathbb{R}^{3+1}$, using ideas from all three of these papers.

Proof of Theorem 1.2: The energy class solution are constructed as the limit of smooth solutions. Via Uhlenbeck's lemma, any finite energy data set can be considered to be in $H^1_{\text{loc}}(\mathbb{R}^3)$. As in [15], Proposition 1.2, one constructs a sequence of C_0^∞ data sets so that

$$(\bar{A}_i^{(n)}, \bar{E}_i^{(n)}, \bar{\phi}_i^{(n)}, \overline{D_0\phi}_i^{(n)}) \quad \to \quad (\bar{A}, \bar{E}, \bar{\phi}, \overline{D_0\phi}) \tag{5.9}$$

in $H^1_{\text{loc}} \times L^2_{\text{loc}} \times H^1_{\text{loc}} \times L^2_{\text{loc}}$.

It suffices to construct finite energy solutions in the domain of influence $I(\Omega)$ of any $\Omega \subset\subset \mathbb{R}^3$. (See figure 5.2.) Indeed the construction below simplifies in the region $J(\Omega)$ if one takes Ω large enough so that

$$\mathcal{E}_0|_{\Omega^c} \quad \leq \quad \epsilon_0.$$

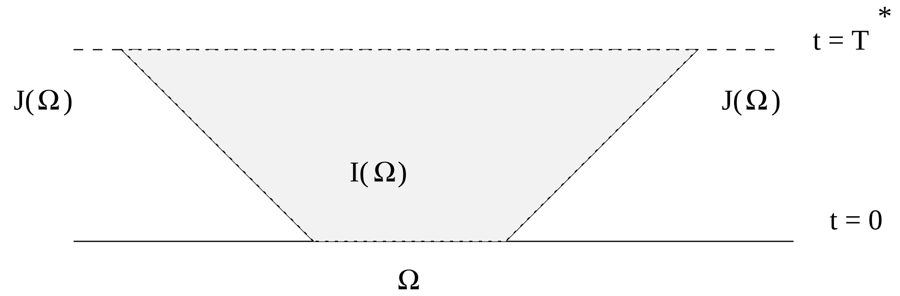

FIGURE 5.2. The intended region of convergence

By compactness, $\exists \delta_0 > 0$ with

$$\mathcal{E}_0|_{B(x,\delta_0)} \quad \leq \quad \epsilon_0 \quad x \in \Omega.$$

Corollary 6.4 proves that the smooth solutions $(A_i^{(n)}, E_i^{(n)}, \phi_i^{(n)}, D_0\phi_i^{(n)})$ converge[31] on the cone with base $B(x, \delta_0)$ and height $\frac{\delta_0}{2}$. These temporal gauge solutions can be pieced together using uniqueness, thereby proving local existence of an energy class solution up to time $\frac{\delta_0}{2}$ as in figure 5.3.

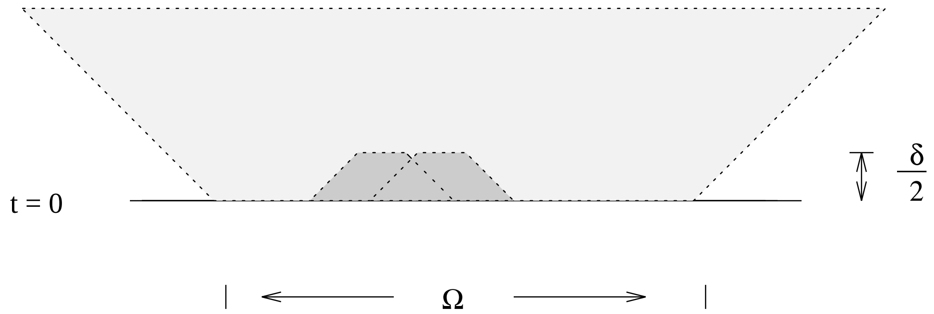

FIGURE 5.3. Pasting the Local Solutions Together

Repeated application of this argument gives[32]

$$A, \phi \quad \in \quad C([0, T^*); H^1_{loc}(\mathbb{R}^3)) \cap C^1([0, T^*); L^2_{loc}(\mathbb{R}^3))$$

[31] In both the Coulomb and temporal gauge.

[32] One adds $\frac{\delta_t}{2}$ to the interval of existence at each step.

with T^* maximal.

Suppose $T^* < \infty$. Then $\delta_t \to 0$ as $t \to T^*$. We arrive at a contradiction by using the following lemma, an energy non-concentration result for the weak solution similar to Lemma (2.3).

Lemma 5.5 (Energy Class Higgs Nonconcentration). *Given*

- *A point p on the hyperplane $t = T^*$,*
- *(A, ϕ) an admissible, weak solution to the critical Yang-Mills-Higgs equations on $[0, T^*)$.*
- *$\epsilon > 0$.*

then there exist δ_1, δ_2 such that for $t \in [T^* - \delta_1, T^*)$

$$\int_{{}_t^{trap}} |\phi|^6 \, dx \quad < \epsilon$$

where ${}_t^{trap}$ is the time $= t$ slice of ${}_{(\delta_1,\delta_2)}$

Proof If the admissible solution enjoys the local energy estimate and the integral equality (3.9), then the proof of this Lemma will follow precisely the same argument as Lemma 3.3. We prove here the local energy estimate. The integral relation (3.9) follows from similar considerations. We follow here the idea written out in [31].

Consider then an admissible solution (A, ϕ) on a domain ${}_{t_0}^{t_1}$ and smooth solutions $(A^{(n)}, \phi^{(n)})$ such that

$$(A^{(n)}, \phi^{(n)}) \to (A, \phi) \quad \text{in} \quad C\left([t_0, t_1]; H^1(\ _t)\right) \cap C^1([t_0, t_1]; L^2(\ _t)).$$

Write $T^{(n)}_{\alpha\beta}$ for the corresponding energy-momentum tensors. Then $T^{(n)}_{\alpha\beta} \to T_{\alpha\beta}$ in $L^2(\ _t)$, so one easily gets

$$\int_{{}_{t_0}^{t_1}} T_{\alpha\beta} \partial^\alpha f dx dt \quad = \quad 0$$

for any piecewise smooth, continuous function f in $C_0(\ {}_{t_0}^{t_1})$.

By a simple coordinate change, assume ${}_{t_0}^{t_1}$ is as in figure 3.1. Define f to be the following cutoff function

$$f = \begin{cases} 1 & \text{on } R_0 \\ -\frac{\eta}{\epsilon} & \text{on } R_1 \\ \frac{t-t_0}{\epsilon} & \text{on } R_2 \\ -\frac{t-t_1}{\epsilon} & \text{on } R_3 \end{cases} \tag{5.10}$$

where $\eta = t + |x|$, ϵ is some positive parameter, and the R are the following regions.

$$\begin{aligned} R_0 &= \left\{(t,x) \,\middle|\, t_0 + \epsilon \le t \le t_1 - \epsilon;\ 0 \le |x| \le -t - \epsilon\right\} \\ R_1 &= \left\{(t,x) \,\middle|\, t_1 - \epsilon \le t \le t_1;\ -t_1 \le |x| \le -t\right\} \\ &\quad \cup \left\{(t,x) \,\middle|\, -\epsilon \le \eta \le 0;\ -t_1 + \eta + \epsilon \le |x| \le 2\eta - t_0\right\} \\ R_2 &= \left\{(t,x) \,\middle|\, t_0 \le t \le t_0 + \epsilon;\ 0 \le |x| \le t_0 - 2t\right\} \\ R_3 &= \left\{(t,x) \,\middle|\, t_0 + \epsilon \le t \le t_1 - \epsilon;\ 0 \le |x| \le -t - \epsilon\right\} \end{aligned}$$

Then

$$\begin{aligned}
0 &= \int\int T_{0\alpha}\partial^{\alpha} f dx dt \\
&= -\frac{1}{2}\int_{R_1}\int T_{03}\partial_{\eta} f dx dt - \int_{R_2}\int T_{00}\partial_t f dx dt \\
&\quad - \int_{R_3}\int T_{00}\partial_t f dx dt \\
0 &= \frac{1}{2\epsilon}\int_{R_1}\int T_{03} - \frac{1}{\epsilon}\int_{R_2}\int T_{00} dx dt \\
&\quad + \frac{1}{\epsilon}\int_{R_3}\int T_{00} dx dt.
\end{aligned}$$

Introducing coordinates $\eta, r = |x|$, and $\omega \in S^2$, we write this as

$$\begin{aligned}
&\int_{t_0}^{t_0+\epsilon} dt \int_{-t_1+\eta+\epsilon}^{2\eta - t_0} dr \int_{S^2} r^2 d\omega T_{00} = \\
&\int_{t_1-\epsilon}^{t_1} dt \int_0^{-t_1} dr \int_{S^2} r^2 d\omega T_{00} + \frac{1}{2}\int_{-\epsilon}^{0} d\eta \int_{-t_1+\eta+\epsilon}^{2\eta-t_0} dr \int_{S^2} r^2 d\omega T_{03} \\
&\qquad + \frac{1}{2}\int_{t_1-\epsilon}^{t_1} dt \int_{-t_1}^{-t} dr \int_{S^2} r^2 d\omega T_{03}.
\end{aligned} \tag{5.11}$$

Differentiating (5.11) with respect to ϵ and setting $\epsilon = 0$, the last term on the right hand side vanishes and we are left with the local energy estimate

$$\int_{t_1} T_{00} dx = \int_{t_0} T_{00} + \frac{1}{2}\int_{/\ \ \backslash_{t_0}^{t_1}} T_{03} dx dt \tag{5.12}$$

as desired. ∎

It remains to show the following nonconcentration of energy: for all

$$p = (T^*, x_p) \in I(\Omega)$$

there exists ${}^{p}_{(\delta_1,\delta_2)}$ so that

$$\mathcal{E}\big|_{{}^{\mathrm{trap}}_{t}} \quad \leq \quad \epsilon_0 \qquad \text{for all } t \in [T^* - \delta_1, T^*). \tag{5.13}$$

Lemma 5.5 gives $\tilde{\delta}_1, \tilde{\delta}_2$ with

$$|||\phi|||_{L^6} \cdot \mathcal{E}_0^3 \quad \leq \quad \epsilon_0 \tag{5.14}$$

where

$$|||\phi|||_{L^6} \equiv \sup_{T^* - \tilde{\delta}_1 \leq t < T^*} \|\phi(t,\cdot)\|_{L^6(\ {}^{\mathrm{trap}}_{t})}.$$

Define $\phi^{\mathrm{L}}, A_i^{\mathrm{L}}$ as the solution of the linear problem

$$\begin{aligned}
\Box\phi^{\mathrm{L}} &= 0 \\
\Box A_i^{\mathrm{L}} &= 0
\end{aligned}$$

with the following data on $^{\text{trap}}_{T^*-\tilde{\delta}_1}$,

$$\begin{aligned} \phi^{\mathrm{L}} &= \phi \qquad & A_i^{\mathrm{L}} &= A_i \\ \partial_t\phi^{\mathrm{L}} &= \partial_t\phi \qquad & \partial_t A_i^{\mathrm{L}} &= \partial_t A_i. \end{aligned}$$

Pick δ_1, δ_2 so that,

- On $^{p}_{(\delta_1,\delta_2)}$,

$$\sup_{T^*-\delta_1\le t\le T^*} \left\|(\phi^{\mathrm{L}}, A_i^{\mathrm{L}})\right\|_{H^1\ (\ ^{\text{trap}}_{t})} \;\le\; \frac{\epsilon_0}{2}.$$

  This is possible since the linear solution's energy doesn't concentrate.
- $\delta_1 \le \tilde{\delta}_1, \delta_2 \le \tilde{\delta}_2$.
- Using (5.14) and Theorem 6.2, pick δ_1 small enough so

$$\int_{T^*-\delta_1}^{T^*} ||\Box\phi||_{L^2(\ ^{\text{trap}}_{s})} + ||\Box A_i||_{L^2(\ ^{\text{trap}}_{s})}\, ds \;\le\; \frac{\epsilon_0}{2}.$$

The bound (5.13) now follows easily from the basic energy estimate applied to

$$(\phi - \phi^{\mathrm{L}}, A_i - A_i^{\mathrm{L}}).$$

6. Bootstrapping the Dynamic Variables A_i and ϕ

My method for getting good bounds for the dynamic variables is taken from the corresponding arguments in [15], Theorem 6.1. We adopt the following notation found in the Klainerman-Machedon paper

$$\begin{aligned}(6.1)\quad \mathcal{R}[A,\phi](t) \;=\; \big(&||\partial A||^2_{L^2(\ _t)} + ||A||^2_{L^2(\ _t)}\, ||\partial\phi||^2_{L^2(\ _t)} \\ &+ ||\phi||^2_{L^2(\ _t)} + \int_0^t \int_{\bigcirc_s} \left|\overline{\partial} A(s,y)\right|^2 + \left|\overline{\partial}\phi(s,y)\right|^2 d\sigma_y ds\big)^{\frac{1}{2}}.\end{aligned}$$

Here $\overline{\partial}$ denotes derivatives tangent to $/\ \ \backslash$. I'll also write $\mathcal{R}_0 \equiv ||(A_i,\phi)||_{H^1(\ _0)}$. It follows from Uhlenbeck's Lemma that, after scaling, $\mathcal{R}_0[A,\phi] < \delta < 1$.

Let $Y(t)$ denote the following combination of norms for A_0,

$$\begin{aligned}(6.2)\quad Y(T) \;=\; &\Big(\int_0^T ||\partial_t A_0||^2_{L^3(\ _t)}\, dt\Big)^{\frac{1}{2}} + \Big(\int_0^T ||A_0||^2_{L^\infty(\ _t)}\, dt\Big)^{\frac{1}{2}} + \\ &\Big(\int_0^T ||\nabla A_0||^2_{L^3(\ _t)}\, dt\Big)^{\frac{1}{2}} + \sup_{0\le t\le T} ||\partial_i A_0||_{L^2(\ _t)} + \sup_{0\le t\le T} ||A_0||_{L^3(\ _t)} + ||A_0||_{L^2(/\ \ \backslash^T)} \\ &+ ||\partial_t A_0||_{L^2(/\ \ \backslash^T)} + ||\partial_i A_0||_{L^2(/\ \ \backslash^T)}\end{aligned}$$

Recall that denotes a truncated cone of height $\frac{1}{2}$ and base radius 1.

Theorem 6.1. *Consider (A_0, A, ϕ) a solution of (C.G.a.)-(C.G.f)*[33] *on* $\triangle^T$.
Then writing

$$X(T) = \int_0^T \|\Box A_i\|_{L^2(\bigcirc_s)} + \|\Box\phi\|_{L^2(\bigcirc_s)}\, ds$$

we have

(a)

$$\|B\|_{L^2(\triangle^T)} \leq C\Big(\|G\|_{L^2(\triangle^T)} + \big(\mathcal{R}_0[A,\phi] + X(T)\big)^2 \Big)$$

(b)

$$\|G\|_{L^2(\triangle^T)} \leq C\Big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + \mathcal{R}_0[A,\phi] \Big)$$

(c)

$$\begin{aligned}\int_0^T \|\Box\phi\|_{L^2(\bigcirc_t)}\, dt &\leq \lambda^2\, \||\phi|\|_{L^6} \cdot \big(\mathcal{R}_0[A,\phi] + X(T)\big)^4 \\ &\quad + C\Big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + Y(T)^4 + \mathcal{R}_0[A,\phi] \Big)\end{aligned}$$

And therefore,

$$\begin{aligned} X(T) &\leq C\big(\|B\|_{L^2(\triangle^T)} + \|G\|_{L^2(\triangle^T)} \big) + \int_0^T \|\Box\phi\|_{L^2(\bigcirc_t)}\, dt \\ &\leq \lambda^2\, \||\phi|\|_{L^6} \cdot \big(\mathcal{R}_0[A,\phi] + X(T)\big)^4 \\ &\quad + C\Big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + Y(T)^4 + \mathcal{R}_0[A,\phi] \Big) \end{aligned}$$

Remark: In the notation above, the standard energy inequality reads,

$$\mathcal{R}[A,\phi](t) \quad \leq \quad C(\mathcal{R}_0[A,\phi] + X(t)).$$

This will be used repeatedly in what follows.
Proof (a). Note first that

$$\begin{aligned} \big(\nabla \times B\big)_i &= \epsilon_{iab} \cdot \big(\partial_a B_b\big) \\ &= \epsilon_{iab} \cdot \big[\partial_a A_j, \partial_b A_j\big] - \epsilon_{iab}\big[\partial_a\phi, \partial_b\phi\big]. \end{aligned}$$

The terms on the right hand side are both linear combinations of null forms. The desired inequality follows[34] from Proposition 2.3 in [15].

(b). Going through the terms in G one by one, the argument is the same as in [15], Theorem 6.1 except for the additional term $\big[\phi, [A_i, \phi]\big]$:

$$\begin{aligned} \big\| \big[\phi, [A_i,\phi]\big] \big\|_{L^2(\bigcirc_s)} &\leq C\, \|A\|_{L^6(\bigcirc_s)} \cdot \|\phi\|^2_{L^6(\bigcirc_s)} \\ &\leq C \cdot \big(\mathcal{R}[A,\phi](T)\big)^3 \end{aligned}$$

[33] *See Section 4, or Section 7 for the derivation of this system.*
[34] The fact that $\nabla \times B$ is the sum of *two* null forms changes only slightly the statement and proof of Proposition 2.3 in [15]. One gets two terms of the form "curl Q" in the expression for $\bar{u}$ on page 45.

Hence

$$\left\| [\phi, [A_i, \phi]] \right\|_{L^2(\blacktriangle_T)} \leq T^{\frac{1}{2}} \cdot C \cdot \Big(\mathcal{R}_0[A, \phi] + X(T) \Big)^3$$

(c). Take the terms in $\Box\phi$ one by one.

$\lambda^2 |\phi|^4 \phi$:

$$\begin{aligned} \int_0^T \left\| \lambda^2 |\phi|^4 \phi \right\|_{L^2(\bullet_t)} dt &\leq \lambda^2 \|\phi\|_{L^6} \cdot \int_0^T \|\phi\|^4_{L^{12}(\bullet_t)} dt \\ &\leq C \cdot \lambda^2 \|\phi\|_{L^6} \left(\mathcal{R}_0[\phi] + X(T) \right)^4. \end{aligned}$$

$[\phi, \partial_t A^0]$:

$$\begin{aligned} \left\| [\phi, \partial_t A^0] \right\|_{L^2(\bullet_t)} &\leq C \|\partial_t A_0\|_{L^3(\bullet_t)} \cdot \|\phi\|_{L^6(\bullet_t)} \\ &\leq C \|\partial_t A_0\|_{L^3(\bullet_t)} \cdot \big(\mathcal{R}_0[A, \phi] + X(T) \big) \end{aligned}$$

Hence

$$\begin{aligned} \int_0^T \left\| [\phi, \partial_t A^0] \right\|_{L^2(\bullet_t)} dt &\leq \big(\mathcal{R}_0[A, \phi] + X(T) \big) T^{\frac{1}{2}} \Big(\int_0^T \|\partial_t A_0\|^2_{L^3(\bullet_t)} dt \Big)^{\frac{1}{2}} \\ &\leq C \big(\mathcal{R}_0[A, \phi] + X(T) \big) \cdot Y(T) \\ &\leq C \big(\mathcal{R}_0[A, \phi] + X(T) \big)^2 + Y(T)^2 \end{aligned}$$

$-2[A^0, \partial_t \phi]$:

$$\begin{aligned} \int_0^T \left\| [A^0, \partial_t \phi] \right\|_{L^2(\bullet_t)} dt &\leq \int_0^T \|A_0\|_{L^\infty(\bullet_t)} \cdot \|\partial_t \phi\|_{L^2(\bullet_t)} dt \\ &\leq T^{\frac{1}{2}} \big(\mathcal{R}_0[A, \phi] + X(T) \big) \cdot \Big(\int_0^T \|A_0\|^2_{L^\infty(\bullet_t)} dt \Big)^{\frac{1}{2}} \\ &\leq C \big(\mathcal{R}_0[A, \phi] + X(T) \big)^2 + C Y(T)^2 \end{aligned}$$

$-2[A^i, \partial_i \phi]$:

$$\begin{aligned} \int_0^T \left\| [A^i, \partial_i \phi] \right\|_{L^2(\bullet_t)} dt &\leq T^{\frac{1}{2}} \left\| [A^i, \partial_i \phi] \right\|_{L^2(\blacktriangle_T)} \\ &\leq C \big(\mathcal{R}_0[A, \phi] + X(T) \big)^2 \end{aligned}$$

by Proposition 2.2 in [15].

$[A_\mu, [A^\mu, \phi]]$:

$$
\begin{aligned}
\int_0^T \|[A_\mu,[A^\mu,\phi]]\|_{L^2(\ \ _t)}\,dt &\le \int_0^T \|A\|^2_{L^6(\ \ _t)}\cdot\|\phi\|_{L^6(\ \ _t)}\,dt \\
&\quad + \int_0^T \|A_0\|^2_{L^\infty}\|\phi\|_{L^6(\ \ _t)}\,dt \\
&\le T\big(\mathcal{R}_0[A,\phi]+X(T)\big)^3 \\
&\quad + Y(T)^2\cdot(\mathcal{R}_0[A,\phi]+X(T)) \\
&\le C\left(\mathcal{R}_0[A,\phi]+X(T)^2+X(T)^3+Y(T)^4\right)
\end{aligned}
$$

$[A_0, [A^0, \phi]]$:

$$
\begin{aligned}
\int_0^T \big\|[A_0,[A^0,\phi]]\big\|_{L^2(\ \ _t)}\,dt &\le \int_0^T \|A_0\|^2_{L^6(\ \ _t)}\|\phi\|_{L^6(\ \ _t)}\,dt \\
&\le C\Big(\int_0^T \|A_0\|^2_{H^1(\ \ _t)}\,dt\Big)\big(\mathcal{R}_0[A,\phi]+X(T)\big) \\
&\le C\left(\mathcal{R}_0[A,\phi]+X(T)^2+Y(T)^4\right)
\end{aligned}
$$

■

In Theorem 7.2 we'll prove

$$
Y(T) \le C\big(X(T)^2+Y(T)^2+X(T)^3+Y(T)^3+\mathcal{R}_0\big)+4\int_0^T \|B\|_{L^2(\ \ _t)}\,dt.
$$

We combine this with Theorem 6.1 to bootstrap X and Y together as follows: put $Z(t)=X(t)+Y(t)$, then

$$
\begin{aligned}
Z(T) &\le \lambda^2\,|\!|\!|\phi|\!|\!|_{L^6}\cdot\big(\mathcal{R}_0[\phi]+X(T)\big)^4 \\
&\quad + C\big(Z(t)^2+Z(t)^3+Z(t)^4+\mathcal{R}_0[A,\phi]\big).
\end{aligned}
\tag{6.3}
$$

C here is universal, independent of λ and the solutions under consideration. In the following discussion C can change from line to line, increasing if necessary. It remains, however, a universal constant.

Theorem 6.2. (A_0, A, ϕ) *a solution of (C.G.a)-(C.G.f) on the space-time region* *satisfying one of the following assumptions.*

(A) $\quad |\!|\!|\phi|\!|\!|_{L^6}\cdot(\mathcal{R}_0[A,\phi])^3 \quad\le\quad \lambda^{-2}\epsilon_0$

or

(B) $\quad \mathcal{R}_0[A,\phi] \le \lambda^{-\frac{1}{2}}\epsilon_0$

Where ϵ_0 *is some small universal constant which is determined in the proof.*

Conclusion: For all $0\le t\le\frac{1}{2}$ *the solution enjoys*

$$
Z(t) \quad\le\quad C\mathcal{R}_0[A,\phi]
$$

Proof At each time t, (6.3) shows that one of the following must be true

(1) $Z(t)\le C\lambda^2\,|\!|\!|\phi|\!|\!|_{L^6}\left(\mathcal{R}_0[A,\phi]+X(T)\right)^4$
(2) $Z(t)\le CZ(t)^2$

(3) $Z(t) \leq CZ(t)^3$
(4) $Z(t) \leq CZ(t)^4$
(5) $Z(t) \leq C\mathcal{R}_0[A, \phi]$

Suppose first that assumption (A) in the Theorem holds. We then split case (1) above into

$$Z(t) \leq C\lambda^2 \, \|\phi\|_{L^6} \cdot \mathcal{R}_0^4 \tag{1a}$$

$$Z(t) \leq C\lambda^2 \, \|\phi\|_{L^6} \, Z(t)^4 \tag{1b}$$

Case (1a) gives a bound for $Z(t)$ in terms of

$$\begin{aligned} C\lambda^2 \, \|\phi\|_{L^6} \cdot \mathcal{R}_0^4 &\leq C\lambda^2 \lambda^{-2} \epsilon_0 \mathcal{R}_0 \\ &= C\epsilon_0 \mathcal{R}_0[A, \phi] \end{aligned}$$

as does case (5). Note that $Z(t)$ is continuous and $Z(0) \leq C\mathcal{R}_0$ (See Proposition 7.1). We aim to conclude that in fact only (1a) and (5) are possible for $t \in [0, T]$, and eliminate (1b) - (4) by showing that each forces a discontinuity in the function $Z(t)$.

Cases (2)-(4) each imply $\frac{1}{5C} \leq Z(t)$. We eliminate this possibility by making our choice of

λ bigger if necessary so that $C\mathcal{R}_0 < \frac{1}{5C}$. (Recall $\mathcal{R}_0$ scales as $\lambda^{-\frac{1}{2}}$.) It remains to exclude (1b).

Case (1b) implies

$$Z(t) \geq \frac{1}{\left(C\lambda^2 \, \|\phi\|_{L^6}\right)^{\frac{1}{3}}}$$

which would be prohibited by the inequality

$$C\mathcal{R}_0[A, \phi] < \frac{1}{\left(C\lambda^2 \, \|\phi\|_{L^6}\right)^{\frac{1}{3}}}$$

or equivalently,

$$C^{\frac{4}{3}} \mathcal{R}_0[A, \phi] \lambda^{\frac{2}{3}} \, \|\phi\|_{L^6}^{\frac{1}{3}} < 1$$

which says

$$C^4 (\mathcal{R}_0[A, \phi])^3 \lambda^2 \, \|\phi\|_{L^6} < 1$$

which is assumption (A) if $\epsilon_0 < \frac{1}{C^4}$.

Suppose now that (B) holds. Then apply Sobolev embedding and the energy inequality to the factor $\|\phi\|_{L^6}$ on the right hand side of case (1) to get

$$Z(t) \leq C\lambda^2 \big(\mathcal{R}_0[A, \phi] + Z(t)\big)^5. \tag{1}$$

We again split this into two cases

$$Z(t) \leq C\lambda^2 \big(\mathcal{R}_0[A, \phi]\big)^5 \tag{1a}$$

$$Z(t) \leq C\lambda^2 \big(Z(t)\big)^5 \tag{1b}$$

Cases (2)-(4) are excluded as before. Case (1b) has

$$\frac{1}{C^{\frac{1}{4}}\lambda^{\frac{1}{2}}} \leq Z(t).$$

The impossibility of this would follow from

$$C\lambda^2\left(\mathcal{R}_0[A,\phi]\right)^5 \quad < \quad \frac{1}{C^{\frac{1}{4}}\lambda^{\frac{1}{2}}}$$

equivalently

$$(\mathcal{R}_0[A,\phi])^5\lambda^{\frac{5}{2}}C^{\frac{5}{4}} \quad < \quad 1$$

equivalently

$$\mathcal{R}_0[A,\phi] \quad < \lambda^{-\frac{1}{2}}\frac{1}{C^{\frac{1}{4}}}$$

which follows from assumption (B) if $\epsilon_0 < \frac{1}{C^{\frac{1}{4}}}$. ∎

The estimates for differences between two solutions follow from the same arguments.

Theorem 6.3. *$(A_0, A, \phi), (A_0', A', \phi')$ two solutions of (C.G.a) - (C.G.b) on . Assume also that the solutions both satisfy either (A) or (B) from Theorem(6.2).*

Then there is a constant C and a constant

$$K = O\big(X(T) + Y(T) + X'(T) + Y'(T) + \mathcal{R}_0 + \mathcal{R}_0'\big)$$

for small values of $X, Y, X', Y', \mathcal{R}_0, \mathcal{R}_0'$ such that

(a)
$$\begin{aligned}\|B - B'\|_{L^2(\,)} &\leq C\,\|G - G'\|_{L^2(\,)} \\ &\quad + K\cdot\big(\mathcal{R}_0[A - A', \phi - \phi'] + X(T, A - A', \phi - \phi')\big)\end{aligned}$$

(b)
$$\begin{aligned}\|G - G'\|_{L^2(\,)} &\leq K\cdot\big(\mathcal{R}_0[A - A', \phi - \phi'] + X(T, A - A', \phi - \phi') \\ &\quad + Y(T, A_0 - A_0')\big)\end{aligned}$$

(c)
$$\begin{aligned}\int_0^T \|\Box(\phi - \phi')\|_{L^2(\,_t)}\,dt &\leq C\lambda^2\,\|\phi - \phi'\|_{L^6}\cdot\left(\|\phi\|^4_{L^{12}_x L^4_t} + \|\phi'\|^4_{L^{12}_x L^4_t}\right) \\ &\quad + K\cdot\big(\mathcal{R}_0[A - A', \phi - \phi'] + X(T, A - A', \phi - \phi') \\ &\quad + Y(T, A_0 - A_0')\big)\end{aligned}$$

The estimate we get in Theorem 7.5 for $A_0 - A_0'$ is

$$Y(A_0 - A_0')(t) \leq C\big(Y(A_0 - A_0')(t) + X(A - A', \phi - \phi')(t) + \mathcal{R}_0[A - A', \phi - \phi']\big).$$

for a small constant C. We use this to prove a bound for

$$Z(A - A', A_0 - A_0', \phi - \phi')(t) \equiv X(A - A', \phi - \phi')(t) + Y(A_0 - A_0')(t)$$

Corollary 6.4. *$(A_0, A, \phi), (A_0', A', \phi')$ two solutions of (C.G.a) - (C.G.f) on with $\|(A_i - A_i', \phi - \phi')\|_{H^1(\bullet_0)} \leq \epsilon$ for some small, universal constant ϵ. Suppose also that one of the following is true*

(A). $\|\phi\|_{L^6} \cdot (\mathcal{R}_0[A, \phi])^3 \leq \lambda^{-2}\epsilon_0$.
Similarly for (A_0', A', ϕ').

(B). $\mathcal{R}_0[A, \phi] \leq \lambda^{-\frac{1}{2}}\epsilon_0$
Similarly for (A_0', A', ϕ').

Where ϵ_0 is the small universal constant found in the proof of Theorem 6.2.

Conclusion: For all $0 \leq t \leq \frac{1}{2}$,

$$Z(A - A', A_0 - A_0', \phi - \phi')(t) \quad \leq \quad C\big(\mathcal{R}_0[A - A', \phi - \phi']\big)$$

Where C is a constant depending only on $\|(A_i, \phi)\|_{H^1(\bullet_0)}$, $\|(A_i', \phi')\|_{H^1(\bullet_0)}$.

7. Estimates for A_0 in a local Coulomb gauge

The strongest estimates for the potential A_0, A_i are proven in a normalized local Coulomb gauge. This means the gauge potential satisfies

$$\begin{aligned} \partial_i A^i &= 0 \quad \text{on} \quad \bullet_0 \\ x \cdot A &= 0 \quad \text{on} \quad \bigcirc_0 \end{aligned}$$

A local Coulomb gauge imposes an elliptic equation on A_0, and therefore includes a boundary condition on A_0. The boundary condition used by Klainerman-Machedon [15] is motivated by the necessary *dynamic* estimates, is shown to be natural with regard to the equations for A_0, and leads to a surprisingly nice integral equation for A_0. We parrot their argument now for Yang-Mills-Higgs, and show that additional, surprising cancellations involving the Higgs field ϕ occur when we derive the integral-differential equations which govern the fields.

Recall from Section 5 that the Yang-Mills-Higgs equations are scaled in order to ensure a Coulomb Gauge initial data set exists. Our field equations are therefore

$$D^\beta F_{\alpha\beta} = -[\phi, D_\alpha \phi] \tag{7.1}$$

$$D_\mu D^\mu \phi = \lambda^2 |\phi|^4 \phi \tag{7.2}$$

where λ is a fixed, positive constant.

7.1. The integral-differential equations. Write (7.1)

$$\partial^\beta F_{\alpha\beta} = J_\alpha \tag{7.3}$$

where the current

$$J_\alpha = \left[F_{\alpha\beta}, A^\beta\right] - [\phi, D_\alpha \phi] \tag{7.4}$$

satisfies $\partial^\alpha J_\alpha = 0$ by the antisymmetry of F. Setting $\alpha = 0$ in (7.3) gives

$$\Delta A_0 = -J_0 + \partial^i [A_0, A_i] \tag{7.5}$$

while $\alpha = i$ yields

$$\Box A_i = -\partial_0 \partial_i A_0 - J_i + \left[\partial^j A_i, A_j\right] + \partial^0 \left[A_i, A_0\right]. \tag{7.6}$$

Equations (7.5) - (7.6) are the Coulomb gauge equations for the potential A_α. In order to apply local versions of the nullform estimates of [13] to the dynamic variables in Section 6, one needs to impose the the variational condition $(\Box A)\cdot\eta = 0$ on (7.6). This can be viewed as the Neumann boundary condition

$$\nabla_\eta \left(\partial_t A_0\right) = -J_\eta + \left[\partial_j A_\eta, A_j\right] - \partial_t \left[A_\eta, A_0\right] \tag{7.7}$$

on $\partial_t A_0$. The compatibility of (7.5), (7.7) is shown using $\partial^\alpha J_\alpha = 0$:

$$\begin{aligned} \Delta \partial_t A_0 &= -\partial_t J_0 + \partial_t \partial^j \left[A_0, A_j\right] \\ &= -\partial^i \Big(J_i - \partial_t \left[A_0, A_i\right] \Big). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\bullet_t} \Delta \partial_t A_0 dx &= \int_{\bullet_t} -\partial^i \Big(J_i - \partial_t \left[A_0, A_i\right] \Big) dx \\ &= \int_{\bigcirc_t} -J_\eta + \partial_t \left[A_0, A_\eta\right] d\sigma \\ &= \int_{\bigcirc_t} -J_\eta + \partial_t \left[A_0, A_\eta\right] - \underbrace{\partial^j \left[A_\eta, A_j\right]}_{\text{Tangential derivatives, integrate to 0}} d\sigma \\ &= \int_{\bigcirc_t} \nabla_\eta \partial_t A_0 d\sigma \end{aligned}$$

by (7.7).

The integral equations for A_0 come from Green's formula: given two smooth $\mathfrak{g}$-valued functions f, g on $\bullet_t$,

$$\int_{\bigcirc_t} \left(\nabla_\eta f\right) g - f\left(\nabla_\eta g\right) d\sigma = \int_{\bullet_t} \left(\Delta f\right) g - f\left(\Delta g\right) dx. \tag{7.8}$$

Put $g = \partial_t A_0$ and $f = G_t(x, y)$, the Neumann Green's function for the ball $\bullet_t \subset \mathbb{R}^3$ of radius a. By definition,

$$\begin{aligned} \Delta_x G_t(x, y) &= \delta(x - y) \quad x, y \in \bullet_t \\ \nabla_{\eta(x)} G_t(x, y) &= \frac{1}{4\pi a^2} \quad x \in \bigcirc_t,\ y \in \bullet_t \end{aligned} \tag{7.9}$$

If we make the assumption

$$\int_{\bigcirc_t} \partial_t A_0 d\sigma = 0 \tag{7.10}$$

part of our Neumann problem for $\partial_t A_0$ then (7.8) reads

(7.11)
$$\partial_t A_0(t, x) = \int_{\bullet_t} G_t(x, y) \Delta \left(\partial_t A_0(t, y)\right) dy - \int_{\bigcirc_t} G_t(x, y) \nabla_\eta \partial_t A_0(t, y) d\sigma_y$$

The problem with (7.11) is the boundary integral: it contains derivatives normal to the cone[35] through the presence of the normal component of the current. (See (7.7), (7.4).) Exactly as in [15][36] we obtain a cancellation by using the equation for $\Delta \partial_t A_0$ and integrating by parts. The result is

$$\begin{aligned} \partial_t A_0(t,x) = &\int_{\bullet_t} \partial^i G_t(x,y)\big(J_i - \partial_t\left[A_0, A_i\right]\big)\, dy \\ &+ \int_{\bigcirc_t} G_t(x,y)\left[A_j, \partial^j A_\eta\right] d\sigma_y \end{aligned} \tag{7.12}$$

which contains only tangential derivatives in the boundary integral. But as in Klainerman-Machedon's treatment of Yang-Mills, the Yang-Mills-Higgs equations allow for even more cancellation! We show this now:

Write

$$\begin{aligned} M &= \Delta A_0(t,x) \\ &= -J_0 + \partial^i\left[A_0, A_i\right] \\ &= -\left[F_{0i}, A^i\right] + \left[\phi, D_0\phi\right] + \partial^i\left[A_0, A_i\right] \\ &= \left[A^i, \partial_0 A_i\right] + 2\partial^i\left[A_0, A^i\right] + \left[A^i, \left[A_0, A_i\right]\right] \\ &\quad + \left[\phi, D_0\phi\right] \end{aligned} \tag{7.13}$$

and we'll write

$$M = M^\dagger + \left[\phi, D_0\phi\right]$$

where

$$M^\dagger = \left[A^i, \partial_0 A_i\right] + 2\partial^i\left[A_0, A^i\right] + \left[A^i, \left[A_0, A_i\right]\right]$$

is the quantity appearing in the uncoupled Yang-Mills theory of [15]. The deeper cancellation comes from looking at the right hand side of (7.11) and writing

$$\begin{aligned} \int_{\bullet_t} G_t(x,y)\partial_t M(t,y)dy = \;&\partial_t\Big(\int_{\bullet_t} G_t(x,y)M(t,y)dy\Big) - \int_{\bullet_t} \partial_t G_t(x,y)M(t,y)dy \\ &+ \int_{\bigcirc_t} G_t(x,y)M(y)d\sigma(y). \end{aligned}$$

Hence

$$\begin{aligned} \partial_t A_0(t,x) = \partial_t &\int_{\bullet_t} G_t(x,y)M(y)dy - \int_{\bullet_t} \partial_s G_s(x,y)M(y)dy \\ &+ \int_{\bigcirc_t} G_t(x,y)\big(M - \nabla_\eta \partial_t A_0\big)d\sigma. \end{aligned} \tag{7.14}$$

Setting $N = M - \nabla_\eta \partial_t A_0$, (7.13), (7.7) give

$$\begin{aligned} N = \;&\left[A^i, \partial_0 A_i\right] + 2\left[\partial^i A_0, A_i\right] + \left[A^i, \left[A_0, A_i\right]\right] \\ &+ \left[\phi, D_0\phi\right] + J_\eta - \left[\partial_j A_\eta, A^j\right] + \partial_t\left[A_\eta, A_0\right]. \end{aligned}$$

[35] Using energy estimates, we can estimate only *tangential* derivatives.

[36] See (1.17a) in the introduction of that paper.

From (7.4), we have,[37]

$$\begin{aligned} J_\eta &= \eta \cdot J \\ &= [\nabla_\eta A_j, A_j] - [\partial_j A_\eta, A_j] + [[A_\eta, A_j], A_j] \\ &\quad - [\phi, D_\eta \phi] + [\nabla_\eta A_0, A_0] - \left[\partial_0 A_\eta, A^0\right] + \left[[A_\eta, A_0], A^0\right]. \end{aligned}$$

Hence

$$\begin{aligned} N &= [\phi, (D_0\phi - D_\eta \phi)] + [A_i, (\partial_0 - \partial_\eta) A_i] + 2[A_j, \partial_j A_\eta] \\ &\quad + 2[\partial_i A_0, A_i] + [A_\eta, \partial_t A_o] - 2[A_0, \partial_t A_\eta] \\ &\quad + [A_0, \partial_\eta A_0] + [A_j, [A_j, A_\eta]] + [A_i, [A_0, A_i]] + [A_0, [A_\eta, A_0]] \end{aligned} \tag{7.15}$$

Let us write $N^\dagger$ for those terms in 7.15 which do not involve the Higgs field ϕ. We record here that

$$N = N^\dagger + [\phi, (\partial_0 - \partial_\eta)\phi] + [\phi, [A_0, \phi]] - [\phi, [A_\eta, \phi]]. \tag{7.16}$$

Integrating (7.14) in time gives the integral equation for A_0

$$\begin{aligned} A_0 &= \int_{\bullet_t} G_t(x,y) M(t,y) dy - \int_{\blacktriangle} \partial_s G_s(x,y) M(s,y) dy ds \\ &\quad + \int_{/\ \backslash^t} G_s(x,y) N(s,y) d\sigma(y) ds. \end{aligned} \tag{7.17}$$

Two remarks are in order here. First, note that only the derivative $\partial_0 - \partial_\eta$ tangential to $/\ \backslash^t$ is applied to ϕ in equation (7.16). This will be a blessing when it comes time to estimate the boundary integral on the right of (7.17).

Second, note that in writing (7.17), we assume that A_0 is given at time zero by the equation

$$A_0(0,x) = \int_{\bullet_0} G(x,y) M(0,y) dy. \tag{7.18}$$

By Lemma 8.1 in the Appendix, this is equivalent to solving the following Neumann problem for A_0 at time zero

$$\begin{aligned} \Delta A_0(0,x) &= M(0,x) \\ \nabla_\eta A_0 \big|_{\bigcirc_0} &= \frac{1}{4\pi} \int_{\bullet_0} M dy \\ \int_{\bigcirc_0} A_0 &= -2 \int_{\bullet_0} M dy \end{aligned}$$

which is straightforward if the right hand side of the boundary conditions can be written in terms of the Cauchy data. We show now that this is possible:

$$\int_{\bullet_0} M dy = \int_{\bullet_0} -J_0 + \partial^i [A_0, A_i] dx. \tag{7.19}$$

[37]When writing down J_η, use $\eta \cdot \left[\partial_j A_i, A^j\right] = \left[\partial_j A_\eta, A^j\right]$.

But the Cauchy data satisfy the compatibility condition

$$\partial^i \bar{E}_i + [\bar{A}_i, \bar{E}_i] = -\left[\bar{\phi}, \overline{D_0\phi}\right].$$

Hence in (7.19)

$$\begin{aligned} -J_0 &= -\left[\bar{E}_i, \bar{A}^i\right] + [\bar{\phi}, \overline{D_0\phi}] \\ &= -\partial^i \bar{E}_i. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\bullet_0} M\,dy &= \int_{\bullet_0} \partial^i\left(-\bar{E}_i + [A_0, \bar{A}_i]\right) \\ &= -\int_{\bigcirc_0} \bar{E}_\eta \, d\sigma \end{aligned}$$

because the data belongs to the normalized Coulomb gauge wherin $\partial_i \bar{A}^i = 0$.

In summary, the equations for (A_0, A_i, ϕ) in what we will term the *localized Coulomb gauge* are the following:

Expanding the right hand side of (7.6) gives the following wave equations for A_i:

(C.G.a)
$$\begin{aligned} \Box A_i &= B_i + G_i \\ B_i &= -\partial_t\partial_i A_0 + [A_j, \partial_i A_j] + [\phi, \partial_i\phi] \\ G_i &= -2[A_j, \partial_j A_i] - [\phi, [\phi, A_i]] + [\partial_t A_0, A_i] \\ &\quad + 2[A_0, \partial_t A_i] + [\partial_i A_0, A_0] + [A_0, [A_0, A_i]] \\ &\quad - [A^j, [A_j, A_i]] \end{aligned}$$

The equation for ϕ comes directly from (7.2) which can be written

(C.G.b)
$$\begin{aligned} \Box\phi &= [\phi, \partial_t A^0] - 2[A^\mu, \partial_\mu\phi] \\ &\quad - [A_\mu, [A^\mu, \phi]] + \lambda^2 |\phi|^4 \phi. \end{aligned}$$

The initial conditions for the dynamic variables A_i, ϕ are

$$\begin{aligned} A_i(0,\cdot) &= \bar{A}_i & \phi(0,\cdot) &= \bar{\phi}(\cdot) \\ \partial_t A_i(0,\cdot) &= \bar{E}_i + \partial_i \bar{A}_0 + [\bar{A}_i, \bar{A}_0] & \partial_t\phi(0,\cdot) &= \overline{D_0\phi} - [\bar{A}_0, \bar{\phi}] \end{aligned}$$

where $\bar{A}_0$ is the solution of A_0 at time zero given below. The number λ appearing in (C.G.b) should be thought of as an arbitrary constant. The data must satisfy the following compatibility condition

(C.G.c)
$$\partial^i \bar{E}_i = -[\bar{\phi}, \overline{D_0\phi}] - \bar{A}\bar{A}_i, \bar{E}_i].$$

The equation for A_0 at time zero is

$$A_0(0,x) = \int_{\bullet_0} G_0(x,y)M(y)dy \tag{C.G.d}$$

that is

$$\Delta \bar{A}_0 = M$$

$$\nabla_\eta \bar{A}_0\big|_{\bigcirc_0} = \frac{1}{4\pi}\int_{\bullet_0} M dy$$

$$\int_{\bigcirc_0} \bar{A}_0 d\sigma = -2\int_{\bullet_0} M dy$$

where

$$\int_{\bullet_0} M dy = -\int_{\bigcirc_0} \bar{E}_\eta d\sigma.$$

The equations for A_0 at later times are

$$\begin{aligned} A_0(t,x) = &\int_{\bullet_t} G_t(x,y)M(t,y)dy - \int_{\blacktriangle_t} \partial_s G_s(x,y)M(s,y)dyds \\ &+ \int_{/\ \backslash_t} G_s(x,y)N(s,y)d\sigma(x)ds \end{aligned} \tag{C.G.e}$$

where

$$\begin{aligned} M &= M^\dagger + [\phi, D_0\phi] \\ &= 2[\partial^i A_0, A_i] + [A_i, \partial_0 A^i] + [A_i,[A_0,A^i]] + [\phi, D_0\phi] \\ N &= N^\dagger + [\phi,(\partial_0-\partial_\eta)\phi] + [\phi,[A_0,\phi]] - [\phi,[A_\eta,\phi]] \\ &= [A_i,(\partial_t-\partial_\eta)A^i] + 2[A_j,\nabla^j A_\eta] + 2[\partial_i A_0, A^i] \\ &\quad - [\partial_t A_0, A_\eta] - 2[A_0,\partial_t A_\eta] + [A_0,\partial_\eta A_0] + [A_j,[A_j,A_\eta]] \\ &\quad - [A_0,[A_0,A_\eta]] + [\phi,(\partial_t-\partial_\eta)\phi] + [\phi,[A_0,\phi]] \end{aligned}$$

and for the time derivative,

$$\begin{aligned} \partial_t A_0(t,x) = &\int_{\bullet_t} \partial_i G_t(x,y)\big(J_i - \partial_t[A_0,A_i]\big)dt \\ &+ \int_{\bigcirc_t} G_t(x,y)[A_j,\partial_j A_\eta]\,d\sigma_y \end{aligned} \tag{C.G.f}$$

7.2. Estimates for A_0. The aim of this section is a Bootstrap-type estimate for the following combination of norms of A_0,

$$\begin{aligned} Y(T) \;=\; &\Big(\int_0^T \|\partial_t A_0\|^2_{L^3(\bullet_t)}dt\Big)^{\frac12} + \Big(\int_0^T \|A_0\|^2_{L^\infty(\bullet_t)}dt\Big)^{\frac12} + \\ &\Big(\int_0^T \|\nabla A_0\|^2_{L^3(\bullet_t)}dt\Big)^{\frac12} + \sup_{0\le t\le T}\|\partial_i A_0\|_{L^2(\bullet_t)} + \sup_{0\le t\le T}\|A_0\|_{L^3(\bullet_t)} + \|A_0\|_{L^3(/\ \backslash^T)} \\ &+ \|\partial_t A_0\|_{L^2(/\ \backslash^T)} + \|\partial_i A_0\|_{L^2(/\ \backslash^T)} \end{aligned} \tag{7.20}$$

The first result bounds A_0 at time zero in terms of the reduced energy norm of the dynamic data, which is defined to be

(7.21)

$$\mathcal{R}_0[A,\phi] \equiv \big(\|A(0,\cdot)\|^2_{L^2(\ _0)} + \|\phi\|^2_{L^2(\ _0)} + \|\partial A(0,\cdot)\|^2_{L^2(\ _0)} + \|\partial\phi(0,\cdot)\|^2_{L^2(\ _0)} \big)^{\frac{1}{2}}.$$

Proposition 7.1. *Suppose the initial data $(\bar{A}_i, \bar{\phi})$ lie in the Coulomb gauge and satisfy*

$$\|\overline{A}\|_{L^3(\ _0)}, \|\overline{\phi}\|_{L^3(\ _0)} \le \epsilon$$

for some universal constant ϵ. Let (A_0, A, ϕ) be an evolution of the data under the system (C.G.a)-(C.G.f).

Then

$$\|A_0(0,\cdot)\|_{L^2(\ _0)} + \|\nabla A_0(0,\cdot)\|_{L^2(\ _0)} \quad \le \quad C\mathcal{R}_0[A,\phi] \tag{7.22}$$

Proof

At time zero we have

$$A_0(0,x) \quad = \quad \int_{\ _0} G_0(x,y)M(y)dy$$

so the elliptic estimate in Lemma 8.4 gives

$$\|A_0(0,\cdot)\|_{L^2(\ _0)} + \|\nabla A_0(0,\cdot)\|_{L^2(\ _0)} \quad \le \quad C\,\|M(0,\cdot)\|_{L^{\frac{6}{5}}(\ _0)}. \tag{7.23}$$

But using the definition of M (see for example C.G.e)

$$\begin{aligned}
\|M(0,\cdot)\|_{L^{\frac{6}{5}}(\ _0)} \quad &\le \quad \big\|[A^i, \partial_0 A_i]\big\|_{L^{\frac{6}{5}}(\ _0)} + 2\,\big\|[\partial^i A_0, A_i]\big\|_{L^{\frac{6}{5}}(\ _0)} \\
&+ \quad \big\|[A^i,[A_0,A_i]]\big\|_{L^{\frac{6}{5}}(\ _0)} + \|[\phi, D_0\phi]\|_{L^{\frac{6}{5}}(\ _0)} \\
&\le \quad C\,\|A\|_{L^3(\ _0)} \big(\|\nabla A_0\|_{L^2(\ _0)} + \|\partial_t A\|_{L^2(\ _0)} \\
&+ \quad \|A\|_{L^3(\ _0)} \|A_0\|_{L^6(\ _0)}\big) + C\,\|\phi\|_{L^3(\ _0)} \|D_0\phi\|_{L^2(\ _0)} \\
&\le \quad C\epsilon\big(\|\nabla A_0\|_{L^2(\ _0)} + \|A_0\|_{L^6(\ _0)} + \mathcal{R}_0\big).
\end{aligned}$$

Since $\|A_0\|_{L^6(\ _0)} \le C\big(\|\nabla A_0\|_{L^2(\ _0)} + \|A_0\|_{L^2(\ _0)}\big)$, we conclude

$$\|M(0,\cdot)\|_{L^{\frac{6}{5}}(\ _0)} \quad \le \quad C\epsilon\big(\|\nabla A_0\|_{L^2(\ _0)} + \|A_0\|_{L^2(\ _0)} + \mathcal{R}_0\big)$$

which, together with (7.23) and the assumptions , gives the result. ∎

The bootstrap argument for $Y(T)$ is contained in the following

Theorem 7.2. *Let (A_0, A, ϕ) be a solution of the system (C.G.a)-(C.G.f) in a standard space-time domain $\ ^T$ with $T \le \frac{1}{2}$ and $\mathcal{R}_0[A,\phi] \le \epsilon$ for a small universal constant ϵ.*

Then for all $0 \le T \le \frac{1}{2}$,

(7.24)

$$\begin{aligned}
&Y(T) \le C\big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + X(T)^4 + Y(T)^4 + \mathcal{R}_0\big) \\
&\qquad + C\big(\|B\|_{L^2(\ ^T)}\big)
\end{aligned}$$

Remarks: Many of terms in (C.G.a) - (C.G.f) are present already in the context of the uncoupled Yang-Mills equations, which is why we write[38]

$$M = M^\dagger + [\phi, D_0\phi] \tag{1.a}$$

$$N = N^\dagger + [\phi, (\partial_0 - \partial_\eta)\phi] + [\phi, [A_0, \phi]] - [\phi, [A_\eta, \phi]] \tag{1.b}$$

$$J_\alpha = J^\dagger_\alpha - [\phi, D_\alpha\phi] \tag{1.c}$$

In what follows, we also use the energy estimate

$$\mathcal{R}[A, \phi](t) \leq C\big(\mathcal{R}_0 + X(t)\big)$$

Recall that $\mathcal{R}_0 = \mathcal{R}_0[A, \phi](t) \leq \epsilon$ and

$$\begin{aligned} \mathcal{R}[A, \phi](t) \ = \ &\big(\|\partial A(t, \cdot\|^2_{L^2(\bullet_t)} + \|A(t, \cdot\|^2_{L^2(\bullet_t)} \|\partial\phi\|^2_{L^2(\bullet_t)} \\ &+ \|\phi\|^2_{L^2(\bullet_t)} + \int_0^t \int_{\bigcirc_s} \big|\overline{\partial} A(s, y)\big|^2 + \big|\overline{\partial}\phi(s, y)\big|^2 \, d\sigma_y ds\big)^{\frac{1}{2}}. \end{aligned} \tag{7.25}$$

Here $\overline{\partial}$ denotes derivatives tangent to / \.

Proof Consider the terms of $Y(T)$ one by one.

First term of Y: $\big(\int_0^T \|\partial_t A_0\|^2_{L^3(\bullet_t)} \, dt\big)^{\frac{1}{2}}$

Equation (C.G.f) reads

$$\begin{aligned} \partial_t A_0(t, x) &= \int_{\bullet_t} \partial_i G_t(x, y)\big(J_i - \partial_t [A_0, A_i]\big) \, dt + \int_{\bigcirc_t} G_t(x, y) \, [A_j, \partial_j A_\eta] \, d\sigma_y \\ &= \int_{\bullet_t} \partial_i G_t(x, y)\big(J_i^\dagger - \partial_t [A_0, A_i]\big) \, dt + \int_{\bigcirc_t} G_t(x, y) \, [A_j, \partial_j A_\eta] \, d\sigma_y \\ &\quad - \int_{\bullet_t} \partial_i G_t(x, y) \, [\phi, D_i\phi] \, dy \\ &= \big(\partial_t A_0\big)^\dagger - \int_{\bullet_t} \partial_i G_t(x, y) \, [\phi, D_i\phi] \, dy \end{aligned} \tag{7.26}$$

say. Arguing *word for word* as [15] pages 90-94, one shows

$$\Big(\int_0^T \Big\|(\partial_t A_0)^\dagger\Big\|^2_{L^3(\bullet_t)} \, dt\Big)^{\frac{1}{2}} \leq C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3\big). \tag{7.27}$$

[38]The terms marked with a † have already been estimated in [15].

To estimate the second term in (7.26) use Lemma 8.4

(7.28)
$$\begin{aligned}\left\| \int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy \right\|_{L^3(\bullet_t)} &\le C\,\|[\phi, D_i\phi]\|_{L^{\frac{3}{2}}(\bullet_t)} \\ &\le C\,\|\phi\|_{L^6(\bullet_t)}\,\|D_i\phi\|_{L^2(\bullet_t)} \\ &\le C\mathcal{R}(t)\cdot\|\partial_i\phi + [A_i,\phi]\|_{L^2(\bullet_t)} \\ &\le C\big(\mathcal{R}(t)^2 + \mathcal{R}(t)^3\big) \\ &\le C\big(\mathcal{R}_0 + X(t)^2 + X(t)^3\big)\end{aligned}$$

Putting together (7.28) and (7.27) gives

(7.29)
$$\Big(\int_0^T \|\partial_t A_0\|^2_{L^3(\bullet_t)}\,dt\Big)^{\frac{1}{2}} \le C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3\big)$$

Second Term of $Y(T)$: $\big(\int_0^T \|A_0\|^2_{L^\infty(\bullet_t)}\,dt\big)^{\frac{1}{2}}$

Write equation (C.G.e) as

$$\begin{aligned}A_0(t,x) = &\int_{\bullet_t} G_t(x,y)M^\dagger(t,y)dy - \int_{\blacktriangle_t} \partial_s G_s(x,y)M^\dagger(s,y)dyds \\ &+ \int_{/\ \backslash_t} G_s(x,y)N^\dagger(s,y)d\sigma(x)ds \\ &+ \int_{\bullet_t} G_t(x,y)\,[\phi, D_0\phi]\,dy - \int_{\blacktriangle_t} \partial_s G_s(x,y)\,[\phi, D_0\phi]\,dyds \\ &+ \int_{/\ \backslash_t} G_s(x,y)\big([\phi,(\partial_0-\partial_\eta)\phi] + [\phi,[A_0,\phi]] - [\phi,[A_\eta,\phi]]\big)d\sigma(x)ds\end{aligned}$$

(7.30)
$$\begin{aligned}&= {A_0}^\dagger(t,x) + \int_{\bullet_t} G_t(x,y)\,[\phi, D_0\phi]\,dy - \int_{\blacktriangle_t} \partial_s G_s(x,y)\,[\phi, D_0\phi]\,dyds \\ &\quad+ \int_{/\ \backslash_t} G_s(x,y)\big([\phi,(\partial_0-\partial_\eta)\phi] + [\phi,[A_0,\phi]] - [\phi,[A_\eta,\phi]]\big)d\sigma(y)ds.\end{aligned}$$

Just as in [15], one finds

(7.31)
$$\Big(\int_0^T \left\|A_0^\dagger\right\|^2_{L^\infty(\bullet_t)}\,dt\Big)^{\frac{1}{2}} \le C\big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + \mathcal{R}_0\big).$$

The second and fourth terms on the right hand side of (7.30) can be estimated in L^∞ at each fixed time as follows.

(7.32)
$$\left|\int_{\bullet_t} G_t(x,y)\,[\phi, D_0\phi]\,dy\right| \le C\Big(\int_{\bullet_t} |G_t(x,y)\phi(t,y)|^2\,dy\Big)^{\frac{1}{2}}\cdot\|D_0\phi\|_{L^2(\bullet_t)}.$$

Lemma 8.4 gives

$$\|G_t(x,\cdot)\phi(t,\cdot)\|_{L^2(\bullet_t)} \le C\mathcal{R}(t)$$

while

$$\|D_0\phi\|_{L^2(\bullet_t)} \le \mathcal{R}(t) + \mathcal{R}(t)Y(t).$$

Hence (7.32) can be written

$$\left|\int_{\bullet_t} G_t(x,y)\,[\phi, D_0\phi]\,dy\right| \le C\mathcal{R}(t)\big(\mathcal{R}(t)+\mathcal{R}(t)Y(t)\big). \tag{7.33}$$

We next bound the fourth term in (7.30). By Hölder's inequality and Lemma 8.4 part (d),

$$\begin{aligned} &\left|\int_{/\ \ \backslash^t} G_s(x,y)\,[\phi,(\partial_0-\partial_\eta)\phi]]\big)\,d\sigma_y ds\right| \\ &\le \left(\int_{/\ \ \backslash^t} |G_s(x,y)\phi(s,y)|^2\,d\sigma_y ds\right)^{\frac12} \left\|\bar\partial\phi\right\|_{L^2(/\ \ \backslash^t)} \\ &\le C\mathcal{R}(t)^2 \end{aligned} \tag{7.34}$$

and by the same reasoning

$$\begin{aligned} &\left|\int_{/\ \ \backslash^t} G_s(x,y)\,[\phi,[A_0,\phi]]\,d\sigma_y ds\right| \\ &\le \left(\int_{/\ \ \backslash^t} |G_s(x,y)A_0(s,y)|^2\,d\sigma_y ds\right)^{\frac12} \left\|\phi^2\right\|_{L^2(/\ \ \backslash^t)} \\ &\le CY(t)\mathcal{R}(t)^2, \end{aligned} \tag{7.35}$$

similarly for $\left|\int_{/\ \ \backslash^t} G_s(x,y)\,[\phi,[A_\eta,\phi]]\,d\sigma_y ds\right|$. Together (7.34) and (7.35) give

$$\begin{aligned} &\left|\int_{/\ \ \backslash^t} G_s(x,y)\big([\phi,(\partial_0-\partial_\eta)\phi]] + [\phi,[A_0-A_\eta,\phi]]\big)\,d\sigma_y ds\right| \\ &\qquad \le C\big(Y(t)^2+\mathcal{R}(t)^2+\mathcal{R}(t)^4\big). \end{aligned} \tag{7.36}$$

The third term on the right hand side of (7.30) has to be estimated directly in space-time. That is, we write this troublesome term as

$$\tilde{A}_0(t,x) = \int_{\blacktriangle_t} \partial_s G_s(x,y)\,[\phi, D_0\phi]\,dyds \tag{7.37}$$

try to bound $\big(\int_0^T \big\|\tilde{A}_0\big\|^2_{L^\infty(\bullet_t)} dt\big)^{\frac{1}{2}}$. By Sobolev's inequality,

$$\int_0^T \big\|\tilde{A}_0\big\|^2_{L^\infty(\bullet_t)} dt\big)^{\frac{1}{2}} \le C\big(\int_0^T \big\|\nabla_x \tilde{A}_0\big\|^2_{L^{3+\epsilon}(\bullet_t)} dt\big)^{\frac{1}{2}} + C\big(\int_0^T \big\|\tilde{A}_0\big\|^2_{L^{3+\epsilon}(\bullet_t)} dt\big)^{\frac{1}{2}}$$
$$\le CT^\delta\big(\big\|\nabla_x \tilde{A}_0\big\|_{L^{3+\epsilon}(\blacktriangle_T)} + \big\|\tilde{A}_0\big\|_{L^{3+\epsilon}(\blacktriangle_T)}\big)$$

where $\epsilon > 0$ will be chosen momentarily and $\delta = \frac{1+\epsilon}{2(3+\epsilon)}$. The space-time norms here are estimated using the four dimensional Hardy-Littlewood-Sobolev theorem of fractional integration and the kernel estimates

$$|\partial_s \partial_x G_s(x,y)| \le \frac{C}{|x-y|^3 + (t-s)^3}$$
$$|\partial_s G_s(x,y)| \le \frac{C}{|x-y|^2 + (t-s)^2}$$

coming from Lemma 8.3 and the fact that $x \in \bullet_t, y \in \bullet_s$. We conclude[39]

$$\big\|\nabla_x \tilde{A}_0\big\|_{L^{3+\epsilon}(\blacktriangle_T)} \le C\,\|[\phi, D_0\phi]\|_{L^p(\blacktriangle_t)} \tag{7.38}$$
$$\big\|\tilde{A}_0\big\|_{L^{3+\epsilon}(\blacktriangle_T)} \le C\,\|[\phi, D_0\phi]\|_{L^p(\blacktriangle_t)}$$

with $\frac{1}{3+\epsilon} = \frac{1}{p} - \frac{1}{4}$. Pick q so that $\frac{p}{2} + \frac{1}{q} = 1$. Then

$$\|[\phi, D_0\phi]\|_{L^p(\blacktriangle_T)} = \Big(\int_0^T \int_{\bullet_t} |[\phi, D_0\phi]|^p \,dxdt\Big)^{\frac{1}{p}}$$
$$\le C\Big(\int_0^T \big(\int_{\bullet_t} |D_0\phi|^2 \,dx\big)^{\frac{p}{2}} \big(\int_{\bullet_t} |\phi|^{pq} \,dx\big)^{\frac{1}{q}} dt\Big)^{\frac{1}{p}}. \tag{7.39}$$

Now

$$\big(\int_{\bullet_t} |D_0\phi|^2 \,dx\big)^{\frac{p}{2}} \le C\big(\int_{\bullet_t} |\partial_0\phi|^2 + |[A_0,\phi]|^2 \,dx\big)^{\frac{p}{2}}$$
$$\le C\big(\mathcal{R}(t)^p + \mathcal{R}(t)^p Y(t)^p\big)$$
$$\le C\big(\mathcal{R}_0 + X(T)\big)^{2p} + CY(T)^{2p} \tag{7.40}$$

and by choosing $\epsilon = \frac{3}{11}$ so that $p = \frac{9}{5}, q = 10$, and $pq = 18$, we have

$$\Big(\int_0^T \big(\int_{\bullet_t} |\phi|^{pq} \,dx\big)^{\frac{1}{q}} ds\Big)^{\frac{1}{p}} \le T^{\left(10-\frac{6}{q}\right)\frac{1}{p}} \cdot \Big(\int_0^T \big(\int_{\bullet_t} |\phi|^{18} \,dx\big)^{\frac{1}{6}} dt\Big)^{\frac{6}{pq}}$$
$$= C\,\|\phi\|_{L^{18}_x L^3_t(\blacktriangle_T)}$$
$$\le C\left(\mathcal{R}_0 + X(T)\right) \tag{7.41}$$

---

[39]In fact, a stronger estimate holds for $\big\|\tilde{A}_0\big\|_{L^{3+\epsilon}(\blacktriangle_T)}$, but this is all we need here.

where I've used the inequality contained in Theorem 2.2. Plugging (7.40) and (7.41) into (7.39) gives

(7.42)
$$\|[\phi, D_0\phi]\|_{L^p(\blacktriangle_T)} \leq C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + X(T)^4 + Y(T)^4\big)$$

as desired.

Third term of $Y(T)$: $\left(\int_0^T \|\nabla A_0\|^2_{L^3(\bullet_t)}\, dt\right)^{\frac{1}{2}}$

As with the second term, use equation (C.G.e) and the same argument as in [15], pages 102-103 to show [40]

$$\Big(\int_0^T \left\|\nabla A_0{}^\dagger\right\|^2_{L^3(\bullet_t)} dt\Big)^{\frac{1}{2}} \leq C\big(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + \mathcal{R}_0\big).$$

We seek similar bounds for the contributions of the other terms in (7.30).

For the second term on the right hand side of (7.30) argue as follows

$$\begin{aligned}
\left\|\int_{\bullet_t} \partial_i G_t(x,y)\, [\phi, D_0\phi]\, dy\right\|_{L^3(\bullet_t)} &\leq C\, \|[\phi, D_0\phi]\|_{L^{\frac{3}{2}}(\bullet_t)} \\
&\leq C\, \|\phi\|_{L^6(\bullet_t)} \|D_0\phi\|_{L^2(\bullet_t)} \\
&\leq C\mathcal{R}(t) \cdot \|\partial_0\phi + [A_0, \phi]\|_{L^2(\bullet_t)} \\
&\leq C\big(\mathcal{R}(t)^2 + Y(t)\mathcal{R}(t)^2\big) \\
&\leq C\big(\mathcal{R}_0 + X(t)^2 + Y(t)^2 + X(t)^4\big)
\end{aligned}$$

For the fourth term on the right of (7.30) use Lemma 8.4.

$$\begin{aligned}
&\left\|\int_{/\ \ \backslash_t} \nabla G_s(x,y)\big([\phi, (\partial_0 - \partial_\eta)\phi] + [\phi, [A_0 - A_\eta, \phi]]\big)\, d\sigma(y) ds\right\|_{L^3(\bullet_t)} \\
&\qquad \leq C\, \|[\phi, (\partial_0 - \partial_\eta)\phi] + [\phi, [A_0 - A_\eta, \phi]]\|_{L^{\frac{3}{2}}(/\ \ \backslash_T)} \\
&\qquad \leq C\big(\mathcal{R}(T)^2 + Y(T)\mathcal{R}(T)^2 + \mathcal{R}(T)^3\big) \\
&\qquad \leq C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 + X(T)^4\big).
\end{aligned}$$

One can use (7.37), (7.38), and (7.42) to estimate immediately the contribution from the third term on the right of (7.30),

$$\Big(\int_0^T \|\nabla A_0\|^2_{L^3(\bullet_t)}\, dt\Big)^{\frac{1}{2}} \leq C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3 \\ X(T)^4 + Y(T)^4\big).$$

Fourth term of $Y(T)$: $\sup_{0\leq t\leq T} \|\partial_i A_0\|_{L^2(\bullet_t)}$

---

[40] See (7.30) for the definition of $A_0{}^\dagger$.

The approach taken in [15] works here as well: In Theorem 6.1 of Section 6, we obtain a space-time L^2 estimate on

$$B_i = -\partial_t \partial_i A_0 + [A_j, \partial_i A_j] - [\phi, \partial_i \phi] .$$

The estimate for the fourth term of $Y(T)$ will rely on this result. Write

$$\begin{aligned} \partial_i A_0(t,x) &= \partial_i A_0(0,x) + \int_0^t \partial_i \partial_s A_0(s,x) ds \\ &= \partial_i A_0(0,x) - \int_0^t B_i \, ds \\ &\quad + \int_0^t [A_j, \partial_i A_j] \, ds - \int_0^t [\phi, \partial_i \phi] \, ds. \end{aligned} \tag{7.43}$$

Hence

$$\begin{aligned} \|\partial_i A_0(t,x)\|_{L^2(\quad_t)} \le \|\partial_i A_0\|_{L^2(\quad_0)} &+ T^{\frac{1}{2}} \|B\|_{L^2(\quad^T)} \\ &+ \left\| \int_0^T [A_j, \partial_i A_j] \, ds \right\|_{L^2(\quad_t)} + \left\| \int_0^T [\phi, \partial_i \phi] \, ds \right\|_{L^2(\quad_t)} . \end{aligned}$$

Using Proposition (2.5) in [15] we conclude

$$\begin{aligned} \left\| \int_0^T [A_j, \partial_i A_j] \, ds \right\|_{L^2(\quad_t)} &\le C\big(\mathcal{R}_0 + X(T)\big)^2 \\ \left\| \int_0^T [\phi, \partial_i \phi] \, ds \right\|_{L^2(\quad_t)} &\le C\big(\mathcal{R}_0 + X(T)\big)^2 . \end{aligned}$$

The first term on the right of (7.44) is bounded using Proposition 7.1. Hence

$$\begin{aligned} \|\nabla A_0(t,\cdot)\|_{L^2(\quad_t)} &\le C\big(\mathcal{R}_0 + (\mathcal{R}_0 + X(T))^2\big) + T^{\frac{1}{2}} \|B\|_{L^2(\quad^T)} \\ &\le C\big(\mathcal{R}_0 + X(T)^2\big) + T^{\frac{1}{2}} \|B\|_{L^2(\quad^T)} . \end{aligned}$$

Fifth and Sixth terms of $Y(T)$: $\sup_{0\le t\le T} \|A_0\|_{L^3(\quad_t)}$, $\|A_0\|_{L^3(/ \quad \backslash^T)}$

We begin the argument here exactly as in [15]:

Write A_0 in terms of its time derivative,

$$A_0(t,x) = A_0(0,x) + \int_0^t \partial_s A_0(s,x) ds \tag{7.44}$$

hence

$$\begin{aligned} \|A_0(t,\cdot)\|_{L^3(\quad_t)} &\le \|A_0(0,\cdot)\|_{L^3(\quad_t)} + \int_0^t \|\partial_s A_0(s,\cdot)\|_{L^3(\quad_s)} \\ &\le \|A_0(0,\cdot)\|_{L^3(\quad_0)} + T^{\frac{1}{2}} \Big(\int_0^t \|\partial_s A_0(s,\cdot)\|^2_{L^3(\quad_s)} \Big)^{\frac{1}{2}} \\ &\le C\mathcal{R}_0 + T^{\frac{1}{2}} Y(T). \end{aligned} \tag{7.45}$$

Since $T < \frac{1}{2}$, the term $T^{\frac{1}{2}}Y(T)$ on the right of (7.45) can be absorbed in the left hand side of (7.24).

Similarly, using (7.44)

$$
\begin{aligned}
\|A_0\|_{L^3(/\ \ \backslash^T)} &\leq C\left\|A_0(0,x) + \int_0^t \partial_s A_0(s,x)ds\right\|_{L^3(/\ \ \backslash^T)} \\
&\leq C\Big(\int_0^T dt \int_{\bigcirc_t} d\sigma_x\, |A_0(0,x)|^3\Big)^{\frac{1}{3}} \qquad (7.46)\\
&+ C\Big(\int_0^T dt \int_{\bigcirc_t} d\sigma_x \left|\int_0^t \partial_s A_0(s,x)ds\right|^3\Big)^{\frac{1}{3}}
\end{aligned}
$$

The first term on the right of (7.46) is clearly bounded by

$$
\begin{aligned}
\Big(\int_0^T dt \int_{\bigcirc_t} d\sigma_x\, |A_0(0,x)|^3\Big)^{\frac{1}{3}} &\leq \|A_0(0,\cdot)\|_{L^3(\bullet_0)} \\
&\leq C\mathcal{R}_0.
\end{aligned}
$$

For the second term, more care is needed. Writing $\chi_{[0,t]}$ for the characteristic function of $[0,t]$,

$$
\begin{aligned}
&\Big(\int_0^T dt \int_{\bigcirc_t} d\sigma_x \left|\int_0^t \partial_s A_0(s,x)ds\right|^3\Big)^{\frac{1}{3}} \qquad (7.47)\\
&\leq \Big(\int_0^T dt \int_{\bigcirc_t} d\sigma_x \Big(\int_0^T ds \left|\partial_s A_0(s,x)\chi_{[0,t]}(s,x)\right|\Big)^3\Big)^{\frac{1}{3}} \\
&= \Big(\int_Y |\int_X F(X,Y)dX|^3 dY\Big)^{\frac{1}{3}}
\end{aligned}
$$

where $Y = (t,x)$ are variables for the cone $/\ \ \backslash^T$, $X = s$ is the variable for the interval $[0,T]$, and $F(X,Y) = \left|\partial_s A_0(s,x)\chi_{[0,t]}(s,x)\right|$. By Minkowski's integral inequality,

$$
\begin{aligned}
\Big(\int_Y |\int_X F(X,Y)dX|^3 dY\Big)^{\frac{1}{3}} &\leq \int_X \Big(\int_Y |F(X,Y)|^3 dY\Big)^{\frac{1}{3}} dX \\
&= \int_0^T ds \Big(\int_0^T dt \int_{\bigcirc_t} d\sigma(x) \left|\partial_s A_0(s,x)\chi_{[0,t]}(s,x)\right|^3\Big)^{\frac{1}{3}} \\
&= \int_0^T ds \Big(\int_s^T dt \int_{\bigcirc_t} d\sigma(x) \left|\partial_s A_0(s,x)\right|^3\Big)^{\frac{1}{3}} \\
&\leq \int_0^T \|\partial_s A_0(s,x)\|_{L^3(\bullet_s)} ds. \\
&\leq C\Big(\int_0^T \|\partial_s A_0(s,x)\|^2_{L^3(\bullet_s)} ds\Big)^{\frac{1}{2}} \\
&\leq CY(T)^2
\end{aligned}
$$

Hence $\|A_0\|_{L^3(\wedge^T)} \leq C(\mathcal{R}_0 + Y(T)^2)$.

Seventh Term of $Y(T)$: $\|\partial_t A_0\|_{L^2(\wedge^t)}$

Here we again use the decomposition

$$\partial_t A_0(t,x) = (\partial_t A_0)^\dagger - \int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy \tag{7.48}$$

and argue exactly as in [15] to prove

$$\begin{aligned}\left\|\partial_t A_0{}^\dagger\right\|_{L^2(\wedge^t)} &\leq C\big(\mathcal{R}(T)^2 + Y(T)^2 + \mathcal{R}(T)^3 + Y(T)^3\big)\\ &\leq C\big(\mathcal{R}_0 + X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3\big).\end{aligned}$$

To estimate the second term in (7.48) use Lemma 8.2:

$$\begin{aligned}\left\|\int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{L^2(\bigcirc_t)} \leq{}& \left\|\nabla_x \int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{L^{\frac{3}{2}}(\bullet_t)}\\ &+ \left\|\int_{\bullet_t} G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{L^{\frac{3}{2}}(\bullet_t)}.\end{aligned} \tag{7.49}$$

By Lemma 8.4,

$$\left\|\nabla_x \int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{L^{\frac{3}{2}}(\bullet_t)} \leq \|[\phi, D_i\phi]\|_{L^{\frac{3}{2}}(\bullet_t)}.$$

But in (7.28) we have already shown

$$\|[\phi, D_i\phi]\|_{L^{\frac{3}{2}}(\bullet_t)} \leq C\left(\mathcal{R}_0 + X(T)^2 + X(T)^3\right).$$

For the second term on the right of (7.49) use again Lemma 8.4,

$$\begin{aligned}\left\|\int_{\bullet_t} G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{\mathrm{L}^{\frac{3}{2}}(\bullet_t)} &\leq C\left\|\int_{\bullet_t} G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|_{L^3(\bullet_t)}\\ &\leq C\,\|[\phi, D_i\phi]\|_{L^{\frac{3}{2}}(\bullet_t)}\\ &\leq C\left(\mathcal{R}_0 + X(T)^2 + X(T)^3\right).\end{aligned}$$

Hence

(7.50)

$$\left(\int_0^T \left\|\int_{\bullet_t} \partial_i G_t(x,y)\,[\phi, D_i\phi]\,dy\right\|^2_{L^2(\bigcirc_t)} dt\right)^{\frac{1}{2}} \leq C\left(\mathcal{R}_0 + X(T)^2 + X(T)^3\right)$$

and

$$\|\partial_t A_0\|_{L^2(\wedge^t)} \quad\leq\quad C(X(T)^2 + Y(T)^2 + X(T)^3 + Y(T)^3).$$

Eighth term of $Y(T)$: $\|\partial_i A_0\|_{L^2(\wedge^T)}$

$$
\begin{aligned}
\|\partial_i A_0\|_{L^2(\diagup\ \diagdown^T)} &= \left\| \partial_i A_0(0,\cdot) + \int_0^t \partial_i \partial_t A_0 \, dt \right\|_{L^2(\diagup\ \diagdown^T)} \\
&\le \|\partial_i A_0\|_{L^2(\bullet_0)} \qquad (7.51)\\
&\quad + \left\| \int_0^t B_i \, ds - \int_0^t [A_j, \partial_i A_j] \, ds + \int_0^t [\phi, \partial_i \phi] \, ds \right\|_{L^2(\diagup\ \diagdown^T)} .
\end{aligned}
$$

For the term involving B on the right of (7.51),

$$
\begin{aligned}
\left\| \int_0^t B_i \, ds \right\|_{L^2(\diagup\ \diagdown^T)} &= \Big(\int_0^T dt \int_{\bigcirc_t} d\sigma(y) \left| \int_0^t B(s,y) \, ds \right|^2 \Big)^{\frac{1}{2}} \\
&\le \Big(\int_0^T ds \int_s^T dt \int_{\bigcirc_t} d\sigma_y \, |B(s,y)|^2 \Big)^{\frac{1}{2}} \\
&\le C \, \|B\|_{L^2(\blacktriangle)} .
\end{aligned}
$$

The remaining terms in (7.51) are handled with a variation on [15], Proposition 2.5. We formulate and prove this variant in Lemma 7.3 below. This result gives

$$
\left\| \int_0^t [\phi, \partial_i \phi] \, ds \right\|_{L^2(\diagup\ \diagdown^T)} \le C \big(\mathcal{R}_0 + X(T) \big)^2
$$

and similarly for the $[A_j, \partial_i A_j]$ term. The conclusion is

$$
\|\partial_i A_0\|_{L^2(\diagup\ \diagdown^T)} \quad \le \quad C \big(\mathcal{R}_0 + X(T) \big)^2 + C \, \|B\|_{L^2(\blacktriangle)} .
$$

This completes the proof of Theorem 7.2. ∎

Lemma 7.3. *Consider two* $\mathfrak{g}$*-valued wave equations*

$$
\Box \phi = \psi \qquad \Box \tilde{\phi} = \tilde{\psi}
$$

on $\blacktriangle^T$ *with initial conditions*

$$
\begin{aligned}
\phi(0,\cdot) = \phi_0(\cdot) \quad & \tilde{\phi}(0,\cdot) = \tilde{\phi}_0(\cdot) \\
\partial_t \phi(0,\cdot) = \phi_1(\cdot) \quad & \partial_t \tilde{\phi}(0,\cdot) = \tilde{\phi}_1(\cdot).
\end{aligned}
$$

Then for all $0 \le t \le T$*,*

$$
\begin{aligned}
\left\| \int_0^t [\phi, \partial \tilde{\phi}] ds \right\|_{L^2(\diagup\ \diagdown^T)} \quad \le \quad & C \left(\|\phi_0\|_{H^1(\bullet_0)} + \|\phi_1\|_{L^2(\bullet_0)} + \int_0^T \|\psi(s,\cdot)\|_{L^2(\bullet_s)} \, ds \right) \\
& \cdot \left(\left\| \tilde{\phi}_0 \right\|_{H^1(\bullet_0)} + \left\| \tilde{\phi}_1 \right\|_{L^2(\bullet_0)} + \int_0^T \left\| \tilde{\psi}(s,\cdot) \right\|_{L^2(\bullet_s)} \, ds \right) .
\end{aligned}
$$

Proof Observe that

$$(7.52)\quad \left\| \int_0^t [\phi, \partial\tilde{\phi}] ds \right\|^2_{L^2(/\ \backslash^T)}$$
$$\leq C \int_{/\ \backslash^T} \left(\int_0^t |\phi(s,x)|^2 \, ds \cdot \int_0^t \left|\partial\tilde{\phi}(s,x)\right|^2 ds \right) d\sigma_t(x) dt$$
$$\leq \left\| \int_0^t |\phi(s,x)|^2 \, ds \right\|_{L^\infty(/\ \backslash^T)} \cdot \left\| \int_0^t \partial\tilde{\phi}(s,x) ds \right\|^2_{L^2(/\ \backslash^T)}.$$

The same argument used to bound the second term in (7.46) shows

$$\left\| \int_0^t \partial\tilde{\phi}(s,x) ds \right\|^2_{L^2(/\ \backslash^T)} \leq \int_0^T \left\| \partial\tilde{\phi}(s,x) \right\|_{L^2(\bullet_s)} ds$$

and the energy inequality implies

$$(7.53)\quad \int_0^T \left\| \partial\tilde{\phi}(s,x) \right\|_{L^2(\bullet_s)} ds$$
$$\leq CT \left(\left\| \nabla\tilde{\phi}_0 \right\|_{L^2(\bullet_0)} + \left\| \tilde{\phi} \right\|_{L^2(\bullet_0)} + \int_0^T \left\| \tilde{\psi}(s,\cdot) \right\|_{L^2(\bullet_s)} ds \right).$$

The requisite bound for the first term on the right of (7.52) is given by the following proposition[41], which completes the proof. ∎

Lemma 7.4. *Consider the $\mathfrak{g}$-valued wave equation in* $\blacktriangle^T$

$$\Box\phi = \psi$$

with initial conditions

$$\phi(0,x) = \phi_0(x) \qquad \partial_t\phi(0,x) = \phi_1(x).$$

Then

$$(7.54)\quad \left\| \int_0^t |\phi(s,x)|^2 \, ds \right\|^{\frac{1}{2}}_{L^\infty(/\ \backslash^T)}$$
$$\leq C \left(\|\phi_0\|_{H^1(\bullet_0)} + \|\phi_1\|_{L^2(\bullet_0)} + \int_0^T \|\psi(s,\cdot)\|_{L^2(\bullet_s)} ds \right).$$

Proof By Duhamel's principle, it suffices to prove the lemma when $\phi_0 = \psi = 0$, in which case the formula for ϕ at fixed $(t,x) \in /\ \backslash^T$ is

$$\phi(t,x) = \frac{1}{4\pi} t \int_{|\omega|=1} \phi_1(x + t\omega) d\omega.$$

[41] Compare to [15], Proposition 2.6

Hence for every fixed $(t,x) \in /\ \ \backslash^T$,

$$\begin{aligned} \int_0^t |\phi(s,x)|^2\, ds &\leq \frac{1}{4\pi} \int_0^t \int_{|\omega|=1} s^2\, |\phi_1(x+s\omega)|^2\, d\omega ds \\ &\leq \frac{1}{4\pi} \|\phi_1\|^2_{L^2(\ \ _0)}. \end{aligned}$$

■

The estimates for differences follow from the same reasoning.

Theorem 7.5. *$(A_0, A, \phi), (A_0', A', \phi')$ two solutions of the Coulomb system C.G.a. - C.G.f. on $\ \ ^T$, $T \leq \frac{1}{2}$. Assume $\mathcal{R}_0[A,\phi], \mathcal{R}_0[A',\phi'] \leq \epsilon$.*

Then there is a constant C and a constant

$$K = O\big(X(T) + Y(T) + X'(T) + Y'(T) + \mathcal{R}_0 + \mathcal{R}_0'\big)$$

for small values of $X, Y, X', Y', \mathcal{R}_0, \mathcal{R}_0'$ such that

(7.55)

$$\begin{aligned} Y(A_0 - A_0')(T) \leq K \cdot \Big(& Y(A_0 - A_0')(T) + X(A - A', \phi - \phi')(T) \\ &+ \mathcal{R}_0[A - A', \phi - \phi'](T)\Big) + C\big(\|B - B'\|_{L^2(\ \ _T)}\big) \end{aligned}$$

8. Appendix: Lemmas for the Neumann Green's Function

For the sake of reference, this appendix contains some of the basic theorems the paper uses from real analysis, focussing mainly on the necessary estimates for the Neumann Green's function of a ball in $\mathbb{R}^3$. All statements here come directly from [15], Sections 1 and 4.

The formula for the Neumann Green's function G of the ball of radius a, center at the origin, is

$$G_t(x,y) \quad = \quad c\Big(\frac{1}{|x-y|} + \frac{a}{|x|\,|x^* - y|} + \frac{1}{a}\log\frac{2a^2}{a^2 - x\cdot y + |x|\,|y - x^*|}\Big)$$

where $x^* = \frac{a^2}{|x|^2}x$ and $c = -\frac{1}{4\pi}$.

Lemma 8.1. *Let $F \in L^2(\ \ )$, $f \in H^{\frac{1}{2}}(\ \ )$ and define*

$$\begin{aligned} \Phi(x) &= \int G(x,y)F(y)dy \\ \Psi(x) &= \int G(x,y)f(y)d\sigma_y. \end{aligned}$$

Then, $\Phi, \Psi \in H^2(\ \ )$. Thus $\Phi, \Psi, \partial_\eta\Phi, \partial_\eta\Psi$ have well defined restrictions to and we have, *i)* *The function* Φ *satisfies the Poisson equation,* $\Delta\Phi = F$ *and the boundary condition* $\partial_\eta \Phi|_{\bigcirc} = \frac{1}{4\pi a^2}\int_{\bullet} F$. *Also, in view of Green's formula*

$$\int_{\bigcirc} \Phi = -2a \int_{\bullet} F.$$

ii) *The function* Ψ *is harmonic in* $\bullet$ *and verifies the boundary condition* $\partial_\eta \Psi(y) = -f(y) + \frac{1}{4\pi a^2}\int_{\bigcirc} f d\sigma$. *In addition* $\int_{\bigcirc}\Psi = -2a\int_{\bigcirc} f$.

iii) *If the compatibility condition* $\int_{\bullet} F = \int_{\bigcirc} f$ *is satisfied then* $U = \Phi - \Psi$ *is the unique solution of the Neumann problem,*

$$\Delta U = F$$
$$\partial_\eta U = f$$
$$\int_{\bigcirc} U = 0.$$

Lemma 8.2. *Let* f *be a sufficiently smooth function in* $\bullet$, *continuous up to the boundary.*

(i) *For* $1 < p < q < \infty$ *satisfying* $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ *we have, with a constant* C *depending only on* p, q,

$$\|f\|_{L^q(\bullet)} \leq C\left(\|\nabla f\|_{L^p(\bullet)} + |a|^{-1}\|f\|_{L^p(\bullet)}\right).$$

(ii) *For* $1 < p < 3$ *the following estimate holds, with* $C > 0$ *depending only on* p,

$$\|f\|_{L^{\frac{2p}{3-p}}(\bigcirc)} \leq C\left(\|\nabla f\|_{L^p(\bullet)} + a^{-1}\|f\|_{L^p(\bullet)}.\right)$$

(iii) *For all* $x \in \bullet$,

$$\int_{\bullet} \frac{|f(y)|^2}{|x-y|^2} dy \leq C\left(\frac{1}{a^2}\|f\|^2_{L^2(\bullet)} + \|\nabla f\|^2_{L^2(\bullet)}\right).$$

Lemma 8.3. *The Green's function* $G = G_a$ *of the ball* $\bullet$ *enjoys the following estimates for* $|x| < b < a$, $|y| < a$,

$$|\partial_a G(x,y)| \leq \frac{C}{\big(|x-y| + (a-b)\big)^2}$$
$$|\partial_x \partial_a G(x,y)| \leq \frac{C}{\big(|x-y| + (a-b)\big)^3}.$$

Lemma 8.4. *Let G be the Neumann Green's function for the ball $\bigcirc$ of radius $a, \frac{1}{2} \leq a \leq 1$. Let $\mathcal{G}$ be the integral operator*

$$\mathcal{G}f(x) = \int_{\bigcirc} G(x,y)f(y)dy.$$

The following inequalities hold true

(a) For any $1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{1}{3}$, we have

$$\|\mathcal{G}f\|_{L^q(\bigcirc)} + \|\nabla(\mathcal{G}f)\|_{L^q(\bigcirc)} \leq C\,\|f\|_{L^p(\bigcirc)}$$

(b) For all $1 < p < \infty$,

$$\left\|\nabla^2(\mathcal{G}f)\right\|_{L^p(\bigcirc)} \leq C\,\|f\|_{L^p(\bigcirc)}$$

(c) Also,

$$\left\|\int_{\bigcirc} |G(x,y)f(y)|^2\, dy\right\|_{L^\infty(\bigcirc)} \leq C\Big(\|f\|^2_{L^2(\bigcirc)} + \|\nabla f\|^2_{L^2(\bigcirc)}\Big).$$

(d) If $/\ \ \backslash^t$ is the boundary of the domain $\triangle^t$ and $G_s(x,y)$ is the Green function of the ball $\bigcirc_s$, of radius $a = 1-s$, we have for all sufficiently smooth functions $f(s,y)$ defined on $/\ \ \backslash^t$, with $0 \leq t \leq \frac{1}{2}$

$$\left\|\int_{/\ \ \backslash^t} |G_s(x,y)f(s,y)|^2 ds d\sigma_y\right\|_{L^\infty(\bigcirc_t)} \tag{8.1}$$
$$\leq C\Big(\|f\|^2_{L^2(/\ \ \backslash^t)} + \left\|\bar{\partial} f\right\|^2_{L^2(/\ \ \backslash^t)} + \|f\|^2_{L^2(\bigcirc_t)} + \|\nabla f\|^2_{L^2(\bigcirc_t)}\Big)$$

where $\left\|\bar{\partial} f\right\|_{L^2(/\ \ \backslash^t)}$ denotes the L^2 norm, on $/\ \ \backslash^t$, of all tangential derivatives of f.

Lemma 8.5. *G as above, $\bigcirc$ the ball of radius $\frac{1}{2} \leq a \leq 1$. Then*

$$\begin{aligned}
\left\|\nabla^3 \int_{\bigcirc} G(x,y)\,F(y)dy\right\|_{L^2(\bigcirc)} &\leq C\,\|F\|_{H^1(\bigcirc)} \\
\left\|\nabla^2 \int_{\bigcirc} G(x,y)f(y)dy\right\|_{L^2(\bigcirc)} &\leq C\,\|f\|_{H^{\frac{1}{2}}(\bigcirc)}
\end{aligned}$$

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