ON GALOIS STRUCTURE INVARIANTS ASSOCIATED TO TATE MOTIVES

D. BURNS AND M. FLACH

ABSTRACT. We establish the equivalence of two definitions of invariants measuring the Galois module structure of K-groups of rings of integers in number fields (due to Chinburg et al on the one hand and the authors on the other). We also make some remarks concerning the possibility of yet another such definition via Lichtenbaum complexes.

1. INTRODUCTION

Let L/K be a finite Galois extension of number fields of group G, and X a smooth projective variety defined over K. If G is abelian then the present authors have shown how the Galois cohomological methods introduced by Bloch, Kato, Fontaine and Perrin-Riou (cf. [1], [15], [10]) allow one to attach to each motive $M = h^n(X)(r)$ (which satisfies certain 'standard' conjectures) a canonical element $\Omega(L/K, M)$ of $\mathrm{Pic}(\mathbb{Z}[G])$ (cf. [5]). These elements combine, in a rather complicated way in general, information about the G-structure of lattices in the de Rham, Betti and motivic cohomology spaces attached to $M_L := h^n(X \times_K L)(r)$. The construction of $\Omega(L/K, M)$ is unconditional in the case of Tate motives $M = \mathbb{Q}(r)$, where r can be any integer, and motives $M = h^1(X)(1)$ where X/K is an abelian variety for which the Tate-Safarevic group of $X \times_K L$ is finite. Moreover, the invariant $\Omega(L/K, \mathbb{Q}(0))$ (respectively $\Omega(L/K, \mathbb{Q}(1))$) was shown to be closely related to an invariant $\Omega(L/K, 3)$ (resp. $\Omega(L/K, 1)$) defined by Chinburg in [6] (resp. [7]).

More recently Chinburg, Kolster, Pappas and Snaith have introduced, for arbitrary groups G and integers $n > 0$, an element $\Omega_n(L/K)$ of the reduced Grothendieck group $\mathrm{Cl}(\mathbb{Z}[G])$ of the category of finitely generated projective $\mathbb{Z}[G]$-modules, which combines information about the $\mathbb{Z}[G]$-module structures of $K_{2n+1}(\mathcal{O}_L)$ and $K_{2n}(\mathcal{O}_L)$ (cf. [9]). This might be considered as a generalization of Chinburg's work to higher K-groups, since $\Omega(L/K, 3)$ is related to the $\mathbb{Z}[G]$-module structure of $\mathcal{O}_L^\times$ and $\mathrm{Pic}(\mathcal{O}_L)$. To reflect this analogy one puts $\Omega_0(L/K) := \Omega(L/K, 3)$.

In this note we concentrate on the case that G is abelian so that there is an identification of $\mathrm{Cl}(\mathbb{Z}[G])$ with $\mathrm{Pic}(\mathbb{Z}[G])$ induced by $\det_{\mathbb{Z}[G]}$. It therefore makes sense to compare the classes defined in [9] to those which arise from Tate motives via the approach of [5]. Let '#' denote the involution on $\mathrm{Pic}(\mathbb{Z}[G])$ induced by changing the G-action on a module by composing with the automorphism $g \mapsto g^{-1}$ of G. Our main result is the identity

$$\Omega(L/K, \mathbb{Q}(-n))^\# = \Omega_n(L/K) \tag{1}$$

for each integer $n \geq 0$. The comparison of the two sides in (1) is somewhat easier for $n > 0$ since then the definitions of both $\Omega_n(L/K)$ and $\Omega(L/K, \mathbb{Q}(-n))$ involve

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perfect complexes naturally arising from étale cohomology. In the case $n = 0$ on the other hand, the definition of $\Omega(L/K, 3)$ is based on complexes representing specific extension classes of $\mathbb{Z}[G]$-modules, and these can be recovered from étale cohomology only after some effort.

After recalling the definition of our invariant $\Omega(L/K, \mathbb{Q}(-n))$ in section 2, we therefore treat the two cases $n = 0$ and $n > 0$ separately in section 3 and 4, respectively. In the final section 5, we briefly indicate how the expected properties of Lichtenbaum's complexes $\Gamma(r)$ for $r \geq 2$ naturally give rise to invariants in $\mathrm{Cl}(\mathbb{Z}[\frac{1}{2}][G])$, and how these invariants relate to $\Omega_{r-1}(L/K)$ (and hence to $\Omega(L/K, \mathbb{Q}(1-r))$).

Meanwhile, in [3], the construction of $\Omega(L/K, M)$ has been generalized from abelian to arbitrary Galois extensions L/K. The methods we develop in this paper also give the identity (1) in the general case.

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2. Definition of $\Omega(L/K, \mathbb{Q}(r))$

We briefly recall from [5] the main steps in the definition of $\Omega(L/K, M)$ in the case $M = \mathbb{Q}(r)$, $r \leq 0$, and set up some notation. Given a set S of places of K, the letter S will also stand for the set of places of L lying above those in S. Confusion is avoided because we denote places of K by v and places of L by w. We put $S_p = S \cup \{v|p\}$, where p is any prime number, and $S_f = S \setminus S_\infty$, where S_∞ is the set of archimedean places of K (so f will never denote a prime number). From now on we fix a finite set S of places of K containing S_∞ and all places which ramify in L/K. We put $E = \mathbb{Q}[G]$, $E_p = E \otimes \mathbb{Q}_p = \mathbb{Q}_p[G]$ and $\mathcal{E}_p = \mathbb{Z}_p[G]$.

Under the assumption that the motivic cohomology groups of M_L are finite dimensional $\mathbb{Q}$-spaces a certain invertible (i.e. free rank 1) E-module $\Xi(M_L)$ was defined in [5, §1.4]. We recall its definition for $M = \mathbb{Q}(r)$ with $r \leq 0$ in which case the necessary finite-dimensionality is known. If $r = 0$

$$\Xi(\mathbb{Q}(0)_L) := \det_E\big(\mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}\big)^\# \otimes_E \det_E^{-1}\mathbb{Q} \otimes_E \bigotimes_{v \in S_\infty} \det_E H^0(K_v, \mathbb{Q}[G]),$$

and if $r < 0$

$$\Xi(\mathbb{Q}(r)_L) := \det_E\big(K_{1-2r}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Q}\big)^\# \otimes_E \bigotimes_{v \in S_\infty} \det_E H^0(K_v, \mathbb{Q}[G](r)).$$

Here the notation $^\#$ indicates a change of G-action on the same underlying abelian group: $gx^\# := g^{-1}x$. The space $\mathbb{Q}[G]$ should be thought of as the Betti realisation of $h^0(\mathrm{Spec}\, L)$, viewed as an étale sheaf of $\mathbb{Q}$-vector spaces on $\mathrm{Spec}\, K_v$ for $v \in S_\infty$. Similarly, after a choice of embedding $\sigma : L \to \bar{K}$, one can identify the p-adic realisation of $h^0(\mathrm{Spec}\, L)$ with $\mathbb{Q}_p[G]$ as a (left) G-module. In this identification the (left) action of $\gamma \in \mathrm{Gal}(\bar{K}/K)$ is given by

$$\gamma(x) := x\pi(\gamma^{-1}), \qquad x \in \mathbb{Q}_p[G] \tag{2}$$

where $\pi : \mathrm{Gal}(\bar{K}/K) \to G$ is defined by $\gamma\sigma(\lambda) = \sigma\pi(\gamma)(\lambda)$ for all $\lambda \in L$.

We let $\mathcal{O}_{L,S}$ denote the ring of S-integers in L. For a sheaf $\mathcal{F}$ on $(\mathrm{Spec}\, \mathcal{O}_{L,S})_{et}$, or more generally for a $\mathbb{Z}_p$ or $\mathbb{Q}_p$-sheaf, we define the cohomology with compact support as

$$R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F}) := \mathrm{Cone}\big(R\Gamma(\mathcal{O}_{L,S},\mathcal{F}) \to \bigoplus_{w\in S} R\Gamma(L_w,\mathcal{F})\big)[-1]. \tag{3}$$

Then there are isomorphisms for any prime number p [5, (1.17)]

$$\vartheta_p : \Xi(\mathbb{Q}(r)_L)\otimes_{\mathbb{Q}}\mathbb{Q}_p \xrightarrow{\sim} \det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r)) \tag{4}$$

which are constructed as follows. Put $(-)^\wedge = \mathrm{Hom}_{\mathbb{Q}_p}(-,\mathbb{Q}_p)$ with contragredient G-action so that one has an isomorphism

$$\det_{E_p} W^\wedge \cong \det^{-1}_{E_p} W^{\#} \tag{5}$$

for any $\mathbb{Q}_p[G]$-space W. The exact triangle [5][(1.11)]

$$R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r)) \to R\Gamma_f(L,\mathbb{Q}_p(r)) \to \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,\mathbb{Q}_p[G](r)) \oplus \bigoplus_{v\in S_\infty} H^0(K_v,\mathbb{Q}_p[G](r)) \to \tag{6}$$

together with the isomorphism [5][p.73/74]

$$H^i_f(L,\mathbb{Q}_p(r)) \cong \begin{cases} \mathbb{Q}_p & i=r=0 \\ (K_{1-2r}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Q}_p)^\wedge & i=2 \\ 0 & \text{otherwise} \end{cases}$$

and (5) gives an isomorphism

$$\Xi(\mathbb{Q}(r)_L)\otimes_{\mathbb{Q}}\mathbb{Q}_p \xrightarrow{\sim} \det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r)) \otimes_{E_p} \bigotimes_{v\in S_{p,f}} \det_{E_p} R\Gamma_f(K_v,\mathbb{Q}_p[G](r)). \tag{7}$$

Now there are several different ways to identify the determinant of $R\Gamma_f(K_v,\mathbb{Q}_p[G](r))$ with E_p and since this is a possible source of confusion we explain the situation here in some detail (see also the remark after [5][(1.16)] to that effect). Recall from [5][(1.5),(1.7)] that the complex $R\Gamma_f(K_v,\mathbb{Q}_p[G](r))$ is naturally quasi-isomorphic to a complex $V_v \xrightarrow{\phi_v} V_v$ where V_v is a projective E_p-module, and $\phi_v \in \mathrm{End}_{E_p}(V_v)$ is moreover 'semisimple at 0' in the following sense.

Let R be a semisimple commutative ring, V a finitely generated R-module and $\phi\in\mathrm{End}_R(V)$. One says that ϕ is semisimple at 0 if the map $\mathrm{id}_\phi : \ker(\phi)\subseteq V \to \mathrm{coker}(\phi)$ is an isomorphism, or equivalently if $V\cong\ker(\phi)\oplus D$ as $R[\phi]$-module. In this situation one defines $\det^*_R(\phi) := \det_R(\phi|D)\in R^\times$ which is independent of the choice of D. Denote by C the complex $V\xrightarrow{\phi} V$. For any isomorphism of R-modules $V\xrightarrow{\psi} W$ let

$$\psi_{triv} : \det^{-1}_R V\otimes_R \det_R W \xrightarrow{\sim} R$$

be the isomorphism obtained by composing $\det^{-1}_R(\psi)\otimes 1$ with the evaluation pairing $\det^{-1}_R W\otimes_R\det_R W \xrightarrow{\sim} R$. Here $\det^{-1}_R(V) = \mathrm{Hom}_R(\det_R(V),R)$ and $\det^{-1}_R(\psi) = \mathrm{Hom}_R(\det_R(\psi),R)^{-1}$.

Lemma 1. *Assume ϕ is semisimple at 0. With the notation just introduced there is a commutative diagram*

$$\begin{array}{ccc} det_R(C) & \xrightarrow{\mathrm{id}_{V,triv}} & R \\ \downarrow & & \downarrow{\scriptstyle \cdot det_R^*(\phi)^{-1}} \\ det_R^{-1}H^0(C)\otimes_R det_R H^1(C) & \xrightarrow{\mathrm{id}_{\phi,triv}} & R \end{array} \tag{8}$$

where the left vertical map is induced by the exact sequence

$$0\to H^0(C)\to V\xrightarrow{\phi} V\to H^1(C)\to 0$$

(this is the canonical identification of [16] *but with the sign convention of* [5]*[0.2]).*

Proof. Immediate from the definitions. □

Coming back to $C_v = R\Gamma_f(K_v,\mathbb{Q}_p[G](r))$ over $R=E_p$ the isomorphism ϑ_p in (4) arises from applying

$$\bigotimes_{v\in S_{p,f}} \mathrm{id}_{V_v,triv}$$

to the last term in (7). If $r<0$ then $R\Gamma_f(K_v,\mathbb{Q}_p[G](r))$ is acyclic, and the long exact sequence induced by (6) gives isomorphisms

$$\bigoplus_{v\in S_\infty} H^0(K_v,\mathbb{Q}_p[G](r)) \xrightarrow{\sim} H^1_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r)) \tag{9}$$

$$H^2_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r)) \xrightarrow{\sim} (K_{1-2r}(\mathcal{O}_L)\otimes_{\mathbb{Z}}\mathbb{Q}_p)^\wedge. \tag{10}$$

If we denote by $\tilde\vartheta_p : \Xi(\mathbb{Q}(r)_L)\otimes_{\mathbb{Q}}\mathbb{Q}_p \xrightarrow{\sim} \det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p(r))$ the isomorphism induced by (9) and (10) then we have implicitly used the diagonal map in (8) and Lemma 1 implies

$$\tilde\vartheta_p = \epsilon_{S_p}(r)\vartheta_p \tag{11}$$

where (see [5][(1.7)] for the definition of C_v)

(12)

$$\epsilon_{S_p}(r) = \prod_{v\in S_{p,f}} \det{}^*_{E_p}(\phi_v)^{-1} = \prod_{v\in S_{p,f}} \det{}_{E_p}(\phi_v)^{-1} = \prod_{v\in S_{p,f}} (1-Nv^{-r}f_v)^{-1} \in E^\times.$$

Here

$$f_v \in \mathbb{Q}[G/I_v] \cong \mathbb{Q}[G]^{I_v} \subseteq E \tag{13}$$

is the Frobenius automorphism at v and $I_v\subseteq G_v\subseteq G$ are the inertia and decomposition group for a place $w|v$. For $r=0$ the situation is slightly more complicated. We have canonical isomorphisms

$$\begin{aligned} H^0_f(K_v,\mathbb{Q}_p[G]) &= H^0_f(L_w,\mathbb{Q}_p[G/G_v]) \cong H^0(L_w,\mathbb{Q}_p[G/G_v]) = \mathbb{Q}_p[G/G_v] \\ H^1_f(K_v,\mathbb{Q}_p[G]) &\cong H^1_f(L_w,\mathbb{Q}_p[G/G_v]) \cong \mathrm{Hom}(\mathrm{Gal}(L_w^{ur}/L_w),\mathbb{Q}_p[G/G_v]) \\ &\cong \mathbb{Q}_p[G/G_v] \end{aligned} \tag{14}$$

where the last map is evaluation at the Frobenius automorphism. Via these maps we can identify $\bigoplus_{v\in S_{p,f}} H^i_f(K_v,\mathbb{Q}_p[G])$ for both $i=0$ and $i=1$ with the free $\mathbb{Q}_p$-space $Y_{S_{p,f}}\otimes_{\mathbb{Z}}\mathbb{Q}_p$ on the set of places of L above those in $S_{p,f}$ (consistently with later notation, Y_S will denote the free abelian group on the places above S). So for $r=0$ the long exact sequence induced by (6) reads

$$0 \to \mathbb{Q}_p \to \bigoplus_{v\in S_\infty} H^0(K_v, \mathbb{Q}_p[G]) \oplus (Y_{S_{p,f}} \otimes_{\mathbb{Z}} \mathbb{Q}_p) \to H^1_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p) \to 0 \tag{15}$$
$$\to Y_{S_{p,f}} \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^2_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p) \to (\mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p)^\wedge \to 0$$

and we denote by $\tilde{\vartheta}_p : \Xi(\mathbb{Q}(0)_L) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p)$ the isomorphism induced by (15) together with the identity map on $Y_{S_{p,f}} \otimes_{\mathbb{Z}} \mathbb{Q}_p$. From diagram (8) we then deduce equation (11) with the factor $\epsilon_{S_p}(0) = \prod_{v\in S_{p,f}} \epsilon_v(0)$ where

$$\epsilon_v(0) := \det\nolimits^*_{E_p}(\phi_v)^{-1}\, \mathrm{id}_{\mathbb{Q}_p[G/G_v],triv} \circ \mathrm{id}^{-1}_{\phi_v,triv} \in E_p^\times.$$

Lemma 2. *Put $G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$ and let v be a finite place of K. We introduce the following notation depending on v. For $x \in \mathbb{Q}^\times$, resp. $e \in E$, let ${}^*x \in E^\times$, resp. $e^* \in E$, be the element such that $\chi({}^*x) = x$, resp. $\chi(e^*) = 0$, if $\chi(G_v) = 1$ and $\chi({}^*x) = 1$, resp. $\chi(e^*) = \chi(e)$ otherwise, for all characters $\chi \in G^\vee$. Then with f_v as in (13)*

$$\epsilon_v(0) = {}^*|G_v/I_v| \cdot (1 - f_v^*)^{-1} \in E^\times. \tag{16}$$

In particular, $\epsilon_v(0)$ is independent of p.

Proof. For simplicity we suppose $G = G_v$ in this proof. The general case easily follows by inducing from G_v to G. We put $V = \mathbb{Q}_p[G_v]$ with its natural $G = G_v$-action and with $\mathrm{Gal}(\bar{K}_v/K_v)$-action given by (2). If $v \nmid p$ we set $\Gamma_v := \mathrm{Gal}(K_v^{ur}/K_v) \supseteq \Gamma_w := \mathrm{Gal}(K_v^{ur}/L_w \cap K_v^{ur})$. Using (2) one verifies that $V^{I_v} = \mathbb{Q}_p[G_v]^{I_v}$ is isomorphic to $\mathrm{Coind}_{\Gamma_w}^{\Gamma_v}\mathbb{Q}_p$ as a Γ_v-module and that the natural map $\pi^0 : \mathrm{Coind}_{\Gamma_w}^{\Gamma_v}\mathbb{Q}_p \to \mathbb{Q}_p$ is given by

$$\pi^0\Big(\sum_{g\in G_v/I_v} \lambda_g g \sum_{i\in I_v} i\Big) = \lambda_1.$$

Consider the following commutative diagram of complexes with vertical differentials

$$\begin{tikzcd}
V^{I_v} \arrow[r, "\varphi_v"] \arrow[d, "1-\varphi_v"'] & V^{I_v} \arrow[r, equal] \arrow[d, "\varphi_v^{-1}-1"'] & V^{I_v} \arrow[r, "\pi^0"] \arrow[d, "\delta^0"'] & \mathbb{Q}_p \arrow[d, "0"'] \\
V^{I_v} \arrow[r, equal] & V^{I_v} & \mathrm{Map}(\Gamma_v, V^{I_v}) \arrow[l, "\alpha"'] \arrow[r, "\pi^1"] \arrow[d, "\delta^1"'] & \mathrm{Map}(\Gamma_w, \mathbb{Q}_p) \arrow[d, "\delta^1"'] \\
 & & \; & \;
\end{tikzcd} \tag{17}$$

Here $\varphi_v \in \Gamma_v$ is the *inverse* Frobenius automorphism, the third (resp. fourth) column is the standard continuous cochain complex for Γ_v (resp. Γ_w) with coefficients in V^{I_v} (resp. $\mathbb{Q}_p$), one has $\delta^0(x) = \gamma \mapsto (\gamma - 1)(x)$, $\alpha(\psi) = \psi(\varphi_v^{-1})$ and π^1 is given by restricting to Γ_w and composing with π^0. By Shapiro's Lemma $\pi^\bullet$ is a quasi-isomorphism and so are the other horizontal maps in (17). By definition [5][(1.7)] the first column in (17) is the complex $C_v = (V_v \xrightarrow{\phi_v} V_v)$. To compute $\mathrm{id}_{\mathbb{Q}_p,triv} \circ \mathrm{id}^{-1}_{\phi_v,triv}$ we must identify $H^0(C_v)$ and $H^1(C_v)$ with $\mathbb{Q}_p$ as indicated in (14), express the map id_{ϕ_v} on $\mathbb{Q}_p$, and take its determinant over E_p. The element $e := \sum_{g\in G_v} g \in H^0(C_v)$ satisfies $\pi^0\varphi_v(e) = \pi^0(e) = 1 \in \mathbb{Q}_p$. On the other hand, $\alpha^{-1}(e)$ is the cocycle ψ with $\psi(\varphi_v^{-1}) = e$. If we evaluate ψ at the Frobenius automorphism of L_w as in (14) we find

$$\psi(\varphi_v^{-|G_v/I_v|}) = (1 + \varphi_v^{-1} + \varphi_v^{-2} + \cdots + \varphi_v^{-|G_v/I_v|+1})\psi(\varphi_v^{-1}) = |G_v/I_v|e$$

and applying π^0 gives $|G_v/I_v| \in \mathbb{Q}_p$. Hence the map id_{ϕ_v} on $\mathbb{Q}_p$ is multiplication with $|G_v/I_v|$ and $\mathrm{id}_{\mathbb{Q}_p,triv} \circ \mathrm{id}_{\phi_v,triv}^{-1}$ is $^*|G_v/I_v| \in E_p^\times$. Furthermore, by (2), φ_v acts like $f_v \in E^\times$ on $\mathbb{Q}_p[G_v]^{I_v}$ so that $\det_{E_p}^*(\phi_v) = \det_{E_p}^*(1 - f_v) = 1 - f_v^*$. This finishes the proof for $v \nmid p$.

To discuss the case $v \mid p$ we make a slight change of notation and put $\Gamma_v := \mathrm{Gal}(\bar{K}_v/K_v)$, $\Gamma_w := \mathrm{Gal}(\bar{K}_v/L_w)$. Then $V \cong \mathrm{Coind}_{\Gamma_w}^{\Gamma_v} \mathbb{Q}_p$ where the natural map $\pi^0 : V \to \mathbb{Q}_p$ is $\pi^0(\sum \lambda_g\, g) = \lambda_1$. Following [5][(1.3)] we denote by $B_v^\bullet(V)$ the complex of continuous Γ_v-modules

$$B_{crys} \otimes V \xrightarrow{1-\Phi\otimes 1} B_{crys} \otimes V$$

which is in fact quasi-isomorphic to V via the inclusion $V \to B_{crys} \otimes V$ in degree 0. Denoting by $V \to C^\bullet(\Gamma_v, V)$ the standard resolution we find maps of complexes of continuous Γ_v-modules

$$B_v^\bullet(V) \to \mathrm{Tot}\, C^\bullet(\Gamma_v, B_v^\bullet(V)) \xleftarrow{\beta} C^\bullet(\Gamma_v, V)$$

with β a quasi-isomorphism. After taking Γ_v-invariants β still induces a quasi-isomorphism and we find the following diagram in degrees 0 and 1

(18)

$$\begin{array}{ccccccc}
\mathrm{Crys}(V) & \to & B_{crys} \otimes V & \leftarrow & V & \xrightarrow{\pi^0} & \mathbb{Q}_p \\
{\scriptstyle 1-\Phi}\downarrow & & {\scriptstyle \Delta^0}\downarrow & & {\scriptstyle \delta^0}\downarrow & & {\scriptstyle 0}\downarrow \\
\mathrm{Crys}(V) & \xrightarrow{\iota} & \mathrm{Map}(\Gamma_v, B_{crys} \otimes V) \oplus B_{crys} \otimes V & \xleftarrow{\beta^1} & \mathrm{Map}(\Gamma_v, V) & \xrightarrow{\pi^1} & \mathrm{Map}(\Gamma_w, \mathbb{Q}_p)
\end{array}$$

where $\mathrm{Crys}(V) = H^0(\Gamma_v, B_{crys} \otimes V)$, $\Delta^0(x) = (\delta^0(x), (1 - \Phi \otimes 1)x)$, $\iota(x) = (0, x)$, $\beta^1(x) = (\text{inclusion} \circ x, 0)$, and δ^0 is as in (17). Again, according to [5][(1.5)], the first column in (18) is the complex $C_v = R\Gamma_f(K_v, V)$. C_v comes equipped with its natural map into $R\Gamma(K_v, V)$ (this the horizontal map into the third column in (18)). If $e := \sum_{g\in G_v} g \in V$ we have $\pi^0(e) = 1$ and the image of e in $\mathrm{Crys}(V)$ is $1 \otimes e$. One can find an element $\lambda \in W(\bar{\mathbb{F}}_p) \subset B_{crys}$ (W the ring of Witt vectors) such that $(1 - \Phi)\lambda = 1$. Then it is easy to see that the 1-cocycle $\psi := \delta^0(\lambda \otimes e)$ takes values in $\mathbb{Q}_p(1 \otimes e) \subseteq V$. Indeed, any $\gamma \in \Gamma_v$ acts on $W(\bar{\mathbb{F}}_p)$ like Φ^d for some $d \in \hat{\mathbb{Z}}$ and acts trivially on e, hence if $d \in \mathbb{N}$

$$\begin{aligned}(19)\quad (\gamma - 1)(\lambda \otimes e) = ((\Phi^d - 1)\lambda) \otimes e =& (1 + \Phi + \Phi^2 + \cdots + \Phi^{d-1})(\Phi - 1)\lambda \otimes e \\ =& (1 + \cdots + \Phi^{d-1})(-1) \otimes e = -d(1 \otimes e)\end{aligned}$$

and the general case follows by writing d as a limit of positive integers. The equality

$$\Delta^0(\lambda \otimes e) = \iota(1 \otimes e) + \beta^1(\psi)$$

shows that $1 \otimes e$ maps to the class of $-\psi$ in $H^1(\Gamma_v, V)$. Evaluating the cocycle $\pi^1(-\psi)$ at a lift γ of the Frobenius of L_w, as required by (14), and using (19) we find the value to be $-\pi^0(-d(1 \otimes e)) = d$ where $d = [L_{w,0} : \mathbb{Q}_p]$ with $L_{w,0} = W(\bar{\mathbb{F}}_p) \cap L_w$. The conclusion is that the map $\mathbb{Q}_p \cong H^0(C_v) \xrightarrow{\mathrm{id}_{\phi_v}} H^1(C_v) \cong \mathbb{Q}_p$ is multiplication with $d = [L_{w,0} : \mathbb{Q}_p]$ and that $\mathrm{id}_{\mathbb{Q}_p,triv} \circ \mathrm{id}_{\phi_v,triv}^{-1}$ equals $^*d \in E_p^\times$.

To compute $\det^*_{E_p}(\phi_v) = \det^*_{E_p}(1-\Phi)$, note that $\mathrm{Crys}(V)$ is isomorphic to $L_{w,0}$ as a G_v-Φ-module via the map $L_{w,0} \ni \lambda \mapsto \sum_{\gamma\in\Gamma_v/\Gamma_w} \gamma\lambda \otimes \pi(\gamma^{-1})$ where π is as in (2). By the normal basis theorem $L_{w,0}$ is a free $\mathbb{Q}_p[G_v/I_v]$-module of rank $\delta = [K_{v,0} : \mathbb{Q}_p]$, where $K_{v,0} := K_v \cap W(\bar{\mathbb{F}}_p)$, Φ is linear with matrix

$$\begin{pmatrix} 0 & \dots\dots & & 0 & f_v \\ 1 & 0 & \dots\dots\dots & & 0 \\ & 1 & 0 & & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{pmatrix}$$

and $1-\Phi$ is semisimple at 0. Choosing a decomposition $L_{w,0} \cong \mathbb{Q}_p \oplus D$ we find

$$1 - f_v X^\delta = \det_{E_p}(1-\Phi\cdot X) = \det^*_{E_p}(1-\Phi\cdot X)(1-1^*\cdot X).$$

Evaluating at $X = 1$ we get $\det^*_{E_p}(\phi_v) = {}^*\delta(1-f_v^*)$ and so

$$\epsilon_v(0) =^* d({}^*\delta)^{-1}(1-f_v^*)^{-1} = {}^*|G_v/I_v|(1-f_v^*)^{-1}.$$

□

For any $p^\nu\mathbb{Z}_p(r) \subset \mathbb{Q}_p(r)$ it was shown in [5, Prop. 1.20] that $R\Gamma_c(\mathcal{O}_{L,S_p}, p^\nu\mathbb{Z}_p(r))$ is a perfect complex of $\mathcal{E}_p$-modules such that

$$\mathcal{I}_p(r) := \det_{\mathcal{E}_p} R\Gamma_c(\mathcal{O}_{L,S_p}, p^\nu\mathbb{Z}_p(r)) \subset \det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(r))$$

is independent of ν. Put

$$\Xi(\mathbb{Q}(r)_L)_{\mathbb{Z}} := \bigcap_p \Xi(\mathbb{Q}(r)_L) \cap \vartheta_p^{-1}\mathcal{I}_p(r). \tag{20}$$

According to [5, Prop. 1.42] this is an invertible $\mathbb{Z}[G]$-module and so it makes sense to define

$$\Omega(L/K, \mathbb{Q}(r)) := \big(\Xi(\mathbb{Q}(r)_L)_{\mathbb{Z}}\big) \in \mathrm{Pic}(\mathbb{Z}[G]). \tag{21}$$

3. The case $n = 0$

3.1. **Definition of $\Omega(L/K, 3)$.** Assume S is large enough so that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$ and put

$$U_S := \mathcal{O}_{L,S}^\times, \quad J_S := \prod_{w\in S} L_w^\times, \quad C_S := J_S/U_S,$$

where U_S is diagonally embedded into J_S. Let Y_S be the free abelian group on the set S (of places of L) and X_S the kernel of the augmentation map $Y_S \to \mathbb{Z}$. All of these groups are naturally G-modules. In [7] it is shown that there exists a commutative diagram with exact row and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U_S \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_S \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & Y_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_S \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{22}
$$

where the terms A, A', A'', B, B', B'' are cohomologically trivial $\mathbb{Z}[G]$-modules and A, B are finitely generated. Henceforth we shall for brevity write 'c.t.' in place of 'cohomologically trivial'. The lower (resp. middle) row represents a (Yoneda) extension class in

$$\mathrm{Ext}^2_G(\mathbb{Z}, C_S) \cong H^2(G, C_S) \xrightarrow{\mathrm{inv}_K} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$$

(resp. $\mathrm{Ext}^2_G(Y_S, J_S)$). This class is specified to be the unique class $c_{L/K}$ with $\mathrm{inv}_K(c_{L/K}) = \frac{1}{|G|}$ (called the global canonical class) for the lower row. As to the middle row, one has decompositions of G-modules

$$Y_S \cong \bigoplus_{v\in S} \mathrm{Ind}^G_{G_v} \mathbb{Z}, \quad J_S \cong \bigoplus_{v\in S} (L\otimes_K K_v)^\times \cong \bigoplus_{v\in S} \mathrm{Ind}^G_{G_v} L^\times_{w(v)} \tag{23}$$

where for each place v of K we have chosen a place $w(v)$ of L above v and G_v denotes the decomposition group for $w(v)$. Hence

$$
\begin{aligned}
\mathrm{Ext}^2_G(Y_S, J_S) &\cong \bigoplus_{(v,v')\in S\times S} \mathrm{Ext}^2_G(\mathrm{Ind}^G_{G_v}\mathbb{Z}, \mathrm{Ind}^G_{G_{v'}} L^\times_{w(v')}) \\
&\supseteq \bigoplus_{v\in S} \mathrm{Ext}^2_G(\mathrm{Ind}^G_{G_v}\mathbb{Z}, \mathrm{Ind}^G_{G_v} L^\times_{w(v)}) \\
&\supseteq \bigoplus_{v\in S} \mathrm{Ext}^2_{G_v}(\mathbb{Z}, L^\times_{w(v)}).
\end{aligned}
$$

Here this last inclusion can be described in two ways. On the one hand it is induced by the natural map $L^\times_{w(v)} \to \mathrm{Res}^G_{G_v} \mathrm{Ind}^G_{G_v} L^\times_{w(v)}$ together with the isomorphism

$$\mathrm{Ext}^2_G(\mathrm{Ind}^G_{G_v}\mathbb{Z}, \mathrm{Ind}^G_{G_v} L^\times_{w(v)}) \cong \mathrm{Ext}^2_{G_v}(\mathbb{Z}, \mathrm{Res}^G_{G_v} \mathrm{Ind}^G_{G_v} L^\times_{w(v)})$$

coming from adjointness. From the point of view of Yoneda extensions this inclusion is given by sending a collection $(C_v)_{v\in S}$ of Yoneda extensions over $(G_v)_{v\in S}$ to the Yoneda extension $\bigoplus_{v\in S} \mathrm{Ind}^G_{G_v} C_v$ over G (here each C_v is supposed to denote the exact sequence specifying a Yoneda extension). We call $\bigoplus_{v\in S} \mathrm{Ext}^2_{G_v}(\mathbb{Z}, L^\times_{w(v)})$ the diagonal subgroup of $\mathrm{Ext}^2_G(Y_S, J_S)$.

The middle row in (22) is then chosen to represent the element $c^{loc}_{L/K}$ of the diagonal subgroup of $\mathrm{Ext}^2_G(Y_S, J_S)$ given by the sum of the local canonical classes $c_{L_{w(v)}/K_v} \in H^2(G_v, L^\times_{w(v)}) \cong \mathrm{Ext}^2_{G_v}(\mathbb{Z}, L^\times_{w(v)})$.

Throughout, we shall identify Yoneda-Ext-groups with derived functor Ext-groups by choosing an injective resolution of the second variable (see [11][Ch. V.9]). One then has the following

Lemma 3. *Suppose* $0 \to A'_i \to A_i \to A''_i \to 0$ *are short exact sequences for* $i = 1, 2$ *and* $0 \to A''_1 \to E_n \to \cdots \to E_1 \to A'_2 \to 0$ *a Yoneda extension with class* $[E] \in \mathrm{Ext}^n(A'_2, A''_1)$*. Then*

$$0 \to A''_1 \to E_n \to \cdots \to E_1 \to A_2 \to A''_2 \to 0$$

represents $\delta[E] \in \mathrm{Ext}^{n+1}(A''_2, A''_1)$ *and*

$$0 \to A'_1 \to A_1 \to E_n \to \cdots \to E_1 \to A'_2 \to 0$$

represents $(-1)^{n+1}\delta[E] \in \mathrm{Ext}^{n+1}(A'_2, A'_1)$.

Proof. See [11][Ch.V.9], in particular the comments after [loc.cit. (9.7)]. □

The uniqueness properties of (22) can be formulated as a statement in the derived category $\mathfrak{D}$ of $\mathbb{Z}[G]$-modules as follows. Denote the complex $A \to B$ with A placed in degree 0 by K_S and define K'_S, K''_S similarly. Then (22) represents an exact triangle

$$K_S \to K'_S \xrightarrow{\gamma} K''_S \to \tag{24}$$

in $\mathfrak{D}$. Recall that, given any complex E with just two nonzero cohomology groups $H^0(E)$ and $H^n(E)$, $n \geq 1$, the complex $\tau^{\leq n}\tau^{\geq 0}E$ represents a Yoneda extension class

$$\mathrm{e}(E) \in \mathrm{Ext}^{n+1}_G(H^n(E), H^0(E)) = \mathrm{Hom}_{\mathfrak{D}}(H^n(E), H^0(E)[n+1]).$$

Here τ is the truncation of complexes preserving cohomology in the indicated degrees. τ extends to a functor on the derived category, so in particular it preserves quasi-isomorphisms. As usual, we view a $\mathbb{Z}[G]$-module (for example $H^i(E)$) as an object of $\mathfrak{D}$ by placing it in degree 0. Given another complex E' and isomorphisms $\theta^i : H^i(E) \cong H^i(E')$ for $i \in \mathbb{Z}$, we obtain an isomorphism $\Theta : \mathrm{Ext}^{n+1}_G(H^n(E), H^0(E)) \cong \mathrm{Ext}^{n+1}_G(H^n(E'), H^0(E'))$. The maps θ^i are induced by an isomorphism $\theta : E \to E'$ in $\mathfrak{D}$ if and only if $\Theta(\mathrm{e}(E)) = \mathrm{e}(E')$. This follows from the definition of equivalence of Yoneda extensions.

Lemma 4. *Suppose given an exact triangle*

$$E \to E' \xrightarrow{\alpha} E'' \to \tag{25}$$

in $\mathfrak{D}$ *such that* $H^i(E) = H^i(E') = H^i(E'') = 0$ *for* $i \neq 0, 1$ *and an isomorphism of* $H^0(E) \to H^0(E') \to H^0(E'')$ *with* $U_S \to J_S \to C_S$ *and of* $H^1(E) \to H^1(E') \to H^1(E'')$ *with* $X_S \to Y_S \to \mathbb{Z}$ *such that* $\mathrm{e}(E') = c^{loc}_{L/K}$ *and* $\mathrm{e}(E'') = c_{L/K}$*. Then (25) is isomorphic to (24) in* $\mathfrak{D}$.

Proof. The assumptions of the lemma, together with our discussion above, yield a diagram

$$
\begin{tikzcd}
K_S \arrow[r] & K'_S \arrow[r, "\gamma"] \arrow[d, "\beta'"'] & K''_S \arrow[r] \arrow[d, "\beta''"'] & {} \\
E \arrow[r] & E' \arrow[r, "\alpha"] & E'' \arrow[r] & {}
\end{tikzcd}
\tag{26}
$$

where β' and β'' are isomorphisms in $\mathfrak{D}$ and such that the two diagrams

$$
\begin{tikzcd}
J_S \arrow[r, "H^0(\gamma)"] \arrow[d, "H^0(\beta')"'] & C_S \arrow[d, "H^0(\beta'')"'] \\
H^0(E') \arrow[r, "H^0(\alpha)"] & H^0(E'')
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_S \arrow[r, "H^1(\gamma)"] \arrow[d, "H^1(\beta')"'] & \mathbb{Z} \arrow[d, "H^1(\beta'')"'] \\
H^1(E') \arrow[r, "H^1(\alpha)"] & H^1(E'')
\end{tikzcd}
$$

commute. If we can show that there is a unique morphism $\gamma \in \mathrm{Hom}_{\mathfrak{D}}(K'_S, K''_S)$ such that $H^0(\gamma)$ and $H^1(\gamma)$ are the maps given in (22), then the diagram (26) must commute and can therefore be completed to a morphism of triangles. This will be an isomorphism, since β' and β'' are isomorphisms, hence the lemma.

From the two exact triangles in $\mathfrak{D}$

$$J_S \to K'_S \to Y_S[-1] \to J_S[1]$$

$$C_S \to K''_S \to \mathbb{Z}[-1] \to C_S[1]$$

we obtain a commutative diagram of abelian groups with exact rows and columns, which are in fact parts of long exact sequences

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], C_S) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], K''_S) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], \mathbb{Z}[-1]) \arrow[d, "f_2"'] \\
\mathrm{Hom}_{\mathfrak{D}}(K'_S, C_S) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak{D}}(K'_S, K''_S) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak{D}}(K'_S, \mathbb{Z}[-1]) \arrow[d] \\
\mathrm{Hom}_{\mathfrak{D}}(J_S, C_S) \arrow[r, "f_1"] & \mathrm{Hom}_{\mathfrak{D}}(J_S, K''_S) \arrow[r] & \mathrm{Hom}_{\mathfrak{D}}(J_S, \mathbb{Z}[-1]).
\end{tikzcd}
$$

Since $\mathrm{Hom}_{\mathfrak{D}}(J_S, \mathbb{Z}[i]) = \mathrm{Ext}^i_G(J_S, \mathbb{Z}) = 0$ for $i < 0$ the maps f_1 and f_2 are isomorphisms. Therefore we obtain an exact sequence

$$\mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], C_S) \to \mathrm{Hom}_{\mathfrak{D}}(K'_S, K''_S) \to \mathrm{Hom}_{\mathfrak{D}}(J_S, C_S) \oplus \mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], \mathbb{Z}[-1]).$$

Moreover

$$\mathrm{Hom}_{\mathfrak{D}}(Y_S[-1], C_S) = \mathrm{Ext}^1_G(Y_S, C_S) = \bigoplus_{v \in S} H^1(G_v, C_S) = 0$$

since C_S is the formation module of a class formation (see Proposition 3.5 below). So this gives the desired uniqueness of $\gamma \in \mathrm{Hom}_{\mathfrak{D}}(K'_S, K''_S)$. $\square$

Clearly K_S is a perfect complex of $\mathbb{Z}[G]$-modules (since every c.t. $\mathbb{Z}[G]$-module has a two-step projective resolution), and hence so is any other complex E as in Lemma 4. By definition then

$$\Omega(L/K, 3) := (K_S) = (A) - (B) \in \mathrm{Cl}(\mathbb{Z}[G]). \tag{27}$$

If G is abelian then with our parity convention for the determinant of a complex, this is equal to

$$\Omega(L/K,3) = \left(\det^{-1}_{\mathbb{Z}[G]} K_S\right) \in \mathrm{Pic}(\mathbb{Z}[G]) \cong \mathrm{Cl}(\mathbb{Z}[G]). \tag{28}$$

Remark: In [7], Chinburg works with $J'_S = \prod_{w\in S} L_w^\times \times \prod_{w\notin S} \mathcal{O}_{L_w}^\times$ rather than J_S (U_S being diagonally embedded). However, since S contains all places which ramify in L/K the natural projection $J'_S \to J_S$ (resp. $J'_S/U_S \to C_S$) induces isomorphisms $\mathrm{Ext}^2_G(Y_S, J'_S) \to \mathrm{Ext}^2_G(Y_S, J_S)$ (resp. $\mathrm{Ext}^2_G(\mathbb{Z}, J'_S/U_S) \to \mathrm{Ext}^2_G(\mathbb{Z}, C_S)$), and so working with either group will lead to quasi-isomorphic complexes K_S, hence to the same $\Omega(L/K,3)$.

3.2. **Another description of $\Omega(L/K,\mathbb{Q}(0))$.** In this section we shall realize the class $\Omega(L/K,\mathbb{Q}(0))^\#$ as the class of a perfect complex Ψ_S of $\mathbb{Z}[G]$-modules, well defined up to quasi-isomorphism. The construction of Ψ_S is valid without the assumption that G is abelian, and so in effect provides a generalization of $\Omega(L/K,\mathbb{Q}(0))^\#$. In the next section we apply Lemma 4 to show that Ψ_S and K_S are in fact quasi-isomorphic.

We denote by $(-)^\vee = R\,\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ the Pontryagin dual of any complex of abelian groups, and put $\hat{X} := X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ for any finitely generated abelian group X. Recall that S always contains the archimedean places and those ramified in L/K.

Proposition 3.1. *Assume S is large enough so that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. Then there exists an exact sequence of $\mathbb{Z}[G]$-modules*

$$0 \to U_S \to \Psi_S^0 \to \Psi_S^1 \to X_S \to 0 \tag{29}$$

with the following properties:

- *Ψ_S^0, Ψ_S^1 are finitely generated, cohomologically trivial $\mathbb{Z}[G]$-modules.*
- *Writing $\tilde{\Psi}_S$ for the complex*

$$\Psi_S^0 \to \Psi_S^1 \to X_S \otimes \mathbb{Q}$$

  where the maps are as in (29), there is a map in $\mathfrak{D}$

$$\tilde{\Psi}_S \to R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3] \tag{30}$$

  inducing an isomorphism on H^i for $i \neq 0$ and the inclusion $U_S \hookrightarrow \hat{U}_S$ for $i = 0$.

We denote by Ψ_S the complex $\Psi_S^0 \to \Psi_S^1$ (which is $\mathbb{Z}[G]$-perfect).

Remark. The following two exact triangles in $\mathfrak{D}$ summarize the relationship between Ψ_S, $\tilde{\Psi}_S$ and $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]$:

$$\tilde{\Psi}_S \to R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3] \to \hat{U}_S/U_S[0] \to \tag{31}$$

$$X_S \otimes \mathbb{Q}[-2] \to \tilde{\Psi}_S \to \Psi_S \to \tag{32}$$

Both $X_S \otimes \mathbb{Q}$ and $\hat{U}_S/U_S$ are uniquely divisible, hence $\mathbb{Q}[G]$-modules. As such they are injective, and because $\mathbb{Z}[G] \to \mathbb{Q}[G]$ is flat, they are also injective $\mathbb{Z}[G]$-modules. So for many purposes the differences between Ψ_S, $\tilde{\Psi}_S$ and $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]$ are inessential (see for example the proof of Proposition 3.2 below).

Proof. There are natural isomorphisms

$$H^i(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]) \cong \begin{cases} \hat{U}_S & i=0 \\ X_S\otimes\mathbb{Q}/\mathbb{Z} & i=2 \\ 0 & i\neq 0,2. \end{cases} \tag{33}$$

This follows from Artin-Verdier duality. More precisely, consider sheaves $\mathcal{F},\mathcal{F}'$ on $(\operatorname{Spec}\mathcal{O}_{L,S})_{et}$ whose restrictions to $(\operatorname{Spec}L_w)_{et}$ we denote by the same letter. The Ext-pairings

$$R\operatorname{Hom}_{\mathcal{O}_{L,S}}(\mathcal{F},\mathcal{F}')\times R\operatorname{Hom}_{\mathcal{O}_{L,S}}(\mathbb{Z},\mathcal{F})\to R\operatorname{Hom}_{\mathcal{O}_{L,S}}(\mathbb{Z},\mathcal{F}')$$
$$R\operatorname{Hom}_{L_w}(\mathcal{F},\mathcal{F}')\times R\operatorname{Hom}_{L_w}(\mathbb{Z},\mathcal{F})\to R\operatorname{Hom}_{L_w}(\mathbb{Z},\mathcal{F}')$$

together with (3) define a bilinear pairing

$$R\operatorname{Hom}_{\mathcal{O}_{L,S}}(\mathcal{F},\mathcal{F}')\times R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F})\to R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F}') \tag{34}$$

which on cohomology induces the pairing from [22][Ch. II, Prop. 2.5.a]. For $\mathcal{F}'=\mathbb{G}_m$ there is a canonical trace map $H^3_c(\mathcal{O}_{L,S},\mathbb{G}_m)\to\mathbb{Q}/\mathbb{Z}$ which lifts to a map $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)\to\mathbb{Q}/\mathbb{Z}[-3]$ since $\mathbb{Q}/\mathbb{Z}$ is injective. Hence we obtain a map

$$R\operatorname{Hom}_{\mathcal{O}_{L,S}}(\mathcal{F},\mathbb{G}_m)\xrightarrow{\mathrm{AV}} R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F})^\vee[-3]. \tag{35}$$

Similar considerations apply to $\tilde{R\Gamma}_c(\mathcal{O}_{L,S},\mathcal{F})$ which is defined as in (3) but with Tate cohomology at each archimedean place w. Denoting by $\iota: R\Gamma_c\to\tilde{R\Gamma}_c$ the natural map and putting $\mathcal{F}=\mathbb{Z}$ we find homomorphisms

$$R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m)\xrightarrow{\tilde{\mathrm{AV}}}\tilde{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]\xrightarrow{\iota^\vee} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]. \tag{36}$$

By [22][Ch.II, Th. 3.1] the map $\tilde{\mathrm{AV}}$ induces an isomorphism on H^i for $i\neq 0$ and the inclusion of $U_S=H^0(\mathcal{O}_{L,S},\mathbb{G}_m)$ into its profinite completion $\hat{U}_S$ for $i=0$. The map $H^i(\iota): H^i_c\to\tilde{H}^i_c$ is an isomorphism for $i\geq 2$ since Tate cohomology agrees with ordinary cohomology in degrees ≥ 1. So $H^i(\iota^\vee)$ is an isomorphism for each $i\leq 1$ and this implies (33) for each $i\leq 1$. On the other hand, by (3) we have $H^i_c=0$ for $i\leq 0$ and an exact sequence

$$0\to\mathbb{Z}\to\bigoplus_{w\in S}\mathbb{Z}\to H^1_c(\mathcal{O}_{L,S},\mathbb{Z})\to 0$$

where the first map is the diagonal embedding whose Pontryagin dual is the sum map. This gives the description of (33) for each $i\geq 2$.

By our discussion before Lemma 4 the complex

$$\tau^{\geq 0}\tau^{\leq 2}R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3]$$

represents an element

$$e_{L/K}\in\operatorname{Ext}^3_G(X_S\otimes\mathbb{Q}/\mathbb{Z},\hat{U}_S)\cong\operatorname{Ext}^3_G(X_S\otimes\mathbb{Q}/\mathbb{Z},U_S)\cong\operatorname{Ext}^2_G(X_S,U_S) \tag{37}$$

where the second (boundary) isomorphism is induced by the short exact sequence obtained by tensoring X_S with

$$0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0, \tag{38}$$

and the first isomorphism holds because $\operatorname{Ext}^n_G(-,\hat{U}_S/U_S)=0$ for $n>0$ (as $\hat{U}_S/U_S$ is injective).

The exact sequence in (29) is then chosen to be a representative of the image of $e_{L/K}$ in $\mathrm{Ext}^2_G(X_S, U_S)$. The modules Ψ^0_S, Ψ^1_S can be chosen finitely generated, because U_S and X_S are finitely generated over $\mathbb{Z}[G]$. They can be chosen c.t. because cup product with $e_{L/K}$ induces a cohomology isomorphism (with a degree shift of 2). This is because $\tau^{\geq 1}\tau^{\leq 3}R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z})$ consists of c.t. G-modules by a standard argument ([5, step 2 in Prop. 1.20] or [14, Prop. and Lemma 2.1], and Lemma 12 below) and the fact that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ preserves cohomological triviality. To prove this last remark note that since $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is exact one can reduce to projective modules by taking a (two-step) projective resolution of a given c.t. module, and then to free modules. But $\mathrm{Hom}_{\mathbb{Z}}(\bigoplus \mathbb{Z}[G], \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \prod \mathbb{Q}/\mathbb{Z})$ is coinduced, and hence certainly c.t..

By Lemma 3 the complex $\tilde{\Psi}_S$ represents the image of the Yoneda class of (29) under the boundary map in (37). The existence of the map (30) follows from the first isomorphism in (37). This finishes the proof of Proposition 3.1. □

Let S be as in Proposition 3.1 and T a finite set of places disjoint from S. For each $v \in T$ let $f_v \in G_v \subseteq G$ be the Frobenius automorphism at v, Y_v the free abelian group on the places of L above v with its natural (left) G-action and Ψ_v the complex

$$\mathbb{Z}[G] \xrightarrow{1-f_v^{-1}} \mathbb{Z}[G]$$

with terms in degree 0 and 1. There are isomorphisms of G-modules

$$H^i(\Psi_v) \xrightarrow{\iota^i_v} Y_v$$

for $i = 0, 1$ defined by $\iota^0_v(x) = |G_v|^{-1}xw(v)$ and $\iota^1_v(x) = xw(v)$ for $x \in \mathbb{Z}[G]$ (note that if x is f_v-invariant $xw(v)$ is divisible by $|G_v|$ in Y_v). The discussion before Lemma 1 applies to $\Psi_v \otimes_{\mathbb{Z}} \mathbb{Q}$. The map $\mathrm{id}_{1-f_v^{-1}}$, when expressed on Y_v after using the identifications ι^i_v, is multiplication with $|G_v|$, hence

$$\mathrm{id}_{Y_v,triv} \circ \mathrm{id}^{-1}_{1-f_v^{-1},triv} = {}^*|G_v| \in E^\times \tag{39}$$

with the notation introduced in Lemma 2.

Proposition 3.2. *There is an exact triangle in* $\mathfrak{D}$

$$\Psi_S \to \Psi_{S\cup T} \to \bigoplus_{v\in T} \Psi_v \to . \tag{40}$$

whose long exact cohomology sequence splits into two short exact sequences

$$0 \to U_S \to U_{S\cup T} \xrightarrow{v} Y_T \to 0 \tag{41}$$

$$0 \to X_S \to X_{S\cup T} \to Y_T \to 0 \tag{42}$$

in which all maps are the natural ones (v is given by taking valuations). Here we have used the maps ι^i_v together with (29) to identify the cohomology of (40).

Proof. For $v \in T$ with residue field $\kappa(v)$ we pick $w|v$ with residue field $\lambda(w)$ and decomposition group G_v. Let Γ_v (resp. Γ_w) be the absolute Galois group of $\kappa(v)$ (resp. $\lambda(w)$) and $F_v \in \Gamma_v$ the Frobenius automorphism. Then $G_v \cong \Gamma_v/\Gamma_w$. There are canonical isomorphisms

$$H^i(R\Gamma(\lambda(w),\mathbb{Z})^\vee[-2]) \cong \begin{cases} \hat{\mathbb{Z}} & i=0 \\ \mathbb{Q}/\mathbb{Z} & i=2 \\ 0 & i\neq 0,2. \end{cases} \tag{43}$$

For $i=0$ this is the Pontryagin-dual of the isomorphism

$$H^2(\lambda(w),\mathbb{Z}) \cong H^1(\lambda(w),\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\Gamma_w,\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \tag{44}$$

where the last map is evaluation at the $F_w := F_v^{|G_v|} \in \Gamma_w$, and for $i=2$ it is the dual of the isomorphism

$$H^0(\lambda(w),\mathbb{Z}) \cong \mathbb{Z}. \tag{45}$$

We shall apply the arguments in the proof of Prop. 3.1 to $R\Gamma(\lambda(w),\mathbb{Z})$ instead of $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})$, except that we can now make this complex much more explicit.

For a profinite group Γ and discrete (left) Γ-module M, denote by $C(\Gamma,M)$ the group of locally constant maps with (left) Γ-action $(\gamma f)(x) = \gamma f(\gamma^{-1}x)$. Suppose $\Gamma \cong \hat{\mathbb{Z}}$ with distinguished generator F. Then for any open subgroup $\Gamma' = n\Gamma \subseteq \Gamma$ the exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to C(\Gamma,\mathbb{Q}/\mathbb{Z}) \xrightarrow{1-F} C(\Gamma,\mathbb{Q}/\mathbb{Z}) \to 0 \tag{46}$$

is a resolution of the trivial module $\mathbb{Q}/\mathbb{Z}$ by Γ'-acyclic modules. Put $G_n = \Gamma/\Gamma'$. After taking Γ'-invariants we obtain an exact sequence of (left) G_n-modules

$$0 \to \mathbb{Q}/\mathbb{Z} \to C(G_n,\mathbb{Q}/\mathbb{Z}) \xrightarrow{1-F} C(G_n,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Q}/\mathbb{Z} \to 0 \tag{47}$$

where $\epsilon(f) = \sum_{g\in G_n} f(g)$. The complex $R\Gamma(\Gamma',\mathbb{Q}/\mathbb{Z})$ identifies with

$$C(G_n,\mathbb{Q}/\mathbb{Z}) \xrightarrow{1-F} C(G_n,\mathbb{Q}/\mathbb{Z}).$$

Lemma 5. *The isomorphism $H^1(\Gamma',\mathbb{Q}/\mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Q}/\mathbb{Z}$ induced by (47) coincides with the chain of isomorphisms*

$$H^1(\Gamma',\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{cont}(\Gamma',\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

where the last map is evaluation at F^n.

Proof. All rows in the following commutative diagram are resolutions of $\mathbb{Q}/\mathbb{Z}$ by Γ'-acyclic modules

$$\begin{tikzcd}
\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, equal] & C(\Gamma,\mathbb{Q}/\mathbb{Z}) \arrow[r, "1-F"] \arrow[d, "\mathrm{res}"] & C(\Gamma,\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{res}\circ(1+F+\cdots+F^{n-1})"] & 0 \\
\mathbb{Q}/\mathbb{Z} \arrow[r] & C(\Gamma',\mathbb{Q}/\mathbb{Z}) \arrow[r, "1-F^n"] & C(\Gamma',\mathbb{Q}/\mathbb{Z}) \arrow[r] & 0 \\
\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[u, equal] & C^0(\Gamma',\mathbb{Q}/\mathbb{Z}) \arrow[r, "\delta^0"] \arrow[u, equal] & C^1(\Gamma',\mathbb{Q}/\mathbb{Z}) \arrow[r, "\delta^1"] \arrow[u, "\alpha"] & C^2(\Gamma',\mathbb{Q}/\mathbb{Z}) \cdots \arrow[u]
\end{tikzcd} \tag{48}$$

The third row is the standard resolution of $\mathbb{Q}/\mathbb{Z}$ over Γ', i.e. $C^i(\Gamma',\mathbb{Q}/\mathbb{Z}) = C((\Gamma')^{i+1},\mathbb{Q}/\mathbb{Z})$ and

$$(\delta^0 f)(\gamma_0,\gamma_1) = f(\gamma_1) - f(\gamma_0) \tag{49}$$

$$(\delta^1 f)(\gamma_0,\gamma_1,\gamma_2) = f(\gamma_1,\gamma_2) - f(\gamma_0,\gamma_2) + f(\gamma_0,\gamma_1), \tag{50}$$

and the map α is given by $(\alpha f)(\gamma) = f(F^{-n}\gamma, \gamma)$. Upon taking Γ'-invariants we find the commutative diagram of abelian groups

$$\begin{array}{ccccc} \mathbb{Q}/\mathbb{Z} & \longrightarrow & C(G_n, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{1-F} & C(G_n, \mathbb{Q}/\mathbb{Z}) \\ \| & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}\,\circ\,(1+F+\cdots+F^{n-1})=\epsilon} \\ \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} & \xrightarrow{0} & \mathbb{Q}/\mathbb{Z} \\ \| & & \| & & \uparrow{\scriptstyle \alpha} \\ \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} & \xrightarrow{0} & \mathrm{Hom}_{cont}(\Gamma', \mathbb{Q}/\mathbb{Z}) \cong \ker(\delta^1)^{\Gamma'} \end{array} \tag{51}$$

where the map res simply becomes evaluation at $1 \in G_n$. If $x \in \mathbb{Q}/\mathbb{Z}$ and $f_x \in \mathrm{Hom}_{cont}(\Gamma', \mathbb{Q}/\mathbb{Z})$ is the unique homomorphism with $f_x(F^n) = x$, then f_x corresponds to the element $\phi_x \in \ker(\delta^1)^{\Gamma'}$ with $\phi_x(\gamma_0, \gamma_1) = \gamma_0 f_x(\gamma_0^{-1}\gamma_1)$ by the standard correspondence between homogenous and inhomogenous cochains. Hence $\alpha(\phi_x)(\gamma) = \phi_x(F^{-n}\gamma, \gamma) = f_x(F^n) = x$ for any $\gamma \in \Gamma'$ which proves the Lemma. $\square$

There is a resolution of the trivial Γ_w-module $\mathbb{Z}$ by Γ_w-acyclic discrete Γ_w-modules

$$0 \to \mathbb{Z} \to \mathbb{Q} \to C(\Gamma_v, \mathbb{Q}/\mathbb{Z}) \xrightarrow{1-F_v} C(\Gamma_v, \mathbb{Q}/\mathbb{Z}) \to 0 \tag{52}$$

The complex $R\Gamma(\lambda(w), \mathbb{Z})$ is obtained by taking Γ_w-invariants of this resolution:

$$\mathbb{Q} \to C(G_v, \mathbb{Q}/\mathbb{Z}) \xrightarrow{1-f_v} C(G_v, \mathbb{Q}/\mathbb{Z}) \tag{53}$$

where f_v is the image of F_v in G_v (this notation is consistent with (13)).

Lemma 6. *The complex $R\Gamma(\lambda(w), \mathbb{Z})^\vee[-2]$ is naturally quasi-isomorphic to*

$$\hat{\mathbb{Z}}[G_v] \xrightarrow{1-f_v^{-1}} \hat{\mathbb{Z}}[G_v] \xrightarrow{\epsilon} \hat{\mathbb{Z}} \otimes \mathbb{Q}$$

where ϵ is as in (47), and the isomorphisms (43) are those induced by the short exact sequence

$$0 \to \hat{\mathbb{Z}} \xrightarrow{\Delta} \hat{\mathbb{Z}}[G_v] \xrightarrow{1-f_v^{-1}} \hat{\mathbb{Z}}[G_v] \xrightarrow{\epsilon} \hat{\mathbb{Z}} \otimes \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

where $\Delta(n) = n\sum_{g \in G_v} g$.

Proof. The first assertion follows by taking the dual of (53). More precisely, we have isomorphisms of (left) G_v-modules

$$C(G_v, \mathbb{Q}/\mathbb{Z})^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_v], \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \hat{\mathbb{Z}}[G_v]$$

where $\hat{\mathbb{Z}}[G_v]$ has its natural left action, and $C(G_v, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_v], \mathbb{Q}/\mathbb{Z})$ the contragredient action (as opposed to the coinduced action). The Pontryagin-dual of a diagonal map is the sum map. Note that if M is a (left) $\mathbb{Z}[G]$-module, the (left) action of G on $M^\vee$ is given by $gx = (g^\vee)^{-1}x$, hence the occurrence of f_v^{-1}.

The map $\mathbb{Q}/\mathbb{Z}[-1] \to \mathbb{Z}$ (representing the extension (38)) in the derived category of discrete Γ_v-modules is isomorphic to the map from the first to the second row in the commutative diagram

$$\begin{array}{ccccc}
 & & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{F_v-1} & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}) \\
 & & \| & & \| \\
\mathbb{Q} & \xrightarrow{-1} & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{F_v-1} & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}) \\
\| & & {\scriptstyle -1}\downarrow & & \| \\
\mathbb{Q} & \longrightarrow & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{1-F_v} & C(\Gamma_v,\mathbb{Q}/\mathbb{Z}).
\end{array}$$

Here we use the acyclic resolutions (46) of $\mathbb{Q}/\mathbb{Z}$. Note the sign change in the differential of (46) caused by the shift [-1]. The middle row is the mapping cone of $\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$ shifted by [-1] and is quasi-isomorphic to the lower row, which is (52). By Lemma 5 the isomorphism (44) is given by ϵ on $C(G_v,\mathbb{Q}/\mathbb{Z})$ whose Pontryagin dual is the map Δ. Similarly, the isomorphism (45) is given by the inclusion $\mathbb{Z}\to\mathbb{Q}$ in (52) whose Pontryagin dual is the natural projection $\hat{\mathbb{Z}}\otimes\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$. $\square$

Define $\tilde\Psi_w$ (resp. Ψ_w) as the second (resp. third) row in the following commutative diagram (with obvious vertical maps)

$$\begin{array}{ccccc}
\hat{\mathbb{Z}}[G_v] & \xrightarrow{1-f_v^{-1}} & \hat{\mathbb{Z}}[G_v] & \xrightarrow{\epsilon} & \hat{\mathbb{Z}}\otimes\mathbb{Q} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{Z}[G_v] & \xrightarrow{1-f_v^{-1}} & \mathbb{Z}[G_v] & \xrightarrow{\epsilon} & \mathbb{Q} \\
\| & & \| & & \downarrow \\
\mathbb{Z}[G_v] & \xrightarrow{1-f_v^{-1}} & \mathbb{Z}[G_v] & \longrightarrow & 0.
\end{array} \tag{54}$$

Then we can rewrite this diagram in the form of two exact triangles of G_v-modules

$$\tilde\Psi_w\to R\Gamma(\lambda(w),\mathbb{Z})^\vee[-2]\to\hat{\mathbb{Z}}/\mathbb{Z}[0]\to \tag{55}$$

$$\mathbb{Q}[-2]\to\tilde\Psi_w\to\Psi_w\to \tag{56}$$

analogous to (31) and (32). Put $\Psi_v=\Psi_w\otimes_{\mathbb{Z}[G_v]}\mathbb{Z}[G]$ and $\tilde\Psi_v=\tilde\Psi_w\otimes_{\mathbb{Z}[G_v]}\mathbb{Z}[G]$.

We claim that there is a commutative diagram in $\mathfrak{D}$ where all rows are exact triangles, and where the left (resp. middle, resp. right) vertical maps are as in (31), (32) (resp. (31), (32) with S replaced by $S'=S\cup T$, resp. (55) and (56)).

$$\begin{array}{ccccccc}
R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3] & \to & R\Gamma_c(\mathcal{O}_{L,S'},\mathbb{Z})^\vee[-3] & \to & \bigoplus_{w\in T} R\Gamma(\lambda(w),\mathbb{Z})^\vee[-2] & \to \\
\uparrow & & \uparrow & & \uparrow & \\
\tilde\Psi_S & \to & \tilde\Psi_{S'} & \to & \bigoplus_{v\in T}\tilde\Psi_v & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\Psi_S & \to & \Psi_{S'} & \to & \bigoplus_{v\in T}\Psi_v & \to .
\end{array} \tag{57}$$

The first row is the dual of the exact triangle [22, Ch. II, Prop 2.3d)]

$$R\Gamma_c(\mathcal{O}_{L,S'},\mathbb{Z}) \to R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}) \to \bigoplus_{w\in T} R\Gamma(\lambda(w),\mathbb{Z}) \to \tag{58}$$

and to fill in the other maps we use the following

Lemma 7. *Let I be an injective $\mathbb{Z}[G]$-module, C a complex of $\mathbb{Z}[G]$-modules and $n\in\mathbb{Z}$.*

a) The natural map $\mathrm{Hom}_{\mathfrak{D}}(C,I[-n]) \to \mathrm{Hom}_G(H^n(C),I)$ is an isomorphism.

b) There is a short exact sequence

$$0 \to \mathrm{Ext}^1_G(I,H^{n-1}(C)) \to \mathrm{Hom}_{\mathfrak{D}}(I[-n],C) \to \mathrm{Hom}_G(I,H^n(C)) \to 0.$$

Proof. Part a) follows very easily by computing the set of homotopy classes of maps of complexes $C \to I[-n]$ which agrees with $\mathrm{Hom}_{\mathfrak{D}}(C,I[-n])$ because I is injective. Part b) follows from the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_G(I,H^q(C)) \Rightarrow H^{p+q}R\,\mathrm{Hom}(I,C)$$

together with the fact that $\mathrm{Hom}_{\mathfrak{D}}(I[-n],C) = H^nR\,\mathrm{Hom}(I,C)$ and $\mathrm{Ext}^p_G(I,-)=0$ for $p\geq 2$ since I is c.t., and hence has a two-step projective resolution. $\square$

The top left square in the diagram (with obvious maps)

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
\hat{U}_S/U_S[0] & \to & \hat{U}_{S'}/U_{S'}[0] & \to & \bigoplus_{v\in T}\hat{Y}_v/Y_v[0] & \to \\
\uparrow & & \uparrow & & \uparrow & & \\
R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^\vee[-3] & \to & R\Gamma_c(\mathcal{O}_{L,S'},\mathbb{Z})^\vee[-3] & \to & \bigoplus_{w\in T} R\Gamma(\lambda(w),\mathbb{Z})^\vee[-2] & \to \\
\uparrow & & \uparrow & & \uparrow & & \\
\tilde{\Psi}_S & \to & \tilde{\Psi}_{S'} & \to & \bigoplus_{v\in T}\tilde{\Psi}_v & \to
\end{array}\tag{59}$$

commutes after taking H^0, hence commutes in $\mathfrak{D}$ by Lemma 7a). So we get a map of triangles from the first column into the second. To argue similarly for the map from the second to the third column we need:

Lemma 8. *The map $\hat{U}_{S'} \to \bigoplus_{v\in T}\hat{Y}_v$ induced by*

$$R\Gamma_c(\mathcal{O}_{L,S'},\mathbb{Z})^\vee[-3] \to \bigoplus_{w\in T} R\Gamma(\lambda(w),\mathbb{Z})^\vee[-2]$$

together with (33) and (43) coincides with $v\otimes\hat{\mathbb{Z}}$ (v given by taking valuations), in particular maps $U_{S'}$ to $\bigoplus_{v\in T} Y_v$.

Proof. Denote the open immersion $\mathrm{Spec}\,\mathcal{O}_{L,S'} \to \mathrm{Spec}\,\mathcal{O}_{L,S}$ by j and let $i: Z \to \mathrm{Spec}\,\mathcal{O}_{L,S}$ be the closed immersion of the complement. We have two short exact sequences of étale sheaves on $\operatorname{Spec}\mathcal{O}_{L,S}$

$$\begin{tikzcd}
0 \arrow[r] & j_!j^*\mathbb{Z} \arrow[r] \arrow[d, "\alpha"'] & \mathbb{Z} \arrow[r] \arrow[d, "\beta"'] & i_*i^*\mathbb{Z} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] & j_*\mathbb{G}_m \arrow[r, "\mathrm{val}"] & i_*i^*\mathbb{Z} \arrow[r] & 0
\end{tikzcd} \tag{60}$$

where val is given by taking valuations (we shall discuss below the possible maps α, β, γ). The exact triangle (58) is obtained by applying $R\Gamma_c$ to the top sequence in (60) (see the proof of [22][Prop.2.3d] for this fact). The Artin-Verdier duality map (35) is functorial in $\mathcal{F}$. Hence the morphism $i_*i^*\mathbb{Z} \xrightarrow{\epsilon} j_!j^*\mathbb{Z}[1]$ induced by the top row in (60), gives a commutative diagram (note $i^*\mathbb{Z} = \mathbb{Z}$)

$$\begin{tikzcd}
R\Gamma_c(\mathcal{O}_{L,S'}, \mathbb{Z})^\vee[-3] \arrow[r] & \bigoplus_{w\in T} R\Gamma(\lambda(w), \mathbb{Z})^\vee[-2] \\
R\operatorname{Hom}_{\mathcal{O}_{L,S}}(j_!j^*\mathbb{Z}, \mathbb{G}_m) \arrow[r, "b"] \arrow[u] & R\operatorname{Hom}_{\mathcal{O}_{L,S}}(i_*\mathbb{Z}, \mathbb{G}_m)[1], \arrow[u]
\end{tikzcd} \tag{61}$$

where $b = R\operatorname{Hom}(\epsilon, \mathbb{G}_m)[1]$ and taking into account the isomorphism

$$R\Gamma_c(\mathcal{O}_{L,S}, i_*\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\mathcal{O}_{L,S}, i_*\mathbb{Z}) \xrightarrow{\sim} \bigoplus_{w\in T} R\Gamma(\lambda(w), \mathbb{Z}). \tag{62}$$

Consider the composite map

$$\mathcal{O}_{L,S'}^\times = \tag{63}$$
$$Hom_{\mathcal{O}_{L,S}}(j_!j^*\mathbb{Z}, \mathbb{G}_m) \xrightarrow{H^0(b)} Ext^1_{\mathcal{O}_{L,S}}(i_*\mathbb{Z}, \mathbb{G}_m) \xleftarrow{H^0(b')} Hom_{\mathcal{O}_{L,S}}(i_*\mathbb{Z}, i_*\mathbb{Z}) \cong \bigoplus_{w\in T} \mathbb{Z}$$

where $b' = R\operatorname{Hom}(i_*\mathbb{Z}, \epsilon')$, $i_*\mathbb{Z} \xrightarrow{\epsilon'} \mathbb{G}_m[1]$ being given by the lower row in (60). If (60) commutes we have $\epsilon \circ \alpha[1] = \gamma \circ \epsilon'$ and hence $H^0(b)(\alpha) = H^0(b')(\gamma)$. Now by adjointness, for any α there is a unique β (and hence a unique γ) making (60) commute. Viewing α (equivalently β) as an element of $O_{L,S'}^\times$, the commutativity of (60) shows that γ is given by multiplication with the valuations at $w \in T$ of α. Putting this together with the fact that $H^0(b')$ is an isomorphism ([22][Ch. II, Rem. 1.7.(b)]) we conclude that (63) is the valuation map.

It remains to be shown that the identification of the target space of (63) with $\bigoplus_{w\in T}\mathbb{Z}$, combined with (61) induces the identification (43) for $i = 0$. Inserting the morphism ϵ' into the $\mathcal{F}'$-variable and putting $\mathcal{F} = i_*\mathbb{Z}$ in (34) we get a commutative diagram of pairings

$$\begin{tikzcd}
R\operatorname{Hom}_{\mathcal{O}_{L,S}}(i_*\mathbb{Z}, \mathbb{G}_m[1]) \times R\Gamma_c(\mathcal{O}_{L,S}, i_*\mathbb{Z}) \arrow[r] & R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m[1]) \\
R\operatorname{Hom}_{\mathcal{O}_{L,S}}(i_*\mathbb{Z}, i_*\mathbb{Z}) \times R\Gamma_c(\mathcal{O}_{L,S}, i_*\mathbb{Z}) \arrow[r] \arrow[u] & R\Gamma_c(\mathcal{O}_{L,S}, i_*\mathbb{Z}). \arrow[u, "\delta"']
\end{tikzcd} \tag{64}$$

It suffices to show that $\delta := R\Gamma_c(\epsilon')$ is compatible with the trace map which we define for the bottom row by combining (62) with (44). Assume for simplicity $T = \{w\}$ and put $U_w = \operatorname{Spec}\mathcal{O}_{L_w}$ (the general case follows by summing over

$w \in T$). There is a commutative diagram of isomorphisms

(65)

$$\begin{array}{ccccccc}
H^3_c(\mathcal{O}_{L,S}, j_!\mathbb{G}_m) & \to & H^3_c(\mathcal{O}_{L,S}, \mathbb{G}_m) & \xleftarrow{H^2(\delta)} & H^2_c(\mathcal{O}_{L,S}, i_*\mathbb{Z}) & \to & H^2(\mathcal{O}_{L,S}, i_*\mathbb{Z}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^3_w(U_w, j_!\mathbb{G}_m) & \to & H^3_w(U_w, \mathbb{G}_m) & \leftarrow & H^2_w(U_w, i_*\mathbb{Z}) & \to & H^2(U_w, i_*\mathbb{Z})
\end{array}$$

in which all maps are the natural ones. According to [22][p.186] the local trace map can be defined either by the isomorphism

$$H^3_w(U_w, j_!\mathbb{G}_m) \leftarrow H^2(L_w, \mathbb{G}_m) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z} \tag{66}$$

or by the isomorphism

$$H^2(U_w, i_*\mathbb{Z}) \cong H^2(\lambda(w), \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

which arises from (44). By [22][p.216]) the global trace map is defined by (65) and (66). This gives the desired compatibility of δ with the trace map and finishes the proof of Lemma 8. $\square$

Therefore, in (59), we have a nine term commutative diagram with exact columns and two exact rows. So the octahedral axiom implies that the third row in (59), i.e. the second row in (57), is also exact. The same argument gives the exactness of the third row in (57) except that we now have to use Lemma 7b). For example, consider the diagram

$$\begin{array}{ccc}
X_S \otimes \mathbb{Q}[-2] & \longrightarrow & \tilde{\Psi}_S \\
\downarrow & & \downarrow \\
X_{S'} \otimes \mathbb{Q}[-2] & \longrightarrow & \tilde{\Psi}_{S'}
\end{array} \tag{67}$$

where the left (resp. right) hand vertical map is the natural one (resp. the one we have just constructed). Applying H^2 to (67) gives a commutative diagram. Since $H^1(\tilde{\Psi}_{S'}) = 0$ we have $\mathrm{Hom}_{\mathfrak{D}}(X_S \otimes \mathbb{Q}[-2], \tilde{\Psi}_{S'}) = \mathrm{Hom}_G(X_S \otimes \mathbb{Q}, H^2(\tilde{\Psi}_{S'}))$ by Lemma 7b), so (67) commutes in $\mathfrak{D}$. This extends to a map of the triangle (32) for S into the triangle (32) for S'.

We have now established that the diagram (57) commutes and has exact rows and this implies Proposition 3.2 (the assertion about v is immediate from Lemma 8 and (54)). $\square$

For a perfect complex of $\mathbb{Z}_p[G]$-modules P we define $P^* = \mathrm{RHom}_{\mathbb{Z}_p}(P, \mathbb{Z}_p)$ with contragredient G-action. Then P^* is again perfect over $\mathbb{Z}_p[G]$ and $P^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ naturally identifies with $(P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\wedge := \mathrm{Hom}_{\mathbb{Q}_p}(P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathbb{Q}_p)$. Recall that S_p denotes the union of S with the set of places above p.

Proposition 3.3. *Let S be as in Prop. 3.1. For any prime number p there exists a quasi-isomorphism*

$$\theta_p : \Psi_{S_p} \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p)^*[-2] \tag{68}$$

such that the sequence (15) is $\mathbb{Q}_p$-dual to the natural long exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p \to U_{S_p} \otimes_{\mathbb{Z}} \mathbb{Q}_p \to Y_{S_{p,f}} \otimes_{\mathbb{Z}} \mathbb{Q}_p \to 0 \\ \to X_{S_p} \otimes_{\mathbb{Z}} \mathbb{Q}_p \to Y_{S_p} \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \mathbb{Q}_p \to 0 \end{aligned} \tag{69}$$

when we identify $X_{S_p}\otimes_{\mathbb{Z}}\mathbb{Q}_p$ with $H^1_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p)^\wedge$ and $U_{S_p}\otimes_{\mathbb{Z}}\mathbb{Q}_p$ with $H^2_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p)^\wedge$ via $\theta_p\otimes\mathbb{Q}_p$. Here we have also identified $\mathbb{Q}_p$ and $Y_S\otimes\mathbb{Q}_p$ with their respective contragredients by mapping the natural basis to its dual basis.

Proof. Denote by C_n (resp. $\tilde{C}_n$), the mapping cone of multiplication by p^n on Ψ_{S_p} (resp. $\tilde{\Psi}_{S_p}$), for each positive integer n. The mapping cone of multiplication by p^n on $R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z})^\vee[-3]$ naturally identifies with $R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^\vee[-2]$. From (31) and (32) we obtain two quasi-isomorphisms

$$C_n \xleftarrow{\phi_{1,n}} \tilde{C}_n \xrightarrow{\phi_{2,n}} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^\vee[-2], \tag{70}$$

compatible with the natural maps $C_{n+1}\to C_n$ etc. This is because multiplication by p^n is an isomorphism on both $X_{S_p}\otimes\mathbb{Q}$ and $\hat{U}_{S_p}/U_{S_p}$.

Recall that $R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)$ is a perfect complex of $\mathbb{Z}_p[G]$-modules. Applying $(-)^*=\mathrm{RHom}_{\mathbb{Z}_p}(-,\mathbb{Z}_p)$ to the exact triangle

$$R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)\xrightarrow{p^n}R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)\to R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})\to$$

produces an exact triangle

$$R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^*\to R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)^*\xrightarrow{p^n}R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)^*\to .$$

From the short exact sequence $0\to\mathbb{Z}_p\to\mathbb{Q}_p\to\mathbb{Q}_p/\mathbb{Z}_p\to 0$ one easily deduces that

$$R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^*\cong R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^\vee[-1],$$

and obtains therefore an exact triangle

$$R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)^*\xrightarrow{p^n}R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)^*\to R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^\vee\to .$$

More concretely then, if we pick a bounded complex of finitely generated projective $\mathbb{Z}_p[G]$-modules $P^\bullet$ quasi-isomorphic to $R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}_p)^*[-2]$, the inverse system $R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Z}/p^n\mathbb{Z})^\vee[-2]$, $n\geq 1$, is isomorphic to $P^\bullet/p^n$.

We can give a similar description for the system C_n. Since Ψ_{S_p} is a perfect complex of $\mathbb{Z}[G]$-modules, we can choose a quasi-isomorphism $\Psi_{S_p}\cong Q^\bullet$ with $Q^\bullet$ a bounded complex of finitely generated projective $\mathbb{Z}[G]$-modules. The inverse system C_n is then quasi-isomorphic to the system of complexes $Q^\bullet/p^n$.

The morphisms $\phi_n:=\phi_{2,n}\phi_{1,n}^{-1}:Q^\bullet/p^n\to P^\bullet/p^n$ from (70) are quasi-isomorphisms such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Q^\bullet/p^n \arrow[r, "\pi_n^Q"] \arrow[d, "\phi_n"'] & Q^\bullet/p^{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \\
\cdots \arrow[r] & P^\bullet/p^n \arrow[r, "\pi_n^P"] & P^\bullet/p^{n-1} \arrow[r] & \cdots
\end{tikzcd} \tag{71}$$

commutes in the derived category. Since however $Q^\bullet/p^n$ consists of projective $\mathbb{Z}/p^n\mathbb{Z}[G]$-modules and $P^\bullet/p^n$ of $\mathbb{Z}/p^n\mathbb{Z}[G]$-modules we can realize each ϕ_n as an actual map of complexes. Moreover, $\phi_{n-1}\pi_n^Q$ will be homotopic to $\pi_n^P\phi_n$, i.e.

$$\phi_{n-1}\pi_n^Q-\pi_n^P\phi_n=dh+hd$$

for some map $h:Q^\bullet/p^n\to P^\bullet/p^{n-1}[-1]$. Since $P^i/p^n\to P^i/p^{n-1}$ is surjective and Q^i/p^n is a projective $\mathbb{Z}/p^n\mathbb{Z}[G]$-module we can lift h to a map $h':Q^\bullet/p^n\to P^\bullet/p^n[-1]$. If we then replace ϕ_n by $\phi_n+dh'+h'd$, the diagram (71) will actually be a commutative diagram of maps of complexes. So by induction we may assume that the ϕ_n are a map of inverse systems of complexes.

Lemma 9. *Let*

$$\cdots \to K_n^\bullet \xrightarrow{t_n} K_{n-1}^\bullet \to \cdots$$

be an inverse system of complexes such that each transition map $t_n^i : K_n^i \to K_{n-1}^i$ is surjective, and each cohomology group $H^i(K_n^\bullet)$ is finite. Then there is a canonical isomorphism

$$H^i(\varprojlim{}_n K_n^\bullet) \xrightarrow{\sim} \varprojlim{}_n H^i(K_n^\bullet)$$

for each $i \in \mathbb{Z}$.

Proof. This follows easily by using the argument of [26, (2.2)] or [12, 1.4-6] and the fact that any inverse system of finite abelian groups $(F_n)_n$ satisfies the Mittag-Leffler condition so that $\varprojlim{}^1_n F_n = 0$. □

This Lemma implies that

$$Q^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}_p = \varprojlim{}_n \big(Q^\bullet/p^n\big) \xrightarrow{\varprojlim_n \phi_n} \varprojlim{}_n \big(P^\bullet/p^n\big) = P^\bullet$$

is a quasi-isomorphism. But $Q^\bullet \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is of course quasi-isomorphic to $\Psi_{S_p} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $P^\bullet$ is quasi-isomorphic to $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p)^*[-2]$. So we have constructed the quasi-isomorphism θ_p in Proposition 3.3. The fact that θ_p identifies the dual of (15) with the natural sequence (69) is a lengthy but straightforward verification (somewhat similar to the proof of Lemma 8) which we leave out for reasons of space. □

The operation $P \mapsto P^\#$ defined in the paragraph before equation (2) induces an involution on $\mathrm{Pic}(\mathbb{Z}[G])$ which we denote by the same symbol. For $e = \sum \lambda_g g \in A[G]$ we put $e^\# = \sum \lambda_g g^{-1}$ where A is any commutative coefficient ring. Note that $^\#$ is a functor from G-modules to G-modules and we have $\phi^\# = \phi$ for any G-homomorphism ϕ. However, if ϕ is multiplication with e then $\phi^\#$ is multiplication with $e^\#$.

Theorem 3.1. *If G is abelian, then $\Omega(L/K, \mathbb{Q}(0))^\# = (\Psi_S) \in \mathrm{Cl}(\mathbb{Z}[G])$.*

Proof. Let S be a finite set of places containing the archimedean places and those ramified in L/K and such that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. There is a long exact sequence of E-modules

$$\begin{aligned} (72)\quad 0 \to \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to U_S \otimes_{\mathbb{Z}} \mathbb{Q} \to Y_{S_f} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0 \\ \to X_S \otimes_{\mathbb{Z}} \mathbb{Q} \to Y_S \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q} \to 0 \end{aligned}$$

where the maps are the natural ones as in (69). If $S = S_p$ the sequence $(72)\otimes\mathbb{Q}_p$ is identical to (69). The $\mathbb{Q}$-dual of (72) induces an isomorphism

$$\begin{aligned} (73)\qquad \kappa(S) : \Xi(\mathbb{Q}(0)_L)^\# &\cong \det{}_E(U_S \otimes \mathbb{Q}) \otimes_E \det{}_E^{-1}(X_S \otimes \mathbb{Q}) \\ &= \det{}_E^{-1}(\Psi_S \otimes_{\mathbb{Z}} \mathbb{Q}) \end{aligned}$$

such that for any prime number p one has an identity of maps

$$(74)\qquad \tilde{\vartheta}_p = \big(\det{}_{E_p}^{-1}(\theta_p \otimes \mathbb{Q}_p)\big)^{-1} \circ (\kappa(S_p) \otimes \mathbb{Q}_p)$$

from $\Xi(\mathbb{Q}(0)_L)^\# \otimes_{\mathbb{Q}} \mathbb{Q}_p$ to $\det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p)^\# = \det_{E_p}^{-1} R\Gamma_c(\mathcal{O}_{L,S_p},\mathbb{Q}_p)^*[-2]$. This is because $\tilde{\vartheta}_p$ was induced by (15) which agrees with the dual of (72)$\otimes\mathbb{Q}_p$ for S_p by Proposition 3.3. Put

$$\tilde{\lambda} := \kappa(S)\circ\kappa(S_p)^{-1} : \det_E^{-1}(\Psi_{S_p}\otimes\mathbb{Q}) \xrightarrow{\sim} \det_E^{-1}(\Psi_S\otimes\mathbb{Q}), \tag{75}$$

and let $\lambda_{\mathbb{Z}}$ be the isomorphism

(76)

$$\det_{\mathbb{Z}[G]}^{-1}\Psi_{S_p} \xrightarrow{\sim} \det_{\mathbb{Z}[G]}^{-1}\Psi_S \otimes_{\mathbb{Z}[G]} \bigotimes_{v\in T} \det_{\mathbb{Z}[G]}^{-1}\big(\mathbb{Z}[G] \xrightarrow{1-f_v^{-1}} \mathbb{Z}[G]\big) \xrightarrow{\sim} \det_{\mathbb{Z}[G]}^{-1}\Psi_S$$

where the first map is induced by the triangle (40) with $T = S_p\setminus S$ and the second is $1\otimes \mathrm{Hom}_{\mathbb{Z}[G]}(\bigotimes_{v\in T}\mathrm{id}_{\mathbb{Z}[G],triv},\mathbb{Z}[G])^{-1}$ in the notation introduced before Lemma 1 with $R=\mathbb{Z}[G]$. Put $\lambda = \lambda_{\mathbb{Z}}\otimes\mathbb{Q}$.

Lemma 10. *We have* $\tilde{\lambda} = \prod_{v\in T}\epsilon_v(0)^{\#,-1}\lambda$ *with* $\epsilon_v(0)\in E^\times$ *as in (16).*

Proof. $\tilde{\lambda}$ is induced by the natural map from the sequence (72) for S to the sequence (72) for S_p, i.e. by the short exact sequences

$$0\to U_S\otimes_{\mathbb{Z}}\mathbb{Q}\to U_{S_p}\otimes_{\mathbb{Z}}\mathbb{Q}\to Y_T\otimes_{\mathbb{Z}}\mathbb{Q}\to 0, \tag{77}$$

$$0\to X_S\otimes_{\mathbb{Z}}\mathbb{Q}\to X_{S_p}\otimes_{\mathbb{Z}}\mathbb{Q}\to Y_T\otimes_{\mathbb{Z}}\mathbb{Q}\to 0 \tag{78}$$

and the identity map on $Y_T\otimes_{\mathbb{Z}}\mathbb{Q} = \bigoplus_{v\in T} Y_v$. By Proposition 3.2 these sequences also arise as the long exact cohomology sequence induced by (40)$\otimes\mathbb{Q}$. This means that $\tilde{\lambda}$ is likewise a composite as in (76)$\otimes\mathbb{Q}$ but with the second map $1\otimes\mathrm{Hom}_E(\bigotimes_{v\in T}\mathrm{id}_{E,triv},E)^{-1}$ replaced by $1\otimes\mathrm{Hom}_E(\bigotimes_{v\in T}\mathrm{id}_{Y_v,triv},E)^{-1}$. To compare the two maps, note that Lemma 1 gives a commutative diagram

$$\begin{array}{ccc}
\bigotimes_{v\in T}\det_E(E\xrightarrow{1-f_v^{-1}}E) & \xrightarrow{\bigotimes_{v\in T}\mathrm{id}_{E,triv}} & E \\
\downarrow & & \downarrow \cdot\prod_{v\in T}(1-f_v^{-1,*})^{-1} \\
\det_E^{-1}(Y_T\otimes\mathbb{Q})\otimes_E\det_E(Y_T\otimes\mathbb{Q}) & \xrightarrow{\bigotimes_{v\in T}\mathrm{id}_{1-f_v^{-1},triv}} & E \\
\| & & \downarrow \cdot\prod_{v\in T}\beta_v \\
\det_E^{-1}(Y_T\otimes\mathbb{Q})\otimes_E\det_E(Y_T\otimes\mathbb{Q}) & \xrightarrow{\bigotimes_{v\in T}\mathrm{id}_{Y_v,triv}} & E
\end{array} \tag{79}$$

where $\beta_v = \mathrm{id}_{Y_v,triv}\circ\mathrm{id}_{1-f_v^{-1},triv}^{-1}\in E^\times$. Applying $\mathrm{Hom}_E(-,E)^{-1}$ to (79) yields

$$\tilde{\lambda} = \prod_{v\in T}(1-f_v^{-1,*})\beta_v^{-1}\,\lambda. \tag{80}$$

Since $\beta_v = {}^*|G_v|$ by (39), $({}^*x)^\# = {}^*x$ and $(e^\#)^* = (e^\#)^*$ in the notation of Lemma 2, Lemma 10 follows. $\square$

For any prime number p, denote the scalar extensions of $\kappa(S)$, $\tilde{\lambda}$ and λ to E_p by the same letter. Now we can compute, using $\epsilon_{S_p}(0) = \epsilon_S(0)\prod_{v\in T}\epsilon_v(0)$

$$\begin{aligned}
\vartheta_p^{-1}(\mathcal{I}_p(0)^{\#}) &= \epsilon_{S_p}(0)^{\#}\tilde{\vartheta}_p^{-1}(\mathcal{I}_p(0)^{\#}) && \text{by (11)}\\
&= \epsilon_{S_p}(0)^{\#}\kappa(S_p)^{-1}\circ \det{}_{E_p}(\theta_p\otimes\mathbb{Q}_p)^{-1}(\mathcal{I}_p(0)^{\#}) && \text{by (74)}\\
&= \epsilon_{S_p}(0)^{\#}\kappa(S_p)^{-1}\big(\det{}_{\mathcal{E}_p}^{-1}(\Psi_{S_p}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\big) && \text{by (68)}\\
&= \epsilon_{S_p}(0)^{\#}\kappa(S)^{-1}\tilde{\lambda}\big(\det{}_{\mathcal{E}_p}^{-1}(\Psi_{S_p}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\big) && \text{by (75)}\\
&= \epsilon_{S}(0)^{\#}\kappa(S)^{-1}\lambda\big(\det{}_{\mathcal{E}_p}^{-1}(\Psi_{S_p}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\big) && \text{by Lemma 10}\\
&= \epsilon_{S}(0)^{\#}\kappa(S)^{-1}\big(\det{}_{\mathcal{E}_p}^{-1}(\Psi_{S}\otimes_{\mathbb{Z}}\mathbb{Z}_p)\big) && \text{by (76)}
\end{aligned}$$

and we obtain

$$\Xi(\mathbb{Q}(0)_L)^{\#}_{\mathbb{Z}} = \epsilon_S(0)^{\#}\kappa(S)^{-1}\big(\det{}_{\mathbb{Z}[G]}^{-1}(\Psi_S)\big). \tag{81}$$

Since $\epsilon_S(0)^{\#}\in E^{\times}$, taking classes in $\mathrm{Pic}(\mathbb{Z}[G])$ and recalling our sign convention for the determinant of a complex, this equality implies that

$$\Omega(L/K,\mathbb{Q}(0))^{\#} = (\det{}_{\mathbb{Z}[G]}^{-1}\Psi_S) = (\Psi_S)\in \mathrm{Cl}(\mathbb{Z}[G]).$$

□

3.3. **Comparison of $\Omega(L/K,3)$ and $\Omega(L/K,\mathbb{Q}(0))^{\#}$.** The idea is to construct an exact triangle

$$\Psi_S\to\Psi'_S\to\Psi''_S\to$$

satisfying the assumptions of Lemma 4. Recall that $\tilde{R\Gamma}_c(\mathcal{O}_{L,S},\mathcal{F})$ is defined as in (3) but with Tate-cohomology at each archimedean place. If we define

$$R\Gamma_{\Delta}(L_w,\mathcal{F}) := \mathrm{Cone}\big(R\Gamma(L_w,\mathcal{F})\to R\Gamma_{\mathrm{Tate}}(L_w,\mathcal{F})\big)[-1] \tag{82}$$

for each archimedean place w, the octahedral axiom gives an exact triangle

$$R\Gamma_c(\mathcal{O}_{L,S},\mathcal{F})\to \tilde{R\Gamma}_c(\mathcal{O}_{L,S},\mathcal{F})\to\bigoplus_{w\in S_\infty}R\Gamma_{\Delta}(L_w,\mathcal{F})\to . \tag{83}$$

Next consider the diagram with exact rows

$$\begin{array}{ccccc}
R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) & \xrightarrow{\tilde{AV}} & \tilde{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{Z})^{\vee}[-3] & \longrightarrow & \hat{U}_S/U_S[0]\\
 & & \downarrow & & \Vert\\
\tilde{\Psi}_S & \longrightarrow & R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z})^{\vee}[-3] & \longrightarrow & \hat{U}_S/U_S[0]
\end{array} \tag{84}$$

where the lower row is (31). By Lemma 7a), the square commutes, so that (84) can be completed to a map of triangles. We now claim that all rows and columns in the following diagram are exact triangles.

(85)
$$\begin{tikzcd}
\bigoplus\limits_{w\in S_\infty} R\Gamma_\Delta(L_w,\mathbb{Z})^\vee[-3] \arrow[r, equal] \arrow[d] & \bigoplus\limits_{w\in S_\infty} R\Gamma_\Delta(L_w,\mathbb{Z})^\vee[-3] \arrow[d, "\alpha"] & \\
R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) \arrow[r] \arrow[d] & \bigoplus\limits_{w\in S} R\Gamma(L_w,\mathbb{G}_m) \arrow[r] \arrow[d] & R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \arrow[d, equal] \\
\tilde\Psi_S \arrow[r] & \tilde\Psi'_S \arrow[r] & \tilde\Psi''_S.
\end{tikzcd}$$

This is obvious for the middle row and follows for the first column by combining (83) with (84) and using the octahedral axiom. $\tilde\Psi'_S$ is defined to make the last row (or alternatively the middle column) exact and the exactness of the middle column (or last row) then again follows from the octahedral axiom.

Proposition 3.4. *$\tilde\Psi'_S$ has a G-equivariant direct sum decomposition*

$$\tilde\Psi'_S \cong \bigoplus_{w\in S_f} R\Gamma(L_w,\mathbb{G}_m) \oplus \bigoplus_{w\in S_\infty} C_w.$$

Proof. An equivalent way of stating the proposition is that the map α in (85) lies in what we have been calling the diagonal subgroup of

$$\mathrm{Hom}_{\mathfrak{D}}\big(\bigoplus_{v\in S_\infty}\mathrm{Ind}_{G_v}^G R\Gamma_\Delta(L_{w(v)},\mathbb{Z})^\vee[-3],\bigoplus_{v\in S}\mathrm{Ind}_{G_v}^G R\Gamma(L_{w(v)},\mathbb{G}_m)\big).$$

Fix an infinite place w_0 of L with decomposition group G_0 and let K' be the fixed field of G_0. Denote by v_0, resp. v_0', the place of K, resp. K', induced by w_0 and by $\mathfrak{D}(\Gamma)$ the derived category of $\mathbb{Z}[\Gamma]$-modules for any group Γ. What we must show then is that the map which α induces in

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{D}(G)}\big(\mathrm{Ind}_{G_0}^G R\Gamma_\Delta(L_{w_0},\mathbb{Z})^\vee[-3],\bigoplus_{v\in S}\mathrm{Ind}_{G_v}^G R\Gamma(L_{w(v)},\mathbb{G}_m)\big)\\
=&\,\mathrm{Hom}_{\mathfrak{D}(G_0)}\big(R\Gamma_\Delta(L_{w_0},\mathbb{Z})^\vee[-3],\bigoplus_{v\in S}\mathrm{Res}_{G_0}^G\mathrm{Ind}_{G_v}^G R\Gamma(L_{w(v)},\mathbb{G}_m)\big)\\
=&\,\mathrm{Hom}_{\mathfrak{D}(G_0)}\big(R\Gamma_\Delta(L_{w_0},\mathbb{Z})^\vee[-3],\bigoplus_{v'\in S}\mathrm{Ind}_{G_{v'}}^{G_0} R\Gamma(L_{w(v')},\mathbb{G}_m)\big)
\end{aligned}$$

factors through the summand corresponding to v_0', i.e. $R\Gamma(L_{w_0},\mathbb{G}_m)$. Here the last sum is over all places v' of K' lying above a place in S and $G_{v'}$ is the decomposition group inside G_0. So we need to show that for each place $v'\neq v_0'$ of K' the map

$$Y := R\Gamma_\Delta(L_{w_0},\mathbb{Z})^\vee[-3]\xrightarrow{\beta}\mathrm{Ind}_{G_{v'}}^{G_0} R\Gamma(L_{w'},\mathbb{G}_m) =: Z$$

induced by α vanishes in $\mathrm{Hom}_{\mathfrak{D}(G_0)}(Y,Z)$. Fixing such a choice of v' we split the discussion into two cases.

Assume first that $G_0=1$ so that $\mathfrak{D}(G_0)=\mathfrak{D}_{ab}$ is the derived category of abelian groups. Morphisms in this category can be profitably studied via the exact sequence

$$0\to\prod_{i\in\mathbb{Z}}\mathrm{Ext}^1(H^i(Y),H^{i-1}(Z))\to\mathrm{Hom}_{\mathfrak{D}_{ab}}(Y,Z)\to\prod_{i\in\mathbb{Z}}\mathrm{Hom}(H^i(Y),H^i(Z))\to 0$$

which is a consequence of the spectral sequence in [27][III.4.6.10]. It is easy to check that Y has cohomology concentrated in degrees ≥ 3, in fact that

$$Y = \begin{cases} \mathbb{Q}/\mathbb{Z}\,[-3] & w_0 \text{ complex} \\ \mathbb{Q}/\mathbb{Z} \xrightarrow{0} \mathbb{Q}/\mathbb{Z} \xrightarrow{2} \mathbb{Q}/\mathbb{Z} \xrightarrow{0} \cdots [-3] & w_0 \text{ real.} \end{cases}$$

If w' is not real one finds that $\mathrm{Hom}_{\mathfrak{D}_{ab}}(Y, Z) = 0$ using that $R\Gamma(L_{w'}, \mathbb{G}_m)$ has cohomology concentrated in degrees ≤ 2 and that $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, Br(L_{w'})) = 0$ since $Br(L_{w'})$ is an injective abelian group. If w' is real on the other hand, we pick a $\lambda \in L = K'$ which is negative at $w' = v'$ and positive at $w_0 = v_0'$, in case w_0 is real. Such a λ exists because of the independence of the valuations v' and v_0'. In the quadratic extension $N = L(\sqrt{\lambda})/K'$ the place v' ramifies whereas v_0' splits. From Lemma 11 below, with L/K replaced by N/L, G by $\Gamma := \mathrm{Gal}(N/L) \cong \mathbb{Z}/2\mathbb{Z}$ and $H = G$, we find that $\beta = R\,\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \beta')$ where

$$\beta' \in \mathrm{Hom}_{\mathfrak{D}(\Gamma)}(\mathrm{Ind}_1^\Gamma Y, R\Gamma(N_{w'}, \mathbb{G}_m)) = \mathrm{Hom}_{\mathfrak{D}_{ab}}(Y, \mathbb{C}^\times[0]) = 0.$$

Hence $\beta = 0$.

If now $G_0 \cong \mathbb{Z}/2\mathbb{Z}$, w_0 is complex and $Y \cong \mathbb{Q}/\mathbb{Z}[-3]$. We first dispose of the case where v' is complex. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{D}(G_0)}(Y, Z) \\ &= \mathrm{Hom}_{\mathfrak{D}(G_0)}(\mathbb{Q}/\mathbb{Z}[-3], \mathrm{Ind}_1^{G_0} \mathbb{C}^\times[0]) = \mathrm{Hom}_{\mathfrak{D}_{ab}}(\mathbb{Q}/\mathbb{Z}[-3], \mathbb{C}^\times[0]) = 0. \end{aligned}$$

If v' is not complex we can choose a quadratic extension $K'(\sqrt{\lambda})/K'$ in which v' ramifies or stays inert whereas v_0' splits. If $L_{w'} \neq K'_{v'}$ we also require that the quadratic extension $K'_{v'}(\sqrt{\lambda})/K'_{v'}$ is isomorphic to $L_{w'}/K'_{v'}$. The existence of λ again follows from the independence of the valuations v' and v_0'. Since v_0' ramifies in L/K' we have $L \neq K'(\sqrt{\lambda})$ and the field $N := L(\sqrt{\lambda})$ has Galois group $\Gamma := G_0 \times G_1 \cong (\mathbb{Z}/2\mathbb{Z})^2$ over K' where $G_1 := \mathrm{Gal}(K'(\sqrt{\lambda})/K')$. Again, from Lemma 11 below, applied to the extension N/K' and the subgroup $1 \times G_1$ of Γ, we have $\beta = R\,\mathrm{Hom}_{\mathbb{Z}[1\times G_1]}(\mathbb{Z}, \beta')$ where

$$\beta' \in \mathrm{Hom}_{\mathfrak{D}(\Gamma)}(\mathrm{Ind}_{G_0\times 1}^\Gamma \mathbb{Q}/\mathbb{Z}[-3],\, \mathrm{Ind}_\Delta^\Gamma R\Gamma(N_{w'}, \mathbb{G}_m)).$$

Note here that $G_0 \times 1$ is the decomposition group of v_0' in N/K' and Δ is by definition the decomposition group of v'. Our assumptions imply that $\Delta \cap G_0 \times 1 = \{1\}$. Indeed, if v' splits in L we have $\Delta = 1 \times G_1$; otherwise Δ is the diagonal subgroup in $G_0 \times G_1$ since then v' does not split in either L or $K'(\sqrt{\lambda})$ and $N_{w'} = K'_{v'}(\sqrt{\lambda})$. By standard formulas for induction and restriction [2][III.5.6] we get

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{D}(\Gamma)}(\mathrm{Ind}_{G_0\times 1}^\Gamma \mathbb{Q}/\mathbb{Z}[-3],\, \mathrm{Ind}_\Delta^\Gamma R\Gamma(N_{w'}, \mathbb{G}_m)) \\ &= \mathrm{Hom}_{\mathfrak{D}(G_0\times 1)}(\mathbb{Q}/\mathbb{Z}[-3],\, \mathrm{Res}_{G_0\times 1}^\Gamma \mathrm{Ind}_\Delta^\Gamma R\Gamma(N_{w'}, \mathbb{G}_m)) \\ &= \mathrm{Hom}_{\mathfrak{D}(G_0\times 1)}(\mathbb{Q}/\mathbb{Z}[-3],\, \mathrm{Ind}_1^{G_0\times 1} \mathrm{Res}_1^\Delta R\Gamma(N_{w'}, \mathbb{G}_m)) \\ &= \mathrm{Hom}_{\mathfrak{D}_{ab}}(\mathbb{Q}/\mathbb{Z}[-3],\, R\Gamma(N_{w'}, \mathbb{G}_m)). \end{aligned}$$

The latter group has been shown to be zero in the discussion of the case $G_0 = 1$, w' not real, above. But by our construction of N we have $N_{w'}/K'_{v'} \cong \mathbb{C}/\mathbb{R}$ if w' is archimedean. We deduce that $\beta' = 0$ and therefore $\beta = 0$ and this finishes the proof of Proposition 3.4. $\square$

Remark. Proposition 3.4 can partly be proven using the compatibility of local and global Artin-Verdier duality. However, this method seems to be incapable of identifying the map β if $G_0 \cong \mathbb{Z}/2\mathbb{Z}$ and v' is a ramified infinite place.

Lemma 11. *Let $H \lhd G$ be a normal subgroup and consider the diagram (85) in $\mathfrak{D} = \mathfrak{D}(G)$. Then the diagram obtained by applying $R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, -)$ to (85) is naturally isomorphic, in the derived category of $\mathbb{Z}[G/H]$-modules, to the corresponding diagram formed with respect to the extension L^H/K.*

Proof. Since $\operatorname{Spec} \mathcal{O}_{L,S} \to \operatorname{Spec} \mathcal{O}_{K,S}$ is a Galois cover, the identities

$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, R\Gamma(\mathcal{O}_{L,S}, \mathcal{F})) = R\Gamma(\mathcal{O}_{L^H,S}, \mathcal{F})$$
$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, R\Gamma(L \otimes_K K_v, \mathcal{F})) = R\Gamma(L^H \otimes_K K_v, , \mathcal{F})$$

and hence

$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})) = R\Gamma_c(\mathcal{O}_{L^H,S}, \mathcal{F})$$

are certainly well known for any étale sheaf $\mathcal{F}$ on $\mathcal{O}_{K,S}$ [21][III.2.20]. More concretely, each of the complexes in the middle row of (85) can be represented by a bounded below complex C of cohomologically trivial G-modules and

$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, C) = C^H \tag{86}$$

gives the corresponding complex over L^H. If now v is an archimedean place of K it is still true that $C := R\Gamma_{\mathrm{Tate}}(L \otimes_K K_v, \mathcal{F})$ can be represented by a complex of c.t. G-modules such that $C^H = R\Gamma_{\mathrm{Tate}}(L^H \otimes_K K_v, \mathcal{F})$ but (86) no longer holds since C may be unbounded in both directions. In any case, in both triangles (82) and (83) each complex is represented by c.t. G-modules and taking H-invariants gives the corresponding complex over L^H. Moreover, both $X = R\Gamma_\Delta(L \otimes_K K_v, \mathcal{F})$ and $X = R\tilde{\Gamma}_c(\mathcal{O}_{L,S}, \mathcal{F})$ are bounded above complexes of c.t. G-modules. Since for any c.t. G-module M the natural map $M_H \to M^H$ is an isomorphism, taking H-invariants actually computes the *homology* of X, i.e. $X^H = X_H = \mathbb{Z} \otimes^L_{\mathbb{Z}[H]} X$. Utilizing the natural isomorphisms

$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, \operatorname{Hom}(X, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}(\mathbb{Z} \otimes^L_{\mathbb{Z}[H]} X, \mathbb{Q}/\mathbb{Z})$$

and

$$R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, \hat{U}_S/U_S) = (\hat{U}_S/U_S)^H = \hat{U}_{L^H,S}/U_{L^H,S}$$

we find that for all terms in the diagrams (84) and (85) the following is true: Application of $R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, -)$ gives the corresponding term over L^H. As to the maps between them these are either obvious or constructed from the duality map $\tilde{\mathrm{AV}}$ in (84). The latter is in turn induced by the G-equivariant pairing of complexes (34) (for $R\tilde{\Gamma}_c$), and the corresponding pairing over L^H just arises by taking H invariants, once the complexes are represented by c.t. G-modules. Hence $R\operatorname{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, \tilde{\mathrm{AV}})$ is the duality map over L^H. $\square$

The exact triangle $\tilde{\Psi}_S \to \tilde{\Psi}'_S \to \tilde{\Psi}''_S \to$ in (85) is essentially our candidate for Lemma 4. Computation of the cohomology of (85) (see Proposition 3.5 below) shows that $\tilde{\Psi}''_S$ represents a class in $\operatorname{Ext}^3_G(\mathbb{Q}/\mathbb{Z}, C_S)$ in a canonical way, and that

$\tilde{\Psi}'_S$ represents a class in $\mathrm{Ext}^3_G(Y_S \otimes \mathbb{Q}/\mathbb{Z}, J_S)$. For example, H^2 of the diagram (85) gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Br}(\mathcal{O}_{L,S}) \arrow[r] \arrow[d] & \bigoplus_{w \in S} \mathrm{Br}(L_w) \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X_S \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & H^2(\tilde{\Psi}'_S) \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{87}$$

which implies that $H^2(\tilde{\Psi}'_S) \cong Y_S \otimes \mathbb{Q}/\mathbb{Z}$, and in fact that the lower row is isomorphic to the right hand column of (22) tensored with $\mathbb{Q}/\mathbb{Z}$.

We can then construct a commutative diagram where all rows and columns are exact triangles, and the right hand column is a candidate for Lemma 4:

$$\begin{tikzcd} X_S \otimes \mathbb{Q}[-2] \arrow[r] \arrow[d] & \tilde{\Psi}_S \arrow[r] \arrow[d] & \Psi_S \arrow[r] \arrow[d] & {} \\ Y_S \otimes \mathbb{Q}[-2] \arrow[r] \arrow[d] & \tilde{\Psi}'_S \arrow[r] \arrow[d] & \Psi'_S \arrow[r] \arrow[d] & {} \\ \mathbb{Q}[-2] \arrow[r] \arrow[d] & \tilde{\Psi}''_S \arrow[r] \arrow[d] & \Psi''_S \arrow[r] \arrow[d] & {} \\ {} & {} & {} & \end{tikzcd} \tag{88}$$

Here the top row is (32) and the middle and bottom row are analogous, that is Ψ'_S (resp. Ψ''_S) represents the image of $e(\tilde{\Psi}'_S)$ (resp. $e(\tilde{\Psi}''_S)$) in $\mathrm{Ext}^2_G(Y_S, J_S)$ (resp. $\mathrm{Ext}^2_G(\mathbb{Z}, C_S)$) under the boundary map induced by (38) tensored with Y_S (resp. (38)). The existence of (88) follows from the commutativity of the left hand squares in (88) which in turn is a consequence of Lemma 7b).

It remains to compute the precise extension classes represented by Ψ'_S and Ψ''_S or, equivalently, $\tilde{\Psi}'_S$ and $\tilde{\Psi}''_S$.

Proposition 3.5. *a) Let K be a local field and L/K a finite Galois extension with group G. Consider the complex of G-modules $R\Gamma(L, \mathbb{G}_m)$. One has $H^i(L, \mathbb{G}_m) = 0$ for $i \neq 0, 2$ and canonical isomorphisms*

$$H^0(L, \mathbb{G}_m) \xleftarrow{\sim} L^\times, \quad H^2(L, \mathbb{G}_m) \xrightarrow[\sim]{\mathrm{inv}_L} \mathbb{Q}/\mathbb{Z},$$

so that $\tau^{\leq 2} R\Gamma(L, \mathbb{G}_m)$ represents a class $r_{L/K} \in \mathrm{Ext}^3_G(\mathbb{Q}/\mathbb{Z}, L^\times) \simeq H^2(G, L^\times)$. Claim: $\mathrm{inv}_K(r_{L/K}) = \frac{1}{|G|}$.

b) Let K be a global field and L/K a finite Galois extension with group G, unramified outside a finite set S of places of K containing S_∞. Consider the complex of G-modules $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$. One has $H^i_c(\mathcal{O}_{L,S}, \mathbb{G}_m) = 0$ for $i \neq 1, 3$ and canonical isomorphisms

$$H^1_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \xleftarrow{\sim} C_S(L), \quad H^3_c(\mathcal{O}_{L,S}, \mathbb{G}_m) \xrightarrow[\sim]{\mathrm{trace}_L} \mathbb{Q}/\mathbb{Z},$$

so that $\tau^{\geq 1} \tau^{\leq 3} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ represents a class $r_{L/K} \in \mathrm{Ext}^3_G(\mathbb{Q}/\mathbb{Z}, C_S(L)) \simeq H^2(G, C_S(L))$. Claim: $\mathrm{inv}_K(r_{L/K}) = \frac{1}{|G|}$.

Here $C_S(L)$ is the group belonging to the S-class formation introduced in [22, Ch.I,§4] *and agrees with the group $C_S = J_S/U_S$ used above if* $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$.

Proof. The computation of the cohomology of $\mathbb{G}_m$ and the construction of the invariant and trace map is standard (see [25, Ch. XIII, Prop. 6], [22, Ch.II,Prop. 2.6]). In the local case the claim essentially follows from [25, Ch. XIII, exercise 2] and the global case follows from the local case. However, we wish to explain this in more detail. In both cases one first shows

Lemma 12. *$r_{L/K}$ is the fundamental class of a class formation, i.e. $r_{L/K}$ generates $H^2(G, L^\times)$ (resp. $H^2(G, C_S(L))$) and for a tower of extensions $L \supseteq L' \supseteq K$ (unramified outside S in the global case) one has*

$$\mathrm{res}^G_H(r_{L/K}) = r_{L/L'} \tag{89}$$

$$\mathrm{inf}^G_{G/H}(r_{L'/K}) = [L : L']r_{L/K} \tag{90}$$

where $H = G(L/L')$.

Proof. Assume K local. Let

$$\mathbb{G}_m \hookrightarrow I^\bullet \tag{91}$$

be an injective resolution over $(\mathrm{Spec}\, K)_{et}$. Then for any finite extension L/K, $R\Gamma(L, \mathbb{G}_m)$ can be identified with $H^0(L, I^\bullet)$ and therefore consists of cohomologically trivial G-modules. It is acyclic in degrees ≥ 3 for any such L/K, hence a standard argument (cf. [5, step 2 in Prop. 1.20] or [14, Prop. and Lemma 2.1]) shows that $\tau^{\leq 2} R\Gamma(L, \mathbb{G}_m)$ still consists of c.t. G-modules. So in the exact sequence representing $r_{L/K}$

$$0 \to L^\times \to A \to B \to C \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{92}$$

A, B, C are c.t. G-modules, which means that cup product with $r_{L/K}$ induces an isomorphism

$$\hat{H}^i(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \hat{H}^{i+3}(G, L^\times)$$

for all $i \in \mathbb{Z}$. The case $i = -1$ of this isomorphism implies that $r_{L/K}$ is a generator of $H^2(G, L^\times)$. The same argument applies to the global case. Indeed, the complex $M^\bullet := \tau^{\leq 3} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ consists of c.t. G-modules and so does $\tau^{\geq 1}\tau^{\leq 3} R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$ because $\tau^{\geq 1}$ only involves replacing M_1 by M_1/M_0 and M_0 by 0 ($M_i = 0$ for $i < 0$).

Formula (89) follows because (92) can be regarded over the subgroup $G(L/L')$ of $G(L/K)$ and represents $r_{L/L'}$ since (91) also gives a resolution of $\mathbb{G}_m$ over $(\mathrm{Spec}\, L')_{et}$. As to (90) the natural map

$$R\Gamma(L', \mathbb{G}_m) \to R\Gamma(L, \mathbb{G}_m)$$

induced by $\mathrm{Spec}\, L \to \mathrm{Spec}\, L'$ gives a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & L'^\times & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle [L':L]} & & \\ 0 & \longrightarrow & L^\times & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array} \tag{93}$$

The top row represents $r_{L'/K}$ and we get $\mathrm{inf}^G_{G/H}(r_{L'/K})$ by regarding the top row as a sequence of G-modules and taking the pushout under the inclusion $L'^\times \hookrightarrow L^\times$, which yields a class in $\mathrm{Ext}^3_G(\mathbb{Q}/\mathbb{Z}, L^\times) = H^2(G, L^\times)$. This pushout agrees with the pullback of the bottom row under multiplication by $[L : L']$ on $\mathbb{Q}/\mathbb{Z}$ because (93) is commutative. But the bottom row represents $r_{L/K}$, hence the formula (90).

The proof in the global case is entirely similar (see [22, Ch.II §3] for the relevant properties of the trace map). $\square$

Lemma 13. *If L/K is an unramified extension of local fields then* $\mathrm{inv}_K(r_{L/K}) = \frac{1}{|G|}$.

Proof. We construct a particular acyclic resolution of $\mathbb{G}_m$ on $(\operatorname{Spec} K)_{et}$ using [25, XIII, ex. 2]. Let K_{ur}/K be the completion of the maximal unramified extension of K, N/K any finite Galois extension and $N_{ur} = NK_{ur}, N_0 = N \cap K_{ur}$ where we consider all fields inside a fixed complete algebraically closed extension of K. Let $F \in G(N_{ur}/K)$ be any choice of a Frobenius automorphism and ϕ (resp. f) its image in $G(N/K)$ (resp. $G(N_0/K)$). Consider the commutative diagram of $G(N/K)$-modules with exact rows

$$\begin{array}{ccccccccc} 1 \to & N^\times & \to & (N \otimes_K K_{ur})^\times & \xrightarrow{1-1\otimes F} & (N \otimes_K K_{ur})^\times & \to & \mathbb{Z} & \to 0 \\ & {\scriptstyle v}\big\downarrow & & {\scriptstyle \tilde{v}}\big\downarrow & & {\scriptstyle \tilde{v}}\big\downarrow & & \| & \\ 0 \to & \mathbb{Z} & \to & C(G(N_0/K), \mathbb{Z}) & \xrightarrow{1-f^{-1}} & C(G(N_0/K), \mathbb{Z}) & \xrightarrow{\varepsilon} & \mathbb{Z} & \to 0. \end{array} \tag{94}$$

Here $C(-, \mathbb{Z})$ is the module of $\mathbb{Z}$-valued functions with $G(N_0/K)$-action explained before Lemma 5. Setting $m = [N_0 : K]$ one has a ring isomorphism

$$N \otimes_K K_{ur} \xrightarrow{\sim} \prod_{i=0}^{m-1} N_{ur}, \quad x \otimes y \mapsto (\phi^i(x)y)_i,$$

and in this description

$$\tilde{v}(y_0, ..., y_{m-1})(f^i) = v(y_{-i})$$

where $v : N_{ur}^\times \to \mathbb{Z}$ is the valuation normalized by $v(\pi) = 1$ for a uniformizer π. The action of $1 \otimes F$ translates into

$$(1 \otimes F)(y_0, ..., y_{m-1}) = (F\phi^{-m}(y_{m-1}), F(y_0), F(y_1), .., F(y_{m-2})).$$

Here ϕ^{-m} lies in the inertia subgroup of $G(N/K)$, so it can be canonically viewed inside $G(N_{ur}/K)$. The commutativity of (94) and the $G(N/K)$-equivariance of $\tilde{v}$ follow easily from these descriptions. The bottom row is a standard exact sequence, the exactness of the top row follows from the commutativity and the surjectivity of $1 - F$ on $\ker(\tilde{v})$ ([25, XIII, Prop. 15]). One has

$$(N \otimes_K K_{ur})^\times \cong \operatorname{Ind}_{G(N/N_0)}^{G(N/K)} N_{ur}^\times$$

so by [25, X, Prop. 11] this is a c.t. $G(N/K)$-module.

If N'/N is a finite extension such that N'/K is Galois the diagram (94) naturally maps into the corresponding diagram for N' where the map on the right hand $\mathbb{Z}$ is multiplication by $[N' : N]$. The map on $C(G(N_0/K), \mathbb{Z})$ is multiplication by the ramification degree $e(N'/N)$ followed by the natural inclusion $C(G(N_0/K), \mathbb{Z}) \to C(G(N_0'/K), \mathbb{Z})$. So after taking the direct limit over all N the top row gives a resolution

$$\mathbb{G}_m \hookrightarrow A \to B \to \mathbb{Q} \to 0 \to \cdots \tag{95}$$

by acyclic, discrete $G(\overline{K}/K)$-modules, at least if we also fix an isomorphism of $\mathbb{Q}$ with the direct limit of the right hand $\mathbb{Z}$'s. We do this by prescribing it to be the inclusion $\mathbb{Z} \subset \mathbb{Q}$ for $N = K$. In this way we obtain an identification of $R\Gamma(N, \mathbb{G}_m)$ with

$$(N \otimes_K K_{ur})^\times \xrightarrow{1-1\otimes F} (N \otimes_K K_{ur})^\times \xrightarrow{w_N} \mathbb{Q}$$

where $w_N = \frac{1}{[N:K]} \varepsilon \tilde{v}$. However, we must still identify the map inv_N.

Lemma 14. *The map*

$$\mathrm{inv}_N : \mathbb{Q}/\operatorname{im}(w_N) = \mathbb{Q}/\frac{1}{[N:K]}\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

is induced by multiplication with $-[N:K]$ *on* $\mathbb{Q}$.

Proof. It suffices to prove this for $N = K$ because of the commutative diagram

$$\begin{tikzcd}
(N \otimes_K K_{ur})^\times \arrow[r, "1-1\otimes F"] & (N \otimes_K K_{ur})^\times \arrow[r, "w_N"] & \mathbb{Q} \arrow[r, "\mathrm{inv}_N"] & \mathbb{Q}/\mathbb{Z} \\
K_{ur}^\times \arrow[u] \arrow[r, "1-1\otimes F"] & K_{ur}^\times \arrow[u] \arrow[r, "w_K"] & \mathbb{Q} \arrow[u, equal] \arrow[r, "\mathrm{inv}_K"] & \mathbb{Q}/\mathbb{Z}. \arrow[u, "{[N:K]}"']
\end{tikzcd}$$

Passing to the limit over all unramified N/K in (94) we get the first two rows of the following commutative diagram of discrete $\Gamma := G(K_{ur}/K)$-modules

$$\begin{tikzcd}
K_{ur}^\times \arrow[r] \arrow[d, "v"'] & (K_{ur} \otimes_K K_{ur})^\times \arrow[r, "1-1\otimes F"] \arrow[d, "\tilde{v}"'] & (K_{ur} \otimes_K K_{ur})^\times \arrow[r] \arrow[d, "\tilde{v}"'] & \mathbb{Q} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z} \arrow[r] \arrow[d] & C(\Gamma, \mathbb{Z}) \arrow[r, "1-F^{-1}"] \arrow[d] & C(\Gamma, \mathbb{Z}) \arrow[r] \arrow[d] & \mathbb{Q} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Q} \arrow[r] \arrow[d] & C(\Gamma, \mathbb{Q}) \arrow[r, "1-F^{-1}"] \arrow[d] & C(\Gamma, \mathbb{Q}) \arrow[r, "\epsilon"] \arrow[d, "-F"] & \mathbb{Q} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Q}/\mathbb{Z} \arrow[r] & C(\Gamma, \mathbb{Q}/\mathbb{Z}) \arrow[r, "1-F"] & C(\Gamma, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0 &
\end{tikzcd} \tag{96}$$

where $\epsilon(f) = n^{-1} \sum_{\gamma \mod n\Gamma} f(\gamma)$ if f factors through $\Gamma/n\Gamma$. All the rows in this diagram are Γ-acyclic resolutions of their left hand terms. The top row has a natural map into (95) inducing the inflation map $H^i(\Gamma, K_{ur}^\times) \to H^i(K, \mathbb{G}_m)$. This is an isomorphism for $i = 2$ which is the first step in the construction of inv_K. One then uses the chain of isomorphisms

$$H^2(\Gamma, K_{ur}^\times) \xrightarrow{v_*} H^2(\Gamma, \mathbb{Z}) \xleftarrow{\delta} H^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\iota} \operatorname{Hom}(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathbb{Q}/\mathbb{Z} \tag{97}$$

where the last map is evaluation at F. All of these steps can be made explicit using (96). Upon taking Γ-invariants in (96) we obtain the following commutative diagram of complexes.

$$
\begin{array}{rcccccc}
R\Gamma(\Gamma, K_{ur}^\times): & K_{ur}^\times & \xrightarrow{1-F} & K_{ur}^\times & \longrightarrow & \mathbb{Q} \\
& \downarrow{\scriptstyle v} & & \downarrow & & \| \\
R\Gamma(\Gamma, \mathbb{Z}): & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\subset} & \mathbb{Q} \\
& \downarrow & & \downarrow & & \| \\
R\Gamma(\Gamma, \mathbb{Q}): & \mathbb{Q} & \xrightarrow{0} & \mathbb{Q} & = & \mathbb{Q} \\
& \downarrow & & \downarrow{\scriptstyle -1} & & \downarrow \\
R\Gamma(\Gamma, \mathbb{Q}/\mathbb{Z}): & \mathbb{Q}/\mathbb{Z} & \xrightarrow{0} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array} \tag{98}
$$

By Lemma 5 applied to $\Gamma' = \Gamma$ we have that $\iota^{-1} \circ \mathrm{ev}^{-1}(x)$ is indeed the constant function with value $x \in \mathbb{Q}/\mathbb{Z}$ (the bottom right group in (98)). The remaining isomorphisms in (97) can then be easily identified in (98), and one obtains the desired description of inv_K. $\square$

To finish the proof of Lemma 13 we look again at (94) for $L = N = N_0$ an unramified extension of K, so that $m = [L : K]$. By our computation of inv_L the class $r_{L/K}$ equals the Yoneda class of the 3-extension of G-modules

$$0 \to L^\times \to (L \otimes_K K_{ur})^\times \xrightarrow{1-1\otimes F} (L \otimes_K K_{ur})^\times \xrightarrow{w_L} \mathbb{Q} \xrightarrow{-[L:K]} \mathbb{Q}/\mathbb{Z} \to 0$$

where $w_L = \frac{1}{[L:K]} \varepsilon \tilde{v}$. So it is easily seen that $-r_{L/K}$ is the image of the element $r'_{L/K} \in \mathrm{Ext}^2_G(\mathbb{Z}, L^\times) = H^2(G, L^\times)$ represented by the top row of (94), under the boundary map induced by the short exact sequence (38). Here we use again Lemma 3. The pushout $r''_{L/K} \in \mathrm{Ext}^2_G(\mathbb{Z}, \mathbb{Z}) = H^2(G, \mathbb{Z})$ of $r'_{L/K}$ under the map v is represented by the second row in (94) and we shall show that $\mathrm{inv}_K(r''_{L/K}) = -\frac{1}{m}$.

Consider the commutative diagram of G-modules

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C(G, \mathbb{Z}) & \xrightarrow{1-f^{-1}} & C(G, \mathbb{Z}) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle \frac{1}{m}\epsilon} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \xrightarrow{(\pi,0)} & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array} \tag{99}
$$

where

$$\beta(\psi) = (\sum_{i=0}^{m-1} \psi(f^i) \frac{i}{m}, \sum_{i=0}^{m-1} \psi(f^i))$$

and the group $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}$ becomes a G-module by

$$g(a, b) = (a - \phi(g)b, b) \tag{100}$$

where $\phi : G \to \mathbb{Q}/\mathbb{Z}$ is the homomorphism such that $\phi(f) = -\frac{1}{m}$. In view of Lemma 3 diagram (99) shows that $r''_{L/K}$ maps to ϕ under the chain of isomorphisms

$$\mathrm{Ext}^2_G(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Ext}^1_G(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}).$$

Note that

$$0 \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

with G-action on the middle term given by (100) represents the 1-extension corresponding to ϕ because (100) is the pushout of the universal 1-extension

$$0 \to I[G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

under the G-homomorphism $\Phi : I[G] \to \mathbb{Q}/\mathbb{Z}$ given by $\Phi(g-1) = \phi(g)$. Following through the definition of inv_K we conclude that $\mathrm{inv}_K(r''_{L/K}) = -\frac{1}{m}$ and this finishes the proof of Lemma 13. □

If now L/K is an arbitrary Galois extension of local fields, there is a (unique) unramified extension N/K with $[N:K] = [L:K]$. Setting $M = NL$, we have

$$\mathrm{inf}_{G(L/K)}^{G(M/K)}(r_{L/K}) = [M:L]r_{M/K} = [M:N]r_{M/K} = \mathrm{inf}_{G(N/K)}^{G(M/K)}(r_{N/K}) \tag{101}$$

by Lemma 12. Since invariants are unchanged under inflation we obtain

$$\mathrm{inv}_K(r_{L/K}) = \mathrm{inv}_K(r_{N/K}) = \frac{1}{|G|}$$

by Lemma 13. This gives part a) of Proposition 3.5.

Given an extension L/K of global fields we can find an extension N/K with $[L:K] = [N:K] = [N_{w_0} : K_{v_0}]$ for some (non-archimedean) place w_0 of N, for example a suitable cyclotomic extension. Here v_0 is the place of K induced by w_0. After enlarging S we can assume that N/K is unramified outside S and that $v_0 \in S$. Enlarging S to S', say, is unproblematic because the natural map

$$R\Gamma_c(\mathcal{O}_{L,S'}, \mathbb{G}_m) \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$$

induces an isomorphism on H^3 [22, Ch.II,§3] and the natural map $\rho : C_{S'}(L) \to C_S(L)$ on H^1. This means that the pushout under ρ of $r_{L/K}$, defined with S', is $r_{L/K}$ defined with S. Moreover ρ induces an isomorphism

$$H^2(G, C_{S'}(L)) \xrightarrow{\sim} H^2(G, C_S(L))$$

which commutes with inv_K (the invariant map for the respective S or S' class formation), and so we can compute $\mathrm{inv}_K(r_{L/K})$ using either S or S'.

Taking such a field N and applying (101) we are reduced to the case that G coincides with the decomposition group G_{w_0} for some place $w_0|v_0 \in S$. So it remains to find a connection between the local and global $r_{L/K}$ in this case. The natural map

$$R\Gamma(L_{w_0}, \mathbb{G}_m) \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$$

gives a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & L_{w_0}^\times & \to & A' & \to & B' & \to & C' & \to & \mathbb{Q}/\mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_S(L) & \to & A & \to & B & \to & C & \to & \mathbb{Q}/\mathbb{Z} & \to & 0 \end{array} \tag{102}$$

which induces for each integer i a commutative diagram

$$\begin{array}{cccc} \cup r_{L_{w_0}/K_{v_0}} : & \hat{H}^i(G_{w_0}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sim} & \hat{H}^{i+3}(G_{w_0}, L_{w_0}^\times) \\ & \downarrow & & \downarrow \\ \cup r_{L/K} : & \hat{H}^i(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sim} & \hat{H}^{i+3}(G, C_S(L)). \end{array}$$

Now take $i = -1$, so that $\hat{H}^{-1}(G_{w_0}, \mathbb{Q}/\mathbb{Z}) \cong \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \cong \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$. Consider the element $\frac{1}{|G_{w_0}|} = \frac{1}{|G|}$ in the top left group. It is mapped to $r_{L_{w_0}/K_{v_0}}$ in the top right group, and to $r_{L/K}$ in the bottom right group. Since the right hand map for $i = -1$ commutes with taking invariants we conclude $\mathrm{inv}_K(r_{L/K}) = \mathrm{inv}_{K_{v_0}}(r_{L_{w_0}/K_{v_0}}) = \frac{1}{|G|}$ by part a) of Proposition 3.5. $\square$

Theorem 3.2. *We have* $\Omega(L/K, 3) = \Omega(L/K, \mathbb{Q}(0))^{\#}$.

Proof. Proposition 3.5b) implies that $e(\Psi''_S) = c_{L/K}$. As to $e(\Psi'_S)$, we know from Proposition 3.4 that $e(\tilde{\Psi}'_S)$ lies in the diagonal subgroup

$$\bigoplus_{v\in S} \mathrm{Ext}^3_{G_v}(\mathbb{Q}/\mathbb{Z}, L^\times_{w(v)}) \subseteq \mathrm{Ext}^3_G(Y_S \otimes \mathbb{Q}/\mathbb{Z}, J_S)$$

and Proposition 3.5a) and Proposition 3.4 imply that the components of $c^{loc}_{L/K}$ and $e(\Psi'_S)$ agree for non-archimedean places v. For archimedean places v on the other hand, the group $\mathrm{Ext}^3_{G_v}(\mathbb{Q}/\mathbb{Z}, L^\times_{w(v)})$ is either trivial or of order 2, the local canonical class $c_{L_{w(v)}/K_v}$ being the only non-zero element in the latter case. To show that the class $r_v \in \mathrm{Ext}^3_{G_v}(\mathbb{Q}/\mathbb{Z}, L^\times_{w(v)})$ represented by $\bigoplus_{w|v} C_w$ is nontrivial for $|G_v| = 2$ we can argue as follows. As in the proof of Proposition 3.5 the map

$$\bigoplus_{w|v} C_w \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)[1]$$

gives a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \to & \bigoplus_{w|v} L^\times_w & \to & A' & \to & B' & \to & C' & \to & \bigoplus_{w|v} \mathbb{Q}/\mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma} & & \\
0 & \to & C_S(L) & \to & A & \to & B & \to & C & \to & \mathbb{Q}/\mathbb{Z} & \to & 0
\end{array}$$

which induces for each integer i a commutative diagram in G-cohomology

$$\begin{array}{rcl}
\cup r_v : \ \hat{H}^i(G_v, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \hat{H}^{i+3}(G_v, L^\times_{w(v)}) \\
\downarrow{\scriptstyle \Sigma_*} & & \downarrow \\
\cup r_{L/K} : \hat{H}^i(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sim} & \hat{H}^{i+3}(G, C_S(L)).
\end{array} \tag{103}$$

The fact that Σ is the sum map is a consequence of (87). We do not know a priori that the top arrow in (103) is an isomorphism but it is easy to check that Σ_* is injective for $i = -1$. So $r_v = 1/2 \cup r_v \in H^2(G_v, L^\times_{w(v)})$ cannot be zero.

We conclude that the right hand column in (88) satisfies the assumptions of Lemma 4 and that K_S and Ψ_S are quasi-isomophic. By Theorem 3.1 and (28) we get $\Omega(L/K, 3) = \Omega(L/K, \mathbb{Q}(0))^{\#}$. $\square$

Remark. Theorem 3.2 gives a positive answer to the first half of Question 1.54 in [5] which asks whether the identity $\Omega(L/K, 3) = \Omega(L/K, \mathbb{Q}(0))^{\#}$ holds. The second half of Question 1.54 (ibid), asking whether Chinburg's invariant $\Omega(L/K, 1)$ equals $-\Omega(L/K, \mathbb{Q}(1))$, will follow from forthcoming work of the first author [4] in conjunction with Theorem 3.2.

Theorem 3.2 is also consistent with, on the one hand, Chinburg's conjecture that $\Omega(L/K,3) = 0$ for abelian G and on the other hand, Conjecture 4 of [5] which predicts the vanishing of $\Omega(L/K, M)$ for any motive M.

4. The case $n > 0$

4.1. **Definition of $\Omega_n(L/K)$.** In this subsection we recall the definition of the class $\Omega_n(L/K)$ from [9]. We fix a finite set S of places as in section 2 and define for any integer $r = n+1 > 1$ and any prime number p

$$K_p(r) := \mathrm{Cone}\big(R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r)) \to \bigoplus_{w|\infty} R\Gamma(L_w, \mathbb{Z}_p(r))\big) \tag{104}$$

Lemma 15. *The complex $K_p(r)$ is quasi-isomorphic to $R\Gamma(\mathcal{O}_{L,S_p}, C_p(r))$ where $C_p(r)$ is the cokernel of the natural injection $\mathbb{Z}_p(r) \to \bigoplus_{w|\infty} i_{w,*}i_w^*\mathbb{Z}_p(r)$ of sheaves on $(\mathrm{Spec}\ \mathcal{O}_{L,S_p})_{et}$. Here $i_w : \mathrm{Spec}\ L_w \to \mathrm{Spec}\ \mathcal{O}_{L,S_p}$ is the natural map.*

Proof. This follows easily from $R^q i_{w,*}\mathbb{Z}_p(r) = 0$ for $q > 0$ (see [21, III 1.13]) which implies that $R\Gamma(\mathcal{O}_{L,S_p}, i_{w,*}i_w^*\mathbb{Z}_p(r)) = R\Gamma(L_w, \mathbb{Z}_p(r))$. □

Remark. We have inserted this lemma for the convenience of the reader because the authors of [9] do not directly refer to the complex $K_p(r)$ but rather work with the cohomology groups of $C_p(r)$.

One shows that $K_p(r) \cong R\Gamma(\mathcal{O}_{L,S_p}, C_p(r))$ is $\mathbb{Z}_p[G]$-perfect (as in [5, Prop. 1.20]), has cohomology concentrated in degrees $0, 1$ (using Artin-Verdier duality) and that there are finitely generated $\mathbb{Z}[G]$-modules K'_{2r-i} with isomorphisms

$$\tau_p^i : K'_{2r-1-i} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong H^i(K_p(r))$$

for $i = 0, 1$ and each prime p ([9]). The group K'_{2r-2} is finite and there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{w|\infty} H^0(L_w, \mathbb{Q}(r)) & \to & K'_{2r-1} \otimes \mathbb{Q} & \to & K_{2r-1}(\mathcal{O}_L) \otimes \mathbb{Q} & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \tau_p^0 \otimes \mathbb{Q}} & & \downarrow{\scriptstyle \mathrm{Chern}} & & \\ 0 & \to & \bigoplus_{w|\infty} H^0(L_w, \mathbb{Q}_p(r)) & \to & H^0(K_p(r) \otimes \mathbb{Q}_p) & \to & H^1(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(r)) & \to & 0 \end{array} \tag{105}$$

The top row here is equation (5) in [9], and the bottom row is induced from (104).

From these facts one shows that there is a perfect complex $K(r)$ of $\mathbb{Z}[G]$-modules and quasi-isomorphisms

$$\tau_p : K(r) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong K_p(r) \tag{106}$$

for each prime number p such that $H^i(\tau_p) = \tau_p^i$. The argument here is similar to the proof of Lemma 17 below. Moreover, the complex $K(r)$ is specified uniquely up to quasi-isomorphism. By definition then

$$\Omega_{r-1}(L/K) := (K(r)) - (\mathbb{Z}\Sigma_\infty) \in \mathrm{Cl}(\mathbb{Z}[G]) \tag{107}$$

where Σ_∞ is the G-set $\mathrm{Hom}(L,\mathbb{C})$. If G is abelian, then with our parity convention for the determinant, this is equal to

$$\begin{aligned}\Omega_{r-1}(L/K) &= \left(\det\nolimits^{-1}_{\mathbb{Z}[G]} K(r)\right) - \left(\det\nolimits^{-1}_{\mathbb{Z}[G]} \mathbb{Z}\Sigma_\infty\right)\\ &= \left(\det\nolimits^{-1}_{\mathbb{Z}[G]} K(r)\right) \in \mathrm{Pic}(\mathbb{Z}[G]) \cong \mathrm{Cl}(\mathbb{Z}[G])\end{aligned} \tag{108}$$

because $\mathbb{Z}\Sigma_\infty$ is a free $\mathbb{Z}[G]$-module.

Remark. The complex $K(r)$ plays a role similar to that of the complex K_S in section 3. Both K_S and $K(r)$ depend on S but their classes in $\mathrm{Cl}(\mathbb{Z}[G])$ do not. But $K(r)\otimes_{\mathbb{Z}}\mathbb{Q}$ is also independent of S which reflects the fact that the rank of $K_{2r-1}(\mathcal{O}_{L,S})$ does not vary with S for $r>1$, as it does for $r=1$.

Another difference is that the class of K_S, taken in $K_0(\mathbb{Z}[G])$, already lies in the direct summand $\mathrm{Cl}(\mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G])$. This is not true for $K(r)$ which is the reason for the term $\mathbb{Z}\Sigma_\infty$ in (107).

4.2. **Comparison with $\Omega(L/K,\mathbb{Q}(1-r))^{\#}$.** We first prove a lemma which is also very useful in other contexts. Define three exact functors from the derived category $\mathfrak{D}_p$ of $\mathbb{Z}_p[G]$-modules to itself

$$\begin{aligned}(-)^\vee &= R\,\mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Q}_p/\mathbb{Z}_p)\\ (-)^\wedge &= R\,\mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Q}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Q}_p)\\ (-)^* &= R\,\mathrm{Hom}_{\mathbb{Z}_p}(-,\mathbb{Z}_p),\end{aligned}$$

all with the contragredient G-action.

Lemma 16. *a) On the full subcategory of $\mathfrak{D}_p$ consisting of objects X such that $H^i(X)$ is finitely generated for each i, we have natural isomorphisms*

$$X \xrightarrow{\sim} X^{\vee\vee} \xrightarrow{\sim} X^{\vee *}[1] \xleftarrow{\sim} X^{**} \tag{109}$$

and

$$X^{\vee\wedge} \xrightarrow{\sim} 0; \quad X^{\wedge *} \xrightarrow{\sim} 0. \tag{110}$$

b) Consider an exact triangle in $\mathfrak{D}_p$

$$X \to X\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to Y \to \tag{111}$$

where X has finitely generated cohomology. Then there is a natural isomorphism

$$Y^\vee \cong X^*.$$

Proof. From the short exact sequence (38), tensored with $\mathbb{Z}_p$, we obtain an exact triangle

$$X^* \to X^\wedge \to X^\vee \to \tag{112}$$

for each object X in $\mathfrak{D}_p$. Consider then the 9-term diagram with exact rows and columns

$$\begin{tikzcd}
X^{\vee *} \arrow[r] \arrow[d] & X^{\wedge *} \arrow[r] \arrow[d] & X^{**} \arrow[r] \arrow[d] & {} \\
X^{\vee\wedge} \arrow[r] \arrow[d] & X^{\wedge\wedge} \arrow[r] \arrow[d, "\beta"'] & X^{*\wedge} \arrow[r] \arrow[d] & {} \\
X^{\vee\vee} \arrow[r] \arrow[d] & X^{\wedge\vee} \arrow[r] \arrow[d] & X^{*\vee} \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd} \tag{113}$$

Since both $\mathbb{Q}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$ are injective $\mathbb{Z}_p$-modules there is an isomorphism

$$H^i(X^{\vee\wedge}) \cong \mathrm{Hom}_{\mathbb{Z}_p}(H^i(X)^\vee, \mathbb{Q}_p).$$

This group is zero because $H^i(X)^\vee$ is a torsion group, hence $X^{\vee\wedge} \xrightarrow{\sim} 0$. Similarly, the map $H^i(\beta)$ identifies with the natural map

$$\mathrm{Hom}_{\mathbb{Z}_p}(W, \mathbb{Q}_p) \to \mathrm{Hom}_{\mathbb{Z}_p}(W, \mathbb{Q}_p/\mathbb{Z}_p)$$

for $W = H^i(X)^\wedge$. But this map is an isomorphism for any finite dimensional $\mathbb{Q}_p$-space W. Therefore β is a quasi-isomorphism and $X^{\wedge *} \xrightarrow{\sim} 0$. The natural map $X \to X^{\vee\vee}$ is a quasi-isomorphism because $M \to M^{\vee\vee}$ is an isomorphism for any finitely generated $\mathbb{Z}_p$-module M. Together with (113) this gives a).

Applying the functor $(-)^*$ to (111) we obtain an exact triangle

$$Y^* \to (X \otimes \mathbb{Q}_p)^* \to X^* \to$$

where the middle term is acyclic, for example because it can be rewritten as $X^{\wedge\wedge *}$ which is zero by a). Hence we deduce an isomorphism $X^* \cong Y^*[1]$. Using finite generation of $H^i(X)$ and the long exact cohomology sequence induced by (111) we find that $H^i(Y)$ is a torsion group and therefore that $Y^\wedge$ is acyclic. From (112) for Y we then obtain an isomorphism $Y^\vee \cong Y^*[1] \cong X^*$, i.e. part b). $\square$

The diagram in the following proposition is the key to the comparison of $\Omega_{r-1}(L/K)$ and $\Omega(L/K, \mathbb{Q}(1-r))$. Recall the definitions of $R\Gamma_\Delta$ and $\tilde{R\Gamma}_c$ from (82).

Proposition 4.1. *There exists a commutative diagram in $\mathfrak{D}_p$ as follows:*

$$\begin{tikzcd}
\Delta_p(r) \arrow[r] \arrow[d] & \bigoplus_{w|\infty} R\Gamma_\Delta(L_w, \mathbb{Z}_p(1-r))^*[-3] \arrow[r, "\alpha_p"] \arrow[d] & \bigoplus_{w|\infty} R\Gamma(L_w, \mathbb{Z}_p(r)) \arrow[d, equal] \\
K_p(r)[-1] \arrow[r] \arrow[d] & R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r)) \arrow[r] \arrow[d] & \bigoplus_{w|\infty} R\Gamma(L_w, \mathbb{Z}_p(r)) \\
R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))^*[-3] \arrow[r, equal] & R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))^*[-3] &
\end{tikzcd} \tag{114}$$

Moreover, all rows and columns in this diagram are exact triangles and all terms in the left hand column are $\mathbb{Z}_p[G]$-perfect .

Proof. If we choose $X = \tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))$ in Lemma 16b) the triangle (111) naturally identifies with the triangle

$$\tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r)) \to \tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(1-r)) \to \tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(1-r)) \to .$$

From Artin-Verdier duality and Lemma 16b) we obtain a quasi-isomorphism

$$\begin{aligned} R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r)) &\cong \tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))^\vee[-3] \\ &\cong \tilde{R\Gamma}_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))^*[-3] \end{aligned}$$

which together with (104) and (83) yields the diagram (114). Since both $K_p(r)$ and $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))$ are $\mathbb{Z}_p[G]$-perfect (see [5, Prop. 1.20]) the same is true for $\Delta_p(r)$. □

Proposition 4.2. *There exists a perfect complex $\Delta(r)$ of $\mathbb{Z}[G]$-modules and quasi-isomorphisms*

$$\sigma_p : \Delta(r) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \Delta_p(r)$$

for each prime p, such that there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^1(\Delta(r)\otimes\mathbb{Q}) & \to & H^0(K(r)\otimes\mathbb{Q}) & \to & K_{2r-1}(\mathcal{O}_L)\otimes\mathbb{Q} & \to 0 \to & \cdots \\ & & \downarrow{\scriptstyle H^1(\sigma_p\otimes\mathbb{Q}_p)} & & \downarrow{\scriptstyle H^1(\tau_p\otimes\mathbb{Q}_p)} & & \downarrow{\scriptstyle (10)^\wedge} & & \\ 0 & \to & H^1(\Delta_p(r)\otimes\mathbb{Q}_p) & \to & H^0(K_p(r)\otimes\mathbb{Q}_p) & \to & H^2_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(1-r))^\wedge & \to 0 \to & \cdots \end{array}$$

$$(115)\qquad \begin{array}{ccccccc} \cdots \to 0 & \to & \bigoplus_{w\in S_\infty} H^0(L_w, \mathbb{Q}(r)) & \to & H^3(\Delta(r)\otimes\mathbb{Q}) & \to & 0 \\ & & \downarrow{\scriptstyle (9)^\wedge} & & \downarrow{\scriptstyle H^3(\sigma_p\otimes\mathbb{Q}_p)} & & \\ \cdots \to 0 & \to & H^1_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(1-r))^\wedge & \to & H^3(\Delta_p(r)\otimes\mathbb{Q}_p) & \to & 0 \end{array}$$

where the lower row is induced by the left vertical triangle in (114), tensored with $\mathbb{Q}_p$.

Proof. Since $\Delta_p(r)$ has cohomology concentrated in degrees $1, 2, 3$, the map α_p in (114) induces a cohomology isomorphism in degrees ≥ 4 so that there is an exact triangle

$$\Delta_p(r) \to \tau^{\leq 4} \bigoplus_{w|\infty} R\Gamma_\Delta(L_w, \mathbb{Z}_p(1-r))^*[-3] \xrightarrow{\tau^{\leq 4}\alpha_p} \tau^{\leq 4} \bigoplus_{w|\infty} R\Gamma(L_w, \mathbb{Z}_p(r)) \to .$$

Moreover, if we denote by $\mathbb{Z}(r)$ the étale sheaf on Spec L_w given by the abelian group $\mathbb{Z}$ on which complex conjugation acts via $(-1)^r$ we have natural isomorphisms

$$(116)\qquad R\Gamma(L_w, \mathbb{Z}_p(r)) \cong R\Gamma(L_w, \mathbb{Z}(r)) \otimes \mathbb{Z}_p$$

and

$$R\Gamma_\Delta(L_w, \mathbb{Z}_p(1-r))^* \cong R\Gamma_\Delta(L_w, \mathbb{Z}(1-r))^* \otimes \mathbb{Z}_p$$

where this last $(-)^*$ means $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$. We set

$$(117)\qquad \begin{aligned} A &:= \tau^{\leq 4} \bigoplus_{w|\infty} R\Gamma_\Delta(L_w, \mathbb{Z}(1-r))^*[-3] \\ B &:= \tau^{\leq 4} \bigoplus_{w|\infty} R\Gamma(L_w, \mathbb{Z}(r)), \end{aligned}$$

so that both A and B are bounded complexes of $\mathbb{Z}[G]$-modules with finitely generated cohomology and for each i either $H^i(A)$ or $H^i(B)$ is torsion.

Lemma 17. *Suppose A and B are bounded complexes of $\mathbb{Z}[G]$-modules with finitely generated cohomology such that for each $i \in \mathbb{Z}$ either $H^i(A)$ or $H^i(B)$ is torsion. Then the natural map*

$$\mathrm{Hom}_{\mathfrak{D}}(A,B) \to \bigoplus_p \mathrm{Hom}_{\mathfrak{D}_p}(A \otimes \mathbb{Z}_p, B \otimes \mathbb{Z}_p)$$

is an isomorphism.

Proof. We first show that for each prime p

$$\mathrm{Hom}_{\mathfrak{D}}(A,B) \otimes \mathbb{Z}_p \cong \mathrm{Hom}_{\mathfrak{D}_p}(A \otimes \mathbb{Z}_p, B \otimes \mathbb{Z}_p). \tag{118}$$

We can pick a quasi-isomorphism $P \cong A$ where P is a complex consisting of finitely generated projective $\mathbb{Z}[G]$-modules so that

$$\mathrm{Hom}_{\mathfrak{D}}(A,B) = H^0(R\,\mathrm{Hom}(A,B)) = H^0(\mathrm{Hom}^\bullet(P,B)). \tag{119}$$

Since B is bounded we have

$$\mathrm{Hom}^\bullet(P,B) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(P^i, B^{\bullet+i}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(P^i, B^{\bullet+i})$$

and therefore

$$\mathrm{Hom}^\bullet(P,B) \otimes \mathbb{Z}_p \cong \mathrm{Hom}^\bullet_{\mathbb{Z}_p[G]}(P \otimes \mathbb{Z}_p, B \otimes \mathbb{Z}_p) \tag{120}$$

since $\mathbb{Z} \to \mathbb{Z}_p$ is flat so that (118) holds for modules, provided A is finitely generated. Again because of flatness we get

$$H^0(\mathrm{Hom}^\bullet(P,B)) \otimes \mathbb{Z}_p = H^0(\mathrm{Hom}^\bullet(P,B) \otimes \mathbb{Z}_p)$$

which together with (119) and (120) gives (118).

The lemma will follow if we show in addition that

$$\mathrm{Hom}_{\mathfrak{D}}(A,B) \cong \bigoplus_p \mathrm{Hom}_{\mathfrak{D}}(A,B) \otimes \mathbb{Z}_p,$$

i.e. that $\mathrm{Hom}_{\mathfrak{D}}(A,B)$ is a torsion group. But this follows easily from the spectral sequence [27][III.4.6.10]

$$E_2^{p,q} = \prod_{i\in\mathbb{Z}} \mathrm{Ext}^p_G(H^i(A), H^{q+i}(B)) \Rightarrow H^{p+q}(R\,\mathrm{Hom}(A,B))$$

together with (119) and the fact that $\mathrm{Ext}^p_G(M,-)$ (resp. $\mathrm{Hom}_G(H^i(A), H^i(B))$) is torsion for finitely generated M and $p > 0$ (resp. $i \in \mathbb{Z}$). $\square$

Lemma 17 shows that there is a unique morphism $\alpha \in \mathrm{Hom}_{\mathfrak{D}}(A,B)$ mapping to $\oplus_p \tau^{\le 4}\alpha_p$. We define $\Delta(r)$ to be the third term in any exact triangle

$$\Delta(r) \to A \xrightarrow{\alpha} B \to .$$

The diagram (115) then follows easily from (105) and (114). $\square$

Theorem 4.1. *If G is abelian and $r > 1$ we have*

$$\Omega(L/K, \mathbb{Q}(1-r))^\# = \Omega_{r-1}(L/K) \in \mathrm{Cl}(\mathbb{Z}[G]).$$

Proof. The left vertical exact triangle in (114) induces an isomorphism

$$\begin{aligned}\lambda_p : \det_{\mathcal{E}_p}^{-1} K_p(r) \otimes_{\mathcal{E}_p} \det_{\mathcal{E}_p}^{-1} \Delta_p(r) &\cong \det_{\mathcal{E}_p}^{-1} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))^* \\ &\cong \det_{\mathcal{E}_p} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(1-r))^{\#}. \end{aligned} \tag{121}$$

The top row in (115) gives an isomorphism

$$\kappa(r) : \Xi(\mathbb{Q}(1-r)_L)^{\#} \cong \det_E^{-1}(K(r)\otimes\mathbb{Q}) \otimes \det_E^{-1}(\Delta(r)\otimes\mathbb{Q})$$

such that

$$\tilde{\vartheta}_p = (\lambda_p \otimes \mathbb{Q}_p) \circ (\det{}^{-1}(\tau_p \otimes \mathbb{Q}_p) \otimes \det{}^{-1}(\sigma_p \otimes \mathbb{Q}_p)) \circ (\kappa(r) \otimes \mathbb{Q}_p)$$

as maps from $\Xi(\mathbb{Q}(1-r)_L)^{\#} \otimes \mathbb{Q}_p$ to $\det_{E_p} R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p(1-r))^{\#}$. This also follows from (115) because $\tilde{\vartheta}_p$ was constructed in section 2 using the maps (9) and (10). By (11) one has

$$\begin{aligned}\vartheta_p^{-1}(\mathcal{I}_p(1-r)^{\#}) &= \epsilon_{S_p}(1-r)^{\#}\tilde{\vartheta}_p^{-1}(\mathcal{I}_p(1-r)^{\#}) \\ &= \epsilon_S(1-r)^{\#}\tilde{\vartheta}_p^{-1}(\mathcal{I}_p(1-r)^{\#})\end{aligned}$$

where the latter equality holds because for $v|p$ ($\Rightarrow p|Nv^{r-1}$) the factor $(1-Nv^{r-1}f_v)$ occurring in (12) is a unit in $\mathcal{E}_p = \mathbb{Z}_p[G]$. Hence

$$\vartheta_p^{-1}(\mathcal{I}_p(1-r)^{\#}) = \epsilon_S(1-r)^{\#}(\kappa(r)\otimes\mathbb{Q}_p)^{-1}\big(\det_{\mathcal{E}_p}^{-1} K_p(r) \otimes_{\mathcal{E}_p} \det_{\mathcal{E}_p}^{-1}\Delta_p(r)\big)$$

and

$$\begin{aligned}\Xi(\mathbb{Q}(1-r)_L)^{\#}_{\mathbb{Z}} &= \bigcap_p \Xi(\mathbb{Q}(1-r)_L)^{\#} \cap \vartheta_p^{-1}(\mathcal{I}_p(1-r)^{\#}) \\ &= \epsilon_S(1-r)^{\#}\kappa(r)^{-1}\big(\det_{\mathbb{Z}[G]}^{-1} K(r) \otimes_{\mathbb{Z}[G]} \det_{\mathbb{Z}[G]}^{-1}\Delta(r)\big).\end{aligned}$$

Since $\epsilon_S(1-r) \in E^\times$ one deduces

$$\Omega(L/K, \mathbb{Q}(1-r))^{\#} = \Omega_{r-1}(L/K) + (\Delta(r)) \in \mathrm{Cl}(\mathbb{Z}[G]).$$

Theorem 4.1 then follows from

Lemma 18. *$\Delta(r)$ is quasi-isomorphic to a complex of free $\mathbb{Z}[G]$-modules, more precisely to*

$$\mathbb{Z}\Sigma_\infty \to \mathbb{Z}\Sigma_\infty \to \mathbb{Z}\Sigma_\infty$$

where the first term is in degree 1.

Proof. One first shows that the map α_p in (114) respects the direct sum decomposition of its source and target, following the proof of Proposition 3.4. Using the same notation as in that proof, one needs to ascertain the vanishing of the map induced by α_p in $\mathrm{Hom}_{\mathfrak{D}_p(G_0)}(Y, Z)$ where

$$Y := R\Gamma_\Delta(L_{w_0}, \mathbb{Z}_p(1-r))^*[-3] = \begin{cases} \mathbb{Z}_p(r-1)[-3] & w_0 \text{ complex} \\ \mathbb{Z}_p \xrightarrow{\delta_1} \mathbb{Z}_p \xrightarrow{\delta_0} \mathbb{Z}_p \xrightarrow{\delta_1} \cdots [-3] & w_0 \text{ real} \end{cases}$$

and $Z = \mathrm{Ind}_{G_{v'}}^{G_0} Z'$ with

$$Z' := R\Gamma(L_{w'}, \mathbb{Z}_p(r)) = \begin{cases} \mathbb{Z}_p(r) & w' \text{ complex} \\ \mathbb{Z}_p \xrightarrow{\delta_0} \mathbb{Z}_p \xrightarrow{\delta_1} \mathbb{Z}_p \xrightarrow{\delta_0} \mathbb{Z}_p \xrightarrow{\delta_1} \mathbb{Z}_p \xrightarrow{\delta_0} \cdots & w' \text{ real.} \end{cases}$$

Here δ_i is multiplication with $1-(-1)^{r+i}$ for $i = 0, 1$ and $\mathfrak{D}_p(\Gamma)$ is the derived category of $\mathbb{Z}_p[\Gamma]$-modules. Again, one has $\mathrm{Hom}_{\mathfrak{D}_p(G_0)}(Y, Z) = 0$ if $G_0 = 1$ and w'

is complex and one reduces to this case by introducing an appropriate extension N/K' as in the proof of Proposition 3.4. The analogue of Lemma 11 to be used here is that $R\operatorname{Hom}_{\mathbb{Z}_p[H]}(\mathbb{Z}_p, -)$ applied to diagram (114) gives the corresponding diagram formed with respect to the extension L^H/K. This is proved along the same lines as Lemma 11. It follows that

$$\Delta(r) \cong \bigoplus_{v \in S_\infty} \operatorname{Ind}_{G_v}^G \Delta(r)_{w(v)}$$

where $\Delta(r)_{w(v)}$ sits in a triangle in $\mathfrak{D}(G_v)$

$$R\Gamma_\Delta(L_{w(v)}, \mathbb{Z}(1-r))^*[-3] \to R\Gamma(L_{w(v)}, \mathbb{Z}(r)) \to \Delta(r)_{w(v)}[1] \to . \tag{122}$$

If $w(v)$ is real, hence $G_v = 1$, the triangle (122) is simply isomorphic to the short exact sequence of complexes of abelian groups

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_1"] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_0"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots \\
\mathbb{Z} \arrow[r, "\delta_0"] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_1"] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_0"] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_1"] \arrow[d] & \mathbb{Z} \arrow[r, "\delta_0"] \arrow[d] & \mathbb{Z} \arrow[r] & \cdots \\
\mathbb{Z} \arrow[r, "\delta_0"] & \mathbb{Z} \arrow[r, "\delta_1"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots &
\end{tikzcd}$$

If $w(v)$ is complex and $G_v = 1$ we have $\Delta(r)_{w(v)}[1] \cong \mathbb{Z}[0] \oplus \mathbb{Z}[-2]$ which can be represented by a complex of abelian groups

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}.$$

Finally, if $w(v)$ is complex and $G_v \cong \mathbb{Z}/2\mathbb{Z}$ the complex $\Delta(r)_{w(v)}[1]$ represents an extension class in $\operatorname{Ext}^3_G(\mathbb{Z}(r-1), \mathbb{Z}(r)) \cong \operatorname{Ext}^3_G(\mathbb{Z}, \mathbb{Z}(1)) \cong \mathbb{Z}/2\mathbb{Z}$. Since we already know that $\Delta(r)_{w(v)}$ is $\mathbb{Z}[G_v]$-perfect this must be the non-trivial class in this group which is represented by a complex

$$\mathbb{Z}[G_v] \to \mathbb{Z}[G_v] \to \mathbb{Z}[G_v].$$

This gives the desired description of $\Delta(r)$. □

Remark. The occurrence of the complex $\Delta(r)$ corresponds to that of $\mathbb{Z}\Sigma_\infty$ in formula (107). Indeed, one has $(K(r)) + (\Delta(r)) = \Omega_{r-1}(L/K)$ in $K_0(\mathbb{Z}[G])$ not merely after projection into $\operatorname{Cl}(\mathbb{Z}[G])$.

5. Lichtenbaum Complexes

In [17] Lichtenbaum has proposed the existence of complexes of étale sheaves $\Gamma(r)$ for $r \geq 0$ and any scheme X which, if X is of finite type over $\mathbb{Z}$, should play an important role in the description of arithmetic properties of X. The model case to think of is the sheaf $\Gamma(1) := \mathbb{G}_m[-1]$ and its appearance in the Artin-Verdier duality theorem for regular one-dimensional X and in the formula for the residue of the Zeta-function of such X.

We have seen how both $\mathbb{G}_m$ and $\Gamma(0) := \mathbb{Z}$ have played a crucial part in our discussion of $\Omega(L/K, \mathbb{Q}(0))$ and $\Omega(L/K, 3)$. In this section we shall indicate briefly how the expected properties of $\Gamma(r)$ for $r \geq 2$ naturally give rise to perfect complexes and therefore to classes related to $\Omega(L/K, \mathbb{Q}(1-r))$. The possibility of such constructions was also noted independently, and in fact a little earlier, by Chinburg, Kolster, Pappas and Snaith (cf. remarks to that effect in [9]). In order to simplify our discussion we shall tensor all occurring complexes with $\Lambda := \mathbb{Z}[\frac{1}{2}]$ and therefore only obtain classes in $\mathrm{Cl}(\Lambda[G])$. In this section we need not assume that G is abelian. If $\Lambda[G] \cong R_1 \times \cdots \times R_m$ where each ring R_i has no non-trivial idempotents we define the class groups $\mathrm{Cl}(\Lambda[G])$ to be the kernel of the rank map

$$K_0(\Lambda[G]) \to \mathbb{Z}^m.$$

Then $\mathrm{Cl}(\Lambda[G])$ is a direct summand of $K_0(\Lambda[G])$ and is isomorphic to $\mathrm{Pic}(\Lambda[G])$ for abelian G.

In the following we shall assume the existence of $\Gamma(r)$ in the derived category of complexes of sheaves of abelian groups on the small étale site of schemes of the type Spec $\mathcal{O}_{L,S}$ considered before. In fact, we assume that $\Gamma(r)$ on Spec $\mathcal{O}_{L,S}$ is the pull back of $\Gamma(r)$ on Spec $\mathbb{Z}$ and moreover that the following properties are satisfied.

1. For any integer N which is invertible in $\mathcal{O}_{L,S}$, there is an exact triangle

$$\Gamma(r) \xrightarrow{N} \Gamma(r) \to \mathbb{Z}/N\mathbb{Z}(r) \to \tag{123}$$

2. The Chern class maps $K_{2r-i}(\mathcal{O}_{L,S}) \to H^i(\mathcal{O}_{L,S}, \mathbb{Z}/N\mathbb{Z}(r))$ factor through $\mathbb{H}^i(\mathcal{O}_{L,S}, \Gamma(r))$ where N is any integer as in 1.
3. The hypercohomology groups $\mathbb{H}^i(\mathcal{O}_{L,S}, \Gamma(r))$ are finitely generated abelian groups for $r \geq 2$ and $i \in \mathbb{Z}$.
4. Let T be a finite set of places disjoint from S. Then there is an exact triangle in $\mathfrak{D}$ for $r \geq 2$

$$\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \to \mathbb{R}\Gamma(\mathcal{O}_{L,S\cup T}, \Gamma(r)) \to \bigoplus_{w\in T} K_{2r-3}(\lambda(w))[-2] \to \tag{124}$$

where $\lambda(w)$ is the residue field.

Remarks. This is only a subset of the axioms for $\Gamma(r)$, adapted to our needs (see for example [22] for a full list). The term $K_{2r-3}(\lambda(w))[-2]$ should be written $\mathbb{R}\Gamma(\lambda(w), \Gamma(r-1))[-1]$ and (124) then represents a purity axiom which also holds for $r = 1$. The finite generation of $\mathbb{H}^i(X, \Gamma(r))$ is not true in general (as the case $X = \mathcal{O}_{L,S}$, $r = 1$, $i = 3$ shows) but one might expect that $\mathbb{H}^i(X, \Gamma(r))$ is finitely generated for all $i \in \mathbb{Z}$ if X is regular of finite type over $\mathbb{Z}$ and if r is large enough with respect to the dimension of X.

The complex $\Gamma(2)$ constructed by Lichtenbaum in [18] satisfies the above list of properties (see [14], [19]), so in the case $r = 2$ our results described below are unconditional.

Proposition 5.1. *a) The complex* $\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda$ *is* $\Lambda[G]$*- perfect for* $r \geq 2$.
b) For each prime $p \neq 2$ *there is a quasi-isomorphism*

$$\mathbb{R}\Gamma(\mathcal{O}_{L,S_p}, \Gamma(r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r))$$

which is compatible with the Chern class maps in 2).

Proof. Part a) can be shown along the lines of the proof of Prop. 1.20 in [5]. The complex $M := \mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r))$ consists of c.t. G-modules, hence the same is true for $M \otimes_{\mathbb{Z}} \Lambda$. Moreover $M \otimes_{\mathbb{Z}} \Lambda$ is acyclic outside degrees $1, 2$ as follows easily by combining properties 1) and 3) above (and this is also true if L is replaced by a field between L and K). Therefore $M \otimes_{\mathbb{Z}} \Lambda$ is quasi-isomorphic to a bounded complex of c.t. $\Lambda[G]$-modules. Perfectness follows because the cohomology of $\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda$ is finitely generated over Λ by property 3).

To demonstrate b) one can copy the proof of Proposition 3.3. For the complex $Q^\bullet$ (resp. $P^\bullet$) one takes a bounded complex of finitely generated projective $\Lambda[G]$-modules (resp. $\mathbb{Z}_p[G]$-modules) quasi-isomorphic to $\mathbb{R}\Gamma(\mathcal{O}_{L,S_p}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda$ (resp. $R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r))$). The perfectness of $R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r))$ over $\mathbb{Z}_p[G]$ for odd primes p follows again as in [5, Prop. 1.20]. Property 1) implies that $Q^\bullet/p^n$ and $P^\bullet/p^n$ are quasi-isomorphic, compatibly with Chern class maps by property 2). One shows that these quasi-isomorphisms can be realized as maps of inverse systems of complexes and concludes with Lemma 9. □

Theorem 5.1. *The image of* $\Omega_{r-1}(L/K)$ *in* $\mathrm{Cl}(\Lambda[G])$ *agrees with the class of* $\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda$. *This class also coincides with the image of* $\Omega(L/K, \mathbb{Q}(1-r))^\#$ *in* $\mathrm{Cl}(\Lambda[G])$ *if* G *is abelian.*

Remark. We understand that Victor Snaith has obtained results along similar lines in the case $r = 2$. He is able to avoid inverting 2 and shows that $\Omega_1(L/K)$ coincides in $K_0(\mathbb{Z}[G])$ with a class defined from the cohomology of $\Gamma(2)$.

Proof. By Theorem 4.1 it suffices to prove the assertion concerning $\Omega_{r-1}(L/K)$. The triangle (124) and Proposition 5.1b) give a quasi-isomorphism

$$\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r)) \tag{125}$$

since for any place $w|p$, the group $K_{2r-3}(\lambda(w))$ is finite of order prime to p [23, §12]. Arguing as in the proof of Lemma 17 one finds a unique map in the derived category of $\Lambda[G]$-modules

$$\beta : \mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda \to \bigoplus_{w|\infty} H^0(L_w, \Lambda(r)) \tag{126}$$

inducing the natural map

$$R\Gamma(\mathcal{O}_{L,S_p}, \mathbb{Z}_p(r)) \to \bigoplus_{w|\infty} H^0(L_w, \mathbb{Z}_p(r))$$

after tensorisation with $\mathbb{Z}_p$ for each odd prime p. Note here that for any odd prime p, and any archimedean place w, the complex $R\Gamma(L_w, \mathbb{Z}_p(r))$ is quasi-isomorphic to $H^0(L_w, \mathbb{Z}_p(r))$ concentrated in degree 0. One then checks that the mapping cone of β satisfies the defining properties of the complex $K(r) \otimes_{\mathbb{Z}} \Lambda$ (The isomorphisms τ_p in (106) are obvious and compatibility with Chern class maps follows from Proposition 5.1b)). The complex $K(r) \otimes_{\mathbb{Z}} \Lambda$ is thus quasi-isomorphic to this mapping cone and so its class in $\mathrm{Cl}(\Lambda[G])$ agrees with

$$\big(\mathbb{R}\Gamma(\mathcal{O}_{L,S}, \Gamma(r)) \otimes_{\mathbb{Z}} \Lambda\big) - \big(\bigoplus_{w|\infty} H^0(L_w, \Lambda(r))\big).$$

But each module

$$\bigoplus_{w|v} H^0(L_w, \Lambda(r)) \cong H^0(K_v, \Lambda[G](r))$$

has trivial class in $\mathrm{Cl}(\Lambda[G])$ because it is a direct factor of $\Lambda[G]$. Hence the theorem. □

Remark. It seems reasonable to speculate that the map β in (126) exists without inverting 2 and indeed has a more natural interpretation as a composite map

$$\mathbb{R}\Gamma(\mathcal{O}_{L,S},\Gamma(r)) \to \bigoplus_{w|\infty} \mathbb{R}\Gamma(L_w,\Gamma(r)) \to \bigoplus_{w|\infty} R\Gamma(L_w,\mathbb{Z}(r)) \tag{127}$$

where the first map is the natural restriction and the last map arises as follows. Recall that Deligne cohomology $R\Gamma_{\mathcal{D}}(\mathbb{C},\mathbb{Z}(r))$ of a point (i.e. Spec $\mathbb{C}$) is concentrated in degree 1 and identifies with $\mathbb{C}/\mathbb{Z}(r)[-1]$ (see [24] for more about Deligne cohomology and the Beilinson regulator map). Moreover, on Spec $\mathbb{C}$ the object $\Gamma(r)$ identifies with a complex of abelian groups. One might conjecture that there is a map in the derived category of abelian groups

$$\rho : \Gamma(r) \to \mathbb{C}/\mathbb{Z}(r)[-1],$$

factoring the Beilinson regulator map after taking H^1:

$$K_{2r-1}(\mathbb{C}) \to H^1(\Gamma(r)) \xrightarrow{H^1(\rho)} \mathbb{C}/\mathbb{Z}(r).$$

The map ρ should also respect the action of complex conjugation, i.e. be a map in the derived category of étale sheaves on Spec $\mathbb{R}$. Composing ρ in the derived category with the obvious map $\mathbb{C}/\mathbb{Z}(r)[-1] \to \mathbb{Z}(r)$ would induce the second map in (127). Finally, one would be led to conjecture that the mapping cone of (127) is cohomologically bounded, hence perfect over $\mathbb{Z}[G]$ by our standard arguments. So one could take the class of this mapping cone in $\mathrm{Cl}(\mathbb{Z}[G])$ to define an invariant. However, by adapting the proof of Theorem 5.1 it would then follow that this class coincides with $\Omega_{r-1}(L/K)$.

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Dept. of Math., King's College London, Strand, London WC2R 2LS, England

Dept. of Math., Caltech, Pasadena CA 91125, USA