SUBSTITUTION DYNAMICAL SYSTEMS: CHARACTERIZATION OF LINEAR REPETITIVITY AND APPLICATIONS

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Abstract. We consider dynamical systems arising from substitutions over a finite alphabet. We prove that such a system is linearly repetitive if and only if it is minimal. Based on this characterization we extend various results from primitive substitutions to minimal substitutions. This includes applications to random Schrödinger operators and to number theory.

1. Introduction

This paper deals with a special class of low complexity subshifts over finite alphabets, viz. subshifts associated to substitutions.

Subshifts over finite alphabets play a role in various branches of mathematics, physics, and computer science. Low complexity or intermediate disorder has been a particular focus of research in recent years. This has been even more the case due to the discovery by Shechtman et al. of special solids [37], later called quasicrystals, which exhibit this form of disorder [2, 23, 38]. Subshifts associated to substitutions and in particular to primitive substitutions are foremost among the models of low complexity subshifts [31, 32, 36].

With the recent work of Durand [18] and Lagarias and Pleasants [26] it became apparent that a key feature to be studied in low complexity subshifts (and their higher-dimensional analogue) is linear repetitivity or linear recurrence. It is known that subshifts associated to primitive substitutions are linearly repetitive [39]. Thus, it is natural to ask:

(Q) Which substitution dynamical systems are linearly repetitive?

The main result of the paper answers this question. Namely, we show that a substitution dynamical system is linearly repetitive if and only if it is minimal, which in turn is the case if and only if one letter (not belonging to a particular subset of the alphabet) appears with bounded gaps. This not only characterizes

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linear repetitivity but also gives an easy to handle condition to verify this feature. This characterization and its proof give very direct methods to

(E) extend results from the framework of primitive substitutions to the framework of minimal substitutions.

We illustrate this extension process with two types of examples. The first type is concerned with the spectral theory of certain Schrödinger operators. The second example deals with number theory. Details will be discussed in the corresponding sections.

The paper is organized as follows: In Section 2 we introduce the necessary notation and state our main result, answering (Q). This result is then proved in Section 3. The following two sections give examples for (E). Section 4 is devoted to a study of Schrödinger operators associated to minimal substitutions. An application to number theory is discussed in Section 5. Finally, in Section 6 we study the unique decomposition property for a special class of nonprimitive substitutions.

2. Notation and Statement of the Main Result

In this section we introduce the necessary notation and present our main result.

Let A be a finite subset of $\mathbb{R}$, called the alphabet. The elements of A will be called letters. In the sequel we will use freely notions from combinatorics on words (see, e.g., [31, 32]). In particular, the elements of the free monoid A^* over A will be called words. The length of a word is the number of its letters; the number of occurrences of $v \in A^*$ in $w \in A^*$ will be denoted by $\#_v(w)$. Moreover, for a word u over A, we let $\mathrm{Sub}(u)$ denote the set of subwords of u.

We can equip A with discrete topology and $A^{\mathbb{Z}}$ with product topology. A pair (Ω, T) is then called a subshift over A if Ω is a closed subset of $A^{\mathbb{Z}}$ which is invariant under $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$, $(Tu)(n) \equiv u(n+1)$. To a subshift (Ω, T) belongs the set $\mathcal{W}(\Omega)$ of finite words given by $\mathcal{W}(\Omega) \equiv \cup_{\omega \in \Omega} \mathrm{Sub}(\omega)$. A word $v \in \mathcal{W}(\Omega)$ is said to occur with bounded gaps if there exists an $L_v > 0$ such that every $w \in \mathcal{W}(\Omega)$ with $|w| \geq L_v$ contains a copy of v. By standard arguments (Ω, T) is minimal (i.e., each orbit is dense) if and only if every $v \in \mathcal{W}(\Omega)$ occurs with bounded gaps. A particular strengthening of minimality is thus the condition of linear repetitivity given as follows: The system (Ω, T) is said to be linearly repetitive if there exists a constant C_{LR} with

$$\#_v(w) \geq 1 \ \text{ whenever } \ |w| \geq C_{\mathrm{LR}} |v| \tag{1}$$

for $v, w \in \mathcal{W}(\Omega)$.

A special way to generate subshifts is given as follows. Consider a map $S : A \longrightarrow A^*$. By definition of A^*, S can be uniquely extended to a morphism

$$S : A^* \longrightarrow A^* \ \text{ by setting } \ S(a_1 \dots a_n) \equiv S(a_1) \dots S(a_n),$$

for arbitrary $a_j \in A$, $j = 1 \dots n$. To such an S, we can associate the set $\mathcal{W}(S) \subset A^*$ given by

$$\mathcal{W}(S) \equiv \{ w \in A^* : w \in \mathrm{Sub}(S^n(a)) \ \text{ for suitable } a \in A \text{ and } n \in \mathbb{N}_0 \ \},$$

and the (possibly empty) set $\Omega(S) \subset A^{\mathbb{Z}}$ given by

$$\Omega(S) \equiv \{ \omega \in A^{\mathbb{Z}} : \mathrm{Sub}(\omega) \subset \mathcal{W}(S) \}.$$

For $\Omega(S)$ to be nonempty, it is necessary and sufficient that there exists an $e \in A$ with

$$|S^n(e)| \longrightarrow \infty, n \longrightarrow \infty. \tag{2}$$

Without loss of generality we can then assume (after possibly removing some letters from A) that

$$\text{for all } a \in A \text{ there exists an } n \in \mathbb{N} \text{ with } \#_a(S^n(e)) \geq 1. \tag{3}$$

We will from now on assume that $S : A \longrightarrow A^*$ satisfies (2) and (3). $(\Omega(S), T)$ is then referred to as a substitution dynamical system. Obviously, $(\Omega(S), T)$ is a subshift over A. Note that every word $v \in \mathcal{W}(S)$ can be extended to a two-sided infinite sequence, that is, $\mathcal{W}(S) = \mathcal{W}(\Omega(S))$.

As mentioned in the introduction, a special class of systems $(\Omega(S), T)$ known to be linearly repetitive are those coming from primitive S. Here, S is called primitive if there exists an $r \in \mathbb{N}$ with $\#_a(S^r(b)) \geq 1$ for arbitrary $a, b \in A$. Our main result characterizes all S with linearly repetitive $(\Omega(S), T)$.

Theorem 1. *Let $(\Omega(S), T)$ be a substitution dynamical system. Then the following are equivalent:*

(i) *There exists an $e \in A$ satisfying* (2) *and* (3) *which occurs with bounded gaps.*
(ii) *$(\Omega(S), T)$ is minimal.*
(iii) *$(\Omega(S), T)$ is linearly repetitive.*

Remark 1. (a) Condition (i) can be easily checked in many concrete cases. For example, it can immediately be seen to be satisfied in the examples of de Oliveira and Lima [35]. Thus, their examples are linearly repetitive and the whole theory developed below applies.

(b) As primitive substitutions can easily be seen to satisfy (i), the theorem contains the well known result (see, e.g., [39]) that these systems are linearly repetitive.

(c) Existence of an arbitrary $a \in A$ occurring with bounded gaps is not sufficient for minimality, as can be seen by considering the example $S : \{0,1\} \longrightarrow \{0,1\}^*$, $0 \mapsto 101$ and $1 \mapsto 1$.

As discussed above, linear repetitivity implies minimality. Moreover, it also implies unique ergodicity as shown by Durand [18] (see [28] for a different proof as well). Thus, we obtain the following corollary.

Corollary 1. *Let $(\Omega(S), T)$ be as above. If $e \in A$ satisfying* (2) *and* (3) *occurs with bounded gaps, then $(\Omega(S), T)$ is uniquely ergodic and minimal.*

Remark 2. (a) A different way of stating the corollary would be to say that for a substitution dynamical system, minimality is equivalent to strict ergodicity. Note that unique ergodicity is not sufficient for minimality, as can be seen by considering the example in (c) of Remark 1.

(b) By (a) of Remark 1, the corollary applies to the examples of [35] and we recover their Proposition 1.

3. Linearly Repetitive Substitutions

This section is devoted to a proof of Theorem 1. It turns out that the hard part in the proof is the implication (ii) $\Longrightarrow$ (iii). Its proof will be split into several parts.

The key issue is to study the growth of $|S^n(a)|$ for $n \to \infty$ and $a \in A$. This will be done by relating $S : A \longrightarrow A^*$ to a suitable other substitution $\widetilde{S} : C \longrightarrow C^*$, which can be shown to be primitive if (ii) is satisfied. As growth properties are well known for primitive substitutions, we obtain the desired results by comparing the growth behavior of S and $\widetilde{S}$.

For notational convenience, we will say that $(\Omega(S), T)$ satisfies the bounded gap condition (for $e \in A$) if

(BG) the letter $e \in A$ satisfies (2) and (3) and occurs with bounded gaps.

Our first result gives some immediate consequence of (BG).

Lemma 3.1. *Let* $(\Omega(S), T)$ *satisfy* (BG). *Then* $(\Omega(S), T)$ *is minimal.*

Proof. It suffices to show that $S^n(e)$ occurs with bounded gaps for arbitrary but fixed $n \in \mathbb{N}$. Set $M \equiv \max\{|S^n(a)| : a \in A\}$. By (BG), there exists $\kappa > 0$ such that every word in $\mathcal{W}(S)$ with length exceeding κ contains e. Consider an arbitrary $w \in \mathcal{W}(S)$ with $|w| \ge (3+\kappa)M$. Now, w is contained in $S^n(a_1 \dots a_s)$ with suitable $a_j \in A$, $j = 1, \dots, s$ with $a_1 \dots a_s \in \mathcal{W}(S)$. By assumption on $|w|$ and definition of κ, we infer that w contains $S^n(a_l)S^n(a_{l+1}) \dots S^n(a_{l+\kappa})$ for a suitable l. By definition of κ, we infer that w contains $S^n(e)$ and the proof is finished. □

We will now introduce the substitution $\widetilde{S}$. Set

$$B \equiv \{a \in A : \limsup_{n\to\infty} |S^n(a)| < \infty\}$$

and

$$C \equiv A \setminus B.$$

Note that S maps B^* into itself. We define

$$\widetilde{S} : C \longrightarrow C^*, \text{ by } \widetilde{S}(x) \equiv \widetilde{S(x)},$$

where for an arbitrary word $w \in A^*$ we define $\widetilde{w}$ to be the word obtained from w by removing every element of B. As B is invariant under S, we infer that

$$\widetilde{S}^n(\widetilde{x}) = \widetilde{S^n(x)}$$

for arbitrary $x \in \mathcal{W}(S)$ and $n \in \mathbb{N}$. This will be used repeatedly in the sequel. Our next aim is to show that $\widetilde{S}$ is primitive if (BG) is satisfied. We need two preparatory lemmas.

Lemma 3.2. *Let* $(\Omega(S), T)$ *satisfy* (BG). *Then, the following are equivalent for* $a \in A$*:*

(i) $|S^n(a)| \longrightarrow \infty$, $n \to \infty$.
(ii) e *is contained in* $S^k(a)$ *for a suitable* $k \in \mathbb{N}$.

Proof. (i) $\Longrightarrow$ (ii). This is clear as e occurs with bounded gaps by (BG).
(ii)$\Longrightarrow$ (i). By (ii), $S^n(e)$ is a subword of $S^{k+n}(a)$ for every $n \in \mathbb{N}$. Now, (i) follows as e satisfies (2) by (BG). □

Lemma 3.3. *Let* $(\Omega(S), T)$ *satisfy* (BG). *Then, there exists* $m \in \mathbb{N}$ *such that* $S^n(e)$ *contains every letter of* A *for every* $n \ge m$.

Proof. By (BG) and (2), there exists $r \in \mathbb{N}$ such that $S^n(e)$ contains e whenever $n \ge r$. By (3), for every $a \in A$, there exists $n(a) \in \mathbb{N}$ such that $S^{n(a)}(e)$ contains a. Then $m = r + \sum_{a\in A} n(a)$ has the desired properties. □

We can now show that $\widetilde{S}$ is primitive, if (BG) holds.

Lemma 3.4. *Let* $(\Omega(S), T)$ *satisfy* (BG). *Then* $\widetilde{S} : C \longrightarrow C^*$ *is primitive.*

Proof. For $c \in C$, we can choose by Lemma 3.2 a number $n(c) \in \mathbb{N}$ such that $S^{n(c)}(c)$ contains e. Moreover, by Lemma 3.3, there exists m such that $S^n(e)$ contains every letter of A whenever $n \geq m$. Let $N \equiv m + \sum_{c \in C} n(c)$. Then, for every $c \in C$, $S^N(c)$ contains every letter of A. In particular, for each $c \in C$, the word $\widetilde{S}^N(c) = \widetilde{S^n(c)}$ contains every letter of C and primitivity of $\widetilde{S}$ is proved. □

As $\widetilde{S}$ is primitive, for $c \in C$, the behavior of $|\widetilde{S}^n(c)|$ for large $n \in \mathbb{N}$ is rather explicit. The next lemma allows us to compare this behavior with the behavior of $|S^n(c)|$.

Lemma 3.5. *Let* $(\Omega(S), T)$ *satisfy* (BG). *There exist constants* $L > 0$ *and* $N \in \mathbb{N}$ *with*

$$\frac{1}{L} \leq \frac{|\widetilde{S}^n(\widetilde{v})|}{|S^n(v)|} \leq 1$$

for arbitrary $n \geq N$ *and* $v \in \mathcal{W}(S)$ *containing at least one letter of* C.

Proof. The inequality $|\widetilde{S}^n(\widetilde{v})| \leq |S^n(v)|$ is obvious. To show the other inequality, note that, by (BG), there exists a constant κ such that every word with length exceeding κ contains a copy of e. This implies, $|\widetilde{v}| \geq \kappa^{-1}|v| - 2$ for arbitrary $v \in \mathcal{W}(S)$. Applying this inequality to $\widetilde{S}^n(\widetilde{v}) = \widetilde{S^n(v)}$, we find

$$|\widetilde{S}^n(\widetilde{v})| \geq \frac{1}{\kappa}|S^n(v)| - 2.$$

By definition of C, there exists $N \in \mathbb{N}$ with

$$|S^n(c)| \geq 4\kappa \text{ for all } c \in C \text{ and } n \geq N.$$

Thus,

$$|\widetilde{S}^n(\widetilde{v})| = |\widetilde{S^n(v)}| \geq \frac{1}{\kappa}|S^n(v)| - 2 \geq \frac{1}{2\kappa}|S^n(v)|$$

for arbitrary $n \geq N$ and $v \in \mathcal{W}(S)$ containing at least one letter of C. □

The key technical result in this section is the following proposition.

Proposition 3.6. *Let* $(\Omega(S), T)$ *satisfy* (BG). *Let* V *be a finite subset of* $\mathcal{W}(S)$ *all of whose elements contain at least one letter of* C. *Then there exist* $\theta > 0$ *and* $\lambda(V), \rho(V) > 0$ *with*

$$\lambda(V)\theta^n \leq |S^n(v)| \leq \rho(V)\theta^n$$

for arbitrary $n \in \mathbb{N}$ *and* $v \in V$.

Proof. Set $\widetilde{V} \equiv \{\widetilde{v} : v \in V\}$. As $\widetilde{S}$ is primitive by Lemma 3.4, there exist $\theta > 0$ and constants $\kappa_1, \kappa_2 > 0$ with $\kappa_1 \leq \theta^{-n}|\widetilde{S}^n(c)| \leq \kappa_2$ for arbitrary $c \in C$ and $n \in \mathbb{N}$. As V is finite and every $v \in V$ contains at least one letter of C, this shows existence of $\nu_1, \nu_2 > 0$ with

$$\nu_1 \leq \frac{|\widetilde{S}^n(\widetilde{v})|}{\theta^n} \leq \nu_2$$

for every $\widetilde{v} \in \widetilde{V}$. Therefore, by Lemma 3.5, there exist $\mu_1, \mu_2 > 0$ and $N \in \mathbb{N}$ with

$$\mu_1\theta^n \leq |S^n(v)| \leq \mu_2\theta^n$$

for arbitrary $n \geq N$ and $v \in V$. Adjusting the constants to fit in the remaining finitely many cases, we conclude the proof. □

With these preparations out of the way, we are now ready to prove Theorem 1.

Proof of Theorem 1. The implications (iii) $\Longrightarrow$ (ii)$\Longrightarrow$ (i) are obvious. The implication (i) $\Longrightarrow$ (ii) is given in Lemma 3.1.

It remains to prove (ii)$\Longrightarrow$ (iii). We will use the notion of return word introduced recently by Durand [17]. Recall that $x \in \mathcal{W}(S)$ is called a return word of $v \in \mathcal{W}(S)$ if $xv \in \mathcal{W}(S)$, xv begins with v, and $\#_v(xv) = 2$. Let $e \in A$ satisfying (BG) be fixed. Such an e exists by minimality. Let V be the set of return words of e. As e satisfies (BG), V is a finite set. Let $U \equiv \{z_1 z_2 : z_1, z_2 \in V, z_1 z_2 \in \mathcal{W}(S)\}$. As V is finite, so is U. By the minimality assumption (ii), there exists $G > 0$ such that every word in $\mathcal{W}(S)$ with length exceeding G contains every word of U. By Proposition 3.6, there exist $\theta, \lambda(V), \rho(V) > 0$ with

$$\lambda(V)\theta^n \leq |S^n(v)| \leq \rho(V)\theta^n \tag{4}$$

for all $v \in V$ and $n \in \mathbb{N}$. Define

$$C_{\mathrm{LR}} \equiv (3 + G)\theta\rho(V)\lambda(V)^{-1}.$$

We will show linear repetitivity of $(\Omega(S), T)$ with this constant. Thus, let $w \in \mathcal{W}(S)$ be given and consider an arbitrary $u \in \mathcal{W}(S)$ with $|u| \geq C_{\mathrm{LR}}|w|$. We have to show that u contains a copy of w. To do so, we will show the following:

- w is contained in $S^n(z_0)$ with suitable $n \in \mathbb{N}$ and $z_0 \in U$,
- u contains all words of the form $S^n(z)$ with $z \in U$.

Here are the details: Let $n \in \mathbb{N}$ be given with

$$\lambda(V)\theta^{n-1} \leq |w| < \lambda(V)\theta^n. \tag{5}$$

Combining this inequality with (4), we see that

$$|w| \leq |S^n(v)| \text{ for every } v \in V. \tag{6}$$

Apparently, we can choose $x = eye \in \mathcal{W}(S)$ such that w is a subword of $S^n(eye)$. Partitioning eye according to occurrences of e, we can write $eye = x_1 \dots x_k e$ with $x_j \in V$, $j = 1, \dots, k$. By (6), and since w is a subword of $S^n(x_1) \dots S^n(x_k)S^n(e)$, we then infer that w is in fact a subword of $S^n(z_1 z_2)$ with $z_1, z_2 \in V$ and $z_1 z_2 \in U$. Let us now turn our attention to u. By $|u| \geq C_{\mathrm{LR}}|w|$, $|w| \geq \lambda(V)\theta^{n-1}$ and the definition of C_{LR}, we infer

$$(3 + G)\rho(V)\theta^n \leq |u|. \tag{7}$$

Of course, as discussed above for w, we can also exhibit u as a subword of $S^n(x_1 \dots x_k)S^n(e)$ with $x_j \in V$, $j = 1, \dots, k$ and $x_1 \dots x_k \in \mathcal{W}(S)$. By (4) and (7), we then conclude that u must contain a word of the form $S^n(v)$ with $|v| \geq G$. By definition of G, the word v then contains $z_1 z_2$. Thus, u contains $S^n(z_1 z_2)$ which contains w. This finishes the proof. □

4. The Associated Schrödinger Operators

In this section we discuss applications to Schrödinger operators.

Recall that to a given subshift (Ω, T) over $A \subset \mathbb{R}$, we can associate the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$, $\omega \in \Omega$ acting by

$$(H_\omega u)(n) \equiv u(n+1) + u(n-1) + \omega(n)u(n). \tag{8}$$

Assume furthermore that (Ω, T) is minimal, uniquely ergodic, and aperiodic (i.e., $T^k\omega \neq \omega$ for every $k \neq 0$ and $\omega \in \Omega$). Denote the unique T-invariant probability measure by μ. Such operators have attracted a lot of attention in recent years (see, e.g., [11, 41] for reviews and below for literature concerning special classes). They arise in the quantum mechanical treatment of (one-dimensional) quasicrystals. The theoretical study of physical features (e.g., conductance) is accordingly performed by investigating the spectral theory of such families. It turns out that the spectral theory of these families is rather interesting. Namely, they exhibit features such as

- purely singular continuous spectrum,
- Cantor spectrum of Lebesgue measure zero,
- anomalous transport.

In the study of these and related properties, two classes of examples have received particular attention. These are Sturmian models (and more generally circle maps) [5, 9, 12, 13, 14, 15, 24, 40] and operators associated to primitive substitutions [3, 4, 6, 8, 10, 29, 30]. The aim of this section is to extend the theory from primitive substitutions to minimal substitutions, thereby giving a precise sense to (E) in this case.

For linearly repetitive systems it was recently shown by one of the authors [29] that their spectrum is a Cantor set if they are not periodic. Thus, we obtain the following result as an immediate corollary to Theorem 1 above and Corollary 2.2 of [29].

Theorem 2. *Let $(\Omega(S), T)$ be an aperiodic minimal substitution dynamical system. Then, there exists a Cantor set $\Sigma \subset \mathbb{R}$ of Lebesgue measure zero with $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$, where $\sigma(H_\omega)$ denotes the spectrum of H_ω.*

Remark 3. (a) If the subshift is periodic, it is well known that the spectrum is a finite union of (nondegenerate) closed intervals. Hence, in this case it is neither a Cantor set nor does it have Lebesgue measure zero.

(b) This theorem contains, in particular, the corresponding result for primitive substitutions obtained in [29] (see also [30] for a different proof).

(c) The theorem covers all the examples discussed in [35].

Next we state our result on singular continuous spectrum.

Theorem 3. *Let $(\Omega(S), T)$ be a minimal substitution dynamical system. If there exists $u \in \mathcal{W}(S)$ starting with $e \in C$ such that $uuue \in \mathcal{W}(S)$, then the operators (H_ω) have purely singular continuous spectrum for μ-almost every $\omega \in \Omega$.*

Remark 4. This also covers all the examples studied by de Oliveira and Lima in [35].

The proof of purely singular continuous spectrum has two ingredients. The first is a proof of absence of absolutely continuous spectrum. This follows by results of Kotani [25] and is in fact valid for every $\omega \in \Omega$ by results of Last and Simon

[27]. Alternatively, this follows by Theorem 2 (whose proof, however, uses Kotani theory [25]). The second ingredient is a proof of absence of eigenvalues. This is based on the so-called Gordon argument going back to [21]. Various variants of this argument have been used in the study of (8) (see [11] for a recent overview). We use it in the following form [11, 16, 24].

Lemma 4.1. *Let* (Ω, T) *be a uniquely ergodic subshift over* A. *Let* (n_k) *be a sequence in* $\mathbb{N}$ *with* $n_k \to \infty$, $k \to \infty$. *Set*

$$\Omega(k) \equiv \{\omega \in \Omega : \omega(-n_k + l) = \omega(l) = \omega(n_k + l),\ 0 \le l \le n_k - 1\}.$$

If $\limsup_{k\to\infty} \mu(\Omega(k)) > 0$, *then* μ*-almost surely,* H_ω *has no eigenvalues.*

The lemma reduces the proof of absence of eigenvalues to establishing the occurrence of sufficiently many cubes. For primitive substitutions, occurrence of many cubes follows from occurrence of one word of the form $uuue$, where e is the first letter of u. This was shown by one of the authors in [10] (see [9] as well). It turns out that this line of reasoning can be carried over to minimal substitutions. Namely, we have the following result.

Lemma 4.2. *Let* $(\Omega(S), T)$ *be a minimal substitution dynamical system. Let* $u \in \mathcal{W}(S)$ *be given starting with* $e \in C$ *such that* $uuue$ *belongs to* $\mathcal{W}(S)$. *Set* $n_k \equiv |S^k(u)|$. *Then,* $\limsup_{k\to\infty} \mu(\Omega(n_k)) > 0$.

Proof. As already mentioned, the proof is modelled after [10]. As $uuue$ occurs in $\mathcal{W}(S)$, so does $S^k(uuue)$ for $k \in \mathbb{N}$. Of course, $S^k(u)$ begins with $S^k(e)$. Thus, each occurrence of $S^k(uuue) = S^k(u)S^k(u)S^k(u)S^k(e)$ gives rise to $|S^k(e)|$ occurrences of cubes and we infer

$$\mu(\Omega(n_k)) \ge \mu(\Omega_{S^k(uuue)}) \times |S^k(e)|, \tag{9}$$

where we set $\Omega_v \equiv \{\omega \in \Omega : \omega(1)\ldots\omega(|v|) = v\}$ for $v \in \mathcal{W}(S)$. By Proposition 3.6, there exist $\lambda, \rho > 0$ and $\theta > 0$ with

$$|S^k(uuue)| \le \rho\,\theta^k \ \text{ and } \ \lambda\,\theta^k \le |S^k(e)|, \tag{10}$$

for every $k \in \mathbb{N}$. By Corollary 1, $(\Omega(S), T)$ is uniquely ergodic and therefore

$$\mu\left(\Omega_{S^k(uuue)}\right) = \lim_{|x|\to\infty} \frac{\#_{S^k(uuue)}(x)}{|x|}.$$

Moreover, by Theorem 1, $(\Omega(S), T)$ is linearly repetitive with some constant C_{LR}. Combining these estimates, we infer

$$\begin{aligned}
\mu(\Omega(n_k)) &\ge \lim_{|x|\to\infty} \frac{\#_{S^k(uuue)}(x)}{|x|}|S^k(e)| \\
&\ge \frac{1}{C_{\mathrm{LR}}|S^k(uuue)|}|S^k(e)| \\
&\ge \frac{\lambda}{C_{\mathrm{LR}}\,\rho}.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 3. By the discussion following the theorem, it suffices to show almost sure absence of point spectrum. This is an immediate consequence of the preceding two lemmas. □

Remark 5. There is another approach to proving absence of eigenvalues which is based on palindromes, rather than cubes. Concretely, Ω is said to be palindromic if $\mathcal{W}(\Omega)$ contains arbitrarily long palindromes. Hof et al. prove in [22] that if Ω is minimal and palindromic, then for a dense G_δ-set of $\omega \in \Omega$, the operator H_ω has empty point spectrum. This gives another method to prove absence of eigenvalues for minimal substitution Hamiltonians in cases where Lemma 4.2 does not apply, but where sufficiently many palindromes occur.

5. Fixed Points of Linearly Repetitive Substitutions

In this section we discuss an application of the results above to number theory.

Recall that every $z \in (0,1)$ has a binary expansion

$$z = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

with $a_n \in \{0,1\}$. For algebraic numbers, this binary expansion is expected to be a "random sequence." Of course, there are various ways to give a precise meaning to "random sequence." One particular way is that this binary expansion should not be a fixed point of a substitution (see [1] for further discussion). The question whether such a binary expansion can be a fixed point has thus attracted attention, and the most general result so far has been obtained by Allouche and Zamboni [1]. Namely, they show that a binary expansion which is a fixed point of either a primitive substitution or a substitution of constant length (i.e., the images of letters all have equal length) can only belong to a rational or transcendental number. We can prove the following result, merely assuming minimality:

Theorem 4. *Suppose $S : \{0,1\} \to \{0,1\}^*$ satisfies* (BG) (*i.e., S induces a minimal, linearly repetitive dynamical system*). *If $u \in \{0,1\}^{\mathbb{N}}$ is an aperiodic fixed point of S and $z \in (0,1)$ is given by*

$$z = \sum_{n=1}^{\infty} \frac{u_n}{2^n},$$

then z is transcendental.

Proof. For primitive S, the assertion was shown in [1]. Let us consider the case where S is nonprimitive. Then the alphabet B is not empty and we have either $S(0) = 0$ or $S(1) = 1$. Let us discuss the case $S(1) = 1$, the other case can be treated in an analogous way. From (BG) we can infer that $S(0)$ contains both 0 and 1 and it begins and ends with 0. By aperiodicity, $S(0)$ cannot be equal to 01^k0 with $k \geq 1$. That is, either $S(0)$ has the form

$$S(0) = 01^k0w0 \tag{11}$$

for some suitable word w, possibly empty, and suitable $k \geq 1$, or $S(0)$ has the form

$$S(0) = 00w0 \tag{12}$$

for some word w containing 1 as a factor.

We first consider the case where $S(0)$ is given by (11). Then

$$S^2(0) = 01^k0w01^k01^k0w0S(w)01^k0w0,$$

where we let $S(\varepsilon) = \varepsilon$ for definiteness. We see that u contains the word 01^k01^k0 and hence, for some prefix p,

$$u = p01^k01^k0\dots$$

Now define

$$U_n = S^n(p),\ V_n = S^n(01^k),\ V_n' = S^n(0).$$

Observe that we have

$$|V_n| \to \infty \text{ as } n \to \infty \tag{13}$$

by (BG),

$$\limsup_{n\to\infty} \frac{|U_n|}{|V_n|} < \infty \tag{14}$$

by Proposition 3.6, and

$$\liminf_{n\to\infty} \frac{|V_n'|}{|V_n|} > 0, \tag{15}$$

again by Proposition 3.6. We can now conclude the proof in this case by applying [20, Proposition 1] since (13)–(15) provide exactly the necessary input for an application of this proposition.

Let us now consider the case where $S(0)$ is given by (12). Then $S^2(0)$, and hence u, contains the factor 0^3. Therefore, $u = p000\dots$, so we can set $U_n = S^n(p)$, $V_n = V_n' = S^n(0)$ and then conclude as above. □

By the same argument one can prove the following extension; see the note added in proof of [1] for the necessary additional input (namely, a result of Mahler [33]).

Theorem 5. *If $z \in (0, 1)$ has a base b expansion ($b \in \mathbb{N}$ and > 1) which is given by an aperiodic fixed point of a substitution on a two-letter alphabet which satisfies* (BG)*, then z is transcendental.*

6. Unique Decomposition Property

In this section we study questions concerning unique decomposition for nonprimitive minimal aperiodic substitutions. For primitive substitutions, such a unique decomposition property has been shown by Mossé [34] (for a study of the higher-dimensional case, we refer to [39]). We are not able to treat the general case but rather restrict our attention to a two-letter alphabet. This case has attracted particular attention recently in the work of de Oliveira and Lima [35]. Thus, we can assume (and will assume throughout this section) that $A = \{a, b\}$ and

$$|S(a)| > 1 \quad \text{and} \quad S(b) = b.$$

For $w \in \mathcal{W}(S)$, an equation

$$w = z_0 z_1 \dots z_n z_{n+1}$$

is called a 1-partition if $z_1, \dots, z_n \in \{S(a), b\}$, z_0 is a suffix of an element in $\{S(a), b\}$ and z_{n+1} is a prefix of an element in $\{S(a), b\}$. Similarly, a 1-partition of $\omega \in \Omega(S)$ consists of a sequence $(E_n) \subset \mathbb{Z}$ with

$$\dots < E_{-2} < E_{-1} < E_0 < E_1 < E_2 < \cdots, \text{ and } \lim_{n\to\infty} E_n = \infty, \quad \lim_{n\to-\infty} E_n = -\infty,$$

such that $\omega(E_n)\dots\omega(E_{n+1} - 1) \in \{S(a), b\}$. We will prove the following theorem.

Theorem 6. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Then, every $\omega \in \Omega(S)$ admits a unique 1-partition. More precisely, there exist $L \in \mathbb{N}$ and words $w_1, \ldots, w_k$ such that the 1-partition $\{E_j\}$ of ω consists of exactly those E with $\omega(E - L) \ldots \omega(E + L) \in \{w_j : j = 1, \ldots, k\}$.*

To prove this result, we need some preparation. The proof of Theorem 6 appears at the end of this section.

Lemma 6.1. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Then $S(a)$ is neither a prefix of $bS(a)$ nor a suffix of $S(a)b$. In particular, two 1-partitions of a finite v which both start with $S(a)$ (end with $S(a)$) must agree up to a suffix (prefix) of length at most $|S(a)b|$.*

Proof. The second statement follows immediately from the first. So assume the first statement is wrong. Then, $S(a) = b^l$ with a suitable $l \in \mathbb{N}$ and periodicity of $(\Omega(S), T)$ follows. □

Lemma 6.2. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Set*

$$L_0 \equiv \max\{|v^n| : v \text{ subword of } S(a) \text{ and } v^n \in \mathcal{W}(S)\}.$$

If v with $|v| > 2|S(a)| + L_0$ admits a 1-partition beginning with $S(a)S(a)$, then every 1-partition of v starts with $S(a)S(a)$.

Proof. Assume the contrary. By Lemma 6.1, there exists then a 1-partition $v = z_0 z_1 z_2 \ldots z_n z_{n+1}$ of v with $0 < |z_0| < |S(a)|$ and $z_1 = S(a)$. This gives $S(a) = v^r$ with a primitive v and $r \in \mathbb{N}$ suitable. By definition of L_0, both this 1-partition of v and the 1-partition beginning with $S(a)S(a)$ contain blocks of the form b. Consider the leftmost of these blocks. Then, we obtain that v is a suffix of vb and thus, $v = b^l$. This in turn yields the contradiction $S(a) = b^{r|b|}$. □

It will be convenient to treat the two cases, where $S(a)$ does or does not contain the word aa, separately. We first consider the case where aa is a subword of $S(a)$ and prove uniqueness of decompositions under this assumption.

Proposition 6.3. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Suppose that aa occurs in $S(a)$. Then, there exists an $L \in \mathbb{N}$ such that all 1-partitions of $v \in \mathcal{W}(S)$ induce the same 1-partition on $v(L) \ldots v(|v| - L)$.*

Proof. By Lemma 6.2, all occurrences of the word $S(a)S(a)$ in 1-partitions of v which begin before the L_0-th position in v are uniquely determined. In particular, they occur at the same places in all 1-partitions. By Lemma 6.1, the 1-partitions must then agree to the right and to the left of such an occurrence up to a boundary term of length not exceeding $|S(a)b|$. Thus, it suffices to show existence of a 1-partition of v containing the blocks $S(a)S(a)$ if v is long enough. This, however, is clear by the assumption that aa occurs in $S(a)$ and minimality of $(\Omega(S), T)$, as every v is contained in a word of the form $S^n(a) = S(S^{n-1}(a))$. □

Next we turn to the case where $S(a)$ does not contain aa as a factor.

Lemma 6.4. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Suppose that aa does not occur in $S(a)$. Let $w \in \mathcal{W}(S)$ be given with*

$$w = S(a)b^{r_1}S(a)b^{r_2}S(a)\cdots S(a)b^{r_n}S(a) = xS(a)b^{s_1}S(a)b^{s_2}S(a)\cdots S(a)b^{s_u}y$$

with suitable $r_1, \dots, r_n \in \mathbb{N}$, $s_1, \dots, s_u \in \mathbb{N}$ and $x, y \in \mathcal{W}(S)$ with $|x| < |S(a)|$ and $|y| \leq |S(a)|$. Then, $u = n$ and $r_1 = s_1 = r_2 = s_2 = \cdots = s_n = r_n$.

Proof. By minimality, $S(a)$ begins and ends with a. As aa does not occur in $S(a)$, we have

$$S(a) = ab^{k_1}a \dots ab^{k_l}a$$

with suitable $k_1, \dots, k_l \in \mathbb{N}$. If $|x| = 0$, the statement now follows easily. Otherwise, we have $x = ab^{k_1}a \dots ab^{k_j}$ with $j < l$ suitable. Consider the blocks of consecutive $b's$ appearing in w. Such a block will be referred to as b-block. By $w = S(a)b^{r_1}\dots$, we infer that b^{r_1} is the $(l+1)$-st b-block appearing in w. By $w = xS(a)\dots$, we then infer that b^{r_1} occurs in $S(a)$ as the $(l+1-j)$-th b-block. Similarly, considering the occurrence of b^{r_2} in $w = S(a)b^{r_1}S(a)b^{r_2}\dots$, we infer that b^{r_2} is the $(2l+2)$-th b-block appearing in w. On the other hand, by $w = xS(a)b^{s_1}S(a)\dots$, we see that b^{r_2} is the $j + l + 1 + t$-th b-block in w, where $t - 1$ is the relative number of b-blocks occurring in $S(a)$ before b^{r_2}. This yields

$$2l + 2 = j + l + 1 + t,$$

and we infer $t = l + 1 - j$. Thus, b^{r_1} and b^{r_2} occur in the corresponding $S(a)$ blocks at the same relative positions. This yields immediately $r_1 = r_2$.

Denote the relative position of $b^{r_1} = b^{r_2}$ in $S(a)$ by p. Thus, $p = |S(a)| - |x| + 1$ by $w = S(a)b^{r_1}\dots = xS(a)\dots$. Now, consider the absolute position h of b^{r_2} in w. Then,

$$h = 2|S(a)| + r_1 + 1 = |x| + |S(a)| + s_1 + p.$$

Putting this together, we infer $r_1 = s_1$. Now, the assertion follows easily by repeating this reasoning. □

Proposition 6.5. *Let $(\Omega(S), T)$ be a minimal, aperiodic, nonprimitive substitution dynamical system over $\{a, b\}$. Suppose that aa does not occur in $S(a)$. Then, there exists $L \in \mathbb{N}$ such that all 1-partitions of $v \in \mathcal{W}(S)$ induce the same 1-partition on $v(L)\dots v(|v| - L)$.*

Proof. As S is minimal and aperiodic, there exists an $N \in \mathbb{N}$ such that $v^n \in \mathcal{W}(S)$ for $v \in \mathcal{W}(S)$ with $|v| \leq 2|S(a)|$ implies $n \leq N$. Set $L = (N+2)2|S(a)|$. Assume that there exists a $w \in \mathcal{W}(S)$ admitting two 1-partitions inducing two different 1-partitions of $w(L)\dots w(|w| - L)$. By Lemma 6.4, we infer existence of an $(N+1)$-power of $v = S(a)b^r$ in $\mathcal{W}(S)$. This contradicts the choice of L. □

Proof of Theorem 6. Let $\omega \in \Omega$ be given. Uniqueness of the 1-partition is clear from Propositions 6.3 and 6.5. Existence of a 1-partition follows by standard compactness-type arguments but can also be shown as follows: For $n \in \mathbb{N}$, set $v_n \equiv \omega(-n)\dots\omega(n)$. Then each v_n admits a 1-partition as it is contained in $S^l(a)$ with a suitable l. These 1-partitions are compatible due to the previous proposition. Thus, they easily induce a 1-partition of ω. These considerations and the previous proposition also imply the last statement of the theorem. □

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