A New Combinatorial Approach To Exponential Sums And The Lindelf Hypothesis:

precise growth orders for a class of sums, semi-natural periods, and the possibility of a critical scaling phenmenon

Hany M. Farag*
California Institute of Technology
253-37, Pasadena, CA 91125

May 9, 2002

Abstract

Recall that the Lindelf hypothesis is a weaker version of the Riemann hypothesis and, among other equivalencies, is equivalent to the statement that the exponential sum $\left|\sum_1^M n^{it}\right| << t^{\epsilon} M^{\frac{1}{2}}$ for all $M \leq \sqrt{t}$. By partial summation and the approximate functional equation, this is also the same as saying that the order of ζ on the critical line is zero. In spite of the introduction of several sophisticated (and diŒcult) approaches to this problem over the past century, there has been very little progress beyond the early classical results. Having made an eœort to thoroughly study previous ideas and attempts on this problem we believe that this is speciøcally due to the fact that all such approaches (to the best of the author's knowledge of the literature to date) have attempted to treat the logarithm (in $e^{it \log n}$) rather crudely in terms of general properties (e.g. ørst few derivatives), well before arriving to a thorough understanding of the distribution of $\{n^{it}\}_1^{\sqrt{t}}$ on the unit circle. Practically one can do this dyadically and study $\{n^{it}\}_{2^{J-1}}^{2^J}$ along with the corresponding sums. This paper reopens the problem from scratch and commences a predominantly combinatorial program of attack on this problem in which we insist on encoding this distribution directly into the method without allowing this information to disintigrate in the course of the analysis. To gain deeper insight on this distribution we believe that computational eŒciency requires that we expand the logarithm in a Taylor series and study the eœects of every term on the distribution (since none can be neglected). Certain heuristics, guided by numerical testing (and vice versa), indicate that, for $t = 3 \times \times 2^{2J}$ for instance, it is the quartic term in the Taylor series that

*The author is partially supported by the National Science Foundation grant #DMS-0070050

produces the most signiøcant oscillatory behavior (embodied in the order $\frac{1}{2}$ of the hypothesis) for the sum $\left|\sum_{2^{J-1}}^{2^J} n^{i2\pi t}\right|$. All other terms, though necessary to keep track of, are expected to have ıless oscillationsȷ. As we go down the dyadic ladder (with t øxed), the responsible terms move towards higher orders (this is all for $t = 2^{2J}$ or some small integer multiple thereof, and for more general t, this gets more complicated). This paper takes the ørst step in this program by subjecting a large class of sums which includes the cubic term to our method via a combinatorial analysis on phase space and we achieve optimal results (i.e. we compute the order exactly). For $t = 3 \times 2^{2J}$, the result is that the order for the exponential sum $\left|\sum_{2^{J-1}}^{2^J} e^{2\pi i t \frac{k^3}{3\times 2^J}}\right|$ is $\frac{2}{3}$, which is in agreement with our heuristics and our preliminary numerical testing. There is an intriguing possibility here: that this order is the reason that $\frac{1}{6} = \frac{2}{3} - \frac{1}{2}$ (from partial summation) has been a signiøcant barrier throughout the past century (and has bearly been penetrated). This is especially notable when we observe that (to the best of our knowledge) all methods to date have bearly been able to go beyond (roughly) third order derivatives of the logarithm in their estimates, at least in any eœective way. As interesting as this issue may be in its own right, we do not pursue this any further since we do not consider it a priority in understanding the heart of the Lindelf problem. Except for being more easily mangeable from a combinatorial perspective there is nothing special about the cubic term which we consider and indeed a whole class containing it is treated in this paper. This problem of Lindelf has a rich oscillatory behavior which has never been uncovered in the past and we believe it can be understood much more deeply via our approach. There is no question however that the present results are only a modest beginning. We take the next step in the near future.

Contents

1 Introduction

Since the problem is well-known to number-theorists we will be rather brief and we will quickly isolate the version on which we will work. We recall ørst (see [Pat] for instance) the Lindelf hypothesis in terms of the zeros of $\zeta(s)$. Let Z denote the nontrivial zeros of $\zeta(s)$. For $\frac{1}{2} < \sigma < 1$, and $T > 0$, let $N(\sigma, T) \equiv Card\{\zeta \in Z : \mathrm{Re}(\zeta) > \sigma, 0 < \mathrm{Im}(\zeta) < T\}$ (which is $\frac{1}{4}$ of the number of nontrivial zeros outside an inønite symmetrized strip of width $2(\sigma - \frac{1}{2})$). We also let $N(T) \equiv N\left(\frac{1}{2}, T\right)$. The Lindelf hypothesis is then the conjecture that, for every $\sigma > \frac{1}{2}$,

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T) \tag{1}$$

as $T \to \infty$. Note that for every such T, we have

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T). \tag{2}$$

It is thus clear that the Riemann hypothesis implies that of Lindelf. It is not yet known whether The latter implies the former however. We thus feel that in an attack on Lindelf, one should aim for optimal methods whenever possible. By that we mean a method which turns the problem into a computation, so that if the estimates fail, one can construct the counter example (i.e. ønd appropriate sequences of M and t which violate the conjectured estimate). The above version of Lindelf is also equivalent (see [Ka&Vo] for a nice treatment of this) to the statement that for every øxed ε, $\left|\zeta\left(\frac{1}{2} + it\right)\right| << t^{\varepsilon}$ as $t \to \infty$. Using the approximate functional equation, and partial summation, the latter version is then reduced to the following estimate

$$\left|\sum_{1 \le n \le x} n^{it}\right| << t^{\varepsilon}\sqrt{x}, \text{ for all } x \le \sqrt{t}. \tag{3}$$

It is this, highly concrete version, that has been the object of the study of exponential sums (see [Gr&Ko] for instance) and it is also the object of our analysis. The literature on the problem is vast, spans a good deal of the past century and we will not survey it. The interested reader is referred to [Hux], the very helpful book [Gr&Ko], and the classic [Tit-HB] for a chronological account. There are numerous contributions by several authors improving the state of knowledge of the order of $\zeta\left(\frac{1}{2} + it\right)$. We fear to list any lest we ønd it hard to draw the line. We will therefore only mention the result [Bo&Iv] which introduced new techniques that allowed improving the order to $\frac{9}{56}$. This has been pushed to $\frac{89}{570} \approx 0.156$ in [Hux]. When compared with the classical $\frac{1}{6} = 0.\overline{16}$ as in [Tit-HB], the diŒculty of the problem is evident. While the purpose of our program is to hopefully provide a method for a complete resolution, we do not attempt to ıpush the epsilonȷ here since we believe that we must ørst study the individual pieces of relevance (speciøcally the individual terms in the Taylor expansion) and achieve optimal insight on each. Only then can we attempt the conclusive step in any reasonable way. This paper is organized as follows: In Section 2 we set up the problem using the dyadic version, and we count the number of terms needed from a ønite Taylor approximation. In Section 3 we state our expectation for the behavior and contribution of general terms in the Taylor approximation and we motivate the appropriate exponents (i.e. growth orders) for the cubic term and other more complicated sums of more general form. In Section 4 we give lower bounds for the exponents using a scaling analysis which allows us to make exact computations (a rather rare occasion!). It seems quite unlikely that we can do exact computations on more complicated terms but it is very reassuring for us to have these lower bounds especially at this early point in our program so that we can then go forward with conødence. In Section 5 we introduce our main method and we subject our class of exponential sums to it and then obtain exact growth orders for this class. A base-2 expansion of the integers is extremely helpful. We obtain our upperbound using a combinatorial partition of the integers based on an algorithm that ıpairs the cancellationsȷ in a natural way. Motivated by the results and discussions in preceeding sections, we conclude the paper in Section 6 with some relevant remarks.

Acknowledgement 1 I would like to thank Roger Heath-Brown and Sid Graham for several helpful communications, Steve Gonek for his kind support and encouragement; Hugh Montgomery, Tom Apostol, and Dinakar Ramakrishna for brief but enlightening conversations. Finally, I am indebted to Barry Simon, the department at Caltech, and my enduring students for the opportunity to teach a graduate topics course on this subject at Caltech during winter 2001.

2 The Setup in Terms of The Taylor Series

Our object is to estimate the sum

$$S(M,t) \equiv \left| \sum_{n=1}^{M} e^{2\pi it\log n} \right|, \text{ for } M \leq \sqrt{2\pi t} \tag{4}$$

(the insertion of 2π in this paper is highly convenient but ultimately unnecessary). A dyadic decomposition of this sum is also very helpful. We thus consider sums like ($2^m \leq M$)

$$S_m(t) = \left| \sum_{2^{m-1}}^{2^m} e^{2\pi it\log n} \right|. \tag{5}$$

In this paper we set $t = 2^{2J}$ for concreteness but we remark that this is not essential to the method in general (though it would require extra work to generalize). The general case will be addressed in a future paper and we do not anticipate this be a big issue. For the time being we remark that, if t has base-2 expansion with a ønite number of 1's, the proofs of the main theorem (on upper bounds) can be readily generalized. It is usually the end point $S_J\left(2^{2J}\right)$ for instance that represents the main diŒculties as the oscillations have to be strong enough in spite of the fact that the derivative of $t\log n$ with respect to n is smallest in this range (if the reader sees this to be a vague statement, we will not contest as it is merely intuition for now, but see e.g. [Hux] for a similar situation). The reader may also wonder if important information may be lost by this dyadic decomposition (e.g. there may be cancellations among such terms which would be missed). To this end we can only say that our heuristics, numerical analysis, and the conjecture itself, seem to indicate that the scale of the cancellations is roughly $\frac{M}{M^{\frac{1}{2}}} = M^{\frac{1}{2}}$, and as long as our interval in the sum is large enough relative to $M^{\frac{1}{2}}$ (in dyadic sums 2^m replaces M), we expect the optimal estimates to hold for either the dyadic decomposition or otherwise (note that this philosophy does not really require the exponent $\frac{1}{2}$ here; any other is good as long as it is less than unity which is trivially true!). So we now wish to consider $S_{J'}\left(2^{2J}\right)$ in general. We expand the logarithm in a Taylor series at $2^{J'}$. For $2^{J'-1} < n \le 2^{J'}$, $\log n = \log\frac{n}{2^{J'}} + \log 2^{J'}$. Let $n = 2^{J'} - k$, so $0 \le k < 2^{J'-1}$, and

$$\log n = \log 2^{J'} + \log\left(1 - \frac{k}{2^{J'}}\right), \tag{6}$$

or

$$\log n - \log 2^{J} = -\sum_{1}^{r}\frac{1}{j}\left(\frac{k}{2^{J'}}\right)^{j} + O\left(\frac{k}{2^{J'}}\right)^{r}. \tag{7}$$

We choose r so that the maximum error $\left(\frac{1}{2}\right)^{r}2^{2J} = O\left(2^{-\frac{J}{2}}\right)$, which is then allowed for the conjectured estimate; i.e.

$$r\log 2 \approx \frac{5}{2}J\log 2, \tag{8}$$

or

$$r \approx \frac{5}{2}J \approx \log t \tag{9}$$

is suŒcient, and we can forget the error term not to mention the minus sign also. Note that this analysis is also valid for general t. Now let us focus on $S_J\left(2^{2J}\right)$. The terms $t\left(\frac{k}{2^J} + \frac{k^2}{2^{2J+1}}\right)$ are basically irrelevant (the ørst is trivial and the second term only gives a minus sign for odd k and is not inÆuential in the analysis; in fact one can sum on odd or even integers independently). In Section 4 we will actually give a lower bound for $\sum_0^{2^J-1} e^{2\pi i\frac{k^j}{2^J}}$ using some kind of multiscale analysis. Our interest of course is mainly in the $j=3$ case but, in fact, when t has more complicated expansion, some of the terms we study including $j=2$ become more relevant. As we will see, even a simple change like $\sum_0^{2^{J-1}-1}$ instead of $\sum_0^{2^J-1}$ already complicates the combinatorial analysis considerably for both upper or lower bounds. For upper bounds especially we must insist as we do in this paper (and succeed) on obtaining upper bounds for both sums and even more general; for otherwise there would be no hope in making the method generalize to the whole Lindelf problem. Having said this however, we remark that there is really no apparent diœerence in the behavior numerically or heuristically. The reason for the nontrivial complications is that we have to dig deeper in the distribution on the unit circle in order to combinatorially exhibit, and sum, the cancellations. It is a rather intriguing phenomenon.

3 General Numerical and Heuristic Indicators

In this section we report on the numerically exhibited behavior for the terms of the Taylor series. Most prominent for us at this point are sums of the form

$$\sum_{0}^{\frac{M}{2}-1} e^{2\pi i t\left(\frac{k}{M}\right)^j}$$

for M a power of 2 and t of the form of the previous section. We have eliminated the factor j in the denominator for reasons which will soon be explained. For the moment we can think of $t = j2^{2J}$ instead of just 2^J.Our numerical work seems to comfortably lean towards the (upper) bounds $M^{\left(1-\frac{2}{j}\right)}$ for $j \geq 4$, with $j = 4$ giving consistently $M^{\frac{1}{2}}$ behavior whether by itself or when combined with other terms. While we do not wish to put too much trust in purely numerical results, our heuristics seem to support this also as we will explain in a moment. First we remark that, although the optimal upper bounds for these sums when $j > 4$ may be better than stated above (though we doubt that), they all seem to be clearly worse than $M^{\frac{1}{2}}$. For $j = 4$, $M^{\frac{1}{2}}$ seems to be numerically stable (consistent). Guided by both numerical works and some actual estimates by the author, the following heuristic explanation for these exponents seems to be insightful (possibly accurate) and robust, though admittedly it was not our ørst guess. Consider for instance ($M = 2^J$)

$$\frac{k^j M^2}{M^j} = \frac{k^j}{M^{j-2}}, \tag{10}$$

and assume that $M^{\frac{2}{j}}$ is an integer (if $M = \alpha^J$ with $\alpha \in \mathbf{N}$, then $\alpha^{[\frac{2J}{j}]+1}$ is good enough). Then integers k of the form $nM^{\frac{j-2}{j}}$ give phases which behave like $2\pi n^j$ which represent the highest reinforcement of the phases. This portion of the sum is then of order $M^{\frac{2}{j}}$. The surprising phenomenon is that it appears that, as we deviate from these speciøc multiples, there seems to be fast decay and the sum seems remain of order $M^{\frac{2}{j}}$ (this was a surprising phenomenon to ønd). This actually works quite well for $j = 3$ and, in fact, will be rigorously seen in our combinatorial analysis for the class we study in Section 5. The main problem with this heuristic seems to be for (at least) the sum from 0 to

$M^{\frac{j-2}{j}}$ (the ørst semi-natural period as we would like to call it). Beyond this point, the cancellations and the ımixingȷ oscillations start to exhibit themselves profoundly. For instance, if we allow j large enough so that $M^{\frac{2}{j}} \leq 2$. then $n = 2$ is not even allowed, and it is intuitively (even rigorously) clear that one can only expect this sum to be of order $M^{\frac{j-2}{j}}$. Thus, in general, we expect the behavior for large M to go like

$$O\left(M^{\max\left(\frac{j-2}{j},\frac{2}{j}\right)}\right). \tag{11}$$

More interestingly, the best bound we can hope for is when $j = 4$ which then yields a magical $M^{\frac{1}{2}}$growth that is very important for the Lindelf problem. For $j = 3$, we expect (and prove) order $\frac{2}{3}$. It is thus no surprise that the bound $\frac{1}{6}$ (after partial summation) has been a major barrier over several decades. This is also consistent with the fact that the 4th Taylor term has not been eœectively incorporated in the literature (neither have we achieved that in this paper). When putting several terms together, the overall numerical behavior is consistent with $\frac{1}{2}$ and we actually expect that the quartic term dominates in some sense even though none of the other terms (or even the factors of j in the denominators) can be neglected in the analysis. The analogy is that of sign waves being multiplied (as in AM transmission); as long as there is one frequency well above the others, the resulting wave averages ıor sumsȷ to a small quantity. Things are much more delicate here of course. The reason we believe that our methods seem to work (at least so far) is that it is actually global. That is we go straight to the distribution on the unit circle rather than attempt any local calculations, for the latter seem to be practically impossible since an accurate estimate of the phase is always required and yet virtually impossible to achieve.

3.1 Generic Semi-natural Periods.

The discussion above motivates the following simple deønition:

Deønition 2 Let l, j, r, m, α, J be positive integers. For sums of the form

$$S(l,j,r,m,\alpha,J) \equiv \sum_{0}^{m^{J-r}} \exp\left(2\pi i \frac{k^j}{\alpha\left(m^J\right)^l}\right),$$

with J running to inønity for each choice of the other integers, we denote by $T(l,j,m,J)$ the generalized semi-natural period of such sum deøned by $T(l,j,m,J) \equiv m^{\left[\frac{Jl}{j}\right]}$.

Note that αdoes not enter the deønition really for reasons we have mentioned and more that we mention later. The intuition (which we do not elevate to a conjecture at this point) is that the growth of these sums as $J \to \infty$ should go like $\max\left(\frac{m^{-r}}{T(l,j,m,J)}, T(l,j,m,J)\right)$. Once again this has a promising minimum of

$M^{\frac{1}{2}} \equiv \left(m^{J-r}\right)^{\frac{1}{2}}$ growth! This is indeed what we prove for a less general class of sums in this paper (the main restriction is that $l = 1$ (and $m = 2$ but the latter is not a problem since we are using a dyadic decomposition). The reader may now wonder about our removal of α from the above considerations. There are two important reasons for that. First of all let us make it clear that the presence of αcan in some cases signiøcantly reduce the growth order from the simpliøed picture we presented above as we will see in studying the cubic term with and without the 3 in the denominator. However, a choice of $t = 3 \times 2^{2J}$makes the order $\frac{2}{3}$ and if $t = 2^{2J}$ we get order no more than $\frac{1}{2}$. Since we are ultimately after the best order for all t,we have to think of the growth order as $\frac{2}{3}$.The second reason is just as important, and that is that the problem is completely absent if α is a power of m for instance and this entire issue is merely an artifact of our use of M a power of 2 as opposed to a power of 3 or 5 etc. Since we ønd base 2 convenient we need to work around these issues. Our deønition of the generic seminatural periods should seem reasonable by now.

4 Multiscale Analysis and Exact Computations

In this section we do exact computations for $\left|\sum_0^{M-1} e^{2\pi i \frac{M^2 k^2}{2M^3}}\right|$, and $\left|\sum_0^{M-1} e^{2\pi i \frac{M^2 k^3}{M^3}}\right|$

for appropriately chosen M of the form 2^J. It should also be clear from the calculations that we now perform that there may be a whole class of sums for which we can perform the same procedure. There is no doubt however that these are still rare and we cannot hope to do exact calculations in any realistic situation. Let us start with the quadratic sum. Assume J is even and write $k = n \times 2^{\frac{J}{2}} + s$, with $0 \leq s, n \leq 2^{\frac{J}{2}} - 1$. Now

$$\frac{k^2}{2 \times 2^J} = \frac{n^2}{2} + \frac{ns}{2^{\frac{J}{2}}} + \frac{s^2}{2 \times 2^J}. \tag{12}$$

Hence the sum can be written as

$$\begin{aligned} &\sum_{s=0}^{2^{\frac{J}{2}}-1} \sum_{n=0}^{2^{\frac{J}{2}}-1} e^{2\pi i\left(\frac{n}{2}+\frac{ns}{2^{\frac{J}{2}}}\right)} e^{2\pi i \frac{s^2}{2\times 2^J}} \qquad (13) \\ &= \sum_{s=0}^{2^{\frac{J}{2}}-1} e^{2\pi i \frac{s^2}{2\times 2^J}} \times \left\{ \begin{array}{l} 2^{\frac{J}{2}}, s = \frac{2^{\frac{J}{2}}}{2} \\ 0 \text{ other wise} \end{array} \right. \\ &= 2^{\frac{J}{2}} \times e^{\frac{2\pi i}{8}}, \end{aligned}$$

and thus the order is $\frac{1}{2}$ for such J. Now we study the cubic sum. We follow the same strategy but we take $J = 2^I$ (so that we can extract lots of square roots). Now

$$\frac{k^3}{2^J} = \frac{3ns^2}{2^{\frac{J}{2}}} + \frac{s^3}{2^J} \bmod 1. \tag{14}$$

Thus the sum can be written as

$$\sum_{s=0}^{2^{\frac{J}{2}}-1}\sum_{n=0}^{2^{\frac{J}{2}}-1} e^{2\pi i\left(\frac{3ns^2}{2^{\frac{J}{2}}}\right)} e^{2\pi i\frac{s^3}{2^J}} \tag{15}$$

$$= \sum_{s=0}^{2^{\frac{J}{2}}-1} e^{2\pi i\frac{s^3}{2^J}} \times \begin{cases} 2^{\frac{J}{2}}, s = n' 2^{\frac{J}{4}}; 0 \le n' \le 2^{\frac{J}{4}} - 1 \\ 0, \text{ other wise} \end{cases}$$

$$= 2^{\frac{J}{2}} \sum_{n'=0}^{2^{\frac{J}{4}}-1} e^{2\pi i\frac{n'^3}{2^{\frac{J}{4}}}}.$$

This iterates for our choice of $J = 2^I$ until we reach a sum over 2 or 4 values. If I is even, we end up with 1 as a factor and the sum is of order

$$2^{\frac{J}{2}\left(1+\frac{1}{4}+\cdots\right)}, \tag{16}$$

so that if we take a sup over J, we get a growth order of $\frac{2}{3}$.

We have thus proved:

Theorem 3 $\mathrm{Sup}_M \frac{\log\left|\sum_0^{M-1} e^{2\pi i\frac{M^2k^2}{2M^3}}\right|}{\frac{1}{2}\log M} \ge 1$ and $\mathrm{Sup}_M \frac{\log\left|\sum_0^{M-1} e^{2\pi i\frac{M^2k^3}{M^3}}\right|}{\frac{2}{3}\log M} \ge 1.$

We do not pursue the case $j > 3$ since our interest in these types of calculations is rather limited. Furthermore, our study of upper bounds will actually reveal many alternate forms of these sums for which exact calculations can be possible using symmetry arguments that only become apparent when we subject the problem to our combinatorial technique. The above analysis can also be generalized to sums of similar form like $\sum_0^{M-1} e^{2\pi i\frac{k^j}{M}\alpha}$, when $(\alpha, M) = 1$. When $(\alpha, M) \neq 1$, this is no longer true but is not hard to analyze. Suppose $(\alpha, M) = 1$ and $\beta|M$. Then

$$\alpha\beta\left(k + \frac{M}{\beta}\right)^j = \alpha\beta\frac{k^j}{M} \bmod 1, \tag{17}$$

and we thus have:

Lemma 4 $\sum_0^{M-1} e^{2\pi i\alpha\beta\frac{k^j}{M}} = \beta\sum_0^{\frac{M}{\beta}-1} e^{2\pi i\alpha\frac{k^j}{\left(\frac{M}{\beta}\right)}}.$

The point of the lemma for our analysis and further possible generalizations should be obvious. In Section 5 we give the estimate $\left|\sum_0^{M-1} e^{2\pi i\frac{k^j}{M}}\right| << M^{\left(\frac{j-1}{j}\right)}$

for $M = 2^J$. Suppose however we consider the dyadic sum with $2^m = \frac{M}{2^r}$ replacing $M = 2^J$ but with $t = M^2$ fixed, i.e.

$$\left| \sum_0^{2^{J-r}-1} e^{2\pi i \frac{2^{jr}k^j}{2^J}} \right|. \tag{18}$$

This then has order $2^{(j-1)r}\left(2^{J-jr}\right)^{\left(\frac{j-1}{j}\right)} = 2^{\left(\frac{j-1}{j}\right)J}$. This suggests that the cancellations provided by this term in the complete exponential sum get worse (i.e. more enforcement in the phase) as we move towards smaller 2^m in the dyadic decomposition. This is also expected to be true of the other terms which we did not consider, so in fact the "dominant" oscillations should move towards higher order terms. Let us work the scaling as it quite insightful. If we assume our first kind of (heuristic) bound $M^{\frac{2}{j}}$, then we expect

$$\left| \sum_0^{2^{J-r}-1} e^{2\pi i \frac{2^{2J}k^j}{j\left(\frac{2^J}{2^r}\right)^j}} \right| << \frac{M}{2^r} \times \frac{1}{\frac{M^{1-\frac{2}{j}}}{2^r}} = M^{\frac{2}{j}}. \tag{19}$$

On the other hand, the other type of bound which is generally worse would give

$$\left|\sum\right| << \frac{M^{1-\frac{2}{j}}}{2^r}. \tag{20}$$

What we would like to get is $\approx \left(\frac{M}{2^r}\right)^{\frac{1}{2}}$ for the Lindelf hypothesis. For $2^r \leq M^{1-\frac{4}{j}}$, the latter dominates and we need $M^{\frac{2}{j}} \approx \left(\frac{M}{2^r}\right)^{\frac{1}{2}}$ which is in fact the same as we would get from either bound. We conclude (on a heuristic basis only) that the term in the Taylor series most likely behind the hypothesis (when $t = 2^{2J}$ or some small integer multiple thereof) should be the one with j related to M via

$$M^{\frac{2}{j}} = \frac{M^{1-\frac{2}{j}}}{2^{\frac{r}{2}}} = \frac{2^{J\left(1-\frac{2}{j}\right)}}{2^{\frac{r}{2}}} \approx \left(\frac{M}{2^r}\right)^{\frac{1}{2}}, \tag{21}$$

or j given via

$$\frac{r}{2} \approx J\left(\frac{1}{2} - \frac{2}{j}\right). \tag{22}$$

In particular, as we go down the dyadic scale (i.e. r increases) j moves towards higher order terms in the Taylor series. While we will not pursue this here, the effect of having t more general in base-2 expansion is expected to go in the opposite direction!

Note that for the Lindelf hypothesis it is enough to consider $\frac{M}{2^r} = 2^J \gtrsim t^{2\varepsilon} = M^{4\varepsilon}$ and thus the range on r is basically $\frac{\log M - 4\varepsilon}{\log 2}$ which is allowed by our equation when j varies in the truncated Taylor series. We will hence forth refer to our equations above as the critical scaling equations so that we can explore their validity in the course of our program. For example, according to these equations the dyadic sum for the Lindelf hypothesis at dyadic scale $2^m = \sqrt{M}$ should only give $2^{\frac{m}{2}} = (M^{\frac{1}{2}})^{\frac{1}{2}}$ growth when the 8th order term in the Taylor series is included. Again, note that in all of this t is initially øxed until the estimates are computed (as it always should be). If nothing else, we hope that by now the reader can see the value of studying the individual terms in the Taylor series, let alone insisting on studying the logarithm in terms of the appropriate Taylor approximation since there is a whole ladder of important seminatural periods coming from each term in the series as we discussed above.

5 Combinatorial Phase Space Analysis and Upper Bounds

This section captures the heart of our approach in which we sum over the integers via their base-2 representation and we show that, away from seminatural periods, the sums exhibit fast decay in terms of how far the base-2 representation is from that of multiples of seminatural periods. This is a surprising phenomenon which we suspect might hold true for general terms in the Taylor expansion of general exponential sums, and it is the reason for our inclusion of the heuristic Section 3 in this paper. We prove the theorem

Theorem 5 For J, j positive integers, we have as $J \to \infty$

1. $\left|\sum_0^{2^{J-1}-1} e^{2\pi i \frac{k^j}{2^J}}\right| << J\left(2^J\right)^{\frac{j-1}{j}}$ with constants depending only on j (explicitly computed in the proof);
2. The same order of growth holds when j is a power of 2 and a factor j appears in the denominator of the phase (i.e. $2\pi i \frac{k^j}{j 2^J}$);
3. For a phase of the form $2\pi i \frac{k^3}{3 \times 2^J}$,the order of growth is $<< J^2 \times 2^{\frac{J}{2}}$.

   Furthermore, in case (1) (for $j > 2$), the results are also sharp.

We start with the proof of (1) and then explain the necessary modiøcations to obtain (2) and (3). We can represent k as (say) $m2^{[\frac{J}{2}]+1} + s$ and expand

$$k^j = \frac{j s^{j-1} m}{2^{J-[\frac{J}{2}]}} + \frac{(j-1)\, m^2 s^{j-2}}{2} + \frac{s^j}{2^J} \bmod 1. \tag{23}$$

We will do this more eŒciently by expressing m in its base-2 expansion and similarly for s. We will øx s and sum over m via this expansion and then show that the absolute value of the sum over m decays as a function of the base-2 expansion of s. Then we sum over s also. Let us ørst do the case $j = 2$ as it is in fact prototypical. Assume J is even since the worst that can happen is that we will lose a factor of 2. We then write

$$k = a_0 2^0 + a_1 2^1 + \cdots + a_{J-1} 2^{J-1}. \tag{24}$$

As mentioned earlier, we are not in general allowed to have $a_{J-1} = 1$ (Lindelf requires this restriction in our setup), and our representation stops at a_{J-2}. We will leave a_{J-1} for the ride to show the reader how much easier it would have been to allow it. We then fix the first half of the $a_i's$ (more precisely up to $a_{[\frac{J}{2}]+1}$ but for simplicity assume J is even) and sum over the remaining half, we show that the sum over the former is of order, $2^{\frac{J}{2}}2^{-\alpha}$ where α is the first index in the representation of s marking the entire last block (of either 1's or zeros), i.e. $a_\alpha = \cdots = a_{\frac{J}{2}} = \varepsilon$, and $a_{\alpha-1} = 1-\varepsilon$, where ε is wither 1 or 0. The sum over s is then easily seen to be of order $J2^{\frac{J}{2}}$. Let us now see how this is done. The phase of the summands in the exponential sum can be expressed as

$$\sum_{\substack{\alpha,\beta \\ \alpha\le\frac{J}{2}, \\ \beta>\frac{J}{2}}} \frac{2\pi 2^\alpha 2^\beta a_\alpha a_\beta}{2^J}. \tag{25}$$

Suppose first that $a_0 = 0$ and $a_1 = 1$. Then if we sum over a_{J-2}, we get a factor of $\sum_{a_{J-2}=0}^{1} e^{2\pi i 2^{-1} a_{J-2}} = 0$, the same can be done for $a_0 = 0 = a_1$ but $a_2 = 1$ etc. We conclude that if s is even but $s \neq 0$, we get exactly zero for the sum (if $s = 0$, the sum is of order $2^{\frac{J}{2}}$). It should now be obvious that if a_{J-1} were allowed to be 1 this result would hold trivially for $s \neq 0$. The situation is much more interesting when $a_0 = 1$ and a_{J-1} is restricted to be zero. This is because none of these nice symmetries are exact any more. They are however approximate and our goal is to estimate their effect. Consider now the base-2 representation of s and divide the $a_i's$ into blocks of 1's and 0's, so that we have (from left to right) y_0 $1'$s, x_1 0's, y_2 1's, etc. (we can usually think of x_0 as infinite). Thus s can be written as

$$2^0 + 2^1 + \cdots + 2^{y_0-1} + 0 + \cdots + 0 + 2^{y_o+x_1} + \cdots + 2^{y_0+x_1+y_1-1} + 0 + \\ \cdots + 2^{y_0+\cdots+y_n+x_1\cdots x_n},$$

where $y_0 + y_1 + \cdots + y_n + x_1 + \cdots + x_n \le \frac{J}{2}$, and y_n denotes the last block of $1'$s (which may or may not be followed by a block of zeros). We now sum over $a_{J-2}, a_{J-3}, \cdots, a_{\frac{J}{2}}$, in that order (only for specificity). First we need a simple but crucial lemma

Lemma 6 $\cos\left(\frac{\Theta}{2}\right) \times \cos\left(\frac{\Theta}{4}\right) \cdots \times \cos\left(\frac{\Theta}{2^i}\right) = \frac{\sin(\Theta)}{2^i \sin\left(\frac{\Theta}{2^i}\right)}$.

The proof is trivial in view of the double angle formula and induction. This result will be important since the required estimates for the combinatorics to work will demand exact computations of certain products. Every time we sum over some a_α we get a factor proportional to $e^{i\theta_\alpha} + e^{i\theta_\alpha + \Delta_\alpha}$, where Δ_α is the phase difference when a_α changes from 0 to 1. Up to an overall phase about which we do not care, we only need to compute the phase difference Δ_α, and in which case we get a factor $2\cos\left(\frac{\Delta_\alpha}{2}\right)$. Thus the sum over all the a_α's will give an overall factor $2^{\frac{J}{2}}\prod_{\beta=\frac{J}{2}+1}^{J-1}\cos\left(\frac{\Delta_\alpha}{2}\right)$. The factor $2^{\frac{J}{2}}$ is what we expect, but the task now is to show that remaining product behaves as we have indicated earlier. We denote the factors in the product by E or F according to the sum over $a_{J-\gamma}$ with $\gamma-1$ an index lying in a block of 1's or 0's (respectively). We also use superscripts denoting the block number and subscripts denoting the term number in the block. So E_j^i denotes the cos factor arising from summing over $a_{J-\gamma}$ with $a_{\gamma-1}=1$ and lying in the block of 1's which has length y_i. Thus $\gamma-1$ is the jth 1 in this block. Similarly for F_j^i (with respect to 0's). We let

$$\overline{E^i} = \prod_{j=1}^{y_i} E_j^i,$$
$$\overline{F^r} = \prod_{m=1}^{x_r} F_m^r. \tag{26}$$

We now drop the usage of the index γ and just use $y_{i's}$ and $x_j's$. Let us compute E_j^0. We note that E_1^0 is only allowed to be 1 since we cannot change a_{J-1} from 0 to 1. However, E_2^0 can be calculated as follows. The phase change which we now denote δ_2^0 is

$$2\pi\left[2^{-1}+2^{-2}\right]=\pi\left[2^0+2^{-1}\right], \tag{27}$$

yielding

$$E_2^0=\cos\frac{\pi}{2}\left(\frac{1}{2}\right), \tag{28}$$

E_3^0 :

$$\delta_3^0=\pi\left[2^0+2^{-1}+2^{-2}\right], \tag{29}$$

and similarly

$$E_3^0=\cos\frac{\pi}{2}\left(2^{-2}\right), \tag{30}$$

...

$$E_{y_0}^0=\cos\frac{\pi}{2}\left(2^{-y_0+1}\right). \tag{31}$$

By Lemma 5,

$$\overline{E^0}=\frac{1}{2^{y_0-1}\sin\left(\pi 2^{-y_0}\right)}. \tag{32}$$

Now we compute F_j^1. F_1^1 : phase diœerence is

$$\pi\left[2^{-1}+2^{-2}+\cdots+2^{-y_0}\right], \tag{33}$$

and

$$F_1^1 = \sin\frac{\pi}{2}\left(2^{-y_0}\right). \tag{34}$$

F_2^1 : phase diœerence is

$$\pi\left[2^{-2} + 2^{-3} + \cdots + 2^{-y_0-1}\right], \tag{35}$$

so

$$F_2^1 = \cos\frac{\pi}{2}\left(\frac{1}{2} - 2^{-y_0-1}\right), \tag{36}$$

$\cdots$,

$$F_{x_1}^1 = \cos\frac{\pi}{2}\left[2^{-x_1+1} - 2^{-x_1-y_0+1}\right], \tag{37}$$

so that

$$\overline{F^1} = \frac{\sin\frac{\pi}{2}\left(2^{-y_0}\right)\sin\frac{\pi}{2}\left(1 - 2^{-y_0}\right)}{2^{x_1-1}\sin\frac{\pi}{2}\left(2^{-x_1+1} - 2^{-x_1-y_0+1}\right)}, \tag{38}$$

and

$$\overline{E^0} \times \overline{F^1} = \frac{1}{2^{y_0-1} \times 2^{x_0}\sin\frac{\pi}{2}\left(2^{-x_1+1} - 2^{-x_1-y_0+1}\right)}. \tag{39}$$

We wish to observe here that the most crucial cancellations come from the transition between blocks (1's to 0's or vice versa). All other sums are still important but in a weaker sense.

We now compute $\overline{E}_1$;

E_1^1 : phase change is $\pi\left\{2^0 + 2^{-(x_1+1)} + 2^{-x_1-2} + \cdots + 2^{-x_1-y_0}\right\}$, so that

$$E_1^1 = \sin\frac{\pi}{2}\left(2^{-x_1} - 2^{-x_1-y_0}\right),$$

E_2^1 : phase change is $\pi\left\{2^0 + 2^{-1} + 2^{-x_1-2} + 2^{-x_1-3} + \cdots\right\}$, so that

$$E_2^1 = \cos\frac{\pi}{2}\left(\frac{1}{2} - 2^{-x_1-2} - 2^{-x_1-3} + \cdots\right),$$

similarly

$$E_3^1 = \cos\frac{\pi}{2}\left(\frac{1}{4} - 2^{-x_1-3} - 2^{-x_1-4} - \cdots\right), \cdots$$

$$E_{y_1}^1 = \cos\frac{\pi}{2}\left(2^{-y_1+1} - 2^{-x_1-y_1} - \cdots\right),$$

hence

$$\overline{E}^1 = \sin\frac{\pi}{2}\left(2^{-x_1} - 2^{-x_1-y_0}\right) \tag{40}$$
$$\times \frac{\sin\frac{\pi}{2}\left(1 - 2^{-x_1-1} - \cdots\right)}{2^{y_1-1}\sin\frac{\pi}{2}\left(2^{-y_1+1} - 2^{-x_1-y_1} - \cdots\right)}$$
$$= \frac{\sin\frac{\pi}{2}\left(2^{-x_1} - 2^{-x_1-y_0}\right)\cos\frac{\pi}{2}\left(2^{-x_1-1} + \cdots\right)}{2^{y_1-1}\sin\frac{\pi}{2}\left(2^{-y_1+1} - 2^{-x_1-y_1} - \cdots\right)}, \tag{41}$$

so that

$$\overline{E}^0 \times \overline{F}^1 \times \overline{E}^1 =$$
$$\frac{\cos\frac{\pi}{2}\left(2^{-x_1-1} + \cdots\right)}{2^{y_0-1}2^{x_0}2^{y_1}\sin\frac{\pi}{2}\left(2^{-y_1+1} - 2^{-x_1-y_1} - \cdots\right)\cos\frac{\pi}{2}\left(2^{-x_1} - 2^{-x_1-y_0}\right)},$$

where if we denote the terms $2^{-x_1-1} + \cdots$ by w, the terms $2^{-y_1+1} - 2^{-x_1-y_1} - \cdots$ represent $2^{-y_1+1}w$. Now observe that $w = 2^{-x_1} - 2^{-x_1-y_0}$, so that

$$\overline{E}^0 \times \overline{F}^1 \times \overline{E}^1 = \frac{1}{2^{-y_0-1}2^{x_0}}\left[\frac{1}{2^{y_1}\sin\frac{\pi}{2}2^{-y_1}}\right]$$
$$\leq \frac{1}{2^{y_0-1}2^{x_0}}.$$

In exactly the same way, it can be seen that, if the last block is y_n, then our overall factor is

$$2^{\left(1-\sum_0^{n-1} y_i - \sum_0^n x_i\right)}. \tag{42}$$

If it is x_m instead, the overall factor is

$$2^{\left(1-\sum_0^{m-1} y_i - \sum_0^{m-1} x_i\right)}. \tag{43}$$

In general this factor can be written as

$$\frac{1}{2^{\text{length of } s \,-\, \text{length of last block}}}. \tag{44}$$

Suppose then that we øx the last block. The cardinality of the set of positive integers s with given last block is bounded by the reciprocal of the above factor. There are J types of last blocks (being either of type x or y, and of length anywhere from 1 to $\frac{J}{2} - 1$). Thus the total sum on k yields at most $J \times 2^{\frac{J}{2}}$ as claimed. Our argument also shows that if we sum over $a_i's$ up to $i = J - r$ only, we still get a $<< J \times 2^{\frac{J}{2}}$ estimate. Now we consider the case $j > 2$. Again, for even s, we have exact cancellations in precisely the same way as before except for one signiøcant diœerence. Namely, if $s = 2^{\alpha_0} + a_{\alpha_0+1}2^{\alpha_0+1} + \cdots$, the cancellation comes from a sum over a_β where $\beta + (j-1)\alpha_0 = J - 1$. Our s

here should actually be thought of as the ørst $[\frac{J}{j}]+1$ portion of the base-2 rep. of k.(assume $j|J$ for simplicity here but the same technique works either way). So α_0 should in fact be at most $[\frac{J}{j}]$ and hence $\beta > [\frac{J}{j}]$ which is what we need if we think of øxing s and summing over the remaining part of k. Note that if this $s=0$, then summing over the remaining part of k is of order $2^{J(\frac{j-1}{j})}$. It is then suŒcient to consider the case s is odd, but now we will go back to our original representation which has s being the ørst half of the rep. of k. In our summation, we ørst øx this s, sum over the remaining part of k, then take absolute values and ønally sum over s. The result is then identical to the case $j=2$. The point is that in (24), we can treat js^{j-1} the same way we treated s when $j=2$. There is one obstruction however; namely that we need to know how many odd integers s have $js^{j-1} = g \bmod 2^{[\frac{J}{j}]}$ for a given g (since our last argument really summed over all possible g). Let $[i]_2$ be the unique integer such that $i = 2^{[i]_2} \times f$ with f odd. We then have (i, J', γ positive integers and α nonnegative integer):

Lemma 7 Given $0 \le g \le 2^{J'}$, we have

$$\mathrm{Card}\left\{ s : [s]_2 = \alpha, 0 \le s \le 2^{J'}, \gamma s^i = g \bmod 2^{J'} \right\} \le \left\{ \begin{array}{c} 2^{i\alpha+[\gamma]_2} \times i(2^{[i]_2}+1), i \text{ even}; \\ 2^{i\alpha+[\gamma]_2},\ i \text{ odd} \end{array} \right. .$$

Assume ørst a, b odd and that $a^i = b^i = g \bmod 2^{J'}$. Write $a^i - b^i = (a-b) \times \left(a^{i-1} + a^{i-2}b + \cdots + b^{i-1}\right)$. If i is odd, then we must have $2^{J'-[\gamma]_2} || (a-b)$. Since $1 \le a, b \le 2^{J'}-1$ we have in this case the cardinality is $2^{[\gamma]_2}$. The case of $\alpha > 0$ is similarly treated by absorbing a of $2^{i\alpha}$ in γ. Now suppose $[i]_2 \ge 1$. Now let us absorb $[\gamma]_2 + i\alpha$ in J' (which has the eœect of multiplying the result by $2^{i\alpha+[\gamma]_2}$ as we indeed have in the statement of the Lemma) so we can now assume (WLOG) $\gamma = 1$ and $\alpha = 0$. Observe that $a^i - b^i = 0 \bmod 2^{J'}$. Suppose we also choose b as the smallest integer satisfying this congruence. Let $a = b + \Delta$. Then $a^i - b^i = \Delta\left(ib^{i-1} + \frac{i(i-1)}{2}\Delta b^{i-2} + \cdots + \Delta^{i-1}\right) \equiv \Delta Q$. We observe , by inspecting the binomial coeŒcients, that if $4|\Delta$, then $[Q]_2 = [i]_2$ (every factor of Δ will give at least one extra factor of 2 than the ørst term). If this is the case then in fact $2^{\frac{J}{2}-[i]_2}|\Delta$. Since $0 \le \Delta \le 2^{J'}$, we have $2^{[i]_2}$ possibilities of such values for such a. We denote the set of such values of a by G_1 which then has cardinality at most $2^{[i]_2}$. If $[\Delta] = 1$, we denote the set of the corresponding values of a by B_1 (G for good, B for bad!). The following argument is a more general version of that in [Dav] (chapter II, section 7). Let $x_1 = b$ and suppose x is a root of the congruence $x^i - x_1^i = 0 \bmod 2^{J'}$ and $x \notin G_1$. Then $x^i - x_1^i = (x - x_1)\, g_1(x) = 0 \bmod 2^{J'}$, where g_1 has degree $i-1$. Now for such x, we have $[x - x_1]_2 = 1$, so that $g_1(x) = 0 \bmod 2^{J'-1}$. Now suppose x_2 satisøes the same congruence and is not in G_1, so that $g_1(x_2) = 0 \bmod 2^{J'-1}$. We can then factor $g_1(x) = (x - x_2)\, g_2(x)$. We now form the sets G_2, B_2 for x_2, and consider x a root of the congruence but $x \notin G_1 \cup G_2$. We can thus factor g_2 again and reduce its degree. This process has to stop with at most i such $x_k's$ and each

g_k is actually monic. At most, we can have i of these roots with corresponding sets G_k and B_k. Thus $Card\ S \leq (i\,[i]_2 + i)$. This completes the proof of the Lemma and hence case (1) of the theorem. Part (2) is almost the same. The eœect of j in the denominator is basically to increase J without increasing the range of the sum, a situation which our method can handle due to its purely combinatorial nature. Basically this disallows us from summing over the last[$j]_2$ $a'_\alpha s$ in the rep. of k. The entire procedure can be applied again in the same way, but now not only odd s is an issue but any s s.t. $[s]_2 < [j]_2 + 1$. In view of Lemma 7 for nonzero α, we still can compute the same growth order albeit with diœerent constants. Case (3) of the theorem is a bit diœerent and arises from the incompatibility of our choice of base-2 and the 3 in the denominator. However we can still rescue our proofs. The main observation is that, if we again (assume J even for simplicity only) express

$$\frac{k^3}{32^J} as \frac{(s+m2^{\frac{J}{2}})^3}{32^J} = \frac{s^3}{32^J} + \frac{s^2 m2^J}{2^J} + \frac{m^3 2^{\frac{3}{2}}}{32^J} \bmod 1.$$

The main new ingredient only comes from the last term which we now analyze. We observe that $m^3 2^{\frac{J}{2}} \bmod 3$ is either $m^3 2^{\frac{3J}{2}} \bmod 3$ or $2m^3 2^{\frac{3J}{2}} \bmod 3$. Assume it is the ørst (the latter is treated identically) and then this term can be written as $\sum_{\alpha=\frac{J}{2}}^{J} a_\alpha^3 2^{3\alpha}$. Since a_a is either 0 or 1, our term can be written as $\sum_{\alpha=\frac{J}{2}}^{J} a_\alpha 2^{3\alpha}$ $\bmod 3$ which, since $2^{2\alpha} = 1 \bmod 3$, we can write as $\sum_{\alpha=\frac{J}{2}}^{J} a_\alpha 2^{\alpha} \bmod 3$. Hence, our term can actually be written as $\frac{m2^{\frac{J}{2}}}{3}$ or, ønally, as $\frac{m2^{\frac{J}{2}}}{32^J}2^J$. The point is that we can now combine $\frac{s^2}{2^J}$ and $\frac{2^{\frac{3J}{2}}}{3}$ as one unit when we perform the sum on a_α (which are now the coeŒcients of the base-2 expansion of $r = \frac{s^2}{2^J} + \frac{2^{\frac{3J}{2}}}{3}$,with $\frac{J}{2} \leq \alpha \leq J$ (or $J-1$) if desired. Now a really useful observation is that we never start with a block of zeros; indeed the fractional part of this new term disallows the case where we would typically have complete re-enforcement of the phase. Our argument is now as follows: if s is odd we have the same situation as in (1),(2) of the theorem which contribute growth $<<J2^{\frac{J}{2}}$. Now suppose $[s]_2 = \alpha$. If $\alpha \geq \frac{J}{4}$, then $[s^2]_2 \geq \frac{J}{2}$ and then only the term $\frac{2^{\frac{3J}{2}}}{3}$ gives cancellations when we sum over a_l with $l \geq \frac{J}{2}$. This sum is easily calculated to be 1 (c.f. $\left|\cos(\frac{2\pi}{3})\right| = \left|\cos(\frac{4\pi}{3})\right| = \frac{1}{2}$). The contribution to the complete sum for terms of this form is then $2^{\frac{J}{4}}$. Suppose however that $\alpha < \frac{J}{4}$, then when we sum over a_l with $J-1-2\alpha \leq \alpha \leq J-1$ we get a factor of 1 It is really the sum over a_l with $\frac{J}{2} \leq \alpha < J-1-2\alpha$ that requires an extra argument. We just have to apply Lemma 7 for this value of α and the cardinality of the relevant set is then of order $2^{2\alpha}$ whereas the sum over a_l (in the representation of r deøned above) with $\frac{J}{2} \leq l \leq J-1-2\alpha$, and then over a_l (precisely as in the proof of part (1) with $2\alpha \leq l < \frac{J}{2}$ yields growth $<<J2^{\frac{J}{2}-2\alpha}$. The total growth (after summing over α as well) is $<< J^2 2^{\frac{J}{2}}$. This completes the proof of the theorem except for the sharpness of (1). This however can be seen in our arguments when $j|J$. Simply the sum over odd k is always of order $2^{\frac{J}{2}}$, the sum over even k cancels exactly whereas the sum over multiples of the semi-natural periods give exactly one term with the precise order of growth. This is all for $j > 2$ (since for $j = 2$ the sum over odd k is of the same order). We remark however that in Section 4 we showed that, for $j = 2$ and a factor of 2 present in the denominator, the result is basically sharp.

6 Concluding Remarks

What we have achieved in this paper so far is certainly a modest beginning. The most important aspect of the problem on which we hope to have shed some light is the crucial importance of a study of the inÆuence of general Taylor terms in the exponential. These seem to represent frequencies that aœect cancellations on diœerent scales (i.e. certain multiples of some integer or say certain multiples of powers of two). The critical scaling equations we presented in Section 4 are only based on the idea of semi-natural periods which in turn is based on the heuristics of Section 3. The results of Section 5 however do prove this principle to be true for the sums we considered there. While still keeping our options open, the author feels strongly that the results are by no means coincidental and that further investigations of this principle should produce stronger and deeper results.

We also made heavy use of dyadic representations. This is certainly not an assumption as we mentioned before but certainly our choice of $t = 2\pi 2^J$(or $j2\pi 2^J$ at times)simpliøed life for the time being. While we expect that generalizing to arbitrary t will require some eœort, we do not believe that there should be any fundamental obstruction to do so. In fact, it seems possible to simply do the same type of analysis with 3-adic or 5-adic representations etc. (and t of the form $2\pi 3^J$, $2\pi 5^J$ resp. etc.), in which case we would decompose the sum for ζ in a similar way (e.g. $\sum_{3^k}^{3^{k+1}}$), and we should be able to get similar results as we found in this paper. We also saw how $j = 3$ with a factor of j in the denominator gave some technical complications but not fundamental ones. We thus feel it is best to consistently work with the simple dyadic representations and $t = 2\pi 2^J$, and only later examine the eœect of more general t. We pursue the next step in our program in a future paper.

References

[Bo&Iv] Bombieri, E. and Iwaniec, H. (1986). On the order of $\zeta\left(\frac{1}{2} + it\right)$, Annali Scoula Normale Superiore - Pisa (4) 13, 449-472.

[Dav] Davenport, H. (1989), Cambridge University Press. The Higher Arithmetic, 5th edition.

[Gr&Ko] Graham, S. and Kolesnik, G. (1991). Van der Corput's Method of Exponential Sums. London Mathematical Society Lecture Note Series 126.

[Hux] Huxley, M. (1993). Exponential sums and the Riemann Zeta Function IV, Proc. London Math. Soc. (3) 66 (1993) 1-40.

[Ka&Vo] Karatsuba, A. and Voronin, S. (1992). The Riemann Zeta-Function. De Gruyter Expositions In Mathematics 5.

[Pat] Patterson, S. (1995). An introduction of the theory of the Riemann Zeta-Function. Cambridge studies in advanced mathematics 14.

[Tit-HB] Titchmarsh, E. (Revised by Heath-Brown, D.) (1999). The Theory of the Riemann Zeta-function, second edition. Oxford Science Publication