# HARMONIC MEASURE AND POLYNOMIAL JULIA SETS

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Abstract. There is a natural conjecture that the universal bounds for the dimension spectrum of harmonic measure are the same for simply connected and for non-simply connected domains in the plane. We prove this fact for domains bounded by polynomial Julia sets. The idea is to consider the coefficients of the dynamical zeta-function of a polynomial as subharmonic functions on a slice of its Teichmüller space, and then apply the maximum principle.

## 1. Harmonic measure

1.1. **Dimension spectra.** For a finite measure $\omega$ in the complex plane, we define

$$f_\omega^+(\alpha) := \dim\{\alpha_\omega(z) \le \alpha\},$$

where $\alpha_\omega(z)$ is the lower pointwise dimension of $\omega$:

$$\alpha_\omega(z) := \liminf_{\delta\to 0} \frac{\log \omega B(z,\delta)}{\log \delta}.$$

The *universal dimension spectra* $\Phi(\alpha)$ and $\Phi_{sc}(\alpha)$ are defined to be the supremum of $f_\omega^+(\alpha)$ taken over all harmonic measures $\omega$ of arbitrary plane domains, and of arbitrary simply connected plane domains respectively. We refer to [13] for the discussion of the universal spectra and related topics.

In the simply connected case, we can express $\Phi_{sc}(\alpha)$ in terms of the universal *integral means* spectrum $B(t)$, see [13]. Namely, if we denote

$$\Pi(t) := B(t) - t + 1,$$

then

$$\Phi_{sc}(\alpha) = \inf_{t\ge 0} \, [\alpha\Pi(t) + t], \qquad (\alpha \ge 1). \tag{1}$$

This work is motivated by the following

**Conjecture.** $\Phi(\alpha) = \Phi_{sc}(\alpha)$ *for all* $\alpha \ge 1$.

In particular, the conjecture would imply the inequality

$$\Phi(\alpha) \le \alpha - c(\alpha-1)^2, \qquad (1 \le \alpha \le 2), \tag{2}$$

with some *absolute* constant $c > 0$, for the corresponding result in the simply connected case follows from (1) and an elementary estimate

$$B(t) \le Ct^2, \qquad (|t| \le 1).$$

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Note that (2) is stronger than the statement

$$\dim\ \omega \leq 1 \qquad \textit{for all plane domains.} \tag{3}$$

Indeed, by the definition of dimension spectrum, we have

$$f^+_\omega(\alpha) = \alpha \qquad \text{for} \quad \alpha = \dim \omega,$$

and so the strict inequality

$$f^+_\omega(\alpha) < \alpha, \qquad (1 \leq \alpha \leq 2),$$

always implies dim $\omega \leq 1$. Theorem (3) was established in [15] for polynomial Julia sets, in [12] for simply connected domains, and in [7] for general planar domains. Also compare [12] and [20].

1.2. **Fractal approximation.** A set $J \subset \mathbb{C}$ is a *conformal Cantor set* if it is generated by some analytic map of the form

$$F : \bigcup D_j \to D, \tag{4}$$

where $\{\bar{D}_j\}$ is a finite collection of pairwise disjoint closed topological discs sitting inside a simply connected domain $D$, and the restriction of $F$ to each $D_j$ is a bijection $D_j \to D$.

If $F$ in (4) is a polynomial, and if $D$ contains all critical points of $F$, then we call $J$ a *polynomial Cantor set*. In this case, the critical points escape to infinity and $J = J_F$ is the usual Julia set of $F$.

For conformal Cantor sets, we have

$$f^+_\omega(\alpha) = \sup\{f_\omega(\alpha') : \ \alpha' \leq \alpha\},$$

where

$$f_\omega(\alpha) := \dim \left\{ z \in J : \ \lim_{\delta \to 0} \frac{\log \omega B(z,\delta)}{\log \delta} \leq \alpha \right\}.$$

It is known, see [4] and [13], that to prove the conjecture it is sufficient to show that

$$f_\omega(\alpha) \leq \Phi_{sc}(\alpha), \qquad (\alpha \geq 1)$$

for arbitrary conformal Cantor sets. We have a partial result in this direction.

**Theorem A.** *If $\omega$ is harmonic measure of a polynomial Cantor set, then*

$$f_\omega(\alpha) \leq \Phi_{sc}(\alpha), \qquad (\alpha \geq 1).$$

The proof depends on the two facts (Theorems B and C), which we state below in this section and verify in Sections 2 and 3. We refer to [3] and [17] for general facts concerning polynomial dynamics.

1.3. **Pressure function and periodic points.** Let $F$ be a polynomial of degree $d$. Fixing a point $z_0$ in the basin of infinity, we define the *pressure function* of $F$ by the formula

$$P_F(t) := \limsup_{n\to\infty} \frac{1}{n} \log_d \sum_{z\in F^{-n}z_0} |F_n'(z)|^{-t}.$$

The following assertions are well known.

**Lemma 1.** *If $J = J_F$ is a polynomial Cantor set, then*

$$f_\omega(\alpha) = \inf_{t\geq 0}\, [t + \alpha P_F(t)], \qquad (\alpha \geq 1).$$

**Lemma 2.** *If $F$ is a polynomial with connected Julia set, then the function*

$$\beta(t) = P_F(t) + t - 1$$

*is the integral means spectrum of the basin of infinity. In particular, by* (1) *we have*

$$P_F(t) \leq \Pi(t). \tag{5}$$

Our goal is to extend the latter inequality (with $t \geq 0$) to polynomial Cantor sets. By (1) and Lemma 1, this will imply Theorem A.

To compare the bounds for the pressure function in the simply connected and non-simply connected cases (and this is our main idea), it is more convenient to work with a version of the pressure function that involves the multipliers of periodic points. Let us denote

$$Z_n(F,t) := \sum_{a\in \mathrm{Fix}(F^n)} |F_n'(a)|^{-t},$$

see [21] regarding the connection with dynamical zeta-function.

**Theorem B.** *If a polynomial $F$ of degree $d$ has no non-repelling cycles, then*

$$P_F(t) \;\geq\; \limsup_{n\to\infty} \frac{1}{n} \log_d Z_n(F,t).$$

It is of course well known that for hyperbolic polynomials we have

$$P_F(t) \;=\; \lim_{n\to\infty} \frac{1}{n} \log_d Z_n(F,t).$$

1.4. **Holomorphic families of polynomials.** The second ingredient in the proof of Theorem A is a construction due to Branner and Hubbard [1], which we recall in Section 2; also see [16] for an interpretation in terms of Teichmüller's spaces. Given a polynomial $F$ with all critical points escaping to infinity, there is a canonical way to embed $F$ in a holomorphic polynomial family:

$$\lambda \in \mathbb{D} \mapsto F_\lambda, \qquad (F_0 = F).$$

In the case of quadratic polynomials, we simply take

$$F_\lambda(z) = z^2 + c(\lambda),$$

where $\lambda \mapsto c(\lambda)$ is a universal covering map of the complement $\mathbb{C} \setminus \mathcal{M}$ of the Mandelbrot set $\mathcal{M}$.

**Theorem C.** *Let $F$ be a polynomial with all critical points escaping to infinity, and let $\{F_\lambda\}$ be its Branner-Hubbard family. Then the following is true for almost every point $\zeta \in S^1$: there exists a limit*

$$F_\zeta(z) := \lim_{r\to 1-} F_{(r\zeta)}(z),$$

*and the polynomial $F_\zeta$ has connected Julia set and has no non-repelling cycles.*

For quadratic polynomials this was proved in [5]. Moreover, it was shown in [2], [6], and [23] that almost every point on the boundary of the Mandelbrot set is a Collet-Eckmann polynomial.

1.5. **Proof of Theorem A (assuming Theorems B and C).** As we mentioned, it is sufficient to show that if all critical points of $F$ escape to infinity, then

$$P_F(t) \le \Pi(t), \qquad (t > 0).$$

Let us introduce the functions

$$s_n(\lambda) := \frac{1}{n} \log_d Z_n(F_\lambda, t), \qquad (\lambda \in \mathbb{D}),$$

and also define

$$s_n(\zeta) := \frac{1}{n} \log_d Z_n(F_\zeta, t)$$

for points $\zeta \in S^1$ satisfying the condition of Theorem C. It is clear that $s_n(\zeta)$ are radial boundary values of the function $s_n(\lambda)$.

Our first observation is that the functions $s_n$ are uniformly bounded by 1. (This is the only place where we use the restriction $t \ge 0$.)

Next we observe that the functions $s_n$ are subharmonic in the unit disc. Indeed, each periodic point $a_\nu \in \mathrm{Fix}(F^n)$ determines a holomorphic function

$$\lambda \mapsto a_\nu(\lambda) \in \mathrm{Fix}(F_\lambda^n), \qquad (a_\nu(0) = a_\nu),$$

and so we have holomorphic functions

$$h_\nu \equiv h_\nu(\lambda) := [(F_\lambda^n)'(a_\nu(\lambda))]^{t/2}.$$

Since

$$s_n = \frac{1}{n}\ \log_d \sum_\nu h_\nu \bar{h}_\nu,$$

we have

$$\Delta s_n = \mathrm{const}\ \frac{\sum |h_\nu|^2 \sum |\partial h_\nu|^2 - |\sum \bar{h}_\nu \partial h_\nu|^2}{(\sum |h_\nu|^2)^2} \ge 0.$$

The boundedness and subharmonicity of $s_n$ imply the inequality

$$s_n(0) \le \frac{1}{2\pi} \int s_n(\zeta) |d\zeta|.$$

Denote

$$\hat{s}_n(\zeta) = \sup_{k\ge n} s_k(\zeta).$$

Applying the dominated convergence theorem, Theorem B and Lemma 1, we have

$$\begin{aligned} P_F(t) = \lim_{n\to\infty} s_n(0) &\leq \lim_{n\to\infty} \frac{1}{2\pi}\int \hat{s}_n(\zeta)|d\zeta| \\ &= \frac{1}{2\pi}\int \limsup_{n\to\infty} s_n(\zeta)|d\zeta| \\ &= \frac{1}{2\pi}\int P_F(\zeta)|d\zeta| \ \leq\ \Pi(t). \end{aligned}$$

## 2. Branner-Hubbard families

In this section we briefly recall the Branner-Hubbard construction [1] of wringing complex structures, and then derive Theorem C from a result of Jan Kiwi [8], [9]. We will use the half plane

$$\mathbb{H} = \{\gamma = \alpha + i\beta :\ \alpha > 0,\ \beta \in \mathbb{R}\},$$

as parameter space of Branner-Hubbard families $\{F_\gamma\}$. The map

$$\lambda(\gamma) = \frac{\gamma - 1}{\gamma + 1}$$

transforms this parameter space into $\mathbb{D}$, the case we considered in Section 1.

### 2.1. Wringing complex structures.

Let $\Gamma$ denote the subgroup of $GL(2, \mathbb{R})$ formed by matrices

$$\gamma = \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix} \qquad \text{with} \quad \alpha > 0,$$

which we will identify with the complex numbers

$$\gamma = \alpha + i\beta \in \mathbb{H}.$$

$\Gamma$ acts on $S^2$ as a group of quasiconformal homeomorphisms

$$A_\gamma(z) := z|z|^{\gamma-1}, \qquad (0 \mapsto 0,\ \infty \mapsto \infty). \tag{6}$$

The Beltrami coefficient of $A_\gamma$ is

$$\mu_\gamma(z) = \lambda(\gamma)\frac{z}{\bar{z}},$$

and the corresponding Beltrami field $\mathcal{E}_\gamma$ of infinitesimal ellipses is invariant with respect to the transformation $T : z \mapsto z^d$.

Let $\mathcal{P} \equiv \mathcal{P}_d$ denote the space of polynomials of degree $d$, and let $\mathcal{S}$ be the subspace of $\mathcal{P}$ which consists of polynomials with all critical points escaping to infinity (the so called *shift locus*). We also use the notation $\mathcal{P}_*$ and $\mathcal{S}_*$ for the corresponding spaces of monic centered polynomials. Clearly,

$$\mathcal{P}_* \cong \mathbb{C}^{d-1},$$

and if we identify equivalent polynomials (two polynomials are equivalent if they are conformally conjugate), then

$$\mathcal{S}/\sim\ \cong\ \mathcal{S}_*/\sim\ \cong\ \mathcal{S}_*/\mathbb{Z}_{d-1},$$

where $\mathbb{Z}_{d-1}$ acts according to the formula $P(z) \mapsto \bar{\eta}P(\eta z)$ , $(\eta^{d-1} = 1)$.

Given a polynomial $F \in \mathcal{S}$, there exists a conformal map ("extended Böttcher function" of $F$)

$$\phi : \Omega^* \to \Delta^*$$

such that

$$T \circ \phi = \phi \circ F,$$

and such that $\Omega^*$ is an open $F$-invariant set of full area measure in $S^2$ and $\Delta^*$ is an open $T$-invariant set of full measure in $\Delta$. Thus the Beltrami fields $\phi^{-1}\mathcal{E}_\gamma$ are defined almost everywhere on $S^2$, and the corresponding family of quasiconformal homeomorphisms

$$R(\gamma, F) : S^2 \to S^2, \qquad (0, 1, \infty) \mapsto (0, 1, \infty),$$

is holomorphic in $\gamma$. It is shown in [1] that the equivalence class $[\gamma F] \in \mathcal{S}/\sim$ of the polynomial

$$\gamma F := R(\gamma, F) \circ F \circ R(\gamma, F)^{-1}$$

depends on $[F]$ and $\gamma$ only, and that the map

$$(\gamma, [F]) \mapsto [\gamma F]$$

is a group action on $\mathcal{S}/\sim$.

Consider now an orbit $O \subset \mathcal{S}_*/\mathbb{Z}_{d-1}$ of $\Gamma$. If $F_1 \in \mathcal{S}_*$ and $[F_1] \in O$, then the map

$$(\gamma \in \mathbb{H}) \;\mapsto\; ([\gamma F_1] \in O)$$

lifts to a map $\mathbb{H} \to \mathcal{P}_*$. Notation:

$$\gamma \mapsto F_\gamma \quad \text{or} \quad \{F_\gamma\}_{\gamma \in \mathbb{H}}.$$

We will call this map the *Branner-Hubbard family of* $F_1$. Note that because of the group action structure,

$$\{F_{\gamma\gamma_0}\}_{\gamma \in \Gamma} \quad \text{is the Branner-Hubbard family of} \quad F_{\gamma_0}. \tag{7}$$

The (monic centered) polynomials $F_\gamma$ depend analytically on $\gamma$. In fact, we have

$$F_\gamma = R_\gamma \circ F_1 \circ R_\gamma^{-1},$$

for some holomorphic family $\{R_\gamma\}$ of quasiconformal automorphisms of $S^2$. We also have the following equation

$$\phi_\gamma := A_\gamma \circ \phi_1 \circ R_\gamma^{-1}, \qquad (\text{near } \infty), \tag{8}$$

for the Böttcher functions $\phi_\gamma$ of $F_\gamma$ satisfying $\phi_\gamma(z) \sim z$ at infinity.

The following lemma describes the boundary behavior of a Branner-Hubbard family.

**Lemma 3.** *Consider a Branner-Hubbard family*

$$F_\gamma(z) = z^d + a_{d-2}(\gamma) z^{d-2} + \cdots + a_0(\gamma), \quad (\gamma \in \mathbb{H}).$$

*The functions $a_j(\gamma)$ have finite angular limits $a_j(i\beta)$ almost everywhere on $\partial\mathbb{H}$. The limit polynomials*

$$F_{i\beta}(z) = z^d + a_{d-2}(i\beta) z^{d-2} + \cdots + a_0(i\beta)$$

*have connected Julia sets.*

*Proof:* For a polynomial $F$, let $G_F(\cdot)$ denote the Green's function of the Julia set $J_F$ with pole of infinity, and let $m(F)$ be the maximal escape rate of the critical set:

$$m(F) \ := \ \max\{G_F(c) : \ c \in \mathrm{Crit}(F)\}.$$

To prove the first statement, we need the following result of Branner and Hubbard [1]:

$$\forall \varrho > 0, \quad \text{the set} \quad \{F \in \mathcal{P}_* : \ m(F) \leq \varrho\} \quad \text{is compact.} \tag{9}$$

By (8), we have

$$G_\gamma \circ R_\gamma = \alpha G_1, \qquad (\alpha := \Re[\gamma]),$$

where $G_\gamma$ denotes the Green's function for $F_\gamma$. Since $R_\gamma$ sends the critical set of $F_1$ onto the critical set of $F_\gamma$, it follows that

$$m(F_\gamma) = \alpha m(F_1). \tag{10}$$

Applying (9), we see that the coefficients $a_j(\gamma)$ are uniformly bounded (if $\alpha < 1$), and so the existence of angular limits follows from Fatou's theorem.

The second statement of the lemma follows from (10) and from the well-known fact that the function $m(F) : \mathcal{P}_* \to \mathbb{R}$ is continuous. □

2.2. **Proof of Theorem C.** The rest of the argument is based on Kiwi's approach in [8], [9].

Let $F_1$ be a monic centered polynomial and let $\{F_\gamma\}$ be its Branner-Hubbard family. For simplicity, let us assume that $F_1$ (and hence every polynomial in the family) is such that the orbits of the critical points are disjoint. We say that $F_\gamma$ is *visible* if for each critical point $c_j$, $(1 \leq j \leq d-1)$, there are precisely two external rays terminating at $c_j$. In this case, let

$$\Theta_j(\gamma) \equiv \Theta_j = \{\theta_j^-, \theta_j^+\} \subset S^1$$

be the set of the corresponding external arguments. The collection of the sets $\Theta_j$,

$$\Theta(F_\gamma) := \{\Theta_1, \dots \Theta_{d-1}\},$$

is called the *critical portrait* of $F_\gamma$. Every critical portrait determines a partition of $S^1$ into $d$ sets of length $1/d$ each. The portrait is *periodic* if the itinerary of each point $\theta_j^\pm$ with respect to this partition is periodic.

The action of the subgroup of $\Gamma$ formed by diagonal matrices determines the flow

$$t \mapsto F_{t\alpha+i\beta}, \qquad (t > 0), \tag{11}$$

in the Branner-Hubbard family. This flow preserves the visibility (or invisibility) of polynomials. The flow also preserves the critical portraits of visible polynomials. As it is mentioned in [8], these assertions with $\beta = 0$ follow from the fact that by (8), the homeomorphism $A_\gamma$ with $\gamma$ real throws the hedgehog of $F_1$ onto the hedgehog of $F_\gamma$. (See [11] regarding hedgehogs and disconnected Julia sets.) On the other hand, we can assume $\beta = 0$ without loss of generality, just by choosing a different uniformization of the Branner-Hubbard family, see (7).

Let us parametrize the orbits of the flow (11) by real numbers $\beta$. It is easy to see that only countably many orbits contain invisible polynomials.

**Lemma 4.** *For a.e. $\beta$, the critical portrait of the corresponding polynomials is aperiodic.*

*Proof:* Using the group action structure, see (7), it is sufficient to show that if $F_1$ is a visible polynomial, then there is a number $\varepsilon > 0$ such that for almost every $\beta \in (-\varepsilon, \varepsilon)$, the critical portrait of $F_{1+i\beta}$ is aperiodic.

Let $\theta_j^{\pm} \in S^1$ be the external angles and $g_j$ the escape rates of the critical points of $F_1$. It is clear from (6) and (8) that for small $\beta$'s the polynomials $F_{1+i\beta}$ are visible and that their external angles $\theta_j^{\pm}(\beta)$ satisfy the equation

$$\theta_j^{\pm}(\beta) = \theta_j^{\pm} + \beta g_j. \tag{12}$$

Fix $j$ and some integer $p$, and consider the set $E$ of $\beta$'s in $(-\varepsilon, \varepsilon)$ such that the itinerary of the point

$$\vartheta \equiv \vartheta(\beta) := T\theta_j^{\pm}(\beta)$$

is $T$-periodic with period $p$. Let $L(\beta)$ denote the element of the partition of $S^1$ corresponding to $F_{1+i\beta}$ such that $\vartheta \in L(\beta)$. If $\varepsilon$ is small enough, we can find an interval $I \subset S^1$ which does not intersect all sets $L(\beta)$. The periodicity of the itinerary implies that

$$\forall n, \quad T^{np}\vartheta \notin I. \tag{13}$$

By ergodicity of $T^p$, the set of $\vartheta$'s satisfying (13) is a set of measure zero , and by (12), the same is true for $E$. □

We can now complete the proof of Theorem C by referring to the following result of Kiwi [8], [9]:

*If a sequence of visible polynomials with the same aperiodic critical portrait tends to a polynomial, then the latter has no non-repelling cycles.*

## 3. Periodic cycles

In this section we prove Theorem B. The proof is preceded by several technical lemmas. For the rest of the paper we will be considering polynomials $F$ with connected Julia sets and *all periodic cycles repelling*. We will also assume that the critical points $c_j$ of $F$ are simple and non-preperiodic.

3.1. **Multiplicity of the kneading map.** A point $b \in J_F$ is a *cut point* if there are at least two external radii landing at $b$. Let $G$ be a finite, forward invariant set which consists of cut points. For a point $z \in J$, which is not in the grand orbit $\mathcal{O}(G)$ of $G$, we denote by $P(z)$ the component of $J \setminus G$ containing $z$. Depending on the context, we use the same notation $P(z)$ for the corresponding *unbounded puzzle piece*, i.e. the component of the complement of external rays landing at $G$, see [18]. Let us number the pieces of the $G$-partition as $P_1, P_2, \ldots, P_N$. The *kneading map*

$$\text{knead}_G : \ J \setminus \mathcal{O}(G) \to \{1, \ldots, N\}^{\mathbb{Z}_+}$$

is the function $z \ \mapsto \ \{i_0(z), i_1(z), \ldots\}$, where $F^{\nu}(z) \in P_{i_\nu(z)}$.

**Lemma 5.** *For any $\varepsilon > 0$, there exists a finite, forward invariant set $G$ such that if $n > n_0(F, \varepsilon)$ and if $a \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$, then*

$$\#\ \{a' \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G) :\ \mathrm{knead}_G(a) = \mathrm{knead}_G(a')\}\ \leq\ e^{\varepsilon n}.$$

*Proof:* Given $\varepsilon > 0$, we choose a large number $m = m(\varepsilon)$ to be specified later. For simplicity of notation, let us assume that the fibers of the critical points have pairwise disjoint orbits, in which case there is a finite, forward invariant set $\tilde{G}$ such that the sets

$$\bigcup_j \tilde{P}(c_j) \quad \text{and} \quad \bigcup_j \bigcup_{k=1}^{m} F^{-k} c_j \quad \text{are disjoint,} \tag{14}$$

where $\tilde{P}(\cdot)$ denote the $\tilde{G}$-pieces. (For an explanation of this fact and for the definition of a fiber, see Appendix at the end of the section.) Replacing each critical piece $\tilde{P}(c_j)$ with components of the $F^{-1}\tilde{G}$-partition, we obtain a new puzzle. Let $G$ denote the corresponding set, i.e.

$$G\ =\ \tilde{G} \bigcup_j [F^{-1}\tilde{G} \cap \tilde{P}(c_j)].$$

The $G$-partition has the following (modified) Markov property (cf. [19], Section 7): *each critical puzzle piece maps onto the corresponding critical value piece by a 2-fold branched covering, while every non-critical piece maps univalently onto a "union" of puzzle pieces.*

A sequence $\{i_0, \ldots, i_{n-1}\}$ is called a *Markov cycle* if

$$P_{i_1} \subset FP_{i_0},\ P_{i_2} \subset FP_{i_1},\ \ldots,\ P_{i_0} \subset FP_{i_{n-1}}.$$

By (14), the number of critical indices in such a sequence does not exceed $n/m$. To each periodic point $z \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$ there corresponds the Markov cycle $\{i_0(z), \ldots, i_{n-1}(z)\}$, and the Markov cycles of two periodic points are equal if and only if the points have the same $G$-kneading. Thus it remains to show that the number of points $z \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$ with the same Markov cycle $\{i_0, \ldots, i_{n-1}\}$ does not exceed $C2^{n/m}$, where $C$ is a constant independent of $n$. This would give an explicit formula for $m = m(\varepsilon)$.

To this end, let us inductively define puzzle pieces $\Pi(k) \subset P_{i_k}$, $(0 \leq k \leq n-1)$, as follows:

$$\Pi(n-1) := P_{i_{n-1}}, \quad \Pi(k) := F_{i_k}^{-1}\Pi(k+1),$$

where $F_{i_k}^{-1}$ denotes preimage under the map

$$F : P_{i_k} \to FP_{i_k} \supset \Pi(k+1).$$

It is clear that the puzzle piece $\Pi(0)$ contains all periodic points with the given Markov cycle.

To bound the number of $n$-periodic points in $\Pi(0)$, we consider the sets $\pi\Pi(k) \subset S^1$, defined as the intersection of $\Pi(k)$ with the "circle at infinity". Each set $\pi\Pi(k)$ consists of finitely many open arcs. The map $T : z \to z^d$ takes $\pi\Pi(k)$ onto $\pi\Pi(k+1)$ homeomorphically if the index $i_k$ is non-critical, and as a two-fold cover if $i_k$ is critical. Let $C$ be a constant such that each set $\pi P_j$ has at most $C$ components. It follows that the number of arcs in $\pi\Pi(0)$ is at most $C2^{n/m}$. It is also clear that each arc in $\pi\Pi(0)$ has at most one $T$-periodic point of period $n$. $\square$

3.2. **Cycles with close orbits.** We will use Lemma 5 to prove the following estimate for polynomials without indifferent periodic points. We don't know if the estimate is true for arbitrary polynomials.

**Lemma 6.** *For any* $\varepsilon > 0$*, there exists a positive number* $\rho = \rho(F, \varepsilon)$ *such that if* $n > n_0(F, \varepsilon)$ *and if* $a \in \mathrm{Fix}(F^n)$*, then*

$$\# \ \{a' \in \mathrm{Fix}(F^n) : \ \forall i, \ |F^i(a) - F^i(a')| \le \rho\} \ \le \ e^{\varepsilon n}.$$

*Proof:* Given $\varepsilon > 0$, we find a finite, invariant set $G$ according to Lemma 5. The argument is based on the notion of the *sector map* $\tau$ associated with $G$. For each $b \in G$, the external rays at $b$ divide the plane into sectors. Since we assumed that $b$ is not a critical point, the polynomial $F$ is a local diffeomorphism identifying sectors $S$ at $b$ with sectors $\tau S$ at $F(b)$:

$$F(S \cap U) \subset \tau S,$$

where $U$ is some small neighborhood of $b$. Denote by $C$ the total number of sectors (considering all points of $G$), and fix a number $m = m(\varepsilon) \gg C$. It follows that if $z$ is sufficiently close to the set $G$, then the initial kneading segment of length $m$ is determined up to $C$ choices by the sector map.

Let us now choose $\rho > 0$ so small that if $|z - z'| < \rho$, then either the points $z$ and $z'$ are in the same component of $J \setminus G$, or they are both close to the set $G$ and so we have the situation described in the previous sentence. If the orbits of periodic points $a$ and $a'$ are $\rho$-close, then their kneading sequences of length $n \gg m$ coincide except for at most $n/m$ segments of length $m$, for which we have $C^{n/m}$ choices. Combining this computation with the estimate of Lemma 5, we complete the proof. □

We will use Lemma 6 in combination with the following statement, which is a special case of Manẽ's lemma [15].

**Lemma 7.** *Given* $\rho > 0$*, there is a positive number* $\delta = \delta(F, \rho)$ *such that if* $n \ge 0$*, and if* $D$ *is a domain such that* $F^n$ *maps* $D$ *univalently onto a disk* $B$ *of radius* $2\delta$*, then*

$$\mathrm{diam}\ \left(F^n|_D\right)^{-1}(B/2) \ < \ \rho. \tag{15}$$

Here and elsewhere, the notation $B/2$ means concentric disc of radius half the radius of $B$.

3.3. **Good and bad cycles.** Let $\delta > 0$. For want of a better name, we say that a cycle $A \in \mathrm{Cycle}(F, n)$ is $\delta$-*good* if there is a periodic point $a \in A$ and a topological disc $D$ containing $a$ such that the restriction of $F^n$ to $D$ is univalent and $F^n(D) = B(a, \delta)$. Otherwise, we say that the cycle is $\delta$-*bad.* The next lemma states that for polynomials without non-repelling periodic points, most of the cycles are "good".

**Lemma 8.** *For any* $\varepsilon > 0$*, there exists a positive number* $\delta = \delta(F, \varepsilon)$ *such that if* $n > n(F, \varepsilon)$*, then*

$$\# \ \{\ \delta\text{-}bad\ n\text{-}cycles\} \ \le \ e^{\varepsilon n}.$$

*Proof:* Fix a large number $m = m(F, \varepsilon)$ to be specified later. For simplicity, we will assume that the orbits of the critical points are pairwise disjoint, and so there is a number $\rho = \rho(F, m)$ such that if $c \neq c'$ are two critical points, then

$$\operatorname{dist}(c, F^k(c')) > 10\rho, \qquad (0 \le k \le m). \tag{16}$$

We can take $\rho$ small enough so that the estimate of Lemma 6 is valid for a given $\varepsilon$. Finally, we choose $\delta > 0$ satisfying the following two conditions

**:** for all $x \in J$ and $k \in [0, m]$, each component of the set $F^{-k}B(x, \delta)$ has diameter less than $\rho$;

**:** the conclusion (15) of Mañé's lemma holds.

Let us estimate the number of $\delta$-bad cycles.

Fix a cover $\mathcal{B}$ of the Julia set with discs of radius $2\delta$. Clearly, we can assume that the concentric discs of radius $\delta$ still cover $J$, and that the multiplicity of the covering is bounded by some absolute constant $M$. Let $n \gg m$. For each periodic point $a \in \operatorname{Fix}(F^n)$, select a disc $B(a) \in \mathcal{B}$ with $a \in B(a)/2$. For $i > 0$, let $B_{-i}(a)$ denote the component of the $F^{-i}B(a)$ containing the point $F^{n-i}(a)$ and define $j(a)$ to be the smallest positive integer such that $B_{-j}(a)$ contains a critical point, which we denote by $c(a)$. Note that if $j(a) > n$, then the cycle of $a$ is $\delta$-good.

We need some further notation. Given $a \in \operatorname{Fix}(F^n)$, we define inductively a sequence of positive integers $j_1, j_2, \dots$ and a sequence of points $a_1 \equiv a,\ a_2, \dots$ in the orbit of $a$ as follows :

$$j_k = j(a_k), \qquad a_{k+1} = F^{n-j_k} a_k.$$

The main observation is that

$$j_k + j_{k+1} > m. \tag{17}$$

Indeed, if $j_k + j_{k+1} < m$, then both $j_k$ and $j_{k+1}$ are $\le m$. By construction, we have $\operatorname{diam} B_{-j_k}(a_k) < \rho$, and so

$$|a_{k+1} - c(a_k)| \ < \ \rho.$$

On the other hand, the disc $B(a_{k+1})$ contains the $j_{k+1}$-th iterate of the critical point $c(a_{k+1})$, and therefore

$$|a_{k+1} - F^{j_{k+1}} c(a_{k+1})| \ < \ 4\delta \ < \ 8\rho.$$

Combining the two inequalities, we get a contradiction with (16).

Define the *schedule* of $a$ to be a finite sequence

$$\operatorname{Sch}(a) := \{j_1(a), j_2(a), \dots, j_l(a)\},$$

where $l$ is the minimal number such that $j_1 + \dots + j_l > n$. By (17), we have

$$l \ \le \ 3n/m. \tag{18}$$

We also consider the corresponding sequence of discs in the cover $\mathcal{B}$, and the corresponding sequence of critical points:

$$\mathcal{B}(a) := \{B(a_1), \dots, B(a_l)\}, \qquad \mathcal{C}(a) := \{c(a_1), \dots, c(a_l)\}.$$

As we mentioned, for $\delta$-bad cycles we have all $j_k \le n$ and therefore

$$n \le \sum_{k=1}^{l} j_k \ \le \ 2n. \tag{19}$$

The lemma now follows from the three observations below.

(i) *The number of sequences* $\{j_1, \ldots, j_l\}$ *satisfying* (17) *and* (19) *is* $\lesssim m^{4n/m}$.

Indeed, consider the numbers $j_1, (j_1 + j_2), \ldots$ as points of the interval $[1, 2n]$. Subdivide the interval into $(2n)/m$ segments of length $m$. Clearly, there are at most two points in each segment, and there are less than $m^2$ choices to select at most two points in any particular segment.

(ii) *Consider all periodic points* $a \in \mathrm{Fix}(F^n)$ *with a given schedule. Then the number of distinct sequences* $\mathcal{B}(a)$ *and* $\mathcal{C}(a)$ *does not exceed* $(dM)^{3n/m}$ *and* $d^{3n/m}$ *respectively.*

This follows from (18) and the fact that the ball $B(a_k)$ must contain the $j_k$-th iterate of a critical point, so the number of such balls is less than $dM$.

(iii) *If two periodic points* $a$ *and* $a'$ *have identical schedules and identical sequences* $\mathcal{B}(a) = \mathcal{B}(a')$ *and* $\mathcal{C}(a) = \mathcal{C}(a')$*, then the orbits of* $a$ *and* $a'$ *are* $\rho$*-close:*

$$\forall i, \ |F^i(a_1) - F^i(a'_1)| \leq \rho.$$

To see this, let $\{j_1, \ldots, j_l\}$ be the schedule and let $B := B(a_1) = B(a'_1)$. By construction, the components of $F^{-j_1}B$ containing the points $F^{n-j_1}a_1$ and $F^{n-j_1}a'_1$ must coincide because both contain the critical point $c(a_1) = c(a'_1)$. It follows that if $n - j_1 < i \leq n$, then the $i$-th iterates of $a_1$ and $a'_1$ belong to the same component of the corresponding preimage of $B$, and this component is mapped univalenly onto $B$. Since $a_1$ and $a'_1$ are in $B/2$, we can apply Lemma 7 to conclude that the iterates of $a_1$ and $a'_1$ are $\rho$-close. Repeat this argument for the discs $B(a_k)$, $k \leq l$.

From (iii) and Lemma 6, it follows that the number of $n$-periodic points with a given schedule and given $\mathcal{B}$- and $\mathcal{C}$-sequences is $\lesssim e^{\varepsilon n}$. On the other hand, by (i) and (ii), the number of possible sequences and schedules satisfying (19) is also $\lesssim e^{\varepsilon n}$, provided that $m = m(\varepsilon)$ is so large that $m^{-1} \log m \ll \varepsilon$. Thus the number of bad cycles is $\lesssim e^{2\varepsilon n}$. □

3.4. **Proof of Theorem B.** Let $F$ be a polynomial with all cycles repelling. Given small $\varepsilon$, we choose $\rho = \rho(F, \varepsilon)$ according to Lemma 5, so the number of $\rho$-close $n$-cycles is $\lesssim e^{\varepsilon n}$. Then we choose a positive number $\delta$ such that

**:** all but $\lesssim e^{\varepsilon n}$ $n$-cycles are $4\delta$-good, see Lemma 8;

**:** the conclusion (15) of Mañe's lemma holds.

Fix $n \gg 1$. In each good $n$-cycle, let us pick a point $a$ such that $F^n$ maps some domain $D_a \ni a$ onto $B(a, 4\delta)$ univalently, and denote by $I$ the set of the points that we picked. Also let $II$ be the set of all periodic points contained in the bad cycles. Then we have

$$\begin{aligned} Z_n(F,t) &= \sum_{a \in \mathrm{Fix}(F^n)} |F_n'(a)|^{-t} \\ &= n \sum_{a \in I} |F_n'(a)|^{-t} + \sum_{a \in II} |F_n'(a)|^{-t} \\ &\leq n \sum_{a \in I} |F_n'(a)|^{-t} + n e^{\varepsilon n}. \end{aligned}$$

To estimate the sum over $I$, cover the Julia set with $\lesssim \delta^{-2}$ discs $B$ of radius $2\delta$. In each $B$, fix a point $z_B \notin J$ so that the points $z_B$ are distinct. Finally, to each $a \in I$ assign one of the discs $B = B(a)$ such that $a \in B(a)/2$. Note that $B(a) \subset B(a, 3\delta)$.

Let $z_a$ denote the preimage of $z_{B(a)}$ under the map $F^n : D_a \to B(a, 4\delta) \supset B(a)$. Since $F^n$ takes both $a$ and $z_a$ inside $B(a, 3\delta)$, by Koebe's lemma we have

$$|F_n'(a)| \asymp |F_n'(z_a)|.$$

Note that if $z_a = z_{a'}$ for some $a, a' \in I$, then the orbits of $a$ and $a'$ are $\rho$-close. Indeed, for $B := B(a) = B(a')$, we have $B \subset B(a, 4\delta) \cap B(a', 4\delta)$, and therefore $F^n$ maps some domain univalently onto $B$ with both $a$ and $a'$ in the preimage of $B/2$, and so we can apply (15).

It follows that the number of points $a$ such that $z_a$ is a given point of $F^{-n} z_B$ is at most $e^{\varepsilon n}$. We have

$$\sum_{a \in I} |F_n'(a)|^{-t} \lesssim e^{\varepsilon n} \sum_B \sum_{z \in F^{-n}(z_B)} |F_n'(a)|^{-t}.$$

Since for each $z_B$, we have

$$P_F(t) = \limsup_{n \to \infty} \frac{1}{n} \log_d \sum_{z \in F^{-n}(z_B)} |F_n'(z)|^{-t}, \tag{20}$$

the theorem follows.

3.5. **Appendix: Fibers.** Let $F$ be a polynomial without non-repelling cycles, and let $z \in J_F$. Following Kiwi [8], consider a sequence of partitions corresponding to the sets

$$G_l(F, z) := \{b \in J \setminus \mathcal{O}(z) : \ b \text{ is a cut point}, \ \ F^l b \text{ is periodic of period } \leq l\}.$$

Let $P_l(\cdot)$ denote the $G_l(F, z)$-pieces. The connected compact set $X(z) := \cap_l \bar{P}_l(z) \subset J$ is called the *fiber* of $z$. (We use the term from a paper of Schleicher [22], see also [**?**].) The fibers satisfy the equation

$$FX(z) = X(Fz).$$

It is also clear that if two points $z_1$ and $z_2$ have infinite orbits, or if they land on the same cycle, then the fibers $X(z_1)$ and $X(z_2)$ are disjoint or coincide.

Our proof of Lemma 5 was based on the following fact mentioned in the proof of Lemma 13.3 of Kiwi's thesis [8]. To make this section self-contained, we reproduce his argument. We will denote by $P_l'(\cdot)$ the puzzle pieces corresponding to $F^{-1} G_l$.

**Lemma 9.** *If $z$ has an infinite orbit, then the fiber of $z$ is wandering.*

*Proof:* (i) Let us first show that if $z$ is a periodic point, then $X(z) = \{z\}$. Since $G_l(F^p, z) \subset G_{lp}(F, z)$, fibers of $F^p$ contain fibers of $F$, and so by replacing $F$ with some iterate, we can assume that $z$ is fixed. For the same reason we can assume that the landing rays at $z$ are all fixed. The latter implies

$$b \in \mathrm{Fix}(F) \cap \bar{P}_2(z) \ \Rightarrow \ \text{rays landing at } b \text{ are fixed.} \tag{21}$$

Indeed, suppose $b$ is not a landing point of some fixed ray. Then $b \in G_1$, and $P_1(z)$ is contained in some sector $S$ at $b$. We have $FP_2(z) \subset P_1(z) \subset S$. Taking some point in $P_2(z)$ close to $b$, we see that $\tau S = S$, where $\tau$ is the sector map, and so the rays at $b$ have to be fixed.

Let $k-1$ be the number of critical points in the fiber $X(z)$. For $l \gg 1$, the map $P_l(z) \to P_l'(z)$ extends to a polynomial-like map $g$ of degree $k$. Observe that $X(z) \subset J_g$, and since the critical points of $g$ belong to $X(z) = gX(z)$, the Julia set $J_g$ of $g$ is connected. It remains to show that $k = 1$. (Then $X(z) \subset J_g = \{z\}$.) The fixed points of $g$ belong to the set $\text{Fix}(F) \cap \bar{P}_2(z)$. By (21), for each fixed point of $g$ there is an $F$-invariant (and therefore $g$-invariant) arc in $J^c \subset J_g^c$ tending to the fixed point.

Let $Q$ be a polynomial of degree $k$ conjugate to $g$. It follows that there are $Q$-invariant arcs in $\mathbb{C} \setminus J_Q$ tending to each of $k$ fixed points of $Q$. Applying the Riemann map, we get $k$ arc tending to $k$ distinct points on the circle invariant with respect to $\zeta \mapsto \zeta^k$. A contradiction.

(ii) Suppose now that $z$ is not preperiodic. Replacing $F$ with some iterate, we can reduce the problem to showing that

$$X(Fz) = X(z) \quad \Rightarrow \quad z \in \text{Fix}(F).$$

Suppose $X(Fz) = X(z)$. Then for every $l$, we have a map $F : P_l(z) \to P_l'(z)$, which extends to a polynomial-like map with Julia set contained in $\bar{P}_l(z)$. It follows that

$$\forall l, \quad \bar{P}_l(z) \cap \text{Fix}(F) \neq \emptyset,$$

and therefore $X(z)$ contains at least one fixed point $b$. Since the partition $G_l(z, F)$ is finer than $G_l(b, F)$, by (i) we have $X(z) \subset X(b) = \{b\}$. $\square$

If the fibers of the critical points have pairwise disjoint orbits, then from Lemma 9 it follows that the statement (14) holds for puzzle pieces $\tilde{P}(c_j) = P_l(c_j)$ with $l$ sufficiently large. This is the fact that we used earlier.

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