Quotients of E^n by $\mathfrak{a}_{n+1}$ and Calabi-Yau manifolds

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Abstract. We give a simple construction, for $n \geq 2$, of an n-dimensional Calabi-Yau variety of Kummer type by studying the quotient Y of an n-fold self-product of an elliptic curve E by a natural action of the alternating group $\mathfrak{a}_{n+1}$ (in $n+1$ variables). The vanishing of $H^m(Y, \mathcal{O}_Y)$ for $0 < m < n$ follows from the lack of existence of (non-zero) fixed points in certain representations of $\mathfrak{a}_{n+1}$. For $n \leq 3$ we provide an explicit (crepant) resolution X in characteristics different from $2, 3$. The key point is that Y can be realized as a double cover of $\mathbb{P}^n$ branched along a hypersurface of degree $2(n+1)$.

Introduction

A *Calabi-Yau manifold* over a field k is a smooth projective variety X of dimension n such that

(CY1) The canonical bundle $\mathcal{K}_X$ is trivial; and
(CY2) $H^m(X, \mathcal{O}_X) = 0$ for all (strictly) positive $m < n$.

The condition (CY2) is equivalent (for smooth X) to requiring that $h^{m,0}(X) = 0$ for all m such that $0 < m < n$. Classically, a Calabi-Yau manifold of dimension $n \geq 2$ is a *complex Kähler n-manifold with finite π_1 (fundamental group) and $SU(n)$-holonomy* ([V]). The equivalence of the definitions is given by a *theorem of S.T. Yau.*

It will be necessary for us to allow X to have mild singularities. By a *Calabi-Yau variety*, we will mean a projective variety X/k on which the canonical bundle $\mathcal{K}_X$ is defined such that the conditions (CY1), (CY2)

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hold. More precisely, we will want such an X to be normal and Cohen-Macaulay, so that the dualizing sheaf $\mathcal{K}_X$ is defined, with the singular locus in codimension at least 2, so that $\mathcal{K}_X$ defines a Weil divisor; finally X should be $\mathbb{Q}$-Gorenstein, so that a power of $\mathcal{K}_X$ will represent a Cartier divisor.

Clearly, every Calabi-Yau manifold of dimension 1 is an elliptic curve, while in dimension 2 it is a $K3$-surface. Abelian varieties, which generalize the elliptic curve in one direction, have trivial canonical bundles but they have non-trivial $h^{m,0}(X)$ for $m < n$.

A classical construction of Kummer associates a $K3$ surface to an abelian surface A by starting with the quotient of A by the involution $\iota : x \to -x$, and then blowing up the sixteen double points, each of which corresponds to a point of order 2 on A. When E is an elliptic curve with CM (short for *complex multiplication*) by $\mathbb{Q}[\sqrt{-3}]$, there is a construction of a Calabi-Yau 3-fold arising as a resolution of a quotient of $E \times E \times E$.

The object of this Note is to present a simple construction of a Calabi-Yau variety *of Kummer type* by starting with an n-fold product $E \times \cdots \times E$ of an elliptic curve E, and then taking a quotient under an action of the alternating group $\mathfrak{a}_{n+1}$. For general n this will lead, under a suitable (crepant) resolution predicted by a standard conjecture, to a Calabi-Yau manifold. We can do this unconditionally for $n \leq 3$, where after getting to a local problem, one can appeal to known results – [**6**], for example. But we take a *direct geometric approach* to arrive at the smooth resolution, and this can at least partly be carried out for arbitrary n. This construction works whether or not E has CM, and it will be used in a forthcoming paper ([**5**]). Already for $n = 2$, it is different from the classical Kummer construction ([**3**]); what we do here is to divide $E \times E$ by the cyclic group of automorphisms of order 3 generated by $(x, y) \to (-x - y, x)$. However, the realization of the $K3$ surface as the double cover of $\mathbb{P}^2$ branched along the dual of a plane cubic has arisen in the previous works of Barth, Katsura, and others.

The construction appears to work for $n = 4$ and also over families of elliptic curves. We plan to take up these matters elsewhere.

For $n = 3$, our Calabi-Yau variety is realized as a double cover of $\mathbb{P}^3$ branched along an *irreducible* octic surface. For other examples of double constructions, with highly reducible branch locus, see [**2**] (and the references therein).

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conference. This Note elaborates on a small part of the actual lecture he gave there, describing the ongoing joint work with the first author.

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1. The construction

Let E be an elliptic curve over a field k with identity 0, and $n \geq 1$ an integer. Put

$$\tilde{Y} := \left\{ \tilde{y} = (y_1, \ldots, y_{n+1}) \in E^{n+1} \mid \sum_{j=1}^{n+1} y_j = 0 \right\} \tag{1.1}$$

Clearly, we have an isomorphism

$$\varphi : \tilde{Y} \rightarrow E^n, \tag{1.2}$$

given by $\tilde{y} \to (y_1, \ldots, y_n)$.

Note that the action of the alternating group $\mathfrak{a}_{n+1}$ on E^{n+1} preserves $\tilde{Y}$. Put

$$Y := \tilde{Y}/\mathfrak{a}_{n+1}. \tag{1.3}$$

This variety is defined over k, but is singular. Denote by $\pi : \tilde{Y} \to Y$ the quotient map and by Z the singular locus in Y. If we set

$$\tilde{Z} := \left\{ \tilde{y} \in \tilde{Y} \mid \exists g \in \mathfrak{a}_{n+1}, g \neq 1, \text{ s.t. } g\tilde{y} = \tilde{y} \right\}, \tag{1.4}$$

namely the set of points in $\tilde{Y}$ with non-trivial stabilizers in $\mathfrak{a}_{n+1}$, we obtain

$$Z \subset \pi(\tilde{Z}). \tag{1.5}$$

If $n = 2$, for example, the action of $\mathfrak{a}_3$ on $E \times E$ (via φ) is generated by $(x, y) \to (-x - y, x)$, which shows that the fixed point set is $\{(x, x) \in E \times E \mid 3x = 0\}$.

Theorem *We have the following (for $n \geq 2$):*
(a) Y is a Calabi-Yau variety i.e., $\mathcal{K}_Y$ is defined with

(i) *$\mathcal{K}_Y$ is trivial*
(ii) *$H^m(Y, \mathcal{O}_Y) = 0$ for all m such that $0 < m < n$*

(b) If $n \leq 3$ and k algebraically closed of characteristic zero or $p \nmid 6$, there exists a smooth resolution $p : X \to Y$ such that X is Calabi-Yau.

Proof of Theorem, part (a): We need the following:

Proposition A *Consider the morphism $\pi : \tilde{Y} \to Y = \tilde{Y}/\mathfrak{a}_{n+1}$. Then π is finite, surjective and separable. Moreover, the natural homomorphism*

$$\mathcal{O}_Y \to \pi_*(\mathcal{O}_{\tilde{Y}})^{\mathfrak{a}_{n+1}}$$

is an isomorphism.

Proof of Proposition A. In view of the Theorem in chap. II, sec. 7 of [Mu], it suffices to prove that for any point $\tilde{y} = (y_1, \ldots, y_{n+1})$ in $\tilde{Y}$, the orbit $O(\tilde{y})$ is contained in an affine open subset of $\tilde{Y}$. (In fact one should properly appeal to this Theorem of Mumford to already know that the algebraic quotient Y exists and is unique.) Now by definition, $y_{n+1} = -\sum_{j=1}^n y_j$ for any $\tilde{y} = (y_1, \ldots, y_{n+1})$. Pick any affine open set U in E which avoids the points $\{y_1, \ldots, y_{n+1}\}$. Then U^n is an affine open subset of E^n, ad the orbit $O(\tilde{y})$ is contained in the affine open subset $\varphi^{-1}(U^n)$ of $\tilde{Y}$. Done.

□

Put

$$W := H^1(E, \mathcal{O}_E) \simeq k \tag{1.6}$$

and

$$W_{m,n} = \Lambda^m(W^{\oplus n}) \simeq H^m(E^n, \mathcal{O}_{E^n}).$$

In view of Proposition A and the isomorphism ϕ, we are led to look for fixed points of the action of $\mathfrak{a}_{n+1}$ on $W_{m,n}$. To be precise, our Theorem will be a consequence of the following

Proposition B *Fix $n \geq 2$. Let k have characteristic zero or $p \nmid (n!/2)$. Then for every integer m such that $0 < m < n$,*

$$W_{m,n}^{\mathfrak{a}_{n+1}} = 0.$$

Proof of Proposition B. First consider the simple case $\mathbf{n = 2}$. Here the only possibility is $m = 1$. The group $\mathfrak{a}_3$ is generated by the 3-cycle $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, which sends $(w_1, w_2) \in W_{1,2}$ to $(-w_1 - w_2, w_1)$ and is represented by the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\mathrm{char}(k) \neq 3$, the eigenvalues are the two non-trivial cube roots of unity, implying that there is no fixed point in $W_{1,2}$.

So we may take $\mathbf{n} \geq \mathbf{3}$ and assume by induction that the Proposition is true for $n-1$. Put

$$(1.7) \qquad W'_{1,n} = \{w = (w_1, \dots, w_{n+1}) \in W^{n+1} \,|\, w_1 = 0, \sum_{j=2}^{n+1} w_j = 0\},$$

$$L = \{w = (w_1, \dots, w_{n+1}) \in W^{n+1} \,|\, w_1 = n, w_j = -1 \ \forall j \geq 2\},$$

and

$$G' := \{g \in \mathfrak{a}_{n+1} \,|\, g(w_1) = w_1\}.$$

Then there are canonical, compatible identifications $W'_{1,n} \simeq W_{1,n-1}$ and $G' \simeq \mathfrak{a}_n$, and so by induction,

$$(1.8) \qquad \Lambda^j(W'_{1,n})^{G'} = 0 \quad \text{if} \quad 0 < j < n-1.$$

Moreover, since $W_{1,n}$ identifies with the $\mathfrak{a}_{n+1}$-space of vectors $(w_1, \dots, w_{n+1}) \in W^{n+1}$ such that $\sum_j w_j = 0$, we get a G'-stable decomposition

$$(1.9) \qquad W_{1,n} = W'_{1,n} \oplus L,$$

with G' acting trivially on the line L. This furnishes, by taking exterior powers, G'-isomorphisms for all positive integers $m \leq n-1$,

$$(1.10) \qquad \Lambda^m(W_{1,n}) \simeq \Lambda^m(W'_{1,n}) \oplus \Lambda^{m-1}(W'_{1,n}) \otimes L.$$

We then get, by the inductive hypothesis,

$$(1.11) \qquad W_{1,n}^{G'} = 0 \quad \text{if} \quad 1 < m < n-1.$$

So it suffices to prove the Proposition for $m = 1$ and $m = n-1$. We will be done once we show the following

Lemma 1.12 *For $n \geq 3$, the representation ρ of $G = \mathfrak{a}_{n+1}$ on $W_{1,n}$ is irreducible. Moreover,*

$$W_{n-1,n} \simeq W_{1,n}^{\vee} \otimes \det(\rho),$$

where the superscript $\vee$ denotes taking the contragredient.

Proof of Lemma. Assume for the moment the irreducibility of ρ. As $\dim W_{1,n} = n$, there is a natural, non-degenerate G-pairing

$$\Lambda^{n-1}(W_{1,n}) \times W_{1,n} \to \Lambda^n(W_{1,n}),\ (\alpha, w) \to \alpha \wedge w,$$

and G acts on the one-dimensional space on the right by $\det(\rho)$, identifying the representation of G on $W_{n-1,n}$ with $\rho^{\vee} \otimes \det(\rho)$.

All that remains now is to check the irreducibility of ρ. For this note that the action of $G = \mathfrak{a}_{n+1}$ on $V = k^{n+1}$ is doubly transitive, and so by [CuR], this permutation representation π, say, decomposes as the direct sum of the trivial representation and an irreducible representation of G, which must be equivalent to ρ. But here is an explicit argument. Since

G' is the stabilizer of $(1, 0, \ldots, 0)$ in V, we see that π is the representation induced by the trivial representation of G'. On the other hand, the double coset space $G' \backslash G / G'$ has exactly two elements, again implying, by Mackey, that the complement of 1 in π is irreducible. Done.

$\square$

Now we turn to the question of triviality of $\mathcal{K}_Y$. As $\tilde{Y}$ is an abelian variety, $\mathcal{K}_{\tilde{Y}}$ is trivial. The quotient Y is Cohen-Macaulay, being a finite group quotient of a smooth variety. It is normal with the singular locus in codimension 2, and is $\mathbb{Q}$-Gorenstein. $\mathcal{K}_Y$ identifies with the line bundle on Y defined by the $\mathfrak{a}_{n+1}$-invariance of $\mathcal{K}_{\tilde{Y}}$. Moreover, there is a section of $\mathcal{K}_{\tilde{Y}}$ which is invariant. This gives a section of $\mathcal{K}_Y$ over Y, showing the triviality of $\mathcal{K}_Y$. (This argument will not work if we divide by the full symmetric group, because then any transposition will act by -1 upstairs, and the section will not be invariant.) Alternatively, we will show below that Y is a double cover of $\mathbb{P}^n$ branched along a hypersurface of degree $2n+2$, again implying that $\mathcal{K}_Y$ is trivial.

We have now proved part (a) of our Theorem.

2. Resolution

Now we will show how to deduce part (b) of Theorem. To begin, since the variety Y constructed above is an orbifold, a standard conjecture predicts that there will be a smooth resolution

$$p : X \to Y$$

which is *crepant*, i.e., that the canonical bundle of X has for image the canonical bundle of Y (under p_*) and is thus trivial. For $n \leq 3$ this can be achieved by making use of [Ro], but we will take a different tack.

Now consider the natural action of the symmetric group $\mathfrak{S}_{n+1}$ on E^{n+1}, the product of $n+1$ copies of E. The addition map $E^{n+1} \to E$ is stable under the action of $\mathfrak{S}_{n+1}$ and thus we obtain a map $\mathrm{Sym}^{n+1}(\mathrm{E}) \to \mathrm{E}$, where the former denotes the quotient of E^{n+1} by the action.

The space $\mathrm{Sym}^{n+1}(\mathrm{E})$ can also be identified with the space of effective divisors of degree $n+1$ on E and under this identification the above map can be understood as follows. Let o denote the origin in E. For each point p in E the fibre of the map consists of all divisors in the linear system $|n[o] + [p]|$. In particular, when $p = o$ we see that the fibre consists of all divisors in the linear system $|(n+1)[o]|$.

From the point of view of quotients the fibre over o is the quotient by the action of $\mathfrak{S}_n$ of the space

$$\tilde{Y} = \{(p_0, \ldots, p_n) | p_0 + \cdots + p_n = 0\}$$

We are interested in the quotient Y of this space by the alternating group $\mathfrak{a}_{n+1}$. Thus, Y can be expressed as a double cover of the linear system $|(n+1)[o]|$ branched along the locus of divisors of the form $2[p] + [p_2] + \cdots + [p_n]$.

When n is at least 2 the linear system $|(n+1)[o]|$ gives an embedding of E into the dual projective space $|(n+1)[o]|^*$. The locus of special divisors as considered above is then identified with the dual variety of E; i. e., the variety that consists of all hyperplanes that are tangent to E. It is well known that this dual variety has degree $2(n+1)$, which follows for example from the Hurwitz genus formula giving the number of ramification points for a map $E \to \mathbb{P}^1$ of degree $n+1$.

Since Y is a double cover of $\mathbb{P}^n$ branched along this hypersurface of degree $2(n+1)$, as claimed above, implying the triviality of $\mathcal{K}_Y$. In order to find a good resolution of Y it is sufficient to understand the singularities of the dual variety.

2.1. The case $n = 2$. Here we have the dual of the familiar embedding of E as a cubic curve in $\mathbb{P}^2$. This curve has 9 points of inflection and no other unusual tangents. It follows from the usual theory that the dual curve is a curve with 9 cusps and no other singularities. Thus Y is the double cover of $\mathbb{P}^2$ branched along such a curve. To resolve Y it is enough to resolve over each cusp individually.

Thus we consider the simpler case of resolving the double cover of $W \to \mathbb{A}^2$ branched along the curve defined by $y^2 - x^3$; the variety W is a closed subvariety of $\mathbb{A}^3$ defined by $z^2 - y^2 + x^3$ with the projection to the (x, y) plane providing the double covering. One checks easily that the blow-up of the maximal ideal (x, y, z) gives a resolution of singularities. Moreover, this blow-up is a double cover of the blow-up of $\mathbb{A}^2$ at the maximal ideal (x, y). Since the exceptional divisor in the first case is a (-1) curve, it follows that the exceptional divisor in the blow-up of W is a (-2) curve.

Let $X \to Y$ be the result of blowing-up the nine singular points in Y that lie over the cusps of the dual curve; as seen above X is smooth. From the adjunction formula we see that $\mathcal{K}_X$ restricts to the trivial divisor on each exceptional divisor; hence $\mathcal{K}_X$ is the pull-back of the $\mathcal{K}_Y$. The usual theory of double covers shows us that $\mathcal{K}_Y$ is trivial and Y is simply-connected. Thus the same properties hold for X as well. In other words we have shown that X is a K3 surface.

2.2. The case $n = 3$. In this case E is embedded as the complete intersection of a pencil of quadrics in $\mathbb{P}^3$. Recall that we have assumed that the characteristic does not divide 6.

A point of the dual variety D corresponds to a plane that contains a tangent line. Thus each point on E determines a pencil of such points. Equivalently, if $P \subset E \times (P^3)^*$ denotes the projective bundle on E that consists of pairs (p, π) where π is a plane in P^3 that is tangent to E at p, D is the image of P under the natural projection to $(P^3)^*$ which is a surface of degree 8. For notational convenience let the origin of the group law on E be chosen to be a point o such that the linear system is $4[o]$. The fibre of P over a point p can then also be described as the collection of all divisors $D = [q]+[r]$ of degree two such that $2p+q+r = o$ in the group law.

Let a be a point of order two in E. Then for each point p in E we can consider the point $2[a - p]$ in the fibre of P over p; this gives a section $\sigma_a : E \to P$ and we denote the image in P as E_a. This gives us four disjoint curves in P. Under the composite map $E_a \to P \to D$, the point $\sigma_a(p)$ and $\sigma_a(a-p)$ are both sent to the hyperplane that intersects E in $2[p] + 2[a - p]$, so the image C_a of E_a in D is the quotient of E_a by the involution $p \mapsto a - p$; thus C_a is isomorphic to P^1. Moreover, D has a transverse ordinary double point along the general point of C_a.

Let p be any point of E. Then $[-3p] + [p]$ is a point in the fibre of P over E; this gives a section $\tau : E \to P$ and we denote the image in P as F. The composite map $F \to P \to D$ is a one-to-one since the hyperplane section of the type $3[p] + [q]$ uniquely determines the point p; let G denote the image of F in D. One notes that D has a transverse cusp along the general point of G.

Let b be a point of order 4 on E and consider the point $a = 2b$ which is a point of order 2 on E. We see that

$$\tau(b) = [-3b] + [b] = 2[b] = 2[a - b] = \sigma_a(b)$$

The curve E_a thus intersects the curve F in the four points of order 4 which are "half" of a. As a varies we obtain 16 points on P lying over the 16 points of order 4 on E. The singularity of D at the image of these sixteen points can be described as follows in suitable local coordinates x, y and z. The curve C_a is described by $y = z + h = 0$ and the curve G is described by $x^3 + y^2 = z + h' = 0$, where h and h' consists of terms of degree at least two and are *distinct* (this is important in the next paragraph). Moreover, the Jacobian ideal of D is the intersection of the ideals $(y, z + h)$ and $(x^3 + y^2, z + h')$ that define these two curves.

Let $f : U \to P^3$ be the the smooth threefold obtained by blowing up along the curves C_a. Let A_a denote the exceptional locus in U over the curve C_a. The canonical divisor of U is $f^*\mathcal{O}(-4) \otimes \mathcal{O}(\sum A_a)$. Since D has an ordinary double point along C_a we see that the strict transform D' of D is linearly equivalent to $f^*\mathcal{O}(8) \otimes \mathcal{O}(-2\sum A_a)$. Moreover, D' is only singular along the strict transform G' of G. Finally, from the above local description it follows that the surface $z + h' = 0$ which G' lies on *is* blown up at the origin (x, y) under f. It follows that G' is smooth and D' has a transverse cusp along it.

Let $g : Z \to U$ be the smooth threefold obtained by blowing up along G'. Let B denote the exceptional locus in Z over G and (by abuse of notation) let A_a again denote the strict transform of the divisors A_a in U. The canonical divisor of Z is $g^* f^*\mathcal{O}(-4) \otimes \mathcal{O}(B + \sum A_a)$. Since D' is a singularity of multiplicity two along G' we see again that its strict transform differs from its total transform by $2B$; thus the strict transform D'' of D' is linearly equivalent to $g^* f^*\mathcal{O}(8) \otimes \mathcal{O}(2B - 2\sum A_a)$. Finally, we see that D'' is smooth as well. Thus the double cover $Y \to Z$ along D'' is a smooth threefold with trivial canonical bundle.

□

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