Algebraic cycles on Hilbert modular fourfolds and poles of L-functions

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To M.S. Raghunathan on his sixtieth birthday, with admiration

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Introduction

Let K be a quartic, totally real number field with ring of integers $\mathfrak{O}_K$ and canonical embedding $K \hookrightarrow \mathbb{R}^4$, given by $x \to (x_\sigma)_{\sigma\in\mathrm{Hom}(K,\mathbb{R})}$. Any congruence subgroup Γ of $\mathrm{SL}(2,\mathfrak{O}_K)$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ with finite covolume. It acts properly discontinuously on the 4-fold product of the upper half plane $\mathcal{H}$, with the quotient $\Gamma\backslash\mathcal{H}^4$ being a complex analytic variety of dimension 4 with at most quotient singularities. Let $\mathbb{A} = \mathbb{R}\times\mathbb{A}_f$ denote the adele ring of $\mathbb{Q}$, C a compact open subgroup of $G(\mathbb{A}_f)$, and $Y = Y_C$ the corresponding Hilbert modular fourfold, which is a quasi-projective, normal complex variety, smooth for C small enough. It comes equipped with the Baily-Borel compactification Y^* and a smooth toroidal compactification $X := \tilde{Y}$, all defined over $\mathbb{Q}$. It is well known that Y is the moduli space of abelian fourfolds A with level C-structure and with $\mathrm{End}(A)\otimes\mathbb{Q}\supset K$. The components of $Y(\mathbb{C})$ are all of the form $\Gamma\backslash\mathcal{H}^4$, defined over an abelian extension of $\mathbb{Q}$. We will consider the cohomology of X in various *avatars* like the singular (Betti) version $H^*_B(X(\mathbb{C}),\mathbb{Q})$ and the étale version $H^*_{et}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$.

For every $q\geq 0$ and for any field $k\supset\mathbb{Q}$, denote by $\mathcal{Z}^q(X_k)$ the $\mathbb{Q}$-vector space generated by the k-rational algebraic cycles of codimension q on $X_{\overline{\mathbb{Q}}}$ modulo homological equivalence, by which we mean the equivalence in $H^{2q}_B(X^*(\mathbb{C}),\mathbb{Q}(j))$. If $\overline{k}$ denotes an algebraic closure of k, the absolute Galois group $\mathfrak{G}_k = \mathrm{Gal}(\overline{k}/k)$ acts on $\mathcal{Z}^q(X_{\overline{k}})$ with $\mathcal{Z}^q(X_k)$ being the $\mathbb{Q}$-subspace of fixed points. It is well known that $\mathcal{Z}^0(X_{\overline{k}})\simeq\mathbb{Q}$, $\mathcal{Z}^q(X_{\overline{k}})\simeq\mathcal{Z}^{4-q}(X_{\overline{k}})$ and $\mathcal{Z}^1(X_k)$ is $\mathrm{NS}(X_k)\otimes\mathbb{Q}$, where $\mathrm{NS}(X_k)$ denotes the Néron-Severi group of k-rational divisors on X modulo algebraic equivalence. The really mysterious one is the group $\mathcal{Z}^2(X_k)$ and there is a dearth of concrete results about codimension 2 cycles which are not generated by intersections of divisors.

Fix a prime ℓ and for every $j\geq 0$ denote by $V_\ell^{(j)}$ the ℓ-adic $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation defined by $H^j_{et}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. For any *number field* k, the restriction of $V_\ell^{(j)}$ to $\mathfrak{G}_k$ defines an L-function

$$L(s,V_\ell^{(j)}/k) = \prod_P \frac{1}{\det(I-(NP)^{-s}Fr_P|{V_\ell^{(j)}}^{I_P})}$$

where P runs over the primes in $\mathfrak{O}_k$ with norm NP, (geometric) Frobenius Fr_P and inertia group I_P. There exists a finite set $S_{k,\ell}$ of primes P containing the primes above ℓ outside which I_P acts trivially on $V_\ell^{(j)}$, and moreover, by Deligne's proof of the Weil conjectures, the eigenvalues of Fr_P on $V_\ell^{(j)}$ for any P outside $S_{k,\ell}$ are of absolute value $(NP)^{j/2}$. This shows that for any finite set S of primes containing $S_{k,\ell}$, the *incomplete L-function* $L^S(s,V_\ell^{(j)}/k)$, defined as the Euler product above except with the factors at S removed, is absolutely convergent in $\Re(s)>\frac{j}{2}+1$. It is not *a priori* clear that this L-function is even defined at the *Tate point* $s=\frac{j}{2}+1$. When $j=2m$, a celebrated *conjecture of Tate* says that the L-function is meromorphic at this point and moreover, that the *order of pole* there is equal to the dimension of $\mathcal{Z}^m(X_k)$. The object of this paper is to provide some non-trivial evidence for it for $m = 2$. Since we will be interested only in the cohomology in degree 4, we will write V_ℓ for $V_\ell^{(4)}$. For any non-zero ideal $\mathfrak{N}$ in $\mathfrak{O}_K$, let $C_0(\mathfrak{N})$, resp. $C_1(\mathfrak{N})$, denote the compact open subgroup of G_f given as the product over the finite places v of $C_{0,v}(P_v^{v(\mathfrak{N})})$, resp. $C_{1,v}(P_v^{v(\mathfrak{N})})$; here $C_{0,v}(P_v^r)$ denotes the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2, \mathfrak{O}_{K_v})$ with $c \equiv 0 \,(\mathrm{mod} P_v^r)$, and $C_{1,v}(P_v^r)$ signifies the subgroup of such matrices with $d \equiv 1 \,(\mathrm{mod} P_v^r)$. (By convention, P_v^0 is $\mathfrak{O}_{K_v}$ and so there is no condition when $r = 0$.) Now let Ψ be any weight 1 Hecke character of a totally imaginary quadratic extension M of K. Then it is well known that Ψ defines a Hilbert modular newform g_Ψ, contributing to V_ℓ relative to a level subgroup C containing $C_1(\mathfrak{N})$ if $\mathfrak{N}$ is divisible by the relative discriminant of M/K times the norm of the conductor of Ψ. By the *non-CM part* of V_ℓ we will mean the quotient V_ℓ' of V_ℓ by the space spanned by the classes attached to such Hecke characters. Let $\mathcal{Z}^2(X_k)'$ denote, for any number field k, the dimension of the non-CM part of $\mathcal{Z}^2(X_k)$. Denote also by $Ta_{\ell,k}(X)$ the space of Tate classes in $V_\ell(2)$ (see section 1 for a definition), and by $Ta_{\ell,k}(X)'$ its non-CM part.

Throughout the paper we will use the classical term *abelian number field* to mean a finite abelian extension of $\mathbb{Q}$.

Theorem A *Let K be a quartic, totally real number field containing a quadratic subfield, C a compact open subgroup of $G(\mathbb{A}_f)$, X a Hilbert modular fourfold of level C, and S a finite set of primes containing $S_{k,\ell}$. Then*

(i) $L^S(s, V_\ell/k)$ extends, for any abelian number field k, to a meromorphic function on all of $\mathbb{C}$ and satisfies, after being multiplied by suitable local factors at S and at the archimedean places, a functional equation relating s to $5 - s$;

(ii) Let $K/\mathbb{Q}$ be Galois. Then we have

$$\dim_{\mathbb{Q}} \mathcal{Z}^2(X_k)' \;=\; -\mathrm{ord}_{s=3} L^S(s, V_\ell'/k),$$

for any abelian number field k, and

$$\dim_{\mathbb{Q}} \mathcal{Z}^2(X_k)' \;=\; \dim_{\mathbb{Q}_\ell} Ta_{\ell,k}(X)'.$$

for any number field k.

The proof will show that there are algebraic cycles of codimension 2 which are not in the $\mathbb{Q}$-linear span of intersections of divisor classes. When $K/\mathbb{Q}$ is non-Galois, it can be shown that the dimension of $\mathcal{Z}^2(X_k)$ is still bounded, for k abelian, by the order of pole of $L^S(s, V_\ell)$ at $s = 3$, and the desired equality will follow from the proof in some cases.

Here are some general philosophical remarks. Given a general smooth, projective variety X, it is hard to find *any* collection of explicit algebraic cycles on it. For Shimura varieties one is fortunate in this regard as there are some natural cycles Z, though not nearly enough, defined by *Shimura subvarieties* and their translates by Hecke correspondences, as well as certain twists. There are also cycles supported on certain rational varieties, and their intersections, occurring in the smooth resolution at infinity. One can often write down explicitly a basis for the (p, p)-cohomology for any p. Still, given a differential form ω representing such a cohomology class and a dimension p algebraic cycle Z, how can one show that the integral of ω over Z is non-zero? One way, and this is the tack we take in this paper, is to express $\int_Z \omega$ as the residue at a pole of some L-function which is non-zero for some reason, for example because of the knowledge of the exact order of pole at that point. The *novelty*, if any, in the situation considered here, is that the L-functions which appear this way are *not* the L-functions of the varieties involved, but rather certain associated ones, and luckily there is a (partial) coincidence of poles for the different L-functions. In certain situations we can also check the Hodge conjecture as seen in the result below:

Theorem B *Let K be a quartic, totally real, Galois number field, and let X (resp. $X(1)$) be a smooth, projective, Hilbert modular fourfold associated to the level subgroup $C_0(\mathfrak{N})$ for some square-free ideal $\mathfrak{N} \neq \mathfrak{O}_K$ (resp. for $\mathfrak{N} = \mathfrak{O}_K$). Then the Hodge cycles on $X_{\mathbb{C}}$ in codimension 2, which are not pull-backs of classes in $X(1)_{\mathbb{C}}$, are all algebraic. For various $\mathfrak{N}$, there are such algebraic cycles Z of codimension 2 on X which are not intersections of divisors.*

For a statement of the Hodge conjecture (and the definition of Hodge cycles), see section 1. The proof of Theorem B will be given in section 10. However, it should be acknowledged that the proof given there depends partially on a joint result with V. Kumar Murty on a comparison of rational Hodge structures (see Theorem 10.4) which we plan to publish elsewhere in a more general setting. If this is not satisfactory to some readers, they can focus just on the proof of Theorem A which is totally self-contained.

This paper provides an extension of the work of Harder, Langlands and Rapoport ([HLR]) on the Tate conjecture for *divisors* on Hilbert modular *surfaces* over abelian number fields. This generalization is *not routine*, however, due to certain subtle problems, the least of which is that we are working with codimension 2 cycles. It is perhaps helpful to elaborate.

The *first difficulty*, which is analytic, is to show that $L(s, V_\ell)$ is meromorphic at $s = 3$, which we will discuss below. Granting that, the *next step* is to look for $\mathbb{Q}$-rational algebraic cycles Z on $\tilde{X}$ not entirely supported on X^∞, with a view to matching their (homological) non-triviality with the possible poles of $L(s, V_\ell)$. Under the hypothesis that K contains a quadratic subfield F, the natural cycles to consider are Hecke translates of the Hilbert modular surface defined by $\mathrm{GL}(2)/F$; these are the analogs of the Hirzebruch-Zagier cycles investigated in [HLR], and earlier – in a concrete form – in [HZ]. It turns out that their non-triviality is linked to the poles of the *Asai* L-functions $L(s, As_{K/F})$ associated to K/F, which are of degree 4 over F and 8 over $\mathbb{Q}$.

More precisely, if π is a cusp form on $\mathrm{GL}(2)/K$ of weight 2, then the π-part of $L(s, As_{K/F})$, written $L(s, \pi; r_{K/F})$, has a pole at the right edge iff the **period integral**

$$\int_{\mathrm{GL}(2,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2,\mathbb{A}_F)} \phi(g)dg$$

is non-zero for some function ϕ in the space of π, where Z denotes the center of $\mathrm{GL}(2)$. This period is simply the residue at the edge of convergence (*Tate point*) of $L(s, \pi; r_{K/F})$, which has a representation as the integral over $\mathrm{GL}(2, F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2, \mathbb{A}_F)$ of ϕ times an Eisenstein series $E(s)$ on $\mathrm{GL}(2)/F$.

The *second difficulty* is that these functions $L(s, \pi; r_{K/F})$ do **not** divide $L(s, V_\ell)$. Fortunately for us we are able to show, and this is a key point, that the poles of $L(s, As_{K/F})$ give rise, over abelian fields k, to poles of $L(s, V_\ell)$. But they do not account for all the poles of $L(s, V_\ell)$. When $K/\mathbb{Q}$ is Galois and π non-CM, however, we are able to show that all such poles are accounted for by suitable twists of (Hecke translates of) embeddings of Hilbert modular surfaces relative to *all* the quadratic fields contained in K.

The *third difficulty* comes up in the biquadratic case when the order of pole of $L(s, \pi; r_{K/F})$ can be 2 for certain cusp forms π. Then it is not enough to consider only the (Hecke translates of) cycles coming from one quadratic subextension, and more importantly, when we consider a pair of cycles coming from two different quadratic subextensions, a key point of the proof is to show that these two, together with a suitable twist of one of them (see section 9), must span a plane in the π_f-component of the homology. The referee has indicated an alternate, elegant way to handle this, which has been described in Remark 9.17, but we have left our proof intact as we think this method could be of use in other situations.

One knows (cf. [BrL], [La1]) that the main part of $L(s, V_\ell)$ is a product, over cusp forms π of $\mathrm{GL}(2)/K$ of weight 2, of certain Asai L-functions $L(s, \pi; r_{K/\mathbb{Q}})$ of degree 16. Under the hypothesis that K contains a quadratic subfield F we establish (see section 7 below) the meromorphic continuation and functional equation for such L-functions. We cannot analyze their poles except, luckily, at the right edge of absolute convergence, which is what is relevant for the Tate conjecture. To be precise, for any such π, and for any Dirichlet character ν of $\mathbb{Q}$, the ν-twisted L-function $L(s, \pi; r_{K/\mathbb{Q}} \otimes \nu)$ turns out to admit a pole at the right edge iff a suitable twist of π is a base change from a quadratic subfield of K; this is as in the case of Hilbert modular surfaces ([HLR]). But more interestingly, we show that such an L-function has a *double pole* at the Tate point iff a twist of π is a base change all the way from $\mathbb{Q}$ *and* $K/\mathbb{Q}$ is biquadratic.

The CM case is not treated in this paper, partly to not make this paper longer, and partly because there are more Tate classes than we can handle in certain situations. In other words, there are exotic ones which cannot be accounted for by the analogs of Hirzebruch-Zagier cycles (and their twists), and such classes can exist even over abelian fields – see Remark 9.18; this was *not* the case for Hilbert modular surfaces. It is likely that a variant of the method of [Mu-Ra] can match such Tate cycles with corresponding Hodge cycles, but since the Hodge conjecture is not known in codimension 2, the analysis grinds to a halt.

Quite generally, given any finite extension K/F of number fields of degree d, and any isobaric automorphic form π on $\mathrm{GL}(n)/K$, one can associate an Asai L-function, denoted $L(s, \pi; r_{K/F})$ (see section 6 for a definition), of degree n^d. This is an analogue of tensor induction on the Galois side. For example, suppose K/F is cyclic and assume the existence of an n-dimensional representation σ (over $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$) of $\mathrm{Gal}(\overline{K}/K)$ with the same L-function as π. For every $g \in G := \mathrm{Hom}(K, \overline{K})$, let $\sigma^{[g]}$ denote the representation of $\mathrm{Gal}(\overline{K}/K)$ defined by $\beta \to \sigma(g\beta g^{-1})$. Then the representation $\otimes_{g\in G}\, \sigma^{[g]}$ is G-invariant and extends to a non-unique n^d-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$. One can define a particular choice of an extension, called the *Asai representation* and denoted $As(\sigma)$. The L-function of $As(\sigma)$ should be the same as $L(s, \pi; r_{K/F})$. The *principle of functoriality* implies that the Asai L-function, whether or not π is associated to any Galois representation, should have meromorphic continuation and a standard functional equation, and this is known in the $(n, 2)$-case for arbitrary n (see section 6 below). Classically, Asai established the requisite properties, and in fact the location of all the poles, for the case $(n, d) = (2, 2)$ when $F = \mathbb{Q}$, K real quadratic and π holomorphic (Hilbert modular) newform over K; the general $(2, 2)$-case was treated in [HLR], and then in [JY] by using the *relative trace formula*. The case $(n, d) = (2, 3)$ was treated in ([RPS]), providing a non-trivial extension of Garrett's construction for the triple product L-function, and a complete location of the poles was then given in [Ik]. The meromorphic continuation and functional equation follows from the work of Shahidi ([Sh])in the $(n, 2)$-case for any n, and the possible poles at the right edge were analyzed in [Fℓ]. The main thing for us is that we can also treat, using [Ra3], the $(2, 4)$-case under the hypothesis that K contains a quadratic subextension F.

For any (n, d), the principle of functoriality implies moreover that there is an isobaric automorphic form $As_{K/F}(\pi)$ on $\mathrm{GL}(n^d)/F$ whose standard L-function coincides with $L(s, \pi; r_{K/F})$. When $(n, d) = (2, 2)$ this was established in [Ra2], and we know of no such result in this direction when $n > 2$ and $d > 1$. This modularity result is crucially used below in deducing the needed properties of $L(s, \pi; r_{K/\mathbb{Q}})$.

I lectured on this material during July 2002 at the conference on Automorphic Forms at Park City, UT, and then later at Columbia University, NY. I benefited from the feedback and interest I received from various people at those places including L. Clozel, H. Jacquet, L. Saper, and S. Zhang. I would also like to thank Jacob Murre for a helpful conversation, Arvind Nair for pointing out the need for elaboration at a point in section 3, and finally the referee for a careful reading of the paper and for making several good comments.

The last section contains some remarks on my earlier papers. The main thing here is to give some details, sought by Joe Shalika, of the proof of the key Lemma 3.4.9 of [Ra2] giving Sobolev inequalities for eigenfunctions of the Casimir operator. It also contains some clarifications/refinements sought by E. Lapid and M. Krishnamurthy on [Ra2] and [Ra3] respectively, as well as some errata for [Ra2,3].

When I began my graduate studies in Mathematics at Columbia University in the Fall of 1975, I had the good fortune to attend Raghunathan's exciting, challenging and (very) fast course on Arithmetic groups. I learnt a lot from that one semester course, which has come in handy at various times in my own work which tends to traverse nearby fields like Automorphic Forms and Arithmetical Geometry. Raghunathan has also been a very encouraging and friendly figure over the years. It is a great pleasure to dedicate this article to him. Thanks are also due to the NSF for financial support through the grant DMS–0100372.

1 The algebraic versus analytic rank

Denote by $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let X be any smooth projective variety of dimension d over $\mathbb{Q}$. A $\overline{\mathbb{Q}}$-rational **algebraic cycle of codimension** $q \leq d$ is a finite formal sum

$$Z = \sum_{i=1}^{m} \alpha_i Z_i, \tag{1.1}$$

with each $\alpha_i \in \mathbb{Q}$ and Z_i a closed, irreducible subvariety of $X_{\overline{\mathbb{Q}}} = X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ of codimension q. Evidently the collection of all such Z forms a $\mathbb{Q}$-vector space, denoted $C^q(X_{\overline{\mathbb{Q}}})$.

Integration of differential forms of degree $2d-2q$ over such a cycle Z defines a class

$$[Z] \in H_{2d-2q}(X(\mathbb{C}), \mathbb{Q}) \simeq H_B^{2q}(X(\mathbb{C}), \mathbb{Q})(q), \tag{1.2}$$

where the subscript B signifies taking the *singular* (or Betti) cohomology, and (q) denotes twisting by $\mathbb{Q}(q) = (2\pi i)^q\mathbb{Q}$. Since $\mathbb{Q}(q)$ is the unique Hodge structure of rank 1, weight $-2q$ and bidegree $(-q,-q)$, $H_B^{2q}(X(\mathbb{C}), \mathbb{Q})(q)$ is a Hodge structure of weight 0. Recall that there is a Hodge decomposition

$$H^{2q}(X(\mathbb{C}), \mathbb{C}) = \oplus_{i+j=2q} H^{i,j}(X), \tag{1.3}$$

where the classes in $H^{i,j}(X)$ are represented by differential forms of bidegree (i,j).

A basic fact is that $[Z]$ lies in the subspace of **Hodge cycles** of codimension q on $X(\mathbb{C})$, defined to be

$$Hg^q(X) = (H_B^{2q}(X(\mathbb{C}), \mathbb{Q}) \cap H^{q,q}(X))(q). \tag{1.4}$$

The **Hodge conjecture** says that every Hodge cycle on $X(\mathbb{C})$ is of the form $[Z]$ for an algebraic cycle Z; it is known to hold, by Lefschetz, for $q = 1$.

Put

$$(1.5) \qquad Z \equiv 0 \Leftrightarrow [Z] = 0.$$

This gives an equivalence, called the **homological equivalence**, for algebraic cycles. Set

$$(1.6) \qquad \mathcal{Z}^q(X_{\overline{\mathbb{Q}}}) = C^q(X_{\overline{\mathbb{Q}}})/\equiv .$$

Since X is defined over $\mathbb{Q}$, the absolute Galois group $\mathcal{G}_{\mathbb{Q}}$ =Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) permutes the closed irreducible subvarieties of $X_{\overline{\mathbb{Q}}}$. Hence we get a Galois action $(\sigma, Z) \to Z^\sigma$ on $C^q(X_{\overline{\mathbb{Q}}})$ given by

$$(1.7) \qquad Z^\sigma = \sum_{i=1}^{m} \alpha_i Z_i^\sigma \quad if \quad Z = \sum_{i=1}^{m} \alpha_i Z_i.$$

Now fix a prime ℓ and consider the ℓ-adic cohomology groups of $X_{\overline{\mathbb{Q}}}$, on which there is a natural action of $\mathcal{G}_{\mathbb{Q}}$. Then we also have ℓ-adic cycle classes $[Z]_\ell$ in $H^{2q}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(q)$, with (q) denoting the tensoring with the Galois representation $\mathbb{Q}_\ell(q) = \mathbb{Z}_\ell(q) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where $Z_\ell(q)$ denotes the inverse limit $\lim\limits_{n} \mu_{\ell^n}^{\otimes q}$ and μ_{ℓ^n} the group of ℓ^n-th roots of unity in $\overline{\mathbb{Q}}$. One knows the following:

There is an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$(1.8) \qquad H^{2q}_B(X(\mathbb{C}), \mathbb{Q})(q) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^{2q}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(q)$$

such that the image of $[Z]$ is $[Z]_\ell$;

The ℓ-adic cycle class map $Z \to [Z]_\ell$ is Galois equivariant, i.e.,

$$(1.9) \qquad [Z^\sigma]_\ell = [Z]_\ell^\sigma, \ \forall \sigma \in \mathcal{G}_{\mathbb{Q}}.$$

It follows that if $Z \equiv 0$, then $Z^\sigma \equiv 0$ for all σ, and this gives us an action of $\mathcal{G}_{\mathbb{Q}}$ on $\mathcal{Z}^q(X_{\overline{\mathbb{Q}}})$. For any field $k \subset \overline{\mathbb{Q}}$, we define the **group of k-rational algebraic cycles** of codimension q *modulo homological equivalence* to be

$$(1.10) \qquad \mathcal{Z}^q_k(X) := \mathcal{Z}^q(X_{\overline{\mathbb{Q}}})^{\mathcal{G}_k},$$

which is finite dimensional because it identifies with a $\mathbb{Q}$-subspace of $H^{2q}(X(\mathbb{C}), \mathbb{Q})(q)$.

The **algebraic rank** of X over k *in codimension* q is defined to be

$$(1.11) \qquad r^q_{\mathrm{alg},k}(X) = \dim_{\mathbb{Q}} \mathcal{Z}^q_k(X).$$

For any $j \leq 2d$, put

$$(1.12) \qquad V_\ell^{(j)} = H^j_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

which is a finite dimensional, continuous representation of $\mathcal{G}_{\mathbb{Q}}$. Let S_k be a finite set of primes P in k such that either $P \mid \ell$ or $V_\ell^{(2q)}$ is ramified at P. The incomplete L**-function** attached to $(V_\ell^{(j)}, k)$ is

$$L^S(s, V_\ell^{(j)}/k) \;=\; \prod_{P \notin S_k} \det(I - Fr_P P^{-s} | V_\ell^{(j)})^{-1}, \tag{1.13}$$

which converges absolutely in the right half plane $\{\Re(s) > j/2+1\}$ by Deligne's proof of the Weil conjectures, which assert that the inverse roots of the geometric Frobenius elements Fr_P are all of absolute value $N(P)^{j/2}$.

Take $j = 2q$. One expects $L^S(s, V_\ell^{(2q)})$ to admit a meromorphic continuation and satisfy, with suitable factors at S_k and infinity, a functional equation relating s and $2q+1-s$. All we need for the Tate conjecture, however, is that this L-function is meromorphic at the *Tate point*, namely the *edge of convergence* $s = q+1$. Admitting this leads to the following definition, for every number field k, of the **analytic rank** of X over k *in codimension* q:

$$r^q_{\mathrm{an},k}(X) \;=\; -\mathrm{ord}_{s=q+1} L^S(s, V_\ell^{(2q)}/k), \tag{1.14}$$

where $V_{\ell,k}$ denotes the restriction of V_ℓ to $\mathcal{G}_k$. We have

Conjecture I (Tate)

$$r^q_{\mathrm{alg},k}(X) \;=\; r^q_{\mathrm{an},k}(X). \tag{1.15}$$

This is true for $q = 0, d$, and there is some positive evidence for divisors $(q = 1)$.

Finally, by the Galois equivariance of the ℓ-adic cycle class map, $[Z]_\ell$ lies in the space of **Tate cycles** over k:

$$Ta^q_{\ell,k}(X) := H^{2q}_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(q))^{\mathcal{G}_k}, \tag{1.16}$$

for all $Z \in \mathcal{Z}^q_k(X)$. The ℓ**-adic cycle rank** of X over k in *codimension* q is then give by

$$r^q_{\ell,k}(X) \;=\; \dim_{\mathbb{Q}_\ell} Ta^q_{\ell,k}(X). \tag{1.17}$$

One also has the following

Conjecture II (Tate)

$$r^q_{\ell,k}(X) \;=\; r^q_{\mathrm{an},k}(X). \tag{1.18}$$

Since $\mathcal{Z}^q_k(X)$ injects into $Ta^q_{\ell,k}(X)$, we get the following

Proposition 1.19

$$r^q_{\mathrm{alg},k}(X) \;\leq\; r^q_{\ell,k}(X).$$

The existence of the **Hodge-Tate decomposition** ([F]) implies the following useful bound (where $h^{q,q}(X)$ denotes the dimension of $H^{q,q}(X_{\mathbb{C}})$):

Proposition 1.20

$$r^q_{\ell,k}(X) \,\le\, h^{q,q}(X).$$

Let $r^q_{Hg}(X)$ denotes the dimension of the space $Hg^q(X_{\mathbb{C}})$ of **Hodge cycles** (of codimension q) on $X_{\mathbb{C}}$. We trivially have

$$r^q_{Hg}(X) \,\le\, h^{q,q}(X), \tag{1.21}$$

and the Hodge conjecture is the statement

$$r^q_{\mathrm{alg}}(X) \,=\, r^q_{Hg}(X_{\mathbb{C}}),$$

where $r^q_{\mathrm{alg}}(X)$ denotes the dimension of algebraic cycles in $Hg^q(X_{\mathbb{C}})$, which, by the comparison theorem (relating the Betti and étale cohomology) and the proper base change theorem (relating the étale cohomology of $X_{\overline{\mathbb{Q}}}$ with that of $X_{\mathbb{C}}$, identifies with $r_{\mathrm{alg},k}(X)$ for k a large enough number field.

If X is an abelian variety over a number field k, it is known by Deligne ([DMOS]) that for any prime ℓ,

$$r^q_{Hg}(X) \,\le\, r^q_{\ell}(X_{\overline{k}}) \tag{1.22}$$

for k large enough. This will be used in section 10.

2 Hilbert modular varieties

Let K be a totally real number field of degree d, with ring of integers $\mathfrak{O}_K$ and adele ring $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$, where the ring $\mathbb{A}_{K,f}$ of finite adeles of K identifies with $K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f}$ with $\mathbb{A}_{\mathbb{Q},f} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$. We will write $\mathbb{A}$, resp. $\mathbb{A}_f$, for $\mathbb{A}_{\mathbb{Q}}$, resp. $\mathbb{A}_{\mathbb{Q},f}$.

Let $\mathcal{H}$ denote the *upper half plane* in $\mathbb{C}$, and let $\mathcal{H}_\pm = \mathbb{C} - \mathbb{R}$. Put

$$G := \, R_{K/\mathbb{Q}}\mathrm{GL}(2)/K,$$

where $R_{K/\mathbb{Q}}$ denotes the Weil restriction of scalars from K to $\mathbb{Q}$. Then G is a reductive algebraic group over $\mathbb{Q}$ with $G(\mathbb{Q}) = \mathrm{GL}(2, K)$, and $G(\mathbb{R}) = \mathrm{GL}(2, K_\infty)$ has a natural action, by fractional linear transformations, on

$$\mathcal{D} := \, K \otimes \mathbb{C} - K \otimes \mathbb{R} \,\simeq\, \mathcal{H}^d_\pm,$$

Define

$$h : \mathbb{C}^* \,\to\, G(\mathbb{R}),\, a + ib \to \delta\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right),$$

where δ denotes the diagonal embeddding of $\mathrm{GL}(2,\mathbb{R})$ in $G(\mathbb{R})$. Let K_∞ denote the centralizer of $h(\mathbb{C}^*)$ in $G(\mathbb{R})$, so that we have the identification

$$\mathcal{D} \,=\, G(\mathbb{R})/K_\infty.$$

Let C be an open compact subgroup of $G_f = G(\mathbb{A}_f)$. Denote by $S_C = S_C(G,h)$ the associated d-dimensional *Shimura variety* over $\mathbb{Q}$, which is quasi-projective and attached to the datum $(G,h;C)$ by the theory of canonical models ([De1]). One has

$$S_C(\mathbb{C}) \,=\, G(\mathbb{Q})\backslash\mathcal{D}\times G_f/C.$$

We will take C to be small enough so that $G(\mathbb{Q})\cap C$ has no elliptic elements, making S_C non-singular.

The standard approximation theorem for $\mathbb{G}_m$ says that we can find a finite set of elements $b_1, b_2, \ldots, b_{h(C)}$ in $\mathbb{A}^*_{K,f}$ such that

$$\mathbb{A}^*_K \,=\, \cup_{j=1}^{h(C)} K^* b_j K^+_\infty \det(C),$$

where K^+_∞ denotes the totally positive elements in $K\otimes\mathbb{R}\simeq\mathbb{R}^d$. Combining this with the strong approximation theorem for SL(2)/K and the finiteness of the class number, and denoting by $G(\mathbb{R})^+$ the subgroup of $G(\mathbb{R})$ consisting of totally positive elements, one obtains the following useful decomposition:

$$G(\mathbb{A}) \,= \cup_{j=1}^{h(C)} G(\mathbb{Q}) x_j G(\mathbb{R})^+ C,$$

with

$$x_j \,=\, \begin{pmatrix} b_j & 0 \\ 0 & 1 \end{pmatrix}$$

Consequently, we get an identification of quasi-projective, complex varieties:

$$S_C(\mathbb{C}) \,=\, \cup_{j=1}^{h(C)} S_{\Gamma_j}(\mathbb{C}),$$

where for each j,

$$S_{\Gamma_j}(\mathbb{C}) \,=\, \Gamma_j\backslash\mathcal{H}^d_\pm,$$

with

$$\Gamma_j \,=\, G(\mathbb{Q})\cap x_j g(\mathbb{R})^+ C x_j^{-1}.$$

Each Γ_j is a discrete subgroup of $G(\mathbb{R})$ and $\Gamma_j\backslash\mathcal{H}^d_\pm$ is a Hilbert modular d-fold in the classical sense, having a canonical model S_{Γ_j} over a finite abelian extension $k(\Gamma_j)$ of $\mathbb{Q}$. $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ acts continuously, but not transitively, on the group $\pi_0(S_C)$ of connected components of $S_C(\mathbb{C})$ ([De1]).

Put

$$S := \lim_{C} S_C,$$

which is a pro-variety over $\mathbb{Q}$ admitting a right G_f-action.

Let S^*_C denote the projective (singular) *Baily-Borel compactification* of S_C over $\mathbb{Q}$. One has

$$S^*_C \,=\, S_C \cup S^\infty_C,$$

where S_C^∞ is a finite set of *cusps*.

Choose and fix a smooth *toroidal compactification* $\tilde{S}_C$ over $\mathbb{Q}$ ([AMRT]), defined by a rational cone decomposition. Let $\tilde{S}_C^\infty$ stand for the inverse image of S_C^∞ in $\tilde{S}_C$, which we can (and we will) arrange to be a divisor with normal crossings. The irreducible components are smooth rational varieties, and this will be important to us. One can construct smooth toroidal compactifications $\tilde{S}_{\Gamma_j}(\mathbb{C})$ of the components $S_{\Gamma_j}(\mathbb{C})$ such that

$$\tilde{S}_C(\mathbb{C}) = \cup_{j=1}^{h(C)} \tilde{S}_{\Gamma_j}(\mathbb{C}).$$

3 Contribution from infinity

Let K be a quartic, totally real extension of $\mathbb{Q}$, and ℓ a prime. Fix a compact open subgroup C as in the section above, together with a smooth toroidal compactification $\tilde{S}_C$ over $\mathbb{Q}$ of the associated Shimura variety S_C. Since we are only interested in the cycles of codimension 2 on these fourfolds, we will write $r_{\rm alg}$ for $r^2_{\rm alg}$ and $r_{\rm an}$ for $r^2_{\rm an}$.

By the *decomposition theorem* of Beilinson, Bernstein and Deligne ([BBD], section 5.4), we have a short exact sequence of $\mathcal{G}_\mathbb{Q}$-modules
(3.1)

$$0 \to IH^4_{et}(\tilde{S}_C \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \to H^4_{et}(S_C^* \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \xrightarrow{b} H^4_{\tilde{S}_C^\infty, et}(\tilde{S}_C \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \to 0,$$

where IH^* denotes the middle intersection cohomology of Goresky, MacPherson and Deligne of the Baily-Borel compactification S_C^*, which is pure by a theorem of Gabber; in other words, the eigenvalues of Frobenius elements Fr_P acting on $IH^4_{et}(S_C^* \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ have, at good primes P, absolute value $N(P)^2$. And the group on the right of (3.1) signifies the cohomology of $\tilde{S}_C \times_\mathbb{Q} \overline{\mathbb{Q}}$ with supports in the divisor $\tilde{S}_C^\infty$.

Though we do not need it here, one knows enough to show explicitly that the sequence (3.1) splits as $\mathcal{G}_\mathbb{Q}$-modules. We have recently learnt that a very general result of this sort for all Shimura varieties has been established by A. Nair ([N]).

Define, for $\nu =$alg, an or ℓ, and for any number field k, the corresponding ranks $r_{\nu,k}(S_C^*)$ and $r_{\nu,k}(\tilde{S}_C^\infty)$ by using the respective $\mathcal{G}_\mathbb{Q}$-modules $IH^4_{et}(\tilde{S}_C \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ and $H^4_{\tilde{S}_C^\infty, et}(\tilde{S}_C \times_\mathbb{Q} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$.

Proposition 3.2 *We have, for any number field k,*

$$(a) \qquad r_{{\rm alg},k}(\tilde{S}_C^\infty) = r_{\ell,k}(\tilde{S}_C^\infty)$$

and

$$(b) \qquad r_{{\rm an},k}(\tilde{S}_C^\infty) = r_{\ell,k}(\tilde{S}_C^\infty)$$

Proof of Proposition 3.2.

(a) It suffices to prove this over a sufficiently large extension of k, which contains in particular the (abelian) fields of definition of the cusps. Since the divisor $\tilde{S}_C^\infty$ is the inverse image of the set S_C^∞, and since the cusps are isolated, we have in étale as well as singular cohomology,

$$(3.3) \qquad H^*_{\tilde{S}_C^\infty}(\tilde{S}_C) \simeq \oplus_{\sigma\in S_C^\infty} H^*_{D_\sigma}(\tilde{S}_C),$$

where each D_σ, the fiber over σ, is a *divisor with normal crossings.* Let $\{D_\sigma^i \,|\, 1 \le i \le r(\sigma)\}$ denote the set of irreducible components of D_σ. One knows (cf. [AMRT]) that each D_σ^i is a smooth toric threefold, and so by purity the cohomology of $\tilde{Y}$ with supports in D_σ^i can be expressed in terms of the cohomology of D_σ^i.

Clearly, it suffices to prove the assertion for each cusp separately. From the geometry of $\tilde{D}_\sigma$ we obtain the following exact sequence

$$(3.4) \qquad \to \oplus_{i=1}^{r(\sigma)} H^2(D_\sigma^i)(1) \to H^4_{D_\sigma}(\tilde{S}_C)(2) \to \oplus_{i\neq j} H^0(D_\sigma^{i,j}) \to,$$

where $D_\sigma^{i,j}$ denotes, for all unequal i,j with $1 \le i,j \le r(\sigma)$, the intersection $D_\sigma^i \cap D_\sigma^j$. It is clear then that the Tate classes of codimension 2 on $\tilde{Y}$ with supports in D_σ are generated by the following:
(3.5)

(i) Tate classes of codimension 1 on the components D_σ^i, *and*

(ii) Classes of the intersections $D_\sigma^{i,j}$.

The classes of type (ii) are obviously algebraic. And so are those of type (i) because the D_σ^i are rational varieties; in fact, for every (σ, i), the entire $H^2(D_\sigma^i)(1)$ is generated by divisor classes. Hence we get (a).

(b) It is evident from (3.3), (3.4) and (3.5) that the Galois representation on $H^4_{\tilde{S}_C^\infty,et}(\tilde{S}_C \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ is *potentially abelian*, and its L-function over any number field k is associated to that of a $\mathbb{C}$-representation of the absolute Weil group W_k. In this case the equality of (b) is well known (cf. [De2]). Done.

□

There is a natural analog of the decomposition (3.1) for the Betti cohomology with $\mathbb{Q}$-coefficients, with $IH^4_B(\tilde{S}_C(\mathbb{C}), \mathbb{Q})$ being a pure $\mathbb{Q}$-Hodge structure of weight 4. So we may define the *Hodge cycle ranks* $r_{Hg}(S_C^*)$ and $r_{Hg}(\tilde{S}_C^\infty)$ by using $IH^4_B(\tilde{S}_C(\mathbb{C}), \mathbb{Q})$ and $H^4_{\tilde{S}_C^\infty,B}(\tilde{S}_C(\mathbb{C}), \mathbb{Q})$ respectively.

Arguing as in the proof of Proposition 3.2, we also get the following

Proposition 3.6 *We have*

$$r_{\mathrm{alg}}(\tilde{S}_C^\infty) = r_{Hg}(\tilde{S}_C^\infty).$$

In view of Propositions 3.2 and 3.6, we see that any Tate or Hodge class of codimension 2 on $\tilde{S}_C$ with supports in $\tilde{S}_C^\infty$ is algebraic with the predicted order of pole (in the Tate case). It then follows from the decomposition (3.1) that to establish the *Tate conjectures* over any k, resp. the Hodge conjecture, for $\tilde{S}_C$, it suffices to prove the following identities:

$$r_{\mathrm{alg},k}(S_C^*) \,=\, r_{\mathrm{an},k}(S_C^*), \tag{3.7 - T1}$$

and

$$r_{\mathrm{alg},k}(S_C^*) \,=\, r_{\ell,k}(S_C^*), \tag{3.7 - T2}$$

resp.

$$r_{\mathrm{alg}}(S_C^*) \,=\, r_{Hg}(S_C^*). \tag{3.7 - Hg}$$

4 Decomposition according to Hecke

Put

$$V_B := IH_B^4(S_C^*(\mathbb{C}), \mathbb{Q}) \tag{4.1}$$

and

$$V_\ell := IH_{et}^4(S_C^* \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell).$$

Let $\mathcal{H}_C$ denote the $\mathbb{Q}$-linear Hecke algebra of level C, which acts semisimply on V_B and V_ℓ. It is generated by the characteristic functions of the double cosets CgC, $g \in G_f$. For every field extension E of $\mathbb{Q}$, we will set

$$\mathcal{H}_{C,E} \,=\, \mathcal{H}_C \otimes_{\mathbb{Q}} E.$$

The elements of $\mathcal{H}_C$ act as algebraic correspondences of finite degree on S_C. To be precise, any g in G_f acts on S on the right, but it does not preserve S_C. But if we put $C_g := C \cap gCg^{-1}$, it is again an open compact subgroup of G_f, and there are two homomorphisms $C_g \to C$ given by the identity and the conjugation by g^{-1}, resulting in a corresponding pair of maps, denoted $R(1)$ and $R(g)$, from S_C into S_{C_g}. This leads to a self-correspondence, called the **Hecke correspondence**, of S_C given by

$$T_g(x) \,=\, R(g)(R(1)^{-1}(x)).$$

It is not hard to see that T_g depends only on the double coset CgC, and this way one gets an isomorphism of the $\mathbb{Q}$-algebra generated by the T_g with $\mathcal{H}_C$. The Hecke correspondences also extend, as ramified correspondences, to the Baily-Borel compactification S_C^*. (They do not extend to the toroidal compactification for a fixed rational cone decomposition, which is the *raison d'etre* for the discussion of the previous section.) Hence they act on the intersection cohomology of S_C^*, and it is natural to decompose according to isotypic subspaces according to the algebra.

It is a basic fact that irreducibles η of $\mathcal{H}_{C,\overline{\mathbb{Q}}}$ are in bijection with $\overline{\mathbb{Q}}$-irreducible admissible representations π_f of G_f which admit a non-zero C-fixed vector, the correspondence being given by $\eta = \pi_f^C$. Let $E = E(\pi_f)$ denote the field of rationality of π_f; it is known that π_f and $\eta = \pi_f^C$ can be realized over E. In sum, irreducibles of $\mathcal{H}_C$ are parametrized by the packets

$$\{\pi_f^\sigma \mid \pi_f^C \neq 0,\, \sigma \in \mathrm{Hom}(E, \overline{\mathbb{Q}})\}$$

One gets the decomposition for $* = B, et$:

$$V_* \simeq \oplus_{\pi_f \in \Sigma_C} V_*(\pi_f)^{m(\pi_f, C)}, \tag{4.2}$$

where Σ_C denotes the set, modulo Galois conjugation, of irreducible admissible $\overline{\mathbb{Q}}$-representations π_f of G_f admitting a non-zero vector fixed by C, and $m(\pi_f, C)$ denotes the dimension of C-invariants. There is an isomorphism of $\mathbb{Q}_\ell$-vector spaces:

$$V_B(\pi_f) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq V_\ell(\pi_f). \tag{4.3--a}$$

Moreover, each $V_B(\pi_f)$ admits an E-action. It is of importance to know the dimension of $V_B(\pi_f)$ over E and we will come to this question below. The ℓ-adic Galois representation is an $E \otimes \mathbb{Q}_\ell$-module and it decomposes as

$$V_\ell(\pi_f) \simeq \oplus_{\lambda \mid \ell} V_\lambda(\pi_f). \tag{4.3--b}$$

One can define in the obvious way, for any number field k, the π_f-components of the algebraic and analytic ranks $r_{\mathrm{alg},k}(S(\pi_f))$ and $r_{\mathrm{an},k}(S(\pi_f))$ respectively. (We want to think of $S(\pi_f)$ as a Grothendieck motive with coefficients in E.)

We will say that π_f is *CM* (or of *CM type*) iff $\pi_f \simeq \pi_f \otimes \nu$, for a non-trivial quadratic character ν. Then π_f is necessarily infinite-dimensional and the quadratic extension $M = M(\nu)$ of F cut out by ν is a CM field. Put, for $* = B, et$,

$$V_*^{CM} = \oplus_{\pi_f \in \Sigma_C,\, \pi_f\, CM} V_*(\pi_f)^{m(\pi_f, C)}, \tag{4.4--a}$$

and

$$V'_* = V_*/V_*^{CM}, \tag{4.4--b}$$

In view of Proposition 3.2 and the decomposition (4.2) above, Theorem A will be a consequence of the following:

Theorem A′ *Suppose K is normal over $\mathbb{Q}$, and k an abelian number field. Let π_f be an irreducible, admissible, $\overline{\mathbb{Q}}$-rational, non-CM representation of $G(\mathbb{A}_f)$, equipped with a non-zero vector in (the space of) π_f fixed by the compact open subgroup C, such that $V_B(\pi_f) \neq 0$. Then we have*

$$r_{\mathrm{alg},k}(S(\pi_f)) = r_{\mathrm{an},k}(S(\pi_f)).$$

Put
$$V_{(2)} \,=\, H^4_{(2)}(S_C(\mathbb{C}), \mathbb{C}),$$
where the group on the right is the L^2-*cohomology* of the open manifold $S_C(\mathbb{C})$. One knows by the proof of Zucker's conjecture by Saper-Stern ([SaSt] and Looijenga ([Lo]), that there is an isomorphism

$$V_B \otimes_{\mathbb{Q}} \mathbb{C} \,\simeq\, V_{(2)}. \tag{4.5}$$

If $\mathfrak{G}_{\mathbb{C}}$ denotes the set of matrices in the complexified Lie algebra of G with purely imaginary trace, there is a standard isomorphism ([BoW])

$$H^*_{(2)}(S_C(\mathbb{C}), \mathbb{C}) \,\simeq\, H^*(\mathfrak{G}_{\mathbb{C}}, K_\infty; L^2_{\mathrm{disc}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})/C)^\infty), \tag{4.6}$$

where $L^2_{\mathrm{disc}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})/C)$ denotes the C-invariants in the discrete spectrum of (the right regular representation) $L^2(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$, and Z the center of G. The superscript ∞ signifies taking the subspace of smooth vectors at infinity. One has a decomposition as unitary $G(\mathbb{A})$-modules

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})) \,\simeq\, L^2_{\mathrm{res}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))\hat{\oplus}L^2_{\mathrm{cusp}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})), \tag{4.7}$$

where the second space on the right is the *space of cusp forms* ([BoJ]), while the first space on the right is spanned by the residual representations, which in this case are precisely the one-dimensional unitary representations π occurring in the discrete spectrum. And $\hat{\oplus}$ signifies taking the Hilbert direct sum.

Using (4.5), (4.6, the complete reducibility of $L^2_{\mathrm{disc}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A}))$, and the fact that this representation is *multiplicity free*, we get

$$V_B \otimes_{\mathbb{Q}} \mathbb{C} \,\simeq\, \hat{\oplus}_\pi \,(H^4(\mathfrak{G}_{\mathbb{C}}, K_\infty; \mathcal{H}(\pi_\infty)) \otimes \pi_f^C), \tag{4.8}$$

where π runs over the irreducible unitary representations of $G(\mathbb{A})$ (up to equivalence) (admitting a C-fixed vector), and $\mathcal{H}_\pi$ denotes the space of smooth vectors of π_∞.

Let $\mathrm{Coh}_{G(\mathbb{R})}$ denote the set of all irreducible unitary representations of $G(\mathbb{R})$ (up to equivalence) with trivial central character and with non-zero $(\mathfrak{G}_{\mathbb{C}}, K)$-cohomology in degree 4, and let $\mathrm{Coh}_G(C)$ denote the set consisting of the π occurring in $L^2_{\mathrm{disc}}(G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})$ such that $\pi_\infty \in\mathrm{Coh}_{G(\mathbb{R})}$ and $\pi_f^C \neq 0$.

By the strong multiplicity one theorem, any cuspidal automorphic representation π is determined by the knowledge of almost all of its local components, in particular by its finite part π_f. The analogous statement about the one-dimensional π is obvious. In view of this one gets, by comparing (4.2) and (4.8) for each $\pi \in\mathrm{Coh}_G(C)$,

$$(V_B(\pi_f) \otimes_{\mathbb{Q}} \mathbb{C})^{\dim \pi_f^C} \,\simeq\, H^4(\mathfrak{G}_{\mathbb{C}}, K_\infty; \mathcal{H}(\pi_\infty) \otimes \pi_f^C. \tag{4.9}$$

The Galois conjugate of any cuspidal , resp. one dimensional, π in $\mathrm{Coh}_G(C)$ is again cuspidal, resp. one dimensional. For $* = B$ or *et*, let us set

$$V_*^{\mathrm{res}} \,\simeq\, \oplus_{\pi\in\mathrm{Coh}_G(C),\dim(\pi)=1}\, V_*(\pi_f)^{m(\pi_f, C)}, \tag{4.10}$$

and

$$V_*^{\mathrm{cusp}} \simeq \oplus_{\pi \in \mathrm{Coh}_G(C), \pi \text{ cuspidal}} V_*(\pi_f)^{m(\pi_f, C)}.$$

Proposition 4.11 *Let $\pi \in Coh_G(C)$ with $dim(\pi) = 1$. Then the Hodge and Tate classes in $V^{\mathrm{res}}(2)$ are algebraic, in fact represented by Chern classes of vector bundles. And we have, for any number field k,*

$$r_{\mathrm{alg}}(S(\pi_f)) = r_{\mathrm{an}}(S(\pi_f)).$$

Proof. Recall from section 2 that S_C is a finite union of connected Hilbert modular varieties S_Γ, whose complex points are quotients of $\mathcal{H}_\pm^4$ by a congruence subgroup Γ of $G(\mathbb{R})$. Restriction of any cohomology class in V_B^{res} to S_Γ comes from the continuous cohomology of $G(\mathbb{R})$, i.e., represented by a $G(\mathbb{R})$-invariant differential form on $\mathcal{H}_\pm^4$. If $z = (z_1, z_2, z_3, z_4)$, $z_j = x_j + iy_j$, represents a point on $\mathcal{H}_\pm^4$, then any such $G(\mathbb{R})$-invariant differential 4-form is spanned by the following forms (with $1 \leq j, r \leq 4$, $j \neq r$):

$$\omega_{j,r} = \eta_j \wedge \eta_r$$

where

$$\eta_j = \frac{dz_j \overline{dz_j}}{y_j^2}.$$

Each η_j is just the volume form on the j-th factor. It is well known that η_j is the Chern class of a line bundle, namely the one given by the tangent bundle on $\mathcal{H}_\pm$. It represents a divisor on $\mathcal{H}_\pm^4$, which descends to one on S_Γ; call it D_j. The intersection of D_j and D_r gives, for $j \neq r$, a codimension 2 cycle $Z_{j,r}$ represented by a class in V_B^{res}. (To be precise, the Chern class lies in $V_B^{\mathrm{res}}(2)$.) The associated Galois representation on V_ℓ^{res} is potentially abelian, and it is easy to match the poles of this L-function (for any k) with the Tate classes, i.,e., the Galois invariants in $V_\ell^{\mathrm{res}}(2)$. Moreover, these Tate cycles, just like the Hodge cycles, are represented by the Chern classes. The Proposition follows. □

So, in order to prove Theorem A' (and hence Theorem A), we may, and we will, concentrate on the *cuspidal case* from here on.

For every ideal $\mathfrak{N}$ of K with prime factorization $\prod_v \mathfrak{P}_v^{f_v}$, denote by $C_1(\mathfrak{N})$ the compact open subgroup of G_f given by

$$C_1(\mathfrak{N}) = \prod_v C_1(\mathfrak{P}_v^{f_v}) \tag{4.12}$$

with

$$C_1(\mathfrak{P}_v^{f_v}) = \{\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \mid c_v, d_v - 1 \in \mathfrak{P}_v^{f_v}\}.$$

One knows that given any cuspidal automorphic representation π, it has a *conductor* $\mathfrak{N}(\pi)$, which is the largest ideal $\mathfrak{N}$ such that the space of π_f has a non-zero vector fixed by $C_1(\mathfrak{N})$. One also knows that the space of $\mathfrak{N}(\pi)$-invariants in π_f is exactly one dimensional. So we get, for any cuspidal π in $\mathrm{Coh}_G(C)$:

$$V_B(\pi_f) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^4(\mathfrak{G}_{\mathbb{C}}, K_\infty; \mathcal{H}(\pi_\infty)) \otimes \pi_f^{C(\pi)}. \tag{4.13}$$

where $C(\pi)$ is shorthand for $C_1(\mathfrak{N}(\pi))$.

It is known that given any cuspidal π contributing to the cohomology in degree 4, we must have

$$\pi_\infty \simeq \mathcal{D}_2^{\otimes 4}, \tag{4.14}$$

where $\mathcal{D}_2$ is the discrete series representation of $\mathrm{GL}(2, \mathbb{R})$ of lowest weight with trivial central character, which contributes to cohomology in degree 1 only. Consequently,

$$H^4(\mathfrak{G}_{\mathbb{C}}, K_\infty; \mathcal{H}(\pi_\infty)) \simeq H^1(\mathfrak{G}l(2, \mathbb{C}), L_\infty; \mathcal{D}_2^\infty)^{\otimes 4}, \tag{4.15}$$

where $L_\infty \simeq \mathrm{SO}(2)\mathbb{R}_+^*$. It is well known that $H^1(\mathfrak{G}l(2, \mathbb{C}), L_\infty; \mathcal{D}_2^\infty)$ is 2-dimensional. Hence it follows that

$$\dim_E V_B(\pi_f) = 16. \tag{4.16}$$

If $V_B(\pi_f)_{\mathbb{C}}^{p,q}$ denotes the Hodge (p,q)-piece of $V_B(\pi_f) \otimes \mathbb{C}$, then we get

$$\mathrm{rk}_{E \otimes \mathbb{C}} V_B(\pi_f)_{\mathbb{C}}^{2,2} = 6. \tag{4.17}$$

Applying Propositions 1.19 and 1.20, we then get (for any number field k)

$$r_{\mathrm{alg},k}(\pi_f) \leq r_{\ell,k}(\pi_f) \leq 6. \tag{4.18}$$

We will see later that for k abelian, and π cuspidal and non-CM), $r_{\ell,k}(\pi_f)$ is at most 2. For π of CM type, however, it could be more than 2 for k abelian, and could be 6 for suitable (non-abelian) k when K is biquadratic.

5 Twisted analogues of Hirzebruch-Zagier cycles

In this section K will denote a totally real number field of degree m and F a subfield. Put $r = [K : F]$. Put

$$G = R_{K/\mathbb{Q}}\mathrm{GL}(2)/K \quad \text{and} \quad H = R_{F/\mathbb{Q}}\mathrm{GL}(2)/F. \tag{5.1}$$

The map h defined in section 2 factors through a map h_0 of $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ into $H_{\mathbb{R}}$. Let S, resp. ${}^H S$, denote the Shimura variety over $\mathbb{Q}$ associated to (G, h), resp. (H, h_0), with Baily-Borel compactifications $S^*, {}^H S^*$ respectively. The natural embedding of H into G leads to an embedding (over $\mathbb{Q}$)

$$\delta : {}^H S \hookrightarrow S,$$

which extends to a map from ${}^H S^*$ into S^*.

Recall that S comes equipped with a G_f-action by right translation R. For any open compact subgroup C of G_f, let

$$p_C : S \to S_C$$

denote the natural projection, which extends to $S^* \to S_C^*$. For any $g \in G_f$, define the corresponding **Hirzebruch-Zagier cycle**, or **HZ-cycle** for short, (relative to H) to be the *algebraic cycle of codimension* $m - r$ given by

$$(5.2) \qquad {}^F Z_{g,C} = p_C(R(g)(\delta({}^H S)))$$

with compactification

$${}^F Z_{g,C}^* = p_C(R(g)(\delta({}^H S^*))).$$

Note that if $x, y \in {}^H S^*$ are in the same orbit under $H_f \cap gCg^{-1}$, which is a compact open subgroup of H_f, then they have the same image in ${}^F Z_{g,C}^*$. Thus one obtains a non-trivial morphism over $\mathbb{Q}$:

$$(5.3) \qquad {}^H S^*_{H_f \cap gCg^{-1}} \to {}^F Z_{g,C}^*.$$

The right translation action of g on S does not descend to a morphism $S_C \to S_C$, but it does define the Hecke correspondence T_g encountered earlier. We can view ${}^F Z_{g,C}$ as the image of the Hilbert modular subvariety ${}^H S_{C \cap H_f}$ under T_g.

These cycles, or rather their classical versions of them, were introduced by Hirzebruch and Zagier ([HZ]) in the case $F = \mathbb{Q}$ and K real quadratic, which were used in [HLR] to prove the Tate conjecture for Hilbert modular surfaces over abelian fields.

Now on to the *twisted versions.* Let μ be any finite order character of (the idele class group of) K of conductor $\mathfrak{c}$. Let $C = C_1(\mathfrak{N})$. Put

$$(5.4) \qquad C[\mu] = C_1(\mathrm{lcm}(\mathfrak{N}, \mathfrak{c}^2))$$

(see (4.12)). Write $\mathfrak{O}_\mathfrak{c}$ for the ring of integers of $F_\mathfrak{c} := \prod_{v|\mathfrak{c}} F_v$, and define a subset of $F_\mathfrak{c}$ by

$$(5.5) \qquad X := \{x = (x_v) \in F_\mathfrak{c} \mid v(x_v) \geq -v(\mathfrak{c}), \forall\, v\}.$$

Let $\tilde{X}$ be a set of representatives in X for X mod $\mathfrak{O}_\mathfrak{c}$, which is a group isomorphic to $\mathfrak{O}_\mathfrak{c}/\mathfrak{c}\mathfrak{O}_\mathfrak{c}$. To each t in $\tilde{X}$, associate the unipotent matrix $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Now recall from section 2 that

$$S_C(\mathbb{C}) = \cup_{j=1}^{h(C)} \Gamma_j \backslash \mathcal{H}_\pm^d,$$

with

$$\Gamma_j = G(\mathbb{Q}) \cap x_j g(\mathbb{R})^+ C x_j^{-1}.$$

Recall that every x in G_f defines a *Hecke correspondence* $T(x)$ of S_C, which does not in general preserve the connected components $S_C^{(j)}$ of S_C. Let $T_j(x)$ denote the restriction of $T(x)$ to $S_C^{(j)}$ when $\det(x) = 1$.

The *twisting correspondence* $R(\mu) \subset S_C \times S_{C[\mu]}$ can now be defined (cf. [MuRa], section 2, for example) as

$$R(\mu) \;=\; \sum_{j=1}^{h(C)} \mu_f(\det(x_j)) R_j(\mu), \tag{5.7}$$

with

$$R_j(\mu) \;=\; \sum_{t \in \tilde{X}} T_j(u_t).$$

It is easy to verify that for all $x \in G_f$,

$$T(x) \circ R(\mu) \;=\; \mu_f(\det(x)) R(\mu) \circ T(x).$$

The twisting correspondence, being algebraic, acts on any cohomology group, Betti or étale, of the fourfold $S = \lim S_C$. The induced operator sends the π_f-component to the $\pi_f \otimes \mu$-component. It may be useful to note that the twisting correspondence $R(\mu)$ is rational over $\mathbb{Q}(\mu_1)$, where μ_1 denotes the restriction of μ to (the ideles of) $\mathbb{Q}$.

Given a H-Z cycle Z on S and a character μ, define the associated *μ-twisted H-Z cycle* $Z(\mu)$ to be the push-forward of Z under $R(\mu)$. It is again algebraic and rational over $\mathbb{Q}(\mu_1)$.

6 Asai L-functions of type (n, d)

Fix $n \geq 1$ and let K/F be an extension of number fields of degree d with Galois closure $\tilde{K}$ (over F). Then $\mathrm{Gal}(\tilde{K}/F)$ acts on $\mathrm{Hom}(K, \mathbb{C})$ and hence on the $\mathbb{C}$-vector space $\mathbb{C}^{\mathrm{Hom}(K,\mathbb{C})} \simeq \mathbb{C}^d$. This in turn induces an action on the group $\mathrm{GL}(n, \mathbb{C})^d$.

Now consider the algebraic F-group

$$G \;=\; R_{K/F}\mathrm{GL}(n)/K, \tag{6.1}$$

where $R_{K/F}$ is the Weil restriction of scalars, so that $G(F) = \mathrm{GL}(n, K)$. Its *dual group* can be taken to be the semidirect product

$${}^L G \;=\; \mathrm{GL}(n, \mathbb{C})^d \rtimes W_F, \tag{6.2}$$

where the absolute Weil group W_F acts via its quotient $W_F/W_{\tilde{K}} \simeq \mathrm{Gal}(\tilde{K}/F)$, the action being the one described above. Then we have the natural representation

$$r_{K/F} : {}^L G \;\to\; \mathrm{GL}((\mathbb{C}^n)^{\otimes d}) \;\simeq\; \mathrm{GL}(n^d, \mathbb{C}), \tag{6.3}$$

given by

$$r_{K/F}((g_\sigma);1)(\otimes_\sigma v_\sigma) \,=\, \otimes_\sigma g_\sigma v_\sigma \tag{6.4}$$

and

$$r_{K/F}((e_\sigma);\tau)(\otimes_\sigma v_\sigma) \,=\, \otimes_\sigma v_{\tau\sigma},$$

for all (g_σ) in $\mathrm{GL}(n,\mathbb{C})^{\mathrm{Hom}(K,\mathbb{C})}$ and $\tau \in W_F$, where σ runs over $\mathrm{Hom}(K,\mathbb{C})$ and e_σ denotes the identity $n\times n$-matrix matrix in the σ-th place.

For any automorphic representation $\pi = \otimes_v' \pi_v$ of $\mathrm{GL}(n,\mathbb{A}_K)$, and for any idele character χ of F, which we may view by class field theory as a character, again denoted χ, of W_F (and hence of ${}^LG$), let $L(s,\pi;r_{K/F}\otimes\chi)$ denote the associated Langlands L-function, which we will call the *χ-twisted Asai L-function of π relative to K/F*, which is of degree n^d over F. It has an Euler product expansion over the places v of F, convergent in a right half plane, with local factors $L_v(s,\pi;r_{K/F}\otimes\chi)$ and $\varepsilon_v(s,\pi;r_{K/F}\otimes\chi)$. If v is a finite place of F such that π_w is unramified at any place w of K above v, then there is a semisimple conjugacy class $A(\pi_v)$ in ${}^LG$ such that

$$L(s,\pi_v;r_{K/F}\otimes\chi_v) \,=\, \det(I-\chi_v(\varpi_v)r_{K/F}(A(\pi_v))Nv^{-s})^{-1}, \tag{6.5}$$

where χ_v denotes the v-component of χ, which is a character of $W_{F_v}\simeq F_v^*$, and ϖ_v denotes the uniformizer at v. Clearly, this L-factor is 1 unless χ_v is also unramified.

In order to describe the local factors at all the places, which was originally done by Langlands at the places which are unramified for the datum, we need some preliminaries. At any place w of K, let W'_{K_w} denote the Weil group W_{K_w} if w is archimedean and $W'_{K_w}\times\mathrm{SL}(2,\mathbb{C})$ if w is non-archimedean. One knows by the *local Langlands conjecture*, established long ago over archimedean fields by Langlands ([La3]), and recently proved for $\mathrm{GL}(n)$ over p-adic fields in the independent works of M. Harris and R.L. Taylor ([Ha-T]) and Henniart ([He]), that π_w is associated to an n-dimensional $\mathbb{C}$-representation σ_w of W'_{K_w}. This association $\pi_w\to\sigma_w$ is functorial for taking contragredients, pairing the central character ω_w of π_w with the determinant of σ_w, such that

$$L(s,\pi_w\times\pi'_w) \,=\, L(s,\sigma_w\otimes\sigma'_w) \tag{6.6}$$

and

$$\varepsilon(s,\pi_w\times\pi'_w) \,=\, \varepsilon(s,\sigma_w\otimes\sigma'_w),$$

for all irreducible admissible representations π'_w of $\mathrm{GL}(m,K_w)$, and for all m-dimensional representations σ'_w, $m\leq n-1$, of W'_{K_w}.

Now let v be a place of F and w a place of K above it. Let $d(w/v)$ denote $[K_w:F_v]$, so that $d=\sum_{w|v} d(w/v)$. Let $M_{K_w}^{F_v}(\sigma_w)$ denote the **tensor induction** (or *multiplicative induction*) of σ_w from W'_{K_w} to W'_{F_v} (see [Cu-R], and also [Mu-P]). It is an $n^{d(w/v)}$-dimensional representation of W'_{F_v}, with the following property:

$$\mathrm{Res}_{K_w}^{F_v}(M_{K_w}^{F_v}(\sigma_w)) \,\simeq\, \otimes_{\tau\in\mathrm{Hom}(K_w,\overline{K}_w)}\sigma_w^\tau, \tag{6.7}$$

where $\tilde{K}_w$ denotes the Galois closure of K_w over F_v. It is not hard to see that the tensor representation on the right of (6.7) extends non-uniquely to a representation, and the key point is that the tensor induction $M_{K_w}^{F_v}(\sigma_w)$ is a *canonical extension.*

Now put

$$As_{K/F}(\sigma)_v \;=\; \otimes_{w|v} M_{K_w}^{F_v}(\sigma_w), \tag{6.8}$$

which is an n^d-dimensional representation of W'_{F_v},

$$L_v(s,\pi; r_{K/F}\otimes\chi) \;=\; L(s, As_{K/F}(\sigma)_v)\otimes\chi_v) \tag{6.9}$$

and

$$\varepsilon_v(s,\pi; r_{K/F}\otimes\chi) \;=\; \varepsilon(s, As_{K/F}(\sigma)_v\otimes\chi_v)).$$

Taking the Euler product of (6.9) over all v, we get the definition of the global χ-twisted Asai L-function $L(s,\pi; r_{K/F}\otimes\chi)$.

Now suppose χ is the restriction of an idele class character $\tilde{\chi}$ of K. (On the Galois side, this corresponds to *transfer.*) Then we have

$$L(s,\pi\otimes\tilde{\chi}; r_{K/F}) \;=\; L(s,\pi; r_{K/F}\otimes\chi). \tag{6.10}$$

One knows nothing in general about the expected properties of these L-functions (for $d>1$) except when $d=2$. Here K/F is a quadratic extension with non-trivial automorphism θ, and if μ is a unitary character of F, then a cuspidal automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_K)$ is said to be *μ-distinguished* ([HLR]) iff the following *μ-period integral* is non-zero for some function f in $\mathcal{V}_\pi$:

$$\mathcal{P}_\mu(f) := \int_{H(F)Z_H(F_\infty)^+\backslash H(\mathbb{A}_F)} \mu(\det(h)f(h)dh,$$

where H denotes $\mathrm{GL}(2)/F$ with center Z_H, and dh is the quotient measure induced by the Haar measure on $H(\mathbb{A}_F)$. When $\mu=1$, we will simply say *distinguished* when such a non-vanishing occurs. One has the following well known result:

Theorem 6.11 *Let K/F be a quadratic extension of number fields, n a positive integer, and π a cuspidal automorphic representation of $GL(n,\mathbb{A}_F)$ with contragredient $\pi^\vee$. Then*

(a) $L(s,\pi; r_{K/F})$ admits a meromorphic continuation to the whole s-plane with a functional equation of the form

$$L(1-s,\pi^\vee; r_{K/F}) \;=\; \varepsilon(s,\pi; r_{K/F})L(s,\pi; r_{K/F}),$$

where $\varepsilon(\pi; r_{K/F})$ is an invertible holomorphic function.

(b) If S is a finite set of places of F containing the archimedean places and the primes where π is ramified, $L^S(s,\pi; r_{K/F})$ has a pole at $s=1$ iff π is distinguished.

Part (a) follows from the work of Shahidi ([Sh]) via the Langlands-Shahidi method, and part (b) was shown by Flicker in [Fℓ] by adapting the Rankin-Selberg method of Jacquet, Piatetski-Shapiro and Shalika ([JPSS]). Note that when $L^S(s,\pi;r_{K/F})$ has a pole at $s=1$, we have $\pi^\vee \simeq \pi\circ\theta$.

7 Zeroing in on the $(2,4)$-case

The main result of this section is the following:

Theorem 7.1 *Let K/F be a quartic extension of number fields such that there is an intermediate field E with $[K:E]=[E:F]=2$. Suppose π is a cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$. Then $L(s,\pi;r_{K/F})$ admits a meromorphic continuation to the whole s-plane with a functional equation of the form*

$$L(1-s,\pi^\vee;r_{K/F}) \,=\, \varepsilon(s,\pi;r_{K/F})L(s,\pi;r_{K/F}),$$

where $\varepsilon(\pi;r_{K/F})$ is an invertible holomorphic function.

Thanks to the identity (6.10), we also get the meromorphic continuation and functional equation of $L(s,\pi;r_{K/F}\otimes\nu)$ for any idele class character ν of F.

When K/F is Galois and π of trivial central character and square-free conductor, it can be shown (see Proposition 8.22 where $F=\mathbb{Q}$) that $L^S(s,\pi;r_{K/F}\otimes\nu)$ admits, for S a finite set of places of F containing the archimedean places and the primes where π is ramified, a pole at $s=1$ iff a suitable twist of π is a base change from a quadratic subfield L containing F. The order of pole is 2 iff a twist of π is a base change from F *and* K/F is biquadratic.

An important ingredient of proof is the following basic result for quadratic extensions, which we established in [Ra3]:

Theorem 7.2 *Let K/E be a quadratic extension of number fields, and let π be a cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$. Then there is an isobaric automorphic representation $As_{K/E}(\pi)$ of $GL(4,\mathbb{A}_E)$ such that*

$$L(s,As_{K/E}(\pi)) \,=\, L(s,\pi;r_{K/E}).$$

One of the steps of the proof of this Theorem in [Ra3] was the integral representation for this Asai L-function when twisted by a cusp form on GL$(2)/E$. Recently, a different proof has been given in [Kr] using instead the Langlands-Shahidi theory of this L-function.

Proof of Theorem 7.1. The simple reason is that the Asai L-functions can be built in stages. To be precise, we have the following:

Proposition 7.3 *Let $K\supset E\supset F$ and π be as in Theorem 7.1, and let $As_{K/E}(\pi)$ be as in Theorem 7.2. Then we have*

$$L(s,\pi;r_{K/F}) \,=\, L(s,As_{K/E}(\pi);r_{E/F}).$$

Proof of Proposition. It suffices to prove the equality of local factors. Fix any place v of F. Let u be a place of E above v, and w a place of K above u. Denote by σ_w the 2-dimensional representation of W'_{K_w} associated to π_w by the local correspondence. Then by the fact that tensor induction can be achieved in steps (cf, [CR]), we get

$$M^{F_v}_{K_w}(\sigma_w) \simeq M^{F_v}_{E_u}(M^{E_u}_{K_w}(\sigma_w)). \tag{7.4}$$

In view of (6.9), this proves immediately the assertion when v is inert in K. Suppose that v is inert or ramified in E with unique divisor u there, but that u splits into w, w' in K. Then $K_w \simeq K_{w'} \simeq E_u$, and by definition (see (6.8), (6.9)),

$$L(s, \pi; r_{K/F}) = L(s, M^{F_v}_{K_w}(\sigma_w) \otimes M^{F_v}_{K_{w'}}(\sigma_w)), \tag{7.5}$$

while by [Ra3],

$$As_{K/E}(\pi_w) \simeq \pi_w \boxtimes \pi_{w'},$$

where $\boxtimes$ denotes the automorphic tensor product on GL(2)×GL(2), constructed in [Ra2]. Since $\pi_w \boxtimes \pi_{w'}$ corresponds to $\sigma_w \otimes \sigma_{w'}$ on the Weil group side, we obtain

$$L(s, As_{K/E}(\pi); r_{E/F}) = L(s, M^{F_v}_{E_u}(\sigma_w \otimes \sigma_{w'})). \tag{7.6}$$

Noting the isomorphism

$$M^{F_v}_{K_w}(\sigma_w) \otimes M^{F_v}_{K_{w'}}(\sigma_w) \simeq M^{F_v}_{E_u}(\sigma_w \otimes \sigma_{w'}), \tag{7.7}$$

we get the assertion of the Proposition when v has a unique divisor u in E, but u splits in K. The remaining cases are similar and are left to the reader.

8 Tate classes and the inequality $\mathbf{r}_{\mathrm{alg}} \leq \mathbf{r}_{\mathrm{an}}$

Let K be a totally real, quartic extension of $\mathbb{Q}$ containing a quadratic subfield F, and C a compact open subgroup of G_f. Suppose $\pi = \pi_\infty \otimes \pi_f$ is a cuspidal automorphic representation of $G(\mathbb{A})$ contributing to $\mathrm{Coh}_G(C)$.

Proposition 8.1 *We have the inequality*

$$r_{\mathrm{alg},k}(S(\pi_f)) \leq r_{\mathrm{an},k}(S(\pi_f)),$$

for any abelian number field k.

In view of Proposition 1.18, this Proposition will be proved once we establish the following

Proposition 8.2 *We have, for any abelian number field k,*

$$r_{\ell,k}(S(\pi_f)) = r_{\mathrm{an},k}(S(\pi_f)).$$

Before beginning the proof, let us note the following fact which will be used later:

Lemma 8.3 *Let $\pi = \pi_\infty \otimes \pi_f$ be a cusp form on $GL(2)/K$ such that π_f admits a non-zero vector fixed by $C_0(\mathfrak{N})$, with $\mathfrak{N}$ square-free.*

(i) If P is a prime dividing the conductor $\mathfrak{c}(\pi)$ of π, then the local component π_P must be an unramified twist of the Steinberg representation of $GL(2, F_P)$.

(ii) If π is moreover of CM type, it must be unramified at every finite place.

Proof. By assumption, $\mathfrak{c}(\pi)$ divides $\mathfrak{N}$; then so does the conductor $\mathfrak{c}(\omega)$ of the central character ω. Suppose ω is ramified. Then if x is a new vector in the space of π_f, the group $C_0(\mathfrak{c}(\omega))$ will act on x by a *non-trivial* character determined by ω, which is trivial on $C_1(\mathfrak{c}(\omega))$. Hence $C_0(\mathfrak{N})$ cannot act trivially on any non-zero vector in π_f, contradiction! So ω must be unramified.

Suppose P is a prime divisor of $\mathfrak{c}(\pi)$. If π_P were supercuspidal, then P^2 would divide the conductor (see [Ge], p.73), so the square-freeness assumption forces π_P to be special or a ramified principal series. In the latter case, it is defined by two local characters μ, ν such that $\mu\nu = \omega_P$. Since ω_P is unramified, $c(\pi_P) = c(\mu)c(\nu) = c(\mu^2)$ will be divisible by P^2, and so this cannot happen. Now part (i) holds because the only special representation having conductor P (cf. *loc. cit.*) is an unramified twist of the Steinberg representation.

When π is dihedral, i.e., when it is automorphically induced by a character Ψ of a quadratic extension M/K, its base change to M becomes Eisensteinian and so no local component π_P can be special. So (i) implies (ii).

□

Recall that the ℓ-adic representation $V_\ell(\pi_f)$ is free of rank 16 over $E \otimes \mathbb{Q}_\ell$, where E is the field of coefficients of π_f. Fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_\ell$. Then we have

$$V_\ell(\pi_f) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \simeq \oplus_{\sigma \in \mathrm{Hom}(E, \overline{\mathbb{Q}})} \overline{V}_\ell(\pi_f^\sigma),$$

where each $\overline{V}_\ell(\pi_f^\sigma)$ is a 16-dimensional $\overline{\mathbb{Q}}_\ell$-representation of $\mathcal{G}_\mathbb{Q}$.

We need to show that for any cuspidal $\pi \in \mathrm{Coh}_G(C)$, every Tate class in $\overline{V}_\ell(\pi_f)$, i.e, a class in $\overline{V}_\ell(\pi_f)(2)$ fixed by $\mathcal{G}_\mathbb{Q}$, contributes to a pole of the L-function of $\overline{V}_\ell(\pi_f)$. The first object is to analyze the structure of $\overline{V}_\ell(\pi_f)$.

Proposition 8.4 *Let $\tilde{K}$ denote the Galois closure of K in $\overline{\mathbb{Q}}$. There exists a 2-dimensional $\overline{\mathbb{Q}}_\ell$-representation $W_\ell(\pi_f)$ of $\mathcal{G}_K$ such that as $\mathcal{G}_{\tilde{K}}$-modules*

$$\overline{V}_\ell(\pi_f) \simeq \otimes_{\tau \in \mathrm{Hom}(K, \overline{\mathbb{Q}})} W_\ell(\pi_f)^{[\tau]},$$

where $X^{[\tau]}$ denotes, for any representation X of $\mathcal{G}_{\tilde{K}}$, the τ-twisted representation $\alpha \to \tau \circ X(\alpha) \circ \tau^{-1}$.

Proof By a theorem of R.L. Taylor ([Ta1,2]), proved independently by D, Blasius and J. Rogawski ([Bℓ-Ro], there is a 2-dimensional representation

$W_\ell(\pi_f)$ of $\mathcal{G}_K$ such that for S a finite set of places of K containing the archimedean places, the places where π_f or K is ramified, and the ones which divide ℓ,

$$L^S(s,\pi) \;=\; L^S(s+1/2, W_\ell(\pi_f)). \tag{8.5}$$

Since K is a quartic extension of $\mathbb{Q}$ containing a quadratic subfield, $\tilde{K}$ is either K or is a quadratic extension of K. So it makes sense to speak about the base change $\pi_{\tilde{K}}$ of π to $\mathrm{GL}(2)/\tilde{K}$. It follows that for every embedding τ of K in $\overline{\mathbb{Q}}$, and any extension $\tilde{\tau}$ of τ as an automorphism of $\tilde{K}$, we have

$$L^{\tilde{S}}(s,\pi_{\tilde{K}}\circ\tau) \;=\; L^{\tilde{S}}(s+1/2, W_\ell(\pi_f)^{[\tau]}).$$

Here $\tilde{S}$ denotes the set of places of $\tilde{K}$ above S along with the places where $\tilde{K}/K$ is ramified.

On the other hand, by definition of the Asai L-function of π (see section 6), one gets immediately that

$$L^S(s,\pi_{\tilde{K}}; r_{K/\mathbb{Q}}) \;=\; L^S(s, \prod_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}})} \pi_{\tilde{K}}\circ\tau), \tag{8.6}$$

where the formally defined Euler product on the right identifies, in view of (8.5), with the L-function

$$L^S(s+2, \otimes_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}})} W_\ell(\pi_f)^{[\tau]}).$$

In view of the Tchebotarev density theorem, it now suffices to know that

$$L^S(s+2, \overline{V}_\ell(\pi_f)) \;=\; L^S(s,\pi_{\tilde{K}}; r_{K/\mathbb{Q}}). \tag{8.7}$$

In fact it is enough to know this over $\tilde{K}$. In any case, (8.7) is known by a theorem of Brylinski and Labesse ([BrL]) if we suitably expand S to a bigger finite set. This suffices for our purposes. However, it should be pointed out that the need to expand S is unnecessary thanks to some recent work ([Bℓ]) of D. Blasius; to be precise, he has proved the identity (8.7) with S being just the primes dividing the conductor of π. We are now done with the proof of Proposition 8.3.

□

Fix an embedding σ of E in $\overline{\mathbb{Q}}_\ell$, and write $\overline{V}_\ell(\pi_f)$ for $\overline{V}_\ell(\pi_f^\sigma)$. For any number field k, let $r_{\ell,k}(\pi_f)$, resp. $r_{\mathrm{an},k}(\pi_f)$, denote the dimension of the k-rational Tate classes in $\overline{V}_\ell(\pi_f)$, resp. the order of pole at the $L(s,\overline{V}_\ell(\pi_f)/k)$ at the edge $s=3$.

Proposition 8.8 *If π is of CM type, we have for any k,*

$$r_{\ell,k}(\pi_f) \;=\; r_{\mathrm{an},k}(\pi_f).$$

Proof. If π is of CM type, then $\overline{V}_\ell(\pi_f)$ is given by a representation of the Weil group W_k, and the identity follows from the results of [De2]. Done.

Proposition 8.9 When π is non-CM, all the Tate classes in $\overline{V}_\ell(\pi_f)$ are rational over an abelian number field k, with

$$r_{\ell,k}(\pi_f) \leq 2.$$

Proof. Let $M \supset K$ be a number field. As π is non-CM, the 2-dimensional Galois representation $W_\ell(\pi_f)$ remains irreducible upon restriction to $\mathcal{G}_M$. Now we make the following

Lemma 8.10 *Let π be non-CM. Then there can be a Tate class over M, i.e., the character χ_ℓ^2 can appear in $\overline{V}_\ell(\pi_f)$, viewed as a $\mathcal{G}_M$-module, iff there is a partition*

$$(*) \qquad \mathrm{Hom}(K, \overline{\mathbb{Q}}) = \{1, \theta\} \cup \{\tau, \eta\},$$

and an ℓ-adic character λ_ℓ of $\mathcal{G}_M$ such that

$$\lambda_\ell \subset W_\ell(\pi_f) \otimes W_\ell(\pi_f)^{[\theta]},$$

and

$$\chi_\ell^2 \lambda_\ell^{-1} \subset W_\ell(\pi_f)^{[\tau]} \otimes W_\ell(\pi_f)^{[\eta]}.$$

Proof of Lemma 8.10. By Proposition 8.4, we see that a Tate class exists over M iff

$$\chi_\ell^2 \subset \otimes_{\tau \in \mathrm{Hom}(K,\overline{\mathbb{Q}})} W_\ell(\pi_f)^{[\tau]}.$$

The *if* part of the Lemma is now clear, and let us assume that there is a Tate class over M (to prove the it only if part). We first claim that if $\tau, \beta \in \mathrm{Hom}(K, \overline{\mathbb{Q}})$ with $\alpha \neq \beta$, the $\mathcal{G}_M$-module

$$Y_\ell(\alpha, \beta) \simeq W_\ell(\pi_f)^{[\alpha]} \otimes W_\ell(\pi_f)^{[\beta]}$$

is reducible iff $W_\ell(\pi_f)^{[\alpha]}$ is a twist, by an ℓ-adic character, of $W_\ell(\pi_f)^{[\beta]}$. Indeed, if this fails, we must have a decomposition (over M) of the form

$$Y_\ell(\alpha, \beta) \simeq Z_\ell \oplus Z'_\ell,$$

with Z_ℓ, Z'_ℓ irreducible of dimension 2. Then the exterior square of $Y_\ell(\alpha, \beta)$ contains the line $\Lambda^2(Z_\ell)$. But we also have

$$\Lambda^2(Y_\ell(\alpha, \beta)) \simeq (\mathrm{sym}^2(W_\ell(\pi_f)^{[\alpha]}) \otimes \omega_\ell^\beta) \otimes (\mathrm{sym}^2(W_\ell(\pi_f)^{[\beta]}) \otimes \omega_\ell^\alpha),$$

where $\omega_\ell^\alpha = \omega^\alpha \chi_\ell$ denotes the determinant of $W_\ell(\pi_f)^{[\alpha]}$. But as π is non-CM, $\mathrm{sym}^2(W_\ell(\pi_f)^{[\alpha]})$ is irreducible upon restriction to any open subgroup (such as $\mathcal{G}_M$) of $\mathcal{G}_{K^\alpha}$. The leads to the desired contradiction, and the claim is proved. Now write

$$\mathrm{Hom}(K, \overline{\mathbb{Q}}) = \{1, \theta, \tau, \eta\}.$$

Suppose $Y_\ell(1, \theta)$ is irreducible. Then the existence of the Tate class implies an isomorphism

$$Y_\ell(1, \theta) \simeq Y_\ell(\tau, \eta)^\vee \otimes \chi_\ell^2,$$

and we must have, up to interchanging $\{\tau, \eta\}$, isomorphisms

$$W_\ell(\pi_f) \simeq W_\ell(\pi_f)^{[\tau]^\vee} \otimes \nu_\ell \quad \text{and} \quad W_\ell(\pi_f)^{[\theta]} \simeq W_\ell(\pi_f)^{[\eta]^\vee} \otimes \nu_\ell^{-1}\chi_\ell^2,$$

for a character ν_ℓ. This gives the Lemma in this case, again up to renaming the embeddings of K. So we may assume that

$$W_\ell(\pi_f)^{[\theta]} \simeq W_\ell(\pi_f) \otimes \mu_\ell,$$

for a character μ_ℓ, so that

$$Y_\ell(1, \theta) \simeq \mathrm{sym}^2(W_\ell(\pi_f)) \otimes \mu_\ell \oplus \omega_\ell\mu_\ell.$$

The existence of the Tate class implies that

$$\mathrm{Hom}_{\mathcal{G}_M}(Y_\ell(1, \theta), Y_\ell(\tau, \eta)^\vee \otimes \chi^2) \neq 0,$$

resulting in an isomorphism $W_\ell(\pi_f)^{[\eta]} \simeq W_\ell(\pi_f)^{[\tau]} \otimes \nu_\ell$, for a suitable character ν_ℓ. So we have

$$Y_\ell(\tau, \eta)^\vee \otimes \chi^2) \simeq (\mathrm{sym}^2(W_\ell(\pi_f)^{(\tau]})^\vee \otimes \nu_\ell^{-1}\chi_\ell^2 \oplus (\omega_\ell^\tau \nu_\ell)^{-1}\chi_\ell^2.$$

Then one of the following must happen:

(i) $\omega_\ell\nu_\ell = (\omega_\ell^\tau\nu_\ell)^{-1}\chi_\ell^2$

(ii) $\mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]}) \simeq \mathrm{sym}^2(W_\ell(\pi_f)) \otimes (\omega_\ell^2\mu_\ell\nu_\ell)^{-1}\chi_\ell^2$

There is nothing to prove when (i) holds, so we may assume the identity (ii). So over a finite extension L of M, $\mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]})$ and $\mathrm{sym}^2(W_\ell(\pi_f))$ are isomorphic. Then, as is well known (see [Ra2] for example, though the situation is much simpler here), $W_\ell(\pi_f)^{[\tau]}$ and $W_\ell(\pi_f)$ will be forced to be twists of each other by a character of $\mathcal{G}_L$. Since the determinants of these two modules differ by a finite order character, they become isomorphic over a finite extension L_1 of M. It then follows that

$$W_\ell(\pi_f)^{[\tau]} \simeq W_\ell(\pi_f) \otimes \xi_\ell,$$

for a character ξ_ℓ of $\mathcal{G}_M$. So $\mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]})$ is isomorphic to $\mathrm{sym}^2(W_\ell(\pi_f)) \otimes \xi_\ell^2$. Comparing this with (ii) and remembering that the symmetric squares do not admit any self-twist, we obtain

$$(ii') \qquad \xi_\ell^2\omega_\ell^2\nu_\ell\mu_\ell\chi_\ell^{-2} = 1.$$

A different way to encode the existence of the Tate class (over M) is to note that

$$\mathrm{Hom}_{\mathcal{G}_M}(Y_\ell(1, \eta), Y_\ell(\theta, \tau)^\vee \otimes \chi^2) \neq 0.$$

Since $W_\ell(\pi_f)^{[\eta]} \simeq W_\ell(\pi_f)^{[\tau]} \otimes \nu_\ell \simeq W_\ell(\pi_f) \otimes \xi_\ell\nu_\ell$, we get

$$Y_\ell(1, \eta) \simeq (\mathrm{sym}^2(W_\ell(\pi_f)) \otimes \xi_\ell\nu_\ell) \oplus \omega_\ell\xi_\ell\nu_\ell.$$

And since $W_\ell(\pi_f)^{[\theta]} \simeq W_\ell(\pi_f) \otimes \mu_\ell$,

$$Y_\ell(\theta,\tau)^\vee \otimes \chi_\ell^2 \;\simeq\; (\mathrm{sym}^2(W_\ell(\pi_f)) \otimes (\mu_\ell\xi_\ell\omega_\ell)^{-1}\chi_\ell^2) \oplus (\mu_\ell\xi_\ell\omega_\ell)^{-1}\chi_\ell^2.$$

In view of (ii'), the characters appearing in $Y_\ell(1,\eta)$ and in $Y_\ell(\theta,\tau)^\vee \otimes \chi_\ell^2$ are the same. This proves the Lemma relative to the partition $\{1,\eta\} \cup \{\theta,\tau\}$ and $\lambda_\ell = \omega_\ell\xi_\ell\nu_\ell$. The Lemma follows.

□

Proof of Proposition 8.9 (contd.) As $W_\ell(\pi_f)$, resp. $W_\ell(\pi_f)^{[\theta]}$, is the restriction of a representation of $\mathcal{G}_K$, resp. $\mathcal{G}_{K^\theta}$, λ_ℓ extends to a character of $\mathcal{G}_{\tilde{K}}$, where $\tilde{K}$ is the Galois closure of K. Hence every Tate class over M is already defined over an abelian extension of $\tilde{K}$.

Since $W_\ell(\pi_f)$ is Hodge-Tate by [Bℓ-Ro], so is its conjugate $W_\ell(\pi_f)^{[\theta]}$. Consequently, λ_ℓ is also Hodge-Tate, locally algebraic, and is therefore attached to an algebraic Hecke character. Comparing weights, we see that

$$\lambda_\ell \;=\; \chi_\ell\beta \tag{8.11}$$

for a *finite order character* β.

Since the dual of $W_\ell(\pi_f)$ is its twist by $(\omega\chi_\ell)^{-1}$, the identities (8.10) imply the following:

$$W_\ell(\pi_f)^{[\theta]} \;\simeq\; W_\ell(\pi_f) \otimes \beta\omega^{-1}, \tag{8.12}$$

and

$$W_\ell(\pi_f)^{[\eta]} \;\simeq\; W_\ell(\pi_f)^{[\tau]} \otimes \beta^{-1}\omega^{-\tau}.$$

Consequently, assuming we have a Tate class over M, we may write

(8.13)

$$\overline{V}_\ell(\pi_f) \;\simeq\; \big(\mathrm{sym}^2(W_\ell(\pi_f)) \otimes \beta\omega^{-1} \oplus \beta\chi_\ell\big) \otimes \big(\mathrm{sym}^2(W_\ell(\pi_f)^\tau) \otimes \beta^{-1}\omega^{-\tau} \oplus \beta^{-1}\chi_\ell\big)\,.$$

Since π is not of CM type, the symmetric square of $W_\ell(\pi_f))$ is irreducible. So there can be a *second Tate class* over M (which is not a multiple of the first one) iff we have a non-zero $\mathcal{G}_M$-homomorphism

$$\phi : \chi_\ell^2 \;\to\; \mathrm{sym}^2(W_\ell(\pi_f)) \otimes \mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]}). \tag{8.14}$$

The irreducibility of $\mathrm{sym}^2(W_\ell(\pi_f))$ implies that there can be at most one such ϕ. Consequently, using the fact any Tate class over a number field k remains a Tate class over $M = kK$, we get

$$r_{\ell,k}(\pi_f) \;\leq\; 2. \tag{8.15}$$

Now we show that all the Tate classes are rational over an *abelian* number field (when π is non-CM). If $r_{\ell,\overline{\mathbb{Q}}}(\pi_f) = 1$, then the fact that $\overline{V}_\ell(\pi_f)$ is a representation of $\mathcal{G}_\mathbb{Q}$ implies that for a finite order character ν of $\mathcal{G}_\mathbb{Q}$ which becomes trivial when restricted to $\mathcal{G}_M$, $\nu\chi_\ell$ must be a summand of $\overline{V}_\ell(\pi_f)$. Consequently, the Tate class is defined over the cyclic extension of $\mathbb{Q}$ cut out by ν.

It remains to consider the case $r_{\ell,\overline{\mathbb{Q}}}(\pi_f) = 2$. Here, by the discussion in (a), we must have (8.10) through (8.14). It follows from (8.14) and the irreducibility of $\mathrm{sym}^2(W_\ell(\pi_f))$ that we must have, as $\mathcal{G}_{\tilde{K}}$-modules:

$$\text{(8.16)} \qquad \mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]})) \simeq \mathrm{sym}^2(W_\ell(\pi_f)) \otimes \nu,$$

Then over a finite extension L (where ν becomes trivial), $\mathrm{sym}^2(W_\ell(\pi_f)^{[\tau]}))$ and $\mathrm{sym}^2(W_\ell(\pi_f))$ are isomorphic. Then, as seen in the proof of Lemma 8.10, we must have (as $\mathcal{G}_L$-modules,

$$\text{(8.17)} \qquad W_\ell(\pi_f)^{[\tau]}) \simeq W_\ell(\pi_f) \otimes \mu,$$

for a character μ of $\mathcal{G}_{\tilde{L}}$. Since π is non-CM, there can be no character other than μ occurring in $W_\ell(\pi_f)^\vee \otimes W_\ell(\pi_f)^{[\tau]})$. Hence μ extends to a character of $\mathcal{G}_{\tilde{K}}$, and (8.17) is valid over $\tilde{K}$. Now putting Lemma 8.10 and (8.17) together, we see that conjugation of $W_\ell(\pi_f)$ by the nontrivial automorphism of K/F, call it θ, is equivalent to a twist of $W_\ell(\pi_f)$. Consequently,

$$\text{(8.18)} \qquad W_\ell(\pi_f) \otimes W_\ell(\pi_f)^{[\theta]} \simeq (\mathrm{sym}^2(W_\ell(\pi_f)) \otimes \alpha) \oplus \alpha\chi_\ell,$$

for a finite order character α. It follows that

$$\text{(8.19)} \qquad As_{K/F}(W_\ell(\pi_f)) \simeq \beta_\ell \oplus \varphi\chi_\ell,$$

where β_ℓ is an irreducible 3-dimensional representation of $\mathcal{G}_F$ and φ is a finite order character. Then β_ℓ must be essentially self-dual, meaning that its symmetric square contains a 1-dimensional summand. We claim that there can be no other one-dimensional summand. Indeed, the restriction of $sym^2(\beta_\ell)$ to $\mathcal{G}_K$ is, up to a twist, isomorphic to $sym^2(sym^2(W_\ell(\pi_f)))$, which splits as $\mathrm{sym}^4(W_\ell(\pi_f))$ and a character. Now since $W_\ell(\pi_f)$ is irreducible upon restriction to any open subgroup of $\mathcal{G}_K$, it follows that $sym^4(W_\ell(\pi_f))$ is irreducible, proving the claim. Now if τ denotes the non-trivial automorphism of F, we get from Lemma 8.10 and (8.14) the consequence that $sym^2(W_\ell(\pi_f)^{[\tau]})$ is a twist of $sym^2(W_\ell(\pi_f))$. Putting these together we get the decomposition

$$\text{(8.20)} \qquad As_{K/F}(W_\ell(\pi_f)) \otimes As_{K/F}(W_\ell(\pi_f)^{[\tau]}) \simeq \sigma_\ell \oplus \varphi\varphi^\tau\chi_\ell^2 \oplus \xi\chi_\ell^2,$$

where ξ is a finite order character, and σ_ℓ is a 14-dimensional representation whose irreducible summands are 5-dimensional or 3-dimensional. Recall that the restriction of $As_{K/Q}(W_\ell(\pi_f))$ to $\mathcal{G}_F$ is isomorphic to $As_{K/F}(W_\ell(\pi_f)) \otimes As_{K/F}(W_\ell(\pi_f)^{[\tau]})$. Clearly $\mathrm{Gal}(F/\mathbb{Q})$ permutes the irreducible constituents on the right of (8.20), and for dimension reasons it must preserve the set $\{\varphi\varphi^\tau, \xi\}$. Since $\varphi\varphi^\tau$ is invariant, it extends to a character φ', say, of $\mathcal{G}_\mathbb{Q}$. This means that ξ must also be τ-invariant and must extend to a character φ'', say, of $\mathcal{G}_\mathbb{Q}$. It follows that the Tate classes are all defined over the compositum of the cyclic extensions of $\mathbb{Q}$ cut out by φ' and φ''. Done.

$\square$

We can write

$$r_{\ell,\mathbb{Q}^{\mathrm{ab}}}(\pi_f) \;=\; \sum_{\nu} r_\ell(\pi_f;\nu), \tag{8.21}$$

where ν runs over the finite order characters of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and $r_\ell(\pi_f;\nu)$ denotes the rank of the ν-isotypic subspace $Ta_\ell(\pi_f;\nu)$ of the space of π_f-Tate classes over $\mathbb{Q}^{\mathrm{ab}}$. Of course $r_\ell(\pi_f;1) \;=\; r_{\ell,\mathbb{Q}}(\pi_f)$. Thanks to Proposition 8.9, we know (for π non-CM) that

$$r_\ell(\pi_f;\nu) \;\le\; r_{\ell,\mathbb{Q}^{\mathrm{ab}}}(\pi_f) \;\le 2.$$

Proposition 8.22 *Let $K/\mathbb{Q}$ be Galois, and π non-CM.*

(a) $r_{\ell,\mathbb{Q}^{\mathrm{ab}}}(\pi_f)$ is non-zero iff a twist of π is a base change from a quadratic subextension of K.

(b) $r_{\ell,\mathbb{Q}^{\mathrm{ab}}}(\pi_f)$ equals 2 iff a twist of π is a base change from $\mathbb{Q}$.

(c) The following are equivalent:

(i) $r_\ell(\pi_f;\nu) \;=\; 2$ for some ν.

(ii) A twist of π is a base change from $\mathbb{Q}$, and $K/\mathbb{Q}$ is biquadratic.

Proof. (a) Suppose $r_\ell(\pi_f,\nu) \neq 0$ for a character ν of the absolute Galois group of $\mathbb{Q}$. Let $\tilde{\nu}$ be a character of the idele class group of K which extends the idele class character of $\mathbb{Q}$ attached to ν by class field theory. Then we know that

$$\overline{V}(\pi_f \otimes \tilde{\nu}^{-1}) \;\simeq\; \overline{V}(\pi_f)\otimes \nu^{-1},$$

so that

$$r_\ell(\pi_f,\nu) \neq 0 \;\Leftrightarrow\; r_\ell(\pi_f\otimes\tilde{\nu}) \neq 0. \tag{8.23}$$

So we may assume, by replacing π be $\pi\otimes\tilde{\nu}^{-1}$, that $r_{\ell,\mathbb{Q}}(\pi_f)$ is itself non-zero, i.e., that χ_ℓ^2 appears in the $\mathcal{G}_{\mathbb{Q}}$-module $\overline{V}_\ell(\pi_f)$. Then by Lemma 8.10, there is a partition $\{1,\theta\}\cup\{\tau,\eta\}$ of $\mathrm{Hom}(K,\overline{\mathbb{Q}})$, and an ℓ-adic character λ_ℓ of the form $\beta\chi_\ell$, with β of finite order, such that $(*)$ holds, with $\beta\beta^\tau \;=\; 1$. Then in the unitary normalization,

$$\beta \;\subset\; \pi\boxtimes\pi^\theta. \tag{8.24}$$

We have to show that a twist of π is a base change from a quadratic subfield.

First consider the *biquadratic case*, when $\theta^2 \;=\; \tau^2 \;=\; 1$ and $\eta \;=\; \theta\tau$. Then $\pi\boxtimes\pi^\theta$ is θ-invariant, and so β^θ is contained in $\pi\boxtimes\pi^\theta$. Put $\mu=\beta^{-1}\beta^\theta$. Then

$$\pi^\theta \simeq \pi^\vee\otimes\beta \simeq \pi^\vee\otimes\beta^\theta, \quad \text{so} \quad \pi^\vee\otimes\mu \simeq \pi^\vee. \tag{8.25}$$

Consequently π admits a self-twist by μ^{-1}, which is impossible as we are in the non-CM case, unless $\mu = 1$. Then $\beta = \beta^\theta$. Hence β comes from a character γ, say, of the quadratic subfield F fixed by θ. Then $As_{K/F}(\pi)$ contains γ as an isobaric summand. If $\tilde{\gamma}$ is an idele class character of K restricting to γ, we know that $As_{K/F}(\pi \otimes \tilde{\gamma}^{-1}) \simeq As_{K/F})(\pi) \otimes \nu^{-1}$, which contains 1. So a twist of π is a base change from F.

Now let $K/\mathbb{Q}$ be *cyclic*. If θ has order 2, the argument of the biquadratic case carries through. So let θ have order 4, so that $\tau = \theta^2$ and $\eta = \tau\theta = \theta^3$. Now β occurring in $\pi \boxtimes \pi^\theta$ implies that π^τ is $\pi^\vee \otimes \mu$, with $\mu = \beta^\theta/\beta$. In other words, μ occurs in $\pi \boxtimes \pi^\tau$, and the rest of the argument is similar.

Conversely, suppose a twist of π is a base change from a quadratic subfield F of K. Then $As_{K/F}(\pi)$ contains a character ξ as an isobaric summand. Then the transfer to $\mathbb{Q}$ of the Galois character defined by ξ is necessarily of the form $\chi_\ell^2\nu$ and occurs in the 16-dimensional $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation, which corresponds to $As_{F/\mathbb{Q}}(As_{K/F}(\pi))$. In other words $r_\ell(\pi_f, \nu) \neq 0$. We are done with this part.

(b) Let $r_\ell(\pi_f, \nu) = 2$ for some ν. Let F be the quadratic extension of $\mathbb{Q}$ contained in K, given by part (a), such that a twist of π is the base change of a cusp form π_0 on $\mathrm{GL}(2)/F$ of central character ω_0. Note that the restriction of α to F will necessarily be the non-trivial automorphism of F. We may replace π by the appropriate twist and assume that it is just $\pi_{0,K}$. It is easy to see that

$$As_{K/F}(\pi_{0,K}) \simeq sym^2(\pi_0) \boxplus \omega_0\delta, \tag{8.26}$$

where $\delta = \delta_{K/F}$. Since a second character occurs in the Galois module, there must exist a character φ of F such that

$$sym^2(\pi_0^\alpha) \simeq sym^2(\pi_0^\vee) \otimes \varphi \simeq sym^2(\pi_0) \otimes \varphi\omega_0^{-2}. \tag{8.27}$$

If we put $\xi = \varphi^\alpha/\varphi$, we see that

$$sym^2(\pi_0) \simeq sym^2(\pi_0) \otimes \xi.$$

Comparing central characters, $\xi^3 = 1$. If ξ is non-trivial, π_0 will become dihedral over a cubic extension of F without being so already over F, and this is not possible. So $\xi = 1$, and φ extends to a character of $\mathbb{Q}$. Furthermore, since the corresponding Tate class over $\mathbb{Q}$ forces the restriction of $\varphi\omega_0^{-2}$ to $\mathbb{Q}$ to be trivial. So we may write $\varphi\omega_0^{-2}$ as γ/γ^α, for some character γ of F. Then $sym^2(\pi_0) \otimes \gamma$ is a base change from $\mathbb{Q}$.

We *claim* that π_0^α and π_0 are themselves twist equivalent. Indeed, if $L = F(\gamma)$, the symmetric squares of $\pi_{0,L}^\alpha$ and $\pi_{0,L}$ are equivalent. Thus, as seen above in the proof of Lemma 8.10, we must have $\pi_{0,L}^\alpha \simeq \pi_{0,L} \otimes \psi$, for a character ψ of L. Then π_0^α and π_0 become isomorphic after base change to the extension $R := L(\psi)$, which implies the claim.

So some character β of F appears in $\pi_0 \boxtimes \pi_0^\alpha$. Now the rest of the proof goes as in the proof of part (a) following (8.24).

It remains to prove the converse.

Proposition 8.28 *Let π' be a cusp form on $GL(2)/\mathbb{Q}$ of weight 2 and central character ω', and let K be a quartic totally real extension containing a quadratic subfield F. Then $As_{K/\mathbb{Q}}(\pi'_K)$ is automorphic with the decomposition*

$$As_{K/\mathbb{Q}}(\pi'_K) \simeq sym^4(\pi') \boxplus (sym^2(\pi') \otimes I_F^{\mathbb{Q}}(\delta) \otimes \omega'\epsilon) \boxplus (sym^2(\pi') \otimes \omega'\epsilon) \boxplus \omega'^2 \boxplus \omega'^2\delta_1,$$

where δ is the quadratic character of F associated to K with restriction δ_1 to $\mathbb{Q}$ and ϵ is the quadratic character of $\mathbb{Q}$ attached to $F/\mathbb{Q}$.

It is easy to see that this Proposition concludes the proof of part (b) of Proposition 8.2.2.

Proof of Proposition 8.28. In what follows we will do everything formally and treat all the representations as admissible representations. Once the identity is proved, however, the fact that the right hand side is automorphic (thanks to Kim [K]) will imply the automorphy of the left hand side. To be precise, we do not really need the automorphy, only the ability to understand the behavior of the relevant L-function at the right edge.

An immediate consequence of the Asai representation is the following identity:

$$As_{K/F}(\pi'_K) \boxplus I_K^F(sym^2(\pi'_K)) \simeq sym^2(I_K^F(\pi'_K)). \tag{8.29}$$

Since the Asai construction is compatible with doing it in stages, the obvious analogue of (8.29) furnishes the isomorphism

(8.30)

$$As_{K/\mathbb{Q}}(\pi'_K) \boxplus I_F^{\mathbb{Q}}(sym^2(sym^2(\pi'_F) \boxplus \omega'_F\delta)) \simeq sym^2(I_F^{\mathbb{Q}}(sym^2(\pi'_F)) \boxplus I_F^{\mathbb{Q}}(\omega'_F\delta)).$$

Since $I_F^{\mathbb{Q}}(sym^2(\pi'_F))$ (resp. $I_F^{\mathbb{Q}}(\omega'_F\delta)$) identifies with $sym^2(\pi') \boxtimes (1 \boxplus \epsilon)$ (resp. $\omega' I_F^{\mathbb{Q}}(\delta)$, and since $sym^2(I_F^{\mathbb{Q}}(\delta))$ is just $1 \boxplus \epsilon \boxplus \delta_1$, the right hand side of (8.30) becomes

$$sym^2(sym^2(\pi') \boxtimes (1 \boxplus \epsilon)) \boxplus (\omega'^2 \otimes (1 \boxplus \epsilon \boxplus \delta_1)) \boxplus (sym^2(\pi') \otimes (1 \boxplus \epsilon) \boxtimes \omega I_F^{\mathbb{Q}}(\delta)).$$

Moreover, $I_F^{\mathbb{Q}}(sym^2(sym^2(\pi'_F))$ identifies with

$$sym^2(sym^2(\pi')) \boxplus (sym^2(sym^2(\pi')) \otimes \epsilon) \boxplus \omega'^2 \boxtimes (1 \boxplus \epsilon) \boxplus (sym^2(\pi') \otimes \omega' \otimes I_F^{\mathbb{Q}}(\delta)).$$

But

$$sym^2(sym^2(\pi') \boxtimes (1 \boxplus \epsilon) \simeq (sym^2(sym^2(\pi')) \boxtimes sym^2(1 \boxplus \epsilon)) \boxplus (\Lambda^2(sym^2(\pi')) \boxtimes \Lambda^2(1 \boxplus \epsilon)),$$

which simplifies as

$$sym^2(sym^2(\pi') \boxtimes (1 \boxplus \epsilon \boxplus 1) \boxplus (sym^2(\pi') \otimes \omega') \otimes \epsilon.$$

Consequently,

(8.31)

$$As_{K/\mathbb{Q}}(\pi'_K) \simeq sym^2(sym^2(\pi')) \boxplus (sym^2(\pi') \otimes \omega'\epsilon) \boxplus \omega'^2\delta_0 \boxplus (sym^2(\pi') \boxtimes \epsilon\omega' \boxtimes I_F^{\mathbb{Q}}(\delta)).$$

The Proposition now follows thanks to the identity

$$sym^2(sym^2(\pi')) \simeq sym^4(\pi') \boxplus \omega'^2.$$

Done.

$\square$

Proof of Proposition 8.22 (contd.)
(c) In view of part (b), it suffices to show that if π is a twist of the base change $(\pi_1)_K$ of a weight 2 cusp form π_1 on $\mathrm{GL}(2)/\mathbb{Q}$, $r_\ell(\pi_f;\nu) = 2$ for some ν iff $K/\mathbb{Q}$ is biquadratic. Thanks to Proposition 8.28, it is then enough to show that

$$\delta_1 = 1 \Leftrightarrow K/\mathbb{Q} \quad \text{biquadratic.}$$

By definition, δ_1 is the restriction to $\mathbb{Q}$ of $\delta = \delta_{K/F}$. When K is biquadratic, we have $K = LF$ for a quadratic extension $L/\mathbb{Q}$, and if δ' denotes the quadratic character of $\mathbb{Q}$ attached to L, $\delta = \delta' \circ N_{F/\mathbb{Q}}$. This gives what we want in this case as for any x in $I_\mathbb{Q}/\mathbb{Q}^*$,

$$\delta_1(x) = \delta'(x^2) = 1.$$

So we may suppose that $K/\mathbb{Q}$ is cyclic. Then if ξ is the quartic character of $\mathbb{Q}$ attached to $K/\mathbb{Q}$, $\delta = \xi \circ N_{F/\mathbb{Q}}$, so that on the idele classes of $\mathbb{Q}$, $\delta_1 = \xi^2 \neq 1$. Done.

$\square$

Proof of Proposition 8.2. Thanks to Proposition 8.8, we may assume that π is not of CM type. The proof of part (a) (resp. (b)) of Proposition 8.22 shows in fact that $r_{\mathrm{an},\mathbb{Q}^{\mathrm{ab}}}(\pi_f)$ is ≥ 1 (resp. $= 2$) iff a twist of π is a base change from a quadratic subfield (resp. from $\mathbb{Q}$). The desired equality of $r_{\mathrm{an},\mathbb{Q}^{\mathrm{ab}}}(\pi_f)$ and $r_{\ell,\mathbb{Q}^{\mathrm{ab}}}(\pi_f)$ then follows from (8.26) and Proposition 8.28, modulo the well known fact that for any isobaric automorphic representation Π on any $\mathrm{GL}(n)$, $L(s,\Pi)$ is non-vanishing at the right edge and has no pole there if 1 does not occur in the isobaric decomposition of Π. We are also using here the modularity results of [Ra3] and [K].

$\square$

As noted already, Proposition 8.1 follows from Proposition 8.2. We are now done with this section.

Remark 8.33: Suppose $K/\mathbb{Q}$ is *non-normal* with an intermediate field F, and π a twist of the base change of a form on $\mathrm{GL}(2)/\mathbb{Q}$. Then Proposition 8.28 still holds, and moreover, it is not hard to see that $\delta_1 \neq 1$. So $r_\ell(\pi_f;\nu)$ is never 2 for such a π. On the other hand, there are π in the non-Galois case with $r_{\mathrm{an}}(\pi_f) \neq 0$, but with $\Pi := As_{K/F}(\pi)$ cuspidal. Since $L(s, As_{F/\mathbb{Q}}(\Pi))$ has a pole at $s = 1$, its period integral over $Z(\mathbb{A}_\mathbb{Q})\mathrm{GL}(4,\mathbb{Q})\backslash\mathrm{GL}(4,\mathbb{A}_\mathbb{Q})$ is non-zero. This should mean, by a conjecture of Jacquet, that Π comes from a unitary group in four variables associated to $F/\mathbb{Q}$. It is not clear (to this author) as to how to use it to deduce the Tate conjecture in that case, whence the Galois assumption in the second half of Theorem A.

9 Matching Poles with algebraic cycles

We can define $r_{\rm alg}(\pi_f;\nu)$ in the obvious way, and the results of the previous section says that

$$r_{\rm alg}(\pi_f;\nu) \,\leq\, r_\ell(\pi_f;\nu) \,=\, r_{\rm an}(\pi_f;\nu) \,\leq\, 2. \tag{9.1}$$

Theorem A′ (and hence Theorem A) will follow once we establish the following

Proposition 9.2 *Let K be a quartic, Galois, totally real number field, $C = C_0(\mathfrak{N})$, and π a non-CM cusp form on $GL(2)/K$ of weight 2 contributing to Coh_C. Then for any Dirichlet character ν, we have*

$$r_{\rm alg}(\pi_f;\nu) \,=\, r_{\rm an}(\pi_f;\nu).$$

Proof. Suppose $r_{\rm an}(\pi_f;\nu) = 1$ for some ν. Then by part (a) of Proposition 8.22, there is a quadratic subfield F of K, a cusp form π_0 on $\mathrm{GL}(2)/F$ with central character ω_0, and a character ξ of K, such that

$$\pi \,\simeq\, (\pi_0)_K \otimes \xi.$$

Let ξ_0 denote the restriction of ξ to (the idele classes of) F. Then we get (using (8.26))

$$As_{K/F}(\pi) \,\simeq\, As_{K/F}((\pi_0)_K) \otimes \xi_0 \,\simeq\, (sym^2(\pi_0) \otimes \xi_0) \boxplus \omega_0\xi_0\delta, \tag{9.3}$$

with $\delta = \delta_{K/F}$. If μ_1 denotes the restriction to $\mathbb{Q}$ of $\mu_0 := \omega_0\xi_0\delta$, then it occurs in $As_{K/\mathbb{Q}}(\pi)$. In other words, $\nu = \overline{\mu}_1$ and

$$\mu_0\chi_\ell^2 \,\subset\, \overline{V}_\ell(\pi_f)$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Choose a finite order character μ of K with restriction μ_0 to F. Then $r_{\rm alg}(\pi_f;\overline{\mu}_1) \,=\, r_{\rm alg}(\pi_f \otimes \overline{\mu})$, we need only show that

$$r_{\rm alg}(\pi_f \otimes \overline{\mu}) \,\neq\, 0. \tag{9.4}$$

But (9.3) implies that $L(s, As_{K/F}(\pi \otimes \overline{\mu}))$ has a pole at the right edge, which implies by the residue formula of [HLR],

$$\int\limits_{\mathrm{GL}(2,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(2,\mathbb{A}_F)} \phi(g)\overline{\mu}(\det(g))dg \,\neq\, 0, \tag{9.5}$$

for some function ϕ in the space of π, with Z denoting the center of $\mathrm{GL}(2)$. In other words, the integral of a $(2,2)$-form η_ϕ on the Hilbert modular fourfold defined by ϕ has non-zero $\overline{\mu}$-twisted period over a Hecke translate of the embedded Hilbert modular surface attached to F. Look at the corresponding twisting self-correspondence of the fourfold (see the end of section 5), which defines (for a suitable g_f in G_f) a $\overline{\mu}$-twisted H-Z cycle $Z(\overline{\mu}) = {}^F Z^*_{g_f,C}(\overline{\mu})$ (of codimension 2). And we obtain

$$\int_{Z(\overline{\mu})} \eta_\phi \neq 0,$$

proving (9.4).

So we may assume that $r_{\rm an}(\pi_f;\nu) = 2$, which implies, thanks to part (c) of Proposition 8.22, that $K/\mathbb{Q}$ is biquadratic and that

$$\pi \simeq (\pi')_K \otimes \psi,$$

for a cusp form π' on $\mathrm{GL}(2)/\mathbb{Q}$ of central character ω', and a character ψ of K (with restriction ψ_1 to $\mathbb{Q}$). Applying Proposition 8.28, we get the embedding

$$\psi_1 {\omega'}^2 \chi_\ell^2 \oplus \psi_1 {\omega'}^2 \chi_\ell^2 \subset \overline{V}_\ell(\pi_f). \tag{9.6}$$

Let μ be a character of K with restriction $\mu_1 := \psi_1 {\omega'}^2$ to $\mathbb{Q}$. Then again $\nu = \overline{\mu}_1$, and we need to show that

$$r_{\rm alg}(\pi_f \otimes \overline{\mu}) = 2. \tag{9.7}$$

Choose two quadratic subfields F, E, say, of K. Then $\pi \otimes \overline{\mu}$ is a base change from both F and E. So we get two twisted, codimension 2 algebraic cycles $Z := {}^F Z(\overline{\mu}; g_f)$ and $Z' := {}^E Z^*_{g'_f,C}(\overline{\mu})$ on $\tilde{S}$, for suitable $g_f, g'_f \in G_f$. These are *homologically non-trivial* because the period integrals of suitable $(2,2)$-forms over these cycles are non-zero, the reason being that they arise as the residues of the associated degree four Asai L-functions.

But these two cycles may be proportional in the π_f-component of the cohomology. So we have to replace one of them with a suitable twisted version.

Let θ, α be the automorphisms of K with respective fixed fields F, E. Then the restriction of α to F is the non-trivial automorphism of F. Fix an embedding $w : K \hookrightarrow \mathbb{R}$ and order the archimedean places of K as $(w, \alpha w, \theta w, \alpha\theta w)$. Given any signature distribution $s = (s_1, s_2, s_3, s_4)$, with each s_j being $+$ or $-$, we can define a *real analytic automorphism* τ_s of $K \otimes \mathbb{C} - K \otimes \mathbb{R} \simeq \mathbb{C}^4 - \mathbb{R}^4$ by

$$\tau_s(z_1, z_2, z_3, z_4) = (\tau_{s_1}(z_1), \tau_{s_2}(z_2), \tau_{s_3}(z_3), \tau_{s_4}(z_4)) \tag{9.8}$$

where τ_{s_j} is the identity, resp. complex conjugation, if s_j is $+$, resp. $-$. Each such involution acts on the Hilbert modular fourfold $\tilde{S}_C$ and its rational Betti cohomology. It also commutes with the Hecke action and we obtain a decomposition

$$V_B(\pi_f) \simeq \oplus_{s\in\Sigma} V_B(\pi_f)^s, \tag{9.9}$$

where Σ runs over all the signature distributions, and $V_B(\pi_f)^s$ denotes the s-eigenspace of $V_B(\pi_f)$. Note that Σ forms a group under componentwise multiplication with identity $(+,+,+,+)$. Also, each eigenspace $V_B(\pi_f)^s$ is one-dimensional.

If ξ is a finite order character of K, its component at any archimedean place will be 1 or the *sign character*, and we will define the *signature* $s(\xi)$ of ξ to be $(s(\xi)_1, s(\xi)_2, s(\xi)_3, s(\xi)_4)$, where $s(\xi)_j$ is the sign of ξ_w, resp. $\xi_{\tau w}$, resp. $\xi_{\theta w}$, resp. $\xi_{\theta\tau w}$, for $j = 1$, resp. $j = 2$, $j = 3$, $j = 4$. (At any archimedean place u, the sign of ξ_u is $+$, resp. $-$, if ξ_u is trivial, resp. non-trivial.) It is easy to check the following:

Lemma 9.10 *The twisting correspondence $R(\xi)$ sends any vector in $V_B(\pi_f)^s$ into $V_B(\pi_f \otimes \xi)^{ss(\xi)}$.*

Lemma 9.11 *There exists a finite order character ξ of K such that*

(i) $s(\xi) = (+, +, -, -)$

(ii) $\xi|_E = 1$.

Proof of Lemma 9.11. Pick any finite order character λ of K of signature $(+, +, +, -)$. Then λ^α has signature $(+, +, -, +)$. Put $\xi = \lambda/\lambda^\alpha$. Then $s(\xi) = (+, +, -, -)$. And ξ also satisfies (i) because E is the fixed field of α. Done.

Proof of Proposition 9.2 (contd.) Pick a ξ as in the Lemma. As $\xi|_{\mathbb{Q}} = 1$,

$$r_{\mathrm{an}}(\pi_f \otimes \xi; \nu) = r_{\mathrm{an}}(\pi_f; \nu). \tag{9.12}$$

And since $\pi \simeq (\pi')_K \otimes \mu$, we have

$$As_{K/F}(\pi \otimes \xi\overline{\mu}) \simeq sym^2(\pi'_F) \otimes \xi|_F \boxplus \xi|_F \tag{9.13}$$

and (since $\xi|_E = 1$)

$$As_{K/E}(\pi \otimes \xi\overline{\mu}) \simeq sym^2(\pi'_E) \boxplus 1. \tag{9.14}$$

So $L(s, As_{K/E}(\pi \otimes \xi\overline{\mu}))$ has a simple pole at the edge. (But $L(s, As_{K/F}(\pi \otimes \xi\overline{\mu}))$ has no pole at the right edge as $\xi|_F$ is non-trivial by construction.) Consequently, if we put

$$Z'' := {}^E Z^*_{g''_f, C_1}(\overline{\mu}\xi),$$

we have (for suitable $g''_f \in G_f$ and compact open C_1,

$$\int_{Z''} \eta_\phi \neq 0, \tag{9.15}$$

for some ϕ in the space of π.

Lemma 9.16 *The space spanned by the classes of Z, Z' and Z'' in $V_B(\pi_f)$ has dimension* 2.

Proof. As we have seen, these three cycles are all homologically non-trivial in the π_f-component. If $[Z]$, $[Z']$ are not proportional, there is nothing to prove. So we may suppose that they span a line L, say, in $V_B(\pi_f)$. Since Z (resp. Z') comes from F (resp. E), $[Z]$ (resp. $[Z']$) has a non-zero component in $V_B(\pi_f)^s$

for some $s = (s_1, s_2, s_3, s_4)$ iff $s_1 = s_3$ and $s_2 = s_3$ (resp. $s_1 = s_2$ and $s_3 = s_4$). So L lies in $Y := V_B(\pi_f)^{(+,+,+,+)} \oplus V_B(\pi_f)^{(-,-,-,-)}$. Now since ξ has signature $(+,+,-,-)$, $[Z'']$ cannot lie in Y, thanks to Lemma 9.10. Done.

$\square$

The Proposition is now proved, as is Theorem A′, which implies Theorem A.

$\square$

Remark 9.17: The referee has suggested the following clever, alternate approach to proving that the algebraic cycles in $V_B(\pi_f)$ span a plane: The fact that the period integral over Z (resp. Z') is non-zero implies that it gives rise to an $\mathrm{SL}(2, \mathbb{A}_F)$-invariant (resp. $\mathrm{SL}(2, \mathbb{A}_E)$-invariant) linear form on the space of π. So if the homology classes of Z and Z' are proportional, then we would get an $\mathrm{SL}(2, \mathbb{A}_K)$-invariant linear form on the space of π, which is impossible as π has no non-trivial intertwining map into an abelian representation. Hence $[Z], [Z']$ are not proportional. Since they are non-trivial, they must span a 2-dimensional vector subspace of $V_B(\pi_f)$.

We will now give a *justification of the remark coming right after the statement of Theorem A* in the Introduction. We have to show that there are codimension 2 cycles which are not intersections of divisors. We *claim* that this is *always* the case for classes coming from cuspidal cohomology, i.e., that the intersection of divisors will never hit $V_B(\pi_f)$ for any cuspidal π_f. Indeed, we can ignore those divisors supported on the boundary, and for the others, the representations of $G(\mathbb{R})$ contributing to $IH^2(S_C^*(\mathbb{C}), \mathbb{C})$ are one-dimensional, and the tensor product of two such cannot contain any discrete series representation, which is what contributes to $H^4_{\mathrm{cusp}} \subset IH^4(S_C^*(\mathbb{C}), \mathbb{C})$. Done.

$\square$

Remark 9.18 It is time to make some comments on the *CM situation* and raise a few questions. Let π be a cohomological cusp form of CM type, say of trivial central character. Then π is the automorphic induction of the unitary version Ψ^u of a weight one Hecke character Ψ of a CM quadratic extension M of K. Let us take K to be Galois over $\mathbb{Q}$. If $\tilde{M}$ denotes the Galois closure of M, then the restriction of $V_\ell(\pi_f)$ to $\mathcal{G}_{\tilde{M}}$ splits as a direct sum of sixteen 1-dimensional representations, six of which could be of Tate type. For any of these 1-dimensionals to define a Tate class (of codimension 2) over an extension field, however, the infinity type of the corresponding Hecke character must be that of the square of norm (or its inverse depending on the normalization). Looking at the different possible CM types one sees that this is possible only if K contains a quadratic subfield, say F, such that M/F is biquadratic. Let us analyze this case when π is a base change to K of a CM cusp form π_0 of F, with π_0 being defined by a weight one Hecke character φ of a CM quadratic extension L of F so that $M = LK$. Let δ (resp. ν) denote the quadratic character of F attached to K (resp. L), and let φ_0 be the restriction of φ to F (which corresponds to the transfer of the Galois character defined by φ). Then π_0 has central character

$\varphi_0\nu$ and its symmetric square is $I_L^F((\varphi^u)^2) \boxplus \varphi_0^u$. Denote by θ the element of $\mathrm{Gal}(\tilde{M}/\mathbb{Q})$ which restricts to the non-trivial automorphism of K/F. Appealing to (8.26) and the compatibility of tensor induction in stages, we get the following decomposition over F:

$$V_\ell(\pi_f)_F \simeq \mathrm{Ind}_L^F(\varphi^2) \otimes \mathrm{Ind}_{L^\theta}^F((\varphi^\theta)^2) \oplus \beta_\ell \oplus \varphi_0\varphi_0^\theta(\nu\nu^\theta \oplus 1 \oplus \nu\delta \oplus \nu^\theta\delta),$$

where β_ℓ is a sum of irreducible 2-dimensional representations. If $L \neq L^\theta$, $\nu\delta \oplus \nu^\theta\delta$ extends to an irreducible of $\mathcal{G}_\mathbb{Q}$, and moreover, $\mathrm{Ind}_L^F(\varphi^2) \otimes \mathrm{Ind}_{L^\theta}^F((\varphi^\theta)^2)$ is irreducible (already as a $\mathcal{G}_F$-module). It follows that $r_\ell(\pi_f;\mu)$ is at most 1 for any Dirichlet character μ, and in this case we can account for all the Tate classes by algebraic cycles coming from suitable twists of embedded Hilbert modular surfaces. So assume that $L = L^\theta$. Then $\nu\nu^\theta = \nu^2 = 1$, and $r_\ell(\pi_f;\mu)$ is at least 2 for a suitable μ. *How is one to account for these Tate classes when $K/\mathbb{Q}$ is cyclic?* Even in the biquadratic case, we can get two independent algebraic classes only if (a twist of) π is a base change all the way from $\mathbb{Q}$, and things become difficult as seen below.

Now suppose π is a base change from $\mathbb{Q}$, i.e., when $\pi = \pi'_K$ with $\pi' = I_E^\mathbb{Q}(\psi^u)$ for a weight one Hecke character ψ of an imaginary quadratic field E. Let ϵ, resp. ν, denote the quadratic character of $\mathbb{Q}$ attached to F, resp. E, and let ψ' denote the transfer of ψ to $\mathcal{G}_\mathbb{Q}$, which is a finite order character times χ_ℓ. As above let δ be the quadratic character of F attached to K, with $K/\mathbb{Q}$ Galois. Then we have $\mathrm{Ind}_F^\mathbb{Q}(\delta) = \gamma \oplus \gamma\epsilon$, where γ is quartic, resp. quadratic, when K is cyclic, resp. biquadratic. Then, using Proposition 8.28, we see that $V_\ell(\pi_f)$ is isomorphic to the following:

$$\mathrm{Ind}_E^\mathbb{Q}(\psi^4) \oplus \Big(\mathrm{Ind}_E^\mathbb{Q}(\psi^2) \otimes (\psi' \oplus \gamma\psi'\epsilon \oplus \gamma\psi' \oplus \psi'\epsilon)\Big) \oplus {\psi'}^2\,(2.\underline{1} \oplus \nu\gamma\epsilon \oplus \nu\epsilon \oplus \nu\gamma \oplus \delta_1)\,,$$

where δ_1 is the transfer of δ to $\mathbb{Q}$. If we write ${\psi'}^2 = \mu\chi_\ell^2$, we then see that $r_\ell(\pi_f;\mu)$ is 2 when K is cyclic, and 3 when K is biquadratic. So in either case we get an *exotic Tate class over an abelian extension of* $\mathbb{Q}$! Note that when K is biquadratic, since M is the compositum of K with E it is a *triquadratic field*, i.e, $M/\mathbb{Q}$ is Galois with group $(\mathbb{Z}/2)^3$. A natural question here (because of Lemma 8.3 (ii)) is this: *Can such a $\pi = (\pi')_K$* (of CM type) *be everywhere unramified with trivial character?*

Finally suppose we are in the triquadratic case, with Ψ a weight one Hecke character of M and $\pi = I_M^K(\Psi^u)$ *not necessarily a base change from anywhere.* We want to point to an interesting Tate class. (There are three such, up to complex conjugation.) Let $\mathrm{Gal}(M/\mathbb{Q})$ be generated by (quadratic elements) ρ, θ, τ, with K be the fixed field of (complex conjugation) ρ and θ restricting to the non-trivial automorphism of K/F. Put

$$\Phi := \{1, \rho\theta, \rho\tau, \theta\tau\},$$

which is a CM type and a subgroup of $\mathrm{Gal}(M/\mathbb{Q})$. If we put

$$\xi = \Psi\Psi^{\rho\theta}\Psi^{\rho\tau}\Psi^{\theta\tau},$$

then it is of Tate type and extends to χ_ℓ^2 times a finite order character λ, say, of $\mathcal{G}_E$, where E is the imaginary quadratic field fixed by Φ. Note that the Tate class is defined over the abelian extension of E cut out by λ, and it is defined over an abelian extension of $\mathbb{Q}$ iff λ extends to a Dirichlet character of $\mathbb{Q}$ (which happens iff $\xi = \xi^\rho$). By starting with a suitable weight 1 Hecke character of an imaginary quadratic field and pulling back to M by norm, it appears that both cases can occur. D. Rohrlich has pointed out to the author that one can construct an *everywhere unramified* weight one Hecke character Ψ even when M/K is unramified. *Can the corresponding λ be extendable to $\mathbb{Q}$?*

10 Algebraicity of some Hodge classes

Now we will prove Theorem B. In view of Propositions 3.6 and 4.11, it suffices to prove the algebraicity of the Hodge classes in $V_B(\pi_f)$ for π cuspidal. By hypothesis, the level $\mathfrak{N} \neq \mathfrak{O}_K$ is square-free. By part (ii) of Lemma 8.3, if π is CM, it will already contrbute at full level. Put $C = C_0(\mathfrak{N})$, $C(1) = C_0(\mathfrak{O}_K)$, $X = \tilde{S}_C$ and $X(1) = \tilde{S}_{C(1)}$, with the associated ramified cover $X \to X(1)$. Since we are interested only in the Hodge classes on $X_\mathbb{C}$ which are not pull-backs of classes in $X(1)_\mathbb{C}$, every relevant π will be non-CM and its conductor will be divisible exactly once by a prime P. Then, by part (i) of Lemma 8.3, the local component π_P must be an unramified twist of the Steinberg representation. Fix such a pair (π, P). Choose a quaternion division algebra B over K which is ramified at P and at three of the infinite places, but nowhere else. Then by the Jacquet-Langlands correspondence, π transfers to a cusp form π' on B^*/K, and π' evidently contributes to the H^1 of the Shimura curve C_B associated to B (at the corresponding level). This curve is proper and smooth over K, and one knows how to associate (by the Eichler-Shimura theory, for example) an abelian variety quotient A_π of the Jacobian of C_B with the same L-function as that of π. This leads to the following identity (for a suitable finite set S of places containing the archimedean ones):

$$L^S(s, V_\ell(\pi_f)) \simeq L^S(s, U_\ell(\pi_f)), \tag{9.1}$$

where

$$U_\ell(\pi_f) = \otimes_\tau H^1_\ell(A^\tau_\pi \times_{K^\tau} \overline{\mathbb{Q}}, \mathbb{Q}_\ell), \tag{10.2}$$

where τ runs over $\mathrm{Hom}(K, \mathbb{R})$ and A^τ_π denotes the τ-conjugate abelian variety. By Tchebotarev this leads to the isomorphism as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules:

$$V_\ell(\pi_f)^{ss} \simeq U_\ell(\pi_f), \tag{10.3}$$

where the superscript *ss* signifies semi-simplification.

We will now accept the truth of the following result, a special case of a more general theorem proved in a joint work with V. Kumar Murty, which will be published elsewhere:

Theorem 10.4 *We have*

$$V_B(\pi_f) \simeq \otimes_\tau H^1_B(A^\tau_\pi(\mathbb{C}), \mathbb{Q}),$$

the isomorphism being one of rational Hodge structures.

Note that the right hand sides of the isomorphisms given by (10.3) and Theorem 10.4 are subspaces of the cohomology (in degree 4) of the product abelian variety $\prod_\tau A^\tau_\pi$.

By the main theorem of [DMOS] (see section 6 of chapter I), the $\mathbb{Q}_\ell$-points of the Mumford-Tate group $MT(H^1(A))$ of any abelian variety A contains the Zariski closure $G(H^1_\ell(A))$ of the image of Galois in the ℓ-adic representation. By (10.1), (10.2), the same holds in our case, i.e., $MT(V_B(\pi_f))(\mathbb{Q}_\ell) \supset G(V_\ell(\pi_f))$. The Hodge cycles are none other than the cohomology classes which are fixed by the Mumford-Tate group; every such class is also fixed by Galois and hence gives rise to a Tate class. Hence the Tate conjecture for the π_f-component of $\tilde{S}_C$ implies the Hodge conjecture for this piece. So Theorem B follows from Theorem A, and we are done.

11 Refinements and errata for the papers [Ra2,3]

In [Ra2] the automorphic tensor product from GL(2)×GL(2) →GL(4) was established. In particular one obtained, given cuspidal automorphic representations π, π' of $\mathrm{GL}(2, \mathbb{A}_F)$, F any number field, an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi') \tag{11.1}$$

and

$$\varepsilon(s, \pi \boxtimes \pi') = \varepsilon(s, \pi \times \pi'),$$

where the functions on the right are respectively the Rankin-Selberg L-function and the associated ε-factor occurring in its functional equation.

In [Ra2], Theorem M, one also finds a *cuspidality criterion*, which is not sufficiently sharp when π and π' are both *dihedral*. Here is the best possible statement, which should be used to supplement Theorem M of [Ra2]:

Theorem 11.2 *Let π, π' be cuspidal automorphic representations of $GL(2, \mathbb{A}_F)$. When exactly one of them is dihedral, $\pi \boxtimes \pi'$ is cuspidal, and when they are both non-dihedral, $\pi \boxtimes \pi'$ is not cuspidal iff π' is of the form $\pi \otimes \chi$ for an idele class character χ of F.*

Suppose π, π' are both dihedral, so that we have

$$\pi = I^F_E(\nu) \quad \text{and} \quad \pi' = I^F_K(\mu),$$

where E, K are quadratic extensions of F, and ν (resp. ν) is an idele class character of E (resp. K). Then the following are equivalent:

(α) $\pi \boxtimes \pi'$ is not cuspidal

(β) $E = K$

And when one of these equivalent conditions holds, we have

$$\pi \boxtimes \pi' \simeq I_K^F(\nu\mu) \boxplus I_K^F(\nu\mu^\tau),$$

where τ is the non-trivial automorphism of K/F.

Proof of Theorem 11.2 When either π or π' is non-dihedral, Theorem M of [Ra2] gives what is asserted here. So assume that both of the representations are dihedral; write them in the form above. By Theorem M of [Ra2], $\pi \boxtimes \pi'$ is non-dihedral iff the base change π_K is cuspidal and admits a self-twist by $\lambda := (\mu \circ \tau)\mu^{-1}$, which is non-trivial on account of π' being cuspidal; we see that λ must be quadratic by comparing the central characters. Hence π_K is cuspidal and dihedral when $\pi \boxtimes \pi'$ is not cuspidal. It suffices to show that π is itself dihedral in such a case. If L is the quadratic extension of K cut out by λ, $\pi_K = I_L^K(\nu_L)$ for a character ν' of L. Then π_K corresponds to the (irreducible) 2-dimensional representation $\rho^K = \mathrm{Ind}_L^K(\nu')$ of the Weil group W_K. Since ρ^K is invariant under $\mathrm{Aut}(K/F)$, it extends to an irreducible 2-dimensional representation ρ of W_F. Since ρ is of solvable type (and 2-dimensional), it corresponds to a cusp form π_1 of $\mathrm{GL}(2, \mathbb{A}_F)$ by Langlands and Tunnell. Moreover, since π_1 and π have the same base change to K, they must differ by at most the quadratic character δ of F corresponding to K/F. Thus, after replacing ρ by $\rho \otimes \delta$ if necessary, we may assume that ρ and π correspond. It then suffices to show that ρ is dihedral. Let τ be the non-trivial automorphism of K/F, extended to an automorphism, again denoted τ, of the Galois closure $\tilde{L}$ of L over F. Since ρ^K is associated to L/K, and since we have

$$\mathrm{End}(\rho_K) = \rho^K \otimes (\rho^K)^\vee = \mathrm{Ad}(\rho^K) \oplus 1,$$

its adjoint square $\mathrm{Ad}(\rho)$ contains the quadratic character λ. There are two cases to consider:

(i) $\lambda = \lambda^\tau$: In this case λ extends to a character of W_F, and so $\mathrm{Ad}(\rho)$ splits as a sum of such an extension plus a 2-dimensional representation. This forces τ to be dihedral. To elaborate, since τ is irreducible, $\mathrm{Ad}(\rho)$ cannot contain 1 by Schur's lemma. Since it is self-dual, if it contains a 1-dimensional summand, it must contain a quadratic character, say ϵ. Then, if M/F is the quadratic extension cut out by ϵ, and if ρ_M denotes the restriction ρ to W_M, then $\mathrm{Ad}(\rho_M)$ contains 1, hence ρ_M is reducible and ρ must be induced from M.

(ii) $\lambda \neq \lambda^\tau$: Since $\mathrm{Ad}(\rho^K)$ extends to W_F, it is invariant under τ, and so we are forced to have the decomposition

$$\mathrm{Ad}(\rho^K) = \lambda \oplus \lambda^\rho \oplus \lambda\lambda^\tau.$$

Now the character $\lambda\lambda^\tau$ extends to W_F, and $\mathrm{Ad}(\rho)$ again contains a one-dimensional summand, implying that ρ is dihedral. Done.

Remark: Erez Lapid has remarked that this can also be proved by appealing to the properties of the symmetric square lifting $\pi \to \mathrm{sym}^2(\pi)$ of Gelbart and Jacquet from GL(2) to GL(3). More precisely, one appeals to the fact that π is dihedral if and only if $\mathrm{sym}^2(\pi)$ is not cuspidal. Moreover, it can be checked that the Gelbart-Jacquet lift is compatible with base change, and that quadratic base change preserves cuspidality for GL(n) for any *odd* n. Lapid has had occasion to use Theorem 11.2 in his elegant article [Lap].

We now move to section 3 of [Ra2], where a key lemma is proved as a preliminary step to achieving **boundedness in vertical strips** for the triple product L-functions. In fact the key lemma there is very general and could be of use in various other situations, giving a bound for the sup norm of an eigenfunction of an elliptic operator (such as the Laplacian) in terms of its L^2-norm and the eigenvalue δ; in fact it also applies to functions which are not eigenfunctions. Joseph Shalika recently asked for a clarification of one of the points of the proof given in [Ra2], and this what we will do right now. First let us restate Lemma 3.4.9 as

Lemma 11.3 *Let Ω be a subset of $\mathbb{R}^N$ contained in the unit ball of radius ϵ, Ω' a subset of Ω with non-empty interior such that $\bar{\Omega}' \subset \Omega$, and Δ an elliptic operator of order 2. Then we have the following:*

- *(1) For any integer $r > N/2$, there exists a constant $C_1 > 0$ depending only on ϵ and Δ such that, for all u in the Sobolev space $\mathcal{H}_r(\Omega)$,*

$$||u||_{\infty,\Omega} \leq C_1 \, ||u||_{(2,r);\Omega}.$$

- *(2) For any integer $i \geq 2$, there is a constant $C_i > 0$ depending only on ϵ and Δ, such that for any $u \in \mathcal{H}_i(\Omega)$:*

$$||u||_{(2,i);\Omega'} \leq C_i \left(||u||_{2,\Omega} + ||\Delta u||_{(2,i-2);\Omega}\right).$$

Here $||.||_{(2,r);X}$ denotes, for any r and $X = \Omega, \Omega'$, the rth L^2-derivative on X so that (for $r \geq 0$) $||u||_{(2,r);X}$ equals $\sum_{|\nu|\leq r} ||\partial^\nu u||_{2,X}$, with the ∂^ν denoting distribution derivatives. Clearly, $||u||_{(2,0);X} = ||u||_{2;X}$. We will henceforth suppress the space X in the subscript. Furthermore, in Lemma 3.4.9 of [Ra2], Δ is (at least) implicitly taken to be the Laplacian, which is not necessary.

The proof of part (1) is as in [Ra2], page 71. Part (2) is proved by induction on $i \geq 2$. Shalika wanted to know the proof for the **starting point** $i = 2$, and here it is. Choose a nice cut-off function ψ with compact support in Ω such that $\psi u = v$ on Ω', with v in $\mathcal{H}_2^0(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $\mathcal{H}_r(\Omega)$. By Theorem 6.2.8 (chapter 6, page 210 of Folland's book [Fo] (second edition –1995), we have (for all $s \in \mathbb{R}$ and $v \in \mathcal{H}_s^0(\Omega')$),

$$||v||_{(2,s)} \leq C \left(||Lu||_{(2,s-k)} + ||v||_{(2,s-1)}\right),$$

where L is an elliptic operator of degree k defined on the closure of Ω' and C a constant. Applying this with $s = 2, L = \Delta, k = 2$ and $v = \psi u$, we get

$$(*) \qquad ||\psi u||_{(2,2)} \leq C \left(||\Delta(\psi u)||_2 + ||\psi u||_{(2,-1)} \right).$$

We can evidently bound the left hand side of $(*)$ from below by a constant times $||u||_{(2,1)}$, and also bound $||\psi u||_{2,-1}$ from above by a constant times $||u||_2$. So to establish part (2) for $n = 2$, we need to bound $||\Delta(\psi u)||_2$ from above. By the Leibnitz rule we can write $\Delta(\psi u)$ as $\Delta(\psi)u + \psi\Delta(u)$ plus a sum of terms of the form $L_1(\Psi)L_2(u)$, where L_1, L_2 are differential operators of order 1. (When Δ is the Laplacian, L_i is the gradient ∇.) Since we can control $L_1(\Psi)$ well, we need only to bound $||L_2(u)||_2$ for any first order operator L_2, or equivalently, we need to bound $||u||_{(2,-1)}$. But this can be bounded by $||u||_2$ since by the definition of these spaces using Fourier transform, $||.||_s \leq ||.||_t$ for all (s,t) with $s \leq t$ (cf. [Fo], page 192). Done.

Here are some **typos in this section (3.4) of [Ra2]** which should be fixed as follows, where $A \to B$ means *change A to B* and $p.x, \ell.y$ means page x, line y, with the understanding that negative line numbers are to be counted from the bottom of the page:

$p.70, \ell. - 13$ (Lemma 3.4.8): *and λ_s such that $\to$ such that for all f*

$p.71, \ell.9$ (part (2) of Lemma 3.4.9): $||u||_{2,\Omega'} \to ||u||_{2,\Omega}$

$p.71, \ell. - 9$: $||\Delta u||_{(2,i-2)} \to ||\Delta u||_{(2,i-2)}$

$p.71, \ell. - 10$: $||\Delta u||_{(2,i-1)} \to ||\Delta u||_{(2,i-1)}$

$p.72, \ell.7$: $\Delta \to$ the Casimir operator

Further, in the ensuing bound (on page 72 of [Ra2]) of the Arthur truncation of $E(f_s)$, it should be noted that the Eisenstein series has a fixed K-type.

Now let K/F be a quadratic extension of number fields. In [Ra3] the **Asai transfer** $\pi \to As(\pi)$ of isobaric automorphic forms from $\mathrm{GL}(2)/K$ to $\mathrm{GL}(4)/F$ was achieved. The **sharp cuspidality criterion** above (Theorem 11.2) has the following **Asai analogue** in the dihedral case, and this could profitably be used to replace part (c) of Theorem 1.4 of [Ra3]:

Theorem 11.4 *Let K/F be a quadratic extension of number fields with nontrivial automorphism θ. Let π be a dihedral cusp form on $GL(2)/K$, i.e., associated to a representation $\sigma = I_M^K(\chi)$ of W_K for a character χ of a quadratic extension M of K. Then the following are equivalent:*

(α) $As(\pi)$ is cuspidal

(β) M is not Galois over F

And when one of these equivalent conditions fails to hold, there exist isobaric automorphic representations η, η' of $GL(2, \mathbb{A}_F)$ such that

$$As(\pi) \simeq \eta \boxplus \eta',$$

and

$$\eta_K \simeq I_M^K(\chi\chi^{\tilde{\theta}}) \quad \text{and} \quad \eta'_K \simeq I_M^K(\chi\chi^{\tilde{\theta}\tau}),$$

where τ is the non-trivial automorphism of M/K and $\tilde{\theta}$ denotes an extension of θ to an automorphism of M over F.

Remark 11.5: When π is non-dihedral (and cuspidal), as proved in part (b) of Theorem 1.4 of [Ra3], $As(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}(4, \mathbb{A}_F)$ iff $\pi \circ \theta$ is not isomorphic to $\pi \otimes \mu$ for any idele class character μ of K. There is no refinement in this case.

Proof. Recall from [Ra3] that the base change $As(\pi)_K$ of $As(\pi)$ to $\mathrm{GL}(4)/K$ satisfies

$$As(\pi)_K \simeq \pi \boxtimes (\pi \circ \theta). \tag{11.6}$$

Since π corresponds to $\sigma = \mathrm{Ind}_M^K(\chi)$, this translates to the following isomorphism of W_K-modules:

$$As(\sigma)_K \simeq \sigma \boxtimes \sigma^{[\theta]}. \tag{11.7}$$

When M is non-Galois, Theorem 11.2 implies that $\pi \boxtimes (\pi \circ \theta)$ is cuspidal. By (11.6), this is the base change of $As(\pi)$ to $\mathrm{GL}(4)/K$. It follows that $As(\pi)$ is itself cuspidal. Hence (β) implies (α).

It was proved in [Ra3] that (α) implies (β). Here is a slightly different proof. Suppose M is Galois over F, i.e., $M = M^{\tilde{\theta}}$. Then we get the following decomposition by Mackey and (11.7):

$$As(\sigma)_K \simeq \mathrm{Ind}_M^K(\chi\chi^{\tilde{\theta}}) \oplus \mathrm{Ind}_M^K(\chi\chi^{\tilde{\theta}\tau}) \tag{11.8}$$

The first representation on the right is evidently invariant under θ and consequently extends to a 2-dimensional representation η, say, of W_F. This means that $As(\sigma)$ is reducible, and so the function $L(s, As(\sigma) \otimes As(\sigma)^\vee)$ has at least a double pole at $s = 1$. Since by [Ra3] this L-function is the same as the Rankin-Selberg L-function $L(s, As(\pi) \times As(\pi)^\vee)$, this automorphic L-function also has a pole of order ≥ 2 at $s = 1$. But then the results of Jacquet and Shalika imply that $As(\pi)$ is *not* cuspidal. Done.

Next we need to make a **correction of incompatible sign conventions used in [Ra3]**. This is important, even though the incompatibilities did not affect the truth of any of the main results in that paper. To elaborate, let K/F be a quadratic extension with non-trivial automorphism θ, and let δ be the quadratic character of $\mathcal{G}_F$ associated to K/F. Given an irreducible 2-dimensional representation σ of $\mathcal{G}_K$) (or W_K), the Asai representation of σ is a choice of an extension $A(\sigma)$, say, to $\mathcal{G}_F$ of the θ-invariant representation $\sigma \otimes \sigma^{[\theta]}$. There are at least two such extensions as one can replace $A(\sigma)$ by its twist by the *sign character* δ; these are the only extensions when $A(\sigma)$ is irreducible. On page 19 (see (4.50)) of [Ra3], we defined the Asai representation to be the summand of $\Lambda^2(\mathrm{Ind}_K^F(\sigma))$ with complement $\mathrm{Ind}_K^F(\det(\sigma))$. This, as well as the analogous definition of $As(\pi)$ for a cuspidal π on $\mathrm{GL}(2)/K$, is fine till we get to the asserted identity (4.64) on page 21 (of [Ra3]), which is off by the sign character δ; we need to twist (exactly) one of the L-functions by δ. The reason is this: Langlands's definition of r is compatible with *tensor induction*, which really occurs in the *symmetric square* of the usual induction; the δ-twist of it occurs in the exterior square. It perhaps makes sense to define $As(\sigma)$ (and $As(\pi)$) using tensor induction. Then we should implement the following **errata** to [Ra3]:

$p.19, \ell. -4$, (4.50): $\quad As(\sigma) \;\to\; As(\sigma) \otimes \delta$

$p.31, \ell. -11$, (6.4):

$$\mathrm{sym}^2(\pi_0) \otimes \delta(\mu\nu)^{-1} \boxplus \delta\mu^{-1} \;\to\; \mathrm{sym}^2(\pi_0) \otimes (\mu\nu)^{-1} \boxplus \mu^{-1}$$

$p.32, \ell.7$: $\quad$ the induction $\quad\to\quad$ the $(\nu_v\delta_v)^{-1}$-twist of the induction

$p.32, \ell.9$, (6.8):

$$\mathrm{sym}^2(\sigma_v(\pi_0)) \otimes \delta_v(\mu_v\nu_v)^{-1} \oplus \delta_v\mu_v^{-1} \;\to\; \mathrm{sym}^2(\sigma_v(\pi_0)) \otimes (\mu_v\nu_v)^{-1} \oplus \mu_v^{-1}$$

$p.37, \ell. -4$, (6.8): $\quad \mu_{1,0}\delta_v \oplus \mu_{2,0}\delta_v \;\to\; \mu_{1,0} \oplus \mu_{2,0}$

$p.37, \ell. -2$, (6.8): $\quad \mu_{1,0}\delta_v \boxplus \mu_{2,0}\delta_v \;\to\; \mu_{1,0} \boxplus \mu_{2,0}$

Finally, it is remarked on page 22 of [Ra3] that the Asai transfer Π to $\mathrm{GL}(4)/F$ satisfies $L(s,\Pi) = L(s,\pi;r)$. But in fact the method of the paper implies more (see sections 7, 8), namely that for any isobaric automorphic representation η of $\mathrm{GL}(2,\mathbb{A}_F)$,

$$L(s,\Pi\times\eta) \;=\; L(s,\pi;r\otimes\eta).$$

In particular, the local component Π_v is at any place v the correct admissible representation of $\mathrm{GL}(4,F_v)$ associated by functoriality.

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