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# INCREASING THE NUMBER OF FIBERED FACES OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We exhibit a closed hyperbolic 3-manifold which satisfies a very strong form of Thurston’s Virtual Fibration Conjecture. In particular, this manifold has finite covers which fiber over the circle in arbitrarily many fundamentally distinct ways. More precisely, it has a tower of finite covers where the number of fibered faces of the Thurston norm ball goes to infinity, in fact faster than any power of the logarithm of the degree of the cover, and we give a more precise quantitative lower bound. The example manifold  $M$  is arithmetic, and the proof uses detailed number-theoretic information, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried’s dynamical characterization of the fibered faces. The origin of the basic fibration  $M \rightarrow S^1$  is the modular elliptic curve  $E = X_0(49)$ , which admits multiplication by the ring of integers of  $\mathbb{Q}[\sqrt{-7}]$ . We first base change the holomorphic differential on  $E$  to a cusp form on  $GL(2)$  over  $K = \mathbb{Q}[\sqrt{-3}]$ , and then transfer over to a quaternion algebra  $D/K$  ramified only at the primes above 7; the fundamental group of  $M$  is a quotient of the principal congruence subgroup of  $\mathcal{O}_D^*$  of level 7. To analyze the topological properties of  $M$ , we use a new practical method for computing the Thurston norm, which is of independent interest. We also give a non-compact finite-volume hyperbolic 3-manifold with the same properties by using a direct topological argument.

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## 1. INTRODUCTION

The most mysterious variant of the circle of questions surrounding Waldhausen’s Virtual Haken Conjecture [Wal1] is:

**1.1. Virtual Fibration Conjecture** (Thurston). *If  $M$  is a finite-volume hyperbolic 3-manifold, then  $M$  has a finite cover which fibers over the circle, i.e. is a surface bundle over the circle.*

This is a very natural question, equivalent to asking whether  $\pi_1(M)$  contains a *geometrically infinite* surface group. However, compared to the other forms of the Virtual Haken Conjecture, there are relatively few non-trivial examples where it is known to hold, especially in the case of closed manifolds (but see [Rei, Lei, Wal2, But] and especially [Ago1]). Moreover, there are indications that fibering over the circle is, in suitable senses, a rare property compared, for example, to simply having non-trivial first cohomology [DT, Mas1].

Despite this, we show here that certain manifolds satisfy Conjecture 1.1 in a very strong way, in that they have finite covers which fiber over the circle in many distinct ways. For a 3-manifold  $M$ , the set of classes in  $H^1(M; \mathbb{Z})$  which can be represented by fibrations over the circle are organized by the Thurston norm on  $H^1(M; \mathbb{R})$ . The unit ball in this norm is a finite polytope where certain top-dimensional faces, called the *fibered faces*, correspond to those cohomology classes coming from fibrations (see Section 2 for details). The number of fibered faces thus measures the number of fundamentally different ways that  $M$  can fiber over the circle. We will sometimes abusively refer to these faces of the Thurston norm ball as “the fibered faces of  $M$ ”.

If  $N \rightarrow M$  is a finite covering map, the induced map  $H^1(M; \mathbb{R}) \rightarrow H^1(N; \mathbb{R})$  takes each fibered face of  $M$  to one in  $N$ ; if  $H^1(N; \mathbb{R})$  is strictly larger than  $H^1(M; \mathbb{R})$ , then it may (but need not) have additional fibered faces. The qualitative form of our main result is:

**1.2. Theorem.** *There exists a closed hyperbolic 3-manifold  $M$  which has a sequence of finite covers  $M_n$  so that the number of fibered faces of the Thurston norm ball of  $M_n$  goes to infinity.*

Moreover, we prove a quantitative refinement of this result (Theorem 1.4) which bounds from below the number of fibered faces of  $M_n$  in terms of the degree of the cover. While it is the closed case of Conjecture 1.1 that is most interesting, we also give an example of a *non-compact* finite-volume hyperbolic 3-manifold with the same property (Theorem 8.3) using a simple topological argument.

The example manifold  $M$  of Theorem 1.2 is arithmetic, and the proof uses detailed number-theoretic information about it, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried’s dynamical characterization of the fibered faces. To state the geometric part of the theorem, we need to introduce the Hecke operators (see Section 3.1 for details). Suppose  $M$  is a closed hyperbolic 3-manifold, and we have a pair of finite covering maps  $p, q: N \rightarrow M$ ; when  $M$  is arithmetic there are many such pairs

1 of covering maps because the commensurator of  $\pi_1(M)$  in  $\text{Isom}(\mathcal{H}^3)$  is very  
 2 large. The associated *Hecke operator* is the endomorphism of  $H^1(M)$  defined by  
 3  $T_{p,q} = q_* \circ p^*$ , where  $q_*: H^1(N) \rightarrow H^1(M)$  is the transfer map. The simplest  
 4 form of our main geometric lemma is the following:

5 **1.3. Lemma.** *Let  $M$  be a closed hyperbolic 3-manifold, and  $p, q: N \rightarrow M$  a pair of*  
 6 *finite covering maps. If  $T_{p,q}(\omega) = 0$  for some  $\omega \in H^1(M; \mathbb{Z})$  coming from a fibration*  
 7 *over the circle, then  $p^*(\omega)$  and  $q^*(\omega)$  lie in distinct fibered faces.*

8 We prove this lemma in Section 3. Then in Section 4, we give a manifold with  
 9 an infinite tower of covers to which Lemma 1.3 applies at each step, thus prov-  
 10 ing Theorem 1.2. In fact, we show the following refined quantitative version:

11 **1.4. Theorem.** *There is a closed hyperbolic 3-manifold  $M$  of arithmetic type, with an*  
 12 *infinite family of finite covers  $\{M_n\}$  of degree  $d_n$ , where the number  $v_n$  of fibered faces*  
 13 *of  $M_n$  satisfies*

$$14 \quad v_n \geq \exp\left(0.3 \frac{\log d_n}{\log \log d_n}\right) \quad \text{as } d_n \rightarrow \infty.$$

15 *In particular, for any  $t < 1$ , there is a constant  $c_t > 0$  such that*

$$16 \quad v_n \geq c_t e^{(\log d_n)^t}.$$

17 Note that this bound for  $v_n$  is slower than any positive power of  $d_n$ , but  
 18 is faster than any (positive) power of  $\log d_n$ . For context, the Betti number  
 19  $b_1(M_n) = \dim H^1(M_n; \mathbb{R})$  is bounded above by (a constant times) the degree  
 20  $d_n$ , and bounded below (see Proposition 7.8), for any  $\varepsilon > 0$ , by (a constant  
 21 times)  $d_n^{1/2-\varepsilon}$  for  $n$  large enough (relative to  $\varepsilon$ ), while  $v_n$  is at least as large as  
 22 (a constant times)  $e^{(\log d_n)^{0.99}}$ . In our non-compact example of Theorem 8.3, the  
 23 number of faces grows exponentially in the degree while  $b_1$  grows linearly.

24 We now describe the basic ideas behind the construction of the manifolds  
 25 in Theorem 1.4. For an arithmetic hyperbolic 3-manifold  $M$ , one has a Hecke  
 26 operator as above associated to each prime ideal of the field of definition. The  
 27 key to applying Lemma 1.3 repeatedly is to have a fibered class  $\omega \in H^1(M)$   
 28 which is killed by infinitely many such Hecke operators. One can produce co-  
 29 homology classes which are killed by infinitely many Hecke operators using  
 30 the special class of CM forms. Fortunately, there is a manifold of manageable  
 31 size whose cohomology contains a CM form coming from base change of an  
 32 automorphic form associated to a certain elliptic curve with complex multipli-  
 33 cation, and that class turns out to fiber over the circle!

34 **1.5. Outline of the arithmetic construction.** Here is a sketch of how the man-  
 35 ifolds  $\{M_n\}$  of Theorem 1.4 are built from arithmetic; for details, see Section 4.  
 36 Let  $E$  be the elliptic curve over  $\mathbb{Q}$  defined by  $y^2 + xy = x^3 - x^2 - 2x - 1$ , which  
 37 has conductor 49 and admits complex multiplication by the ring of integers  
 38 of the imaginary quadratic field  $\mathbb{Q}[\sqrt{-7}]$ . It corresponds to a holomorphic  
 39 cusp form of weight 2 for the congruence subgroup  $\Gamma_0(49)$  of  $\text{SL}(2, \mathbb{Z})$  act-  
 40 ing on the upper half plane  $\mathcal{H}$ , given by  $f(z) = \sum_{n \geq 1} a_n q^n$  with  $q = e^{2\pi iz}$ ,  
 41 where for every prime  $p \neq 7$ , the eigenvalue of  $f$  under the Hecke operator

1  $T_p$  is  $a_p = p + 1 - |E(\mathbb{F}_p)|$ . The differential  $f(z)dz$  is invariant under  $\Gamma_0(49)$ ,  
 2 and hence defines a holomorphic 1-form on  $Y_0(49) := \Gamma_0(49) \backslash \mathcal{H}$ , and by the  
 3 cuspidality of  $f$ , this differential extends to the natural (cusp) compactifica-  
 4 tion of  $Y_0(49)$ . Put  $K = \mathbb{Q}[\sqrt{-3}]$ , in which the prime 7 splits as  $Q\bar{Q}$ , with  
 5  $Q = \left(2 + (1 + i\sqrt{3})/2\right)$ . Let  $f_K$  denote the base change of  $f$  to  $K$ , which is a  
 6 cusp form on the hyperbolic 3-space  $\mathcal{H}^3$  of “weight 2” and level  $Q^2\bar{Q}^2$  for the  
 7 group  $\mathrm{GL}(2, \mathcal{O}_K)$ . One can associate to  $f_K$  a cuspidal automorphic representa-  
 8 tion  $\pi'$  of  $\mathrm{GL}(2, \mathbb{A}_K)$  with trivial central character and conductor  $Q^2\bar{Q}^2$ ; here  
 9  $\mathbb{A}_K$  denotes the adèle ring of  $K$ . By a basic property of base change, we have,  
 10 for every degree 1 prime  $P$  of  $\mathcal{O}_K$  above a rational prime  $p \neq 7$  unramified in  
 11  $K$ , the  $P^{\mathrm{th}}$  Hecke eigenvalue  $a_P(f_K)$  of  $f_K$  equals the  $p^{\mathrm{th}}$  Hecke eigenvalue of  $f$ ,  
 12 namely  $a_p = p + 1 - |E(\mathbb{F}_p)|$ .

13 Let  $D$  be the quaternion division algebra over  $K$  which is ramified exactly at  
 14 the primes  $Q$  and  $\bar{Q}$ . Fortunately for us, the local components of  $\pi'$  at  $Q$  and  
 15  $\bar{Q}$  are both supercuspidal (see Lemma 5.3); this has to do with the fact that  $E$   
 16 does not acquire good reduction over an abelian extension of  $\mathbb{Q}_7$ , this fact being  
 17 controlled, thanks to a useful criterion of D. Rohrlich [Roh], by the valuation  
 18 (at  $Q$  and  $\bar{Q}$ ) of the discriminant  $\Delta$  of  $E$ . (After writing this paper, we learned  
 19 of an earlier proof of the supercuspidality at 7 in [GL], where the argument  
 20 is somewhat different.) Therefore, by the Jacquet-Langlands correspondence,  
 21 there is an associated cusp form  $h$  of weight 2 on  $\mathcal{H}^3$  relative to a congruence  
 22 subgroup  $\Gamma$  of the units in a maximal order  $\mathcal{O}_D$ , such that for every unramified  
 23 prime  $P \neq Q, \bar{Q}$ , the  $P^{\mathrm{th}}$  Hecke eigenvalues of  $f_K$  and of  $h$  coincide. Such a  $\Gamma$  is  
 24 co-compact and turns out to be the principal congruence subgroup of level 7 =  
 25  $Q\bar{Q}$ . Our base manifold in Theorem 1.4 is  $M = X(7) = \Gamma \backslash \mathcal{H}^3$ . While  $H^1(M)$   
 26 is 3-dimensional, we show that the new subspace of  $H^1(M)$  is 2-dimensional  
 27 and, as a module under the Hecke algebra of correspondences, is isotypic of  
 28 type  $\eta = \eta_h$ , the cohomology class defined by  $h$ . A difficult computation shows  
 29 that  $M$  fibers over the circle, and moreover, this can be associated with a class  
 30 of type  $\eta$ .

31 Now consider the set  $\mathcal{P}$  of rational primes  $p \neq 7$  which are inert in  $\mathbb{Q}[\sqrt{-7}]$ ,  
 32 but are split in  $K$  as  $P\bar{P}$ ; it has density  $1/4$ . Then, if we put

$$33 \quad \mathcal{P}_K = \{P \mid \mathcal{N}_{K/\mathbb{Q}}(P) = p \in \mathcal{P}\},$$

34 the Hecke operators  $T_P$  act by zero on  $\eta$ . Each  $P$  in  $\mathcal{P}_K$  gives a covering  
 35  $M(P)$  of  $M$  of degree  $\mathcal{N}(P) + 1$  which defines the associated Hecke opera-  
 36 tor. Lemma 1.3 allows us, since  $T_P\eta = 0$ , to conclude that the two natural  
 37 transforms  $\eta_1, \eta_P$  of  $\eta$  define cohomology classes of  $M(P)$  which lie on two dif-  
 38 ferent fibered faces. If we order  $\mathcal{P}_K$  according to the rational primes  $p$  defin-  
 39 ing them, then we inductively build a covering  $M_n = M(P_1 \dots P_n)$  of degree  
 40  $d_n := \prod_{j=1}^n (1 + p_j)$  such that there are at least  $2^n$  distinct fibered faces on  $M_n$ .  
 41 Using the density of  $\mathcal{P}$ , we get the lower bound for  $\nu_n \geq 2^n$  given in Theo-  
 42 rem 1.4 in terms of  $d_n$ .

1 To understand why we chose to look at this particular example, it may be  
 2 helpful to note the following. Suppose  $D_0$  is an indefinite quaternion algebra  
 3 over  $\mathbb{Q}$ , and  $\Delta$  a congruence subgroup of  $D_0^*$  with associated *Shimura curve*  $S =$   
 4  $S_\Delta$  over  $\mathbb{C}$ , which is a compact Riemann surface. For any imaginary quadratic  
 5 field  $K$  such that  $D = D_0 \otimes_{\mathbb{Q}} K$  is still a division algebra, we may consider the  
 6 hyperbolic 3-manifold  $M = \Gamma \backslash \mathcal{H}_3$ , for a congruence subgroup  $\Gamma$  of  $D^*$ . When  
 7  $\Gamma \cap D_0^* = \Delta$ , the surface  $S$  embeds in  $M$ , and since it is totally geodesic, the  
 8 cohomology class defined by  $S$  cannot give rise to any fibering of  $M$  over the  
 9 circle [Thu2]. This suggests that we will fail to construct cohomology classes  
 10 on  $M$  with the desired fibering property if we transfer to  $D^*$  a cusp form on  
 11  $GL(2)/K$  which is the base change  $f_K$  of an elliptic modular cusp form  $f$  of  
 12 weight 2 which will transfer to such a  $D_0^*$ . It is not an accident that we chose  
 13 our example above where the  $f$  of interest does not transfer to any indefinite  
 14 quaternion algebra over  $\mathbb{Q}$ .

15 Finally, it should be noted that if one starts with a non-CM elliptic curve  $E$   
 16 over  $\mathbb{Q}$ , then one knows, by a theorem of Elkies, that there are infinitely many  
 17 primes  $p$  for which  $a_p$  is zero [Elk]. But for our method to work we would also  
 18 need an example where this property holds for an infinite set  $\mathcal{P}$  of primes  $p$   
 19 which split in a suitable imaginary quadratic field  $K$ . Even then it would only  
 20 give a qualitative result as  $\mathcal{P}$  would have density zero. Our quantitative result  
 21 (Theorem 1.4) depends on the density of the corresponding  $\mathcal{P}$  in the CM case  
 22 being  $1/4$ .

23 **1.6. Moral.** For an *arithmetic* hyperbolic 3-manifold, the commensurator of its  
 24 fundamental group is very large, in fact dense in  $\text{Isom}^+(\mathcal{H}^3) \cong \text{PSL}(2, \mathbb{C})$ .  
 25 Recently, there has been much important work which exploits this density *geo-*  
 26 *metrically*, see [LLR, CLR2, Ven, Ago2]. On the number-theoretic side, the the-  
 27 ory of automorphic forms tells us a great deal about the cohomology of arith-  
 28 metic hyperbolic 3-manifolds as it relates to virtual Haken type questions (see  
 29 e.g. [Clo, CD]). To the best of our knowledge, our work here is the first time  
 30 that *more refined* automorphic information has been combined with geometric  
 31 arguments and yields, for example, a geometric/topological interpretation of  
 32 the vanishing of the Hecke eigenvalues. Thus we hope for deeper connections  
 33 between these two areas in the future. In particular, it would be very interest-  
 34 ing to answer the following:

35 **1.7. Question.** *Is there an automorphic criterion which implies that certain cohomol-*  
 36 *ogy classes of arithmetic hyperbolic 3-manifold give fibrations over the circle?*

37 **1.8. Practical methods for computing the Thurston norm.** The example man-  
 38 ifold of Theorem 1.4 is quite complicated from a topological point of view; its  
 39 hyperbolic volume is about 100 and triangulations of it need some 130 tetrahe-  
 40 dra. Despite this, we were able to compute its Thurston norm and check that  
 41 it fibers over the circle, which is necessary for the proof of Theorem 1.4. To do  
 42 this, we used new methods for both these tasks. While loosely based on normal  
 43 surfaces, these techniques eschew guaranteed termination in favor of quick re-  
 44 sults. The basic idea is to consider only normal surfaces representing elements

1 of  $H^1(M)$  that are “obvious” with respect to the triangulation, and then ran-  
 2 domize the triangulation until a minimal norm surface is found. These same  
 3 techniques have been useful in many other examples and are of independent  
 4 interest. See Sections 6.6 and 6.10 for a complete description of our method,  
 5 which can often determine whether a manifold made up of several hundred  
 6 tetrahedra fibers. Computing the Thurston norm is more subtle, and in the  
 7 case of our particular  $M$ , we had to heavily exploit its symmetries, but we also  
 8 suggest a general method, as of yet untested, for attacking this.

9 **1.9. Improvements.** In subsequent work, Long and Reid have considerably  
 10 strengthened Lemma 1.3 and its extension Theorem 3.12 by showing that the  
 11 hypothesis on the Hecke operators can be dispensed with:

12 **1.10. Theorem** (Long and Reid [LR]). *Let  $M$  be a closed arithmetic hyperbolic 3-*  
 13 *manifold. If  $M$  fibers over the circle, then  $M$  has finite covers whose Thurston norm*  
 14 *balls have arbitrarily many fibered faces.*

15 In addition to the work of Fried [Fri] on which Lemma 1.3 hinges, the proof  
 16 of Theorem 1.10 uses work of Cooper, Long, and Reid on suspension pseudo-  
 17 Anosov flows [CLR1], as strengthened by Masters [Mas2]. In Section 9 we give  
 18 a simplified proof of Theorem 1.10 using only Fried’s theorem; a different, but  
 19 equally concise, simplification was given by Agol [Ago1]. In any case, the soft  
 20 nature of its proof mean that Theorem 1.10 cannot be used to prove quantitative  
 21 results such as Theorem 1.4. However, Theorem 1.10 does have the advantage  
 22 that it is much easier to apply.

23 In a major breakthrough, Agol has just shown there are infinitely many com-  
 24 mensurability classes of arithmetic hyperbolic 3-manifolds which fiber over the  
 25 circle [Ago1]. Combining with Theorem 1.10, this means the qualitative behav-  
 26 ior of Theorem 1.2 actually occurs in an infinite number of examples, providing  
 27 further evidence for Conjecture 1.1.

28 **1.11. Paper outline.** In Section 2, we review the basics of the Thurston norm  
 29 and Fried’s dynamical characterization of the fibered faces. In Section 3, we  
 30 discuss Hecke operators and congruence covers in the context of arithmetic  
 31 hyperbolic 3-manifolds, and then prove Lemma 1.3 and its generalization The-  
 32 orem 3.12 which underpins Theorem 1.4. We give the precise description of the  
 33 manifold used in Theorem 1.4 in Section 4. We then analyze the automorphic  
 34 and topological properties of this manifold in Sections 5 and 6 respectively. In  
 35 Section 7 we assemble the pieces and prove Theorem 1.4. Section 8 gives our  
 36 example of this phenomenon in the case of hyperbolic 3-manifolds with cusps.  
 37 Finally, Section 9 gives our simplified proof of Theorem 1.10.

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1 result (Theorem 1.4), and Danny Calegari for suggesting that we look at fami-  
 2 lies of link complements in the cusped case, which led us to Theorem 8.3. We  
 3 also thank Darren Long and Alan Reid for sending us an early version of [LR],  
 4 and Benedict Gross for informing us of an earlier proof of Lemma 5.3 in [GL].  
 5 Finally, we thank the authors of the software packages [W, G, Cre, BC, SAGE],  
 6 without which this paper would not have been possible.

7 2. FIBERED FACES OF THE THURSTON NORM BALL

8 Let  $M$  be a closed orientable hyperbolic 3-manifold. When a cohomology  
 9 class  $\omega \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$  can be represented by a fibration  
 10  $M \rightarrow S^1$ , we say that it *fibers*. In this section, we review the work of Thurston  
 11 and Fried on the structure of the set of fibered classes in  $H^1(M; \mathbb{Z})$ . It is not  
 12 hard to see that  $\omega \in H^1(M; \mathbb{Z})$  fibers if and only if it can be represented by  
 13 a nowhere vanishing 1-form: a fibration gives rise to such a form by pulling  
 14 back the standard orientation 1-form on  $S^1$ , and conversely such a form can be  
 15 integrated (since its periods are integers) into a map to  $S^1$  which is a fibration.  
 16 If we pass to real coefficients, the latter condition clearly defines an *open* sub-  
 17 set  $U$  of  $H^1(M; \mathbb{R}) \setminus \{0\}$ ; a fundamental result of Thurston shows that  $U$  must  
 18 have the following very restricted form. This set is prescribed by the Thurston  
 19 norm, which measures the simplest surface that represents the Poincaré dual  
 20 of a cohomology class.

21 More precisely, for  $\omega \in H^1(M; \mathbb{Z})$  define its *Thurston norm* by

$$22 \quad \|\omega\| = \min \{ -\chi(\Sigma) \mid \Sigma \text{ is an embedded orientable surface dual to } \omega \},$$

23 where  $\chi(\Sigma)$  is the Euler characteristic of  $\Sigma$ , and we further require that  $\Sigma$  has no  
 24 2-sphere components. Thurston showed that this gives a norm on  $H^1(M; \mathbb{Z})$   
 25 which extends continuously to one on  $H^1(M; \mathbb{R})$ ; see [Thu1] for details of the  
 26 assertions made in this paragraph. Moreover, the unit ball  $B$  of this norm  
 27 is a bounded convex polytope, i.e. the convex hull of finitely many points.  
 28 Moreover, there are top-dimensional faces of  $B$ , called the *fibered faces*, so that  
 29  $\omega \in H^1(M; \mathbb{Z})$  fibers if and only if it lies in the cone over the interior of a  
 30 fibered face, that is, the ray from the origin through  $\omega$  intersects the interior of  
 31 such a face. We will say that such a fibered  $\omega$  *lies in* the corresponding fibered  
 32 face. Finally, as the map  $\omega \mapsto -\omega$  preserves fibering, the fibered faces come  
 33 in pairs interchanged by this symmetry of the Thurston norm ball. When de-  
 34 termining the number of fibered faces, we often count in terms of these fibered  
 35 face pairs, and say that fibered classes  $\alpha$  and  $\beta$  lie in *genuinely distinct* fibered  
 36 faces if both  $\alpha$  and  $-\alpha$  are not in the fibered face of  $\beta$ .

37 **2.1. Behavior of fibered faces under covers.** Now suppose  $p: N \rightarrow M$  is a fi-  
 38 nite covering map. The natural map  $p^*: H^1(M; \mathbb{R}) \rightarrow H^1(N; \mathbb{R})$  is an embed-  
 39 ding, and a deep theorem of Gabai shows that  $p^*$  preserves the Thurston norm  
 40 [Gab1, Cor. 6.18]. Equivalently, if we denote the unit Thurston norm balls of  
 41  $M$  and  $N$  by  $B_M$  and  $B_N$  respectively, we have  $p^*(B_M) = B_N \cap p^*(H^1(M; \mathbb{R}))$ .  
 42 By work of Stallings, a class  $\omega \in H^1(M; \mathbb{Z})$  represents a fibration if and only if

1  $p^*(\omega)$  does [Sta1]. In particular, if  $N$  fibers but  $M$  does not, then  $H^1(N)$  is larger  
 2 than  $H^1(M)$ . Thus each fibered face of  $B_M$  gives rise to a distinct fibered face  
 3 of  $B_N$ ; if  $N$  has additional cohomology, we can hope that  $B_N$  has new fibered  
 4 faces, but the interiors of these must be disjoint from  $p^*(H^1(M))$ .

5 **2.2. Fried's work.** We now turn to Fried's dynamical characterization of when  
 6 two fibrations lie in the same fibered face, which is in terms of a certain flow  
 7 that is transverse to the fibers of the fibration  $M \rightarrow S^1$ . Suppose  $\phi$  is a self-  
 8 homeomorphism of a closed surface  $\Sigma$  of genus at least 2, and consider the  
 9 mapping torus with monodromy  $\phi$ :

$$10 \quad M_\phi = \Sigma \times [0, 1] / (s, 1) \sim (\phi(s), 0)$$

11 Thurston proved that  $M_\phi$  is hyperbolic if and only if  $\phi$  is what is called pseudo-  
 12 Anosov [Thu2, Ota]. The latter means that  $\phi$  is isotopic to a homeomorphism  
 13 which preserves a pair of foliations of  $\Sigma$  in a controlled way. Henceforth, we  
 14 always assume that  $\phi$  has been isotoped to such a preferred representative.  
 15 Now,  $M_\phi$  has a natural suspension flow  $\mathcal{F}_\phi$  which is transverse to the circle  
 16 fibers, where a point moves at unit speed in the  $[0, 1]$ -direction. We will call  
 17  $\mathcal{F}_\phi$  the *transverse pseudo-Anosov flow*.

18 Conversely, given  $\omega \in H^1(M; \mathbb{Z})$  coming from a fibration, the monodromy  
 19  $\phi$  of the bundle structure is well-defined up to isotopy. Thus there is a cor-  
 20 responding transverse pseudo-Anosov flow  $\mathcal{F}_\omega$ , which is well-defined up to  
 21 isotopy. Fried's first result is:

22 **2.3. Theorem** (Fried [Fri]). *Let  $M$  be a closed orientable hyperbolic 3-manifold. Then*  
 23 *two fibrations of  $M$  over the circle lie over the same fibered face if and only if the*  
 24 *corresponding transverse pseudo-Anosov flows are isotopic.*

25 Fried also provided the following characterization of those  $\omega$  lying over a  
 26 fibered face. Let  $\mathcal{F}$  be the flow associated to a particular fibered face  $F$ , and  
 27 let  $D \subset H_1(M; \mathbb{R})$  be the set of *homology directions* of  $\mathcal{F}$ , namely the set of all  
 28 accumulation points of the homology classes of long, nearly closed orbits of  $\mathcal{F}$ .  
 29 Fried showed that the dual cone to  $D$ ,

$$30 \quad C = \left\{ \omega \in H^1(M; \mathbb{R}) \mid \omega(v) > 0 \text{ for all } v \in D \right\},$$

31 is precisely the cone on the interior of the fibered face  $F$  [Fri, Thm. 7].

32 One kind of element of  $D$  is the homology class of a closed flowline of  $\mathcal{F}$ .  
 33 There are always closed flowlines, for instance coming from the orbits of the  
 34 finitely many singular points of the invariant foliations, which are permuted  
 35 among themselves by the monodromy. The particular consequence of Fried's  
 36 work that we will use here, which is immediate from our discussion, is:

37 **2.4. Lemma** (Fried). *Let  $M$  be a closed orientable hyperbolic 3-manifold, with fibered*  
 38 *classes  $\alpha, \beta \in H^1(M; \mathbb{Z})$ . Let  $c$  be a closed orbit of the transverse pseudo-Anosov flow*  
 39 *for  $\alpha$ . If  $\beta(c) = 0$ , then  $\alpha$  and  $\beta$  lie over genuinely distinct fibered faces.*

1

## 3. THE GEOMETRIC THEOREM

2 **3.1. Hecke operators.** From a geometric point of view, a Hecke operator is  
 3 the map on cohomology induced from the following setup. Suppose  $M$  is a  
 4 topological space, and we have a pair of *finite* covering maps  $p, q: N \rightarrow M$ .  
 5 The map on singular chains  $C_*(M) \rightarrow C_*(N)$  which takes a singular sim-  
 6 plex to the sum of its inverse images under  $q$  induces transfer homomorphisms  
 7  $q^*: H_*(M) \rightarrow H_*(N)$  and  $q_*: H^*(N) \rightarrow H^*(M)$  (see e.g. [Hat, §3.G] for de-  
 8 tails). The Hecke operator  $T_{p,q}$  of this pair of covering maps is the endomor-  
 9 phism of  $H^*(M)$  defined by  $q_* \circ p^*$ .

10 **3.2. Remark.** In this paper,  $M$  will always be a 3-manifold and we will be in-  
 11 terested in the Hecke operator on  $H^1(M)$ . While the argument below is given  
 12 purely in terms of cohomology, the geometrically minded reader may prefer to  
 13 contemplate the Poincaré dual group  $H_2(M)$ . There, the Hecke operator  $T_{p,q}$   
 14 on  $H_*(M)$  is the composite  $q_* \circ p^*$ , where  $p^*$  is the transfer map. Such Hecke  
 15 operators commute with the Poincaré duality isomorphism  $H^1(M) \cong H_2(M)$ ,  
 16 and so it makes no difference whether one takes the homological or cohomolo-  
 17 gical point of view. The action of  $T_{p,q}$  on a class  $\omega \in H_2(M)$  is particularly  
 18 simple to think of geometrically: If an embedded surface  $\Sigma \subset M$  represents  $\omega$ ,  
 19 then the immersed surface  $q(p^{-1}(\Sigma))$  represents  $T_{p,q}(\omega)$ .

20 **3.3. Main geometric idea.** When  $M$  is arithmetic, there are many manifolds  $N$   
 21 which cover it in distinct ways. Before getting into this, let us give the central  
 22 topological idea of Theorem 1.2 in its simplest form from the introduction:

23 **1.3. Lemma.** *Let  $M$  be a closed hyperbolic 3-manifold, and suppose  $\omega \in H^1(M; \mathbb{Z})$   
 24 comes from a fibration of  $M$  over the circle. Further assume that  $p, q: N \rightarrow M$  are a  
 25 pair of finite covering maps. If  $T_{p,q}(\omega) = 0$  then  $p^*(\omega)$  and  $q^*(\omega)$  lie in genuinely  
 26 distinct fibered faces.*

27 *Proof.* We will apply Lemma 2.4 to justify our claim. Downstairs in  $M$ , let  $c$  be  
 28 a closed orbit of the pseudo-Anosov flow associated to the fibration  $\omega$ . Then  
 29  $q^{-1}(c)$  is a closed orbit of the flow associated to the fibration  $q^*(\omega)$ . To calculate  
 30  $p^*(\omega)(q^{-1}(c))$ , note that for any  $\alpha \in H^1(N)$  one has  $\alpha(q^*(c)) = (q_*(\alpha))(c)$  and  
 31 hence

$$32 \quad p^*(\omega) \left( q^{-1}(c) \right) = p^*(\omega) (q^*(c)) = q_* (p^*(\omega)) (c) = T_{p,q}(\omega)(c) = 0$$

33 as required to apply the lemma. □

34 In the case of a tower of covers of  $M$ , the following strengthening of the  
 35 previous lemma will be needed to work inductively:

36 **3.4. Lemma.** *Let  $M$  be a closed hyperbolic 3-manifold, and  $p, q: N \rightarrow M$  be a pair of  
 37 finite covering maps. Suppose  $\omega_1, \omega_2, \dots, \omega_n \in H^1(M; \mathbb{Z})$  lie in genuinely distinct  
 38 fibered faces. If  $T_{p,q}(\omega_i) = 0$  for all  $i$ , then  $N$  has at least  $2n$  pairs of fibered faces.  
 39 More precisely,  $\{p^*(\omega_1), \dots, p^*(\omega_n), q^*(\omega_1), \dots, q^*(\omega_n)\}$  lie in genuinely distinct  
 40 fibered faces.*

1 *Proof.* By the discussion in Section 2.1, it is clear that the  $p^*(\omega_i)$  lie in distinct  
 2 fibered faces, as do the  $q^*(\omega_j)$ . Thus it remains to distinguish the face of  $p^*(\omega_i)$   
 3 from that of  $q^*(\omega_j)$ . As before, let  $c$  be a closed orbit of the pseudo-Anosov  
 4 flow associated to  $\omega_j$ , so that  $q^{-1}(c)$  is an orbit of the flow associated to  $q^*(\omega_j)$ .  
 5 Then

$$6 \quad p^*(\omega_i) \left( q^{-1}(c) \right) = p^*(\omega_i) (q^*(c)) = q_* (p^*(\omega_i)) (c) = T_{p,q}(\omega_i)(c) = 0$$

7 and so Lemma 2.4 applies as needed.  $\square$

8 **3.5. Sources of Hecke operators.** Now we turn to the source of such multiple  
 9 covering maps. For a hyperbolic 3-manifold  $M$ , let  $\Gamma$  be its fundamental group,  
 10 thought of as a lattice in  $\mathrm{PSL}(2, \mathbb{C})$ . The commensurator of  $\Gamma$  is the subgroup

$$11 \quad \mathrm{Comm}(\Gamma) = \{g \in \mathrm{PSL}(2, \mathbb{C}) \mid g^{-1}\Gamma g \cap \Gamma \text{ is finite index in both } \Gamma \text{ and } g^{-1}\Gamma g\}.$$

12 When  $M$  is arithmetic,  $\mathrm{Comm}(\Gamma)$  is dense in  $\mathrm{PSL}(2, \mathbb{C})$ , and Margulis showed  
 13 that the converse is true as well; indeed, if  $M$  is not arithmetic, then  $\Gamma$  has finite  
 14 index in  $\mathrm{Comm}(\Gamma)$ .

15 Regardless,  $g \in \mathrm{Comm}(\Gamma)$  can be associated to a Hecke operator as follows.  
 16 Let  $\Gamma_g = g^{-1}\Gamma g \cap \Gamma$ , and  $M_g = \Gamma_g \backslash \mathcal{H}^3$  be the corresponding closed hyperbolic  
 17 3-manifold. Consider the finite covering map  $p_g: M_g \rightarrow M$  induced by the  
 18 inclusion  $\Gamma_g \rightarrow \Gamma$ . We will define a second such covering map by considering  
 19  $\Gamma_{g^{-1}} = g\Gamma g^{-1} \cap \Gamma$ , and analogously  $p_{g^{-1}}: M_{g^{-1}} \rightarrow M$ . Now, as  $g\Gamma_g g^{-1} = \Gamma_{g^{-1}}$ ,  
 20 the action of  $g$  on  $\mathcal{H}^3$  induces an isometry  $\tau_g: M_g \rightarrow M_{g^{-1}}$  giving us the  
 21 picture

$$22 \quad \begin{array}{ccc} M_g & \xrightarrow{\tau_g} & M_{g^{-1}} \\ & \searrow p_g & \swarrow p_{g^{-1}} \\ & M & \end{array}$$

23 Combining, we get a covering map  $q_g: M_g \rightarrow M$  by taking  $q_g = p_{g^{-1}} \circ \tau_g$ . On  
 24 the level of groups, the covering map  $q_g$  corresponds to the homomorphism  
 25  $\Gamma_g \rightarrow \Gamma$  given by  $\gamma \mapsto g\gamma g^{-1}$ . By definition, the Hecke operator associated to  $g$   
 26 is  $T_{p_g, q_g}$  in the notation of Section 3.1.

27 **3.6. Standard congruence covers.** In number theory, one usually restricts to a  
 28 subset of all the Hecke operators described above which have nice collective  
 29 properties. To describe these, we first review the general arithmetic construc-  
 30 tion of a lattice in  $\mathrm{Isom}^+(\mathcal{H}^3)$ ; for details see e.g. [Vig, MR]. Begin with a  
 31 number field  $K$  with exactly one complex place, and choose a quaternion al-  
 32 gebra  $D$  over  $K$  which is ramified at all real places of  $K$ . Now at the complex  
 33 place of  $K$ , the algebra  $D \otimes_K \mathbb{C}$  must be isomorphic to  $M_2(\mathbb{C})$ , the algebra of  
 34  $2 \times 2$  matrices. Taking the units gives a homomorphism  $D^* \hookrightarrow \mathrm{GL}(2, \mathbb{C})$ ; di-  
 35 viding out by the respective centers embeds  $D^*/K^*$  as a dense subgroup of  
 36  $\mathrm{PGL}(2, \mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathcal{H}^3)$ . Now if  $\mathcal{O}_D$  is a maximal order of  $D$ ,  
 37 then the image  $\Gamma$  of the units  $\mathcal{O}_D^*$  in  $\mathrm{PGL}(2, \mathbb{C})$  is a lattice, which is cocompact

1 provided  $D$  is a division algebra rather than  $M_2(K)$ . We will denote the quo-  
 2 tient hyperbolic 3-orbifold as  $X = \Gamma \backslash \mathcal{H}^3$ .

3 For each ideal  $A$  of  $\mathcal{O}_K$ , we can define a corresponding congruence orbifold  
 4  $X(A)$  covering  $X$  as follows. Since we will only need that case, we restrict  
 5 to when  $A$  is square-free. The key case is that of a prime ideal  $P$ . Let  $K_P$  be  
 6 the local completion of  $K$  at the place  $P$ ; its valuation ring is denoted  $\mathcal{O}_P$  with  
 7 maximal ideal  $\mathcal{P}$  and residue field  $\mathbb{F} = \mathcal{O}_P / \mathcal{P}$ . The cover  $X(P)$  is constructed  
 8 using the local algebra  $D_P = D \otimes_K K_P$ .

9 First, suppose  $D$  does not ramify at  $P$ , i.e.  $D_P \cong M_2(K_P)$ . The cover  $X(P)$  is  
 10 the congruence cover of “ $\Gamma_0$ -type” built as follows. The order  $\mathcal{O}_{D_P} := \mathcal{O}_D \otimes_{\mathcal{O}_K}$   
 11  $\mathcal{O}_P$  is maximal in  $D_P$ , and so is conjugate to  $M_2(\mathcal{O}_P)$ . Thus we get

$$12 \quad \mathcal{O}_D^* \hookrightarrow \mathcal{O}_{D_P}^* \cong \mathrm{GL}(2, \mathcal{O}_P) \rightarrow \mathcal{O}_{D_P}^* / 1 + \mathcal{P} \mathcal{O}_{D_P} \cong \mathrm{GL}(2, \mathbb{F})$$

13 which induces a homomorphism  $\Gamma \rightarrow \mathrm{PGL}(2, \mathbb{F})$ . By strong approximation,  
 14 the image of  $\Gamma$  acts transitively on  $P^1(\mathbb{F})$ . By definition,  $X(P) = \Gamma_0(P) \backslash \mathcal{H}^3$   
 15 where  $\Gamma_0(P)$  is the  $\Gamma$ -stabilizer of a point in  $P^1(\mathbb{F})$ . This gives a cover  $X(P) \rightarrow X$   
 16 of degree  $|P^1(\mathbb{F})| = N(P) + 1$ , where  $N(P) = |\mathbb{F}|$  is the norm of  $P$ .

17 Suppose instead  $P$  is one of the finitely many primes where  $D$  ramifies. Then  
 18  $D_P$  is the unique quaternion division algebra over  $K_P$ , and so is a local skew-  
 19 field whose valuation ring is  $\mathcal{O}_{D_P}$  with a unique maximal bi-ideal  $\mathcal{Q}$ . If  $q = |\mathbb{F}|$   
 20 we have

$$21 \quad \mathcal{O}_D^* \hookrightarrow \mathcal{O}_{D_P}^* \rightarrow \mathcal{O}_{D_P}^* / 1 + \mathcal{Q} \cong \mathbb{F}_{q^2}^*$$

22 See e.g. [MR, §6.4] for details. Set  $\Gamma(P) \triangleleft \Gamma$  to be the projectivization of the  
 23 kernel of  $\mathcal{O}_D^* \rightarrow \mathbb{F}_{q^2}^*$ , and let  $X(P) = \Gamma(P) \backslash \mathcal{H}^3$ . The precise degree of  $X(P) \rightarrow$   
 24  $X$  depends on  $K$  and  $D$ , as for instance  $\mathcal{O}_D^* \rightarrow \mathbb{F}_{q^2}$  is rarely onto.

25 More generally, suppose  $A = P_1 P_2 \cdots P_n$  is a square-free ideal of  $\mathcal{O}_K$ . Then  
 26  $X(A)$  is defined as the common cover of the  $X(P_i)$ . If  $D$  does not ramify at any  
 27  $P_i$ , strong approximation implies that the degree of  $X(A) \rightarrow X$  is the product  
 28 of the degrees of  $X(P_i) \rightarrow X$ .

29 **3.7. Standard Hecke operators.** The Hecke operators for these covers are de-  
 30 fined as follows. Let  $\mathrm{ram}(D)$  denote the set of primes where  $D$  is ramified.  
 31 Suppose  $A$  divides a square-free ideal  $B$ . Then we have the natural covering  
 32 map

$$33 \quad \phi_1 = \phi_{1, B/A}: X(B) \rightarrow X(A).$$

34 Moreover, if  $B/A$  is prime to  $\mathrm{ram}(D)$ , then for each ideal  $J$  dividing  $B/A$ , there  
 35 is a certain covering map:

$$36 \quad \phi_J = \phi_{J, B/A}: X(B) \rightarrow X(A),$$

37 which is just  $\phi_1$  when  $J$  is the unit ideal. These maps are defined below. In  
 38 particular, for a prime  $P \notin \mathrm{ram}(D)$  not dividing  $A$ , we have two covering  
 39 maps:

$$40 \quad \phi_1, \phi_P: X(AP) \rightarrow X(A)$$

41 Then the Hecke operator for  $P$  on  $H^*(X(A))$  is defined as  $T_P = T_{\phi_1, \phi_P}$  in the  
 42 notation of Section 3.1. These have the following key properties:

1 **3.8. Proposition.** *Let  $A$  be a square-free ideal of  $\mathcal{O}_K$ .*

2 (1) *The standard Hecke algebra for  $X(A)$  is generated by the  $T_P$  for  $p \nmid A$  and*  
 3  *$p \notin \text{ram}(D)$ . For prime ideals  $P$  and  $Q$ , the Hecke operators  $T_P$  and  $T_Q$*   
 4 *commute.*

5 (2) *Consider a cover  $\phi_J: X(B) \rightarrow X(A)$ , with  $B$  square-free, and the associated*  
 6 *degeneracy map*

$$7 \quad \phi_J^*: H^i(X(A)) \rightarrow H^i(X(B)).$$

8 *If  $P$  is a prime ideal not dividing  $B$ , then*

$$9 \quad \phi_J^* \circ T_P^A = T_P^B \circ \phi_J^*$$

10 *where the superscripts on the Hecke operators indicate which orifolds' coho-*  
 11 *mology is being acted upon.*

12 We will now give a definition of the maps  $\phi_J$ , and indicate how to deduce  
 13 the properties above. For this it is convenient to work in the adèlic setup.

14 Let  $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$  be the adèle ring of  $K$ , with  $K_\infty$  denoting the prod-  
 15 uct of the archimedean completions of  $K$  and  $\mathbb{A}_{K,f}$  being the ring of finite  
 16 adèles. Put  $G = D^*$ , which is a reductive algebraic group over  $K$ , with cen-  
 17 ter  $Z$ . Then  $G(K)$  is a discrete subgroup of the locally compact group  $G(\mathbb{A}_K) =$   
 18  $G_\infty \times G(\mathbb{A}_{K,f})$ . Since  $K$  has a unique complex place and  $D$  is a quaternion divi-  
 19 sion algebra ramified at all the real places, we have  $G_\infty \simeq \text{GL}(2, \mathbb{C}) \times \mathbb{H}^{[K:\mathbb{Q}]-2}$ ,  
 20 where  $\mathbb{H}$  is Hamilton's quaternion algebra over  $\mathbb{R}$ ; consequently,  $Z_\infty \backslash G_\infty$  is  
 21  $\text{PGL}(2, \mathbb{C}) \times \text{SU}(2)^{[K:\mathbb{Q}]-2}$ . Moreover, since  $G$  is anisotropic, Godement's com-  
 22 pactness criterion implies that the adèlic quotient

$$23 \quad X_{\mathbb{A}} := G(K)Z(\mathbb{A}_K) \backslash \mathcal{H}^3 \times G(\mathbb{A}_{K,f})$$

24 is a compact space, equipped with a right action by  $G(\mathbb{A}_{K,f})$ . We may view  $X_{\mathbb{A}}$   
 25 as the projective limit of closed hyperbolic 3-orbifolds

$$26 \quad X_U := X_{\mathbb{A}}/U,$$

27 as  $U$  varies over a cofinal system of compact open subgroups of  $G(\mathbb{A}_{K,f})$ . These  
 28 orbifolds are in fact manifolds for deep enough  $U$ . The reduced norm on  $D^*$   
 29 induces a map  $\text{Nrd}: G(\mathbb{A}_{K,f}) \rightarrow \mathbb{A}_{K,f}^*$ , and if  $U$  is any compact open subgroup  
 30 of  $G(\mathbb{A}_{K,f})$ , we can write  $G(\mathbb{A}_K)$  as a finite union

$$31 \quad G(\mathbb{A}_K) = \bigcup_{j=1}^{h(U)} G(K)G_\infty x_j U,$$

32 for any elements  $x_1 = 1, x_2, \dots, x_{h(U)}$  in  $G(\mathbb{A}_{K,f})$  such that  $\{\text{Nrd}(x_j)\}$  is a  
 33 complete set of representatives for  $K^* \text{Nrd}(U) \backslash \mathbb{A}_{K,f}^*$ . We may choose these  
 34  $x_j$  to have components 1 at the primes in  $\text{ram}(D)$ , and such that for finite  
 35  $v \notin \text{ram}(D)$ ,

$$36 \quad x_{j,v} = \begin{pmatrix} a_{j,v} & 0 \\ 0 & 1 \end{pmatrix},$$

1 for some  $a_{j,v}$  in  $K_v^*$ , of course with  $a_{j,v} \in \mathcal{O}_{K,v}^*$  for almost all  $v$ . When  $U$  is the  
 2 maximal compact subgroup  $G(\mathcal{O}_{K,f}) = \prod_P G(\mathcal{O}_{K,P})$ , the number of compo-  
 3 nents  $h(U)$  is just the class number  $h$  of  $K$ .

4 We will use the following cofinal system of compact open subgroups  $U_A$  for  
 5 ideals  $A = \prod_P P^{m(P)}$ , given by:

$$6 \quad U_A = \left( \prod_{P \in \text{ram}(D)} U_P(P^{m_P}) \right) \times \left( \prod_{P \notin \text{ram}(D)} U_{0,P}(P^{m_P}) \right),$$

7 where for  $P \in \text{ram}(D)$ , the subgroup  $U_P(P^{m_P})$  is the principal congruence  
 8 subgroup of  $G(K_P)$  of level  $P^{m_P}$ , while for  $P \notin \text{ram}(D)$ ,

$$9 \quad U_{0,P}(P^{m_P}) = \left\{ g \in G(\mathcal{O}_{K,P}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{P^{m_P}} \right\}.$$

10 Note that  $\text{Nrd}(U_{0,P}(P^{m_P}))$  is  $\mathcal{O}_{K,P}^*$ , implying that

$$11 \quad (A, \text{ram}(D)) = 1 \implies h(A) := h(U_A) = h.$$

12 For the main example of this paper  $K = \mathbb{Q}[\sqrt{-3}]$  for which  $h = 1$ .

13 The space  $X_{U_A}$  has  $h(A)$  connected components. More precisely, if we put

$$14 \quad \Gamma_j(A) = G(K) \cap x_j U_A x_j^{-1},$$

15 we get

$$16 \quad X_{U_A} = \bigcup_{j=1}^{h(A)} \Gamma_j(A) \backslash \mathcal{H}^3.$$

17 Here  $\Gamma_1(A) \backslash \mathcal{H}^3$  is the orbifold  $X(A)$  that we defined in Section 3.6.

18 **3.9. Degeneracy maps.** We now turn to the degeneracy maps discussed in  
 19 Proposition 3.8(2). If  $P$  is a prime ideal not in  $\text{ram}(D)$ , choose a uniformizer  
 20  $\varpi$  at  $P$  and define  $g(P) \in G(\mathbb{A}_K)$  to be  $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$  at  $P$  and 1 at all other places. We  
 21 then define a map via right multiplication:

$$22 \quad X_{\mathbb{A}} \rightarrow X_{\mathbb{A}} \quad \text{given by } \xi \mapsto \xi g(P).$$

23 Note that  $g(P)$  has only one non-trivial component, namely at  $P$ , and that com-  
 24 ponent is diagonal. At the finite levels we get the induced maps

$$25 \quad \phi'_P: X_{U_A \cap g(P) U_A g(P)^{-1}} \rightarrow X_{U_A}.$$

26 It is immediate that

$$27 \quad P \nmid A \implies U_A \cap g(P) U_A g(P)^{-1} = U_{AP}.$$

28 We let  $\phi'_P$  denote again the restriction to the connected component  $X(AP)$ . It  
 29 is easy to extend this to square-free ideals  $J$  away from  $A$  and  $\text{ram}(D)$ , and  
 30 if  $J$  divides  $B/A$  for another square-free ideal  $B$ , then we may compose the  
 31 natural map  $\phi_1: X(B) \rightarrow X(AJ)$  with  $\phi_J: X(AJ) \rightarrow X(A)$  to obtain the map  
 32  $\phi_J$  discussed above.

33 For each prime  $P \nmid A$ ,  $P \notin \text{ram}(D)$ , the Hecke correspondence  $T_P: X(A) \rightarrow$   
 34  $X(A)$ , defined above by pulling back along  $\phi_1$  and then pushing down by  $\phi_P$ ,

1 also acts on the cohomology of  $X(A)$  in the obvious way. This Hecke opera-  
 2 tor does not depend on our choice of uniformizer, because if we replace  $\varpi$  by  
 3 another uniformizer  $\varpi'$  at  $P$ , then  $u := \varpi'\varpi^{-1}$  is a unit at  $P$ , and right multi-  
 4 plication by  $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$  acts trivially on  $X(A)$ . So the action of  $T_P$  on  $H^*(X(A))$  is  
 5 invariantly defined for any such  $P$ , and is denoted as  $T_P^A$  if we want to keep  
 6 track of the level  $A$ .

7 Note that right multiplication by  $g(P)$  on the adèlic space  $X_{\mathbb{A}}$  induces a fam-  
 8 ily of maps

$$9 \quad g(P)^A : X_{U_P} \rightarrow X_U,$$

10 as  $U = U_A$  runs over congruence subgroups of level  $A$  prime to  $P$ . Also,  
 11 the right multiplication by  $g(P)$  and  $g(Q)$  on  $X_{\mathbb{A}}$  evidently commute for two  
 12 distinct primes  $P, Q$ , since the matrices  $g(P)$  and  $g(Q)$  themselves commute.  
 13 One gets the identities, for any compact open subgroup  $U$  of level  $A$  prime to  
 14  $PQ$ ,

$$\begin{aligned} U_{PQ} &= U \cap \left( g(P)g(Q)Ug(Q)^{-1}g(P)^{-1} \right) \\ &= U \cap \left( g(Q)g(P)Ug(P)^{-1}g(Q)^{-1} \right) = U_{QP}. \end{aligned}$$

15 Moreover

$$16 \quad U_P \cap \left( g(Q)U_Pg(Q)^{-1} \right) = U_{PQ} \quad \text{and} \quad U_Q \cap \left( g(P)U_Qg(P)^{-1} \right) = U_{PQ}.$$

17 and one also gets a commutative diagram

$$18 \quad \begin{array}{ccc} & X(APQ) & \\ g(Q)^{AP} \swarrow & & \searrow g(P)^{AQ} \\ X(AP) & & X(AQ) \\ g(P)^{AQ} \swarrow & & \searrow g(Q)^A \\ & X(A) & \end{array}$$

19 From this, and the definition of  $T_P^A$ , we get assertion (1) of Proposition 3.9. For  
 20 assertion (2) we note that  $T_P$  descends compatibly to levels  $A$  and  $B$  with  $A \mid B$ ,  
 21 as long as  $P \nmid B$ . Compatibility with  $\phi_j^*$  follows.

22 **3.10. Newforms and oldforms.** The cohomology of  $X(A)$  decomposes natu-  
 23 rally into oldforms and newforms. The former, denoted  $H^i(X(A))^{\text{old}}$  is by  
 24 definition the subspace of  $H^i(X(A))$  generated by the images of  $\phi_{J,B/A}^*$  as  $B$   
 25 varies over all the ideals properly containing  $A$  and  $J$  varies over divisors of  
 26  $B/A$  which are prime to  $\text{ram}(D)$ . Note that an oldform is therefore fixed by a  
 27 conjugate of a congruence subgroup of smaller level.

28 As we saw above, the adèlic quotient  $X(U_A)$  is not connected, and we have

$$29 \quad H^3(X(U_A)) = H^3(X_{\mathbb{A}})^{U_A} \simeq \mathbb{Q}^{h(A)},$$

30 and this holds even when  $X(U_A)$  is an orbifold. As  $X(A)$  is the connected  
 31 component, it follows that

$$32 \quad H^3(X(A)) \simeq \mathbb{Q}.$$

1 Let us focus on  $H^1$ . There is a complementary *new subspace*  $H^1(X(A))^{\text{new}}$ ,  
 2 which consists of cohomology classes which are genuine to the level at hand,  
 3 and it can be defined as the annihilator of  $H^2(X(A))^{\text{old}}$  under the cup product  
 4 pairing

$$5 \quad \cup: H^1(X(A)) \times H^2(X(A)) \rightarrow H^3(X(A)) \simeq \mathbb{Q}.$$

6 Since the elements of the Hecke algebra act as correspondences on  $X(A)$ , this  
 7 pairing is also functorial for the Hecke action.

8 **3.11. The main geometric theorem.** Now we turn to

9 **3.12. Theorem.** *Let  $M = X(A)$  be an arithmetic hyperbolic 3-manifold defined as*  
 10 *above from a quaternion algebra  $D/K$  and an ideal  $A$  of  $\mathcal{O}_K$ . Let  $P_1, P_2, \dots, P_n$  be*  
 11 *prime ideals of  $\mathcal{O}_K$  coprime to  $A$  at which  $D$  does not ramify. Consider the correspond-*  
 12 *ing congruence cover  $M(P_1 P_2 \cdots P_n) \rightarrow M$ , which has degree  $\prod (1 + N_{K/\mathbb{Q}}(P_i))$ .*  
 13 *Suppose  $\omega \in H^1(M)$  comes from a fibration over the circle, and that  $T_{P_i}(\omega) = 0$  for*  
 14 *each  $P_i$ . Then  $M(P_1 P_2 \cdots P_n)$  has at least  $2^n$  pairs of genuinely distinct fibered faces.*

15 Theorem 1.2 follows from Theorem 3.12 once we exhibit in Section 4 an  $M$   
 16 and an  $\omega$ , together with an *infinite* set of primes  $\{P_i\}$  such that  $T_{P_i}(\omega) = 0$ , by  
 17 considering all the covers  $M(P_1 P_2 \cdots P_m)$ .

18 *Proof.* We claim inductively that  $M_m = M(P_1 P_2 \cdots P_m)$  has  $2^m$  fibered classes  
 19  $\omega_i$  lying in genuinely distinct fibered faces which are killed by  $T_{P_k}$  for all  $k > m$ .  
 20 To simplify notation, set  $P = P_{m+1}$ . By the discussion above, we have two  
 21 covering maps

$$22 \quad \phi_1, \phi_P: M_{m+1} \rightarrow M_m$$

23 and let  $\tilde{\omega}_j$  be the  $2^{m+1}$  classes which are the pull-backs of the  $\omega_i$  by  $\phi_1$  and  $\phi_P$ .  
 24 As  $T_P$  kills all the  $\omega_i$ , Lemma 3.4 implies that the  $\tilde{\omega}_j$  lie in genuinely distinct  
 25 fibered faces. The commutativity property of Hecke operators and degeneracy  
 26 maps implies that  $T_{P_k}(\tilde{\omega}_j) = 0$  for all  $j$  and any  $k > m + 1$ , completing the  
 27 induction. □

#### 28 4. THE BASE MANIFOLD $M$

29 Here is the arithmetic description of our example in the framework of Sec-  
 30 tion 3.6. Consider the imaginary quadratic field  $K = \mathbb{Q}[\sqrt{-3}]$ , with ring of  
 31 integers  $\mathcal{O}_K$ . The non-trivial automorphism of  $K$  can be identified with com-  
 32 plex conjugation via embedding  $K$  into  $\mathbb{C}$ , and will be denoted as such. The  
 33 rational prime 7, or rather the ideal  $7\mathcal{O}_K$ , splits in  $K$  as  $Q\bar{Q}$ . Let  $D$  be the unique  
 34 quaternion division algebra over  $K$  which ramifies exactly at  $Q$  and  $\bar{Q}$ . Let  $\mathcal{O}_D$   
 35 be a maximal order of  $D$ ; this order is unique up to conjugation as  $K$  is qua-  
 36 dratic and has restricted class number  $h_\infty = 1$ . The corresponding hyperbolic  
 37 3-orbifold is  $X = \mathcal{O}_D^\times \backslash \mathcal{H}^3$ . Our example manifold  $M$  is the (principal) con-  
 38 gruence cover of  $X$  of level  $Q\bar{Q}$ . To state its key properties, we need one more  
 39 piece of notation. Let  $\mathcal{P}$  be the set of rational primes  $p \neq 7$  which are inert in  
 40  $\mathbb{Q}[\sqrt{-7}]$  but are split in  $K$ , and consider the set of prime ideals of  $\mathcal{O}_K$  given by  
 41  $\mathcal{P}_K = \{P \mid N_{K/\mathbb{Q}}(P) = p \in \mathcal{P}\}$ . We will show:

1 **4.1. Theorem.** *Let  $M$  be the arithmetic hyperbolic 3-manifold described above.*

- 2 (1) *The new subspace  $V = H^1(M; \mathbb{Q})^{\text{new}}$  is 2-dimensional and isotypic under the*  
 3 *Hecke action. Moreover, for each prime  $P$  in the set  $\mathcal{P}_K$  described above,  $V$  is*  
 4 *in the kernel of the Hecke operator  $T_P$ .*  
 5 (2) *There is an  $\omega \in V$  coming from a fibration of  $M$  over the circle.*

6 The proofs of parts (1) and (2) are essentially independent, and we tackle  
 7 them separately in the next two sections. The main result of this paper, Theo-  
 8 rem 1.4, follows from considering the congruence covers  $M(P_1 P_2 \cdots P_n) \rightarrow M$   
 9 where  $P_j \in \mathcal{P}_K$ . Theorem 4.1 implies that  $M(P_1 P_2 \cdots P_n)$  satisfies the hypothe-  
 10 ses of Theorem 3.12, and hence has at least  $2^{2n}$  fibered faces. Using that  $\mathcal{P}$   
 11 consists of  $1/4$  of all rational primes, we then calculate a lower bound on the  
 12 number of fibered faces in terms of the degree of the cover. The details of this  
 13 proof of Theorem 1.4 are given in Section 7.

14 The manifold  $M$  is the only one we could find with a CM form coming from  
 15 a fibration over the circle. Because CM forms are fairly rare, there are very few  
 16 potential examples where one can computationally examine the topology in  
 17 order to check fibering, even with the improved methods we introduce here.  
 18 We were very fortunate that  $M$  does indeed have the desired properties, since  
 19 there are probably only one or two more potential examples that are within  
 20 reach. The next example to look at would have been starting with the elliptic  
 21 curve  $y^2 + y = x^3$  over  $\mathbb{Q}$  which has conductor 27 and is CM by  $\mathbb{Q}(\sqrt{-3})$  and  
 22 then base-changing to  $\mathbb{Q}(\sqrt{-2})$ .

## 23 5. THE AUTOMORPHIC STRUCTURE OF THE COHOMOLOGY OF $M$

24 In this section, we prove part (1) of Theorem 4.1. The reader is urged to refer  
 25 back to Section 1.5 for an outline before proceeding.

26 **5.1. A cusp form form on  $GL(2, \mathbb{Q})$ .** Let  $E$  be the elliptic curve over  $\mathbb{Q}$  defined  
 27 by

$$28 \quad y^2 + xy = x^3 - x^2 - 2x - 1.$$

29 It has conductor 49, admitting complex multiplication by an order in the imag-  
 30 inary quadratic field  $L := \mathbb{Q}[\sqrt{-7}]$ . For any rational prime  $p$ , we set, as usual,

$$31 \quad a_p(E) := p + 1 - N_p(E),$$

32 where  $N_p(E)$  is the number of points of  $E$  modulo  $p$ . The  $L$ -function of  $E$  is  
 33 given by

$$34 \quad L(s, E) := \prod_p \left( 1 - a_p(E)p^{-s} + p^{1-2s} \right)^{-1}$$

35 which, by a theorem of Deuring (see e.g. [Gro]), equals

$$36 \quad L(s, \Psi) := \prod_P (1 - \Psi(P) \mathcal{N}_{L/\mathbb{Q}}(P)^{-s})^{-1} = \prod_p \left( \prod_{P|p} (1 - \Psi(P) \mathcal{N}_{L/\mathbb{Q}}(P)^{-s}) \right)^{-1}$$

1 where  $\Psi$  is a Hecke character of  $L$  of weight 1, and  $\mathcal{N}_{L/\mathbb{Q}}$  denotes the norm  
 2 from  $L$  to  $\mathbb{Q}$ . Equating, for each  $p$ , the corresponding Euler  $p$ -factors of  $L(s, E)$   
 3 and  $L(s, \Psi)$ , we see that

$$4 \quad p \in \mathcal{O}_L \text{ prime} \implies a_p(E) = 0.$$

5 One knows by Hecke, or by using the converse theorem, that  $\Psi$  defines a  
 6 holomorphic cusp form of weight 2 for the congruence subgroup  $\Gamma_0(49)$  of  
 7  $\mathrm{SL}(2, \mathbb{Z})$ , given by  $f(z) = \sum_{n \geq 1} a_n q^n$  with  $q = e^{2\pi iz}$  which is new for that level.  
 8 Here, we normalize  $f$  so that  $a_1(f) = 1$  and  $a_p(f)$  is the eigenvalue of the  $p^{\mathrm{th}}$   
 9 Hecke operator  $T_p$ . One obtains, for every prime  $p \neq 7$ , that  $a_p(f) = a_p(E)$ .  
 10 Thus  $a_p(f)$  is zero whenever  $p$  is inert in  $L$ .

11 **5.2. Ramification at 7.** The cusp form  $f$  is ramified at 7, but to make this pre-  
 12 cise, and to see what happens when we base change to  $K = \mathbb{Q}[\sqrt{-3}]$ , it is  
 13 necessary for us to consider the situation adèlically.

14 For any number field  $k$ , let  $\mathbb{A}_k$  denote the topological ring of adèles of  $k$ ,  
 15 which is a restricted direct product  $\prod'_v \mathbb{Q}_v$ , as  $v$  runs over the places of  $k$  and  
 16  $k_v$  denotes the completion of  $k$  at  $v$ . When  $k$  has only one archimedean place  
 17 up to complex conjugation, e.g.  $k = \mathbb{Q}$  or  $k$  imaginary quadratic, this place is  
 18 denoted by  $\infty$ . Given any algebraic group over  $k$ , it makes sense to consider  
 19 the locally compact topological group  $G(\mathbb{A}_k) = \prod'_v G(k_v)$ .

20 One knows that the cusp form  $f$  generates a unitary, cuspidal automorphic  
 21 representation  $\pi = \otimes'_v \pi_v$  of  $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$  with trivial central character, with  $\pi_{\infty}$   
 22 being the lowest discrete series representation of  $\mathrm{GL}(2, \mathbb{R})$  ([Gel]), such that  
 23  $L(s - 1/2, \pi) = L(s, f)$ . By construction,  $\pi_p$  is unramified at every prime  $p \neq$   
 24  $7$ , which translates to the elliptic curve  $E$  having good reduction at  $p$ . More  
 25 precisely, the conductor  $c(\pi)$  of  $\pi$  satisfies

$$26 \quad c(\pi) = c(\pi_7) = 49.$$

27 We need to identify the local representation at 7, and in fact to make sure that  
 28 it is not a principal series representation.

29 **5.3. Lemma.** *Let  $f, \pi$  be as above. Then  $\pi_7$  is a supercuspidal representation of*  
 30  *$\mathrm{GL}(2, \mathbb{Q}_7)$  of conductor 49.*

31 After this paper was written, we learnt from B. Gross of an earlier, different  
 32 proof of this Lemma in [GL].

33 *Proof.* Let us first note a few basic facts. Since  $\pi$  is defined by a Hecke character  
 34 of a quadratic extension, or equivalently since  $E$  has complex multiplication,  
 35 the local representation  $\pi_p$  at any prime  $p$  is either in the principal series or  
 36 is supercuspidal. In either case, we claim that there is a finite extension  $F/\mathbb{Q}_p$   
 37 over which (the base change of)  $\pi_p$  becomes an unramified principal series rep-  
 38 resentation. Indeed, by the local Langlands correspondence for  $\mathrm{GL}(2)$  [Kut], at  
 39 any  $p$  the local representation  $\pi_p$  is attached to a semisimple 2-dimensional  $\mathbb{C}$ -  
 40 representation  $\sigma_p$  of the local Weil group  $W_{\mathbb{Q}_p}$ , whose determinant corresponds  
 41 to the central character of  $\pi_p$  (by local class field theory), which is trivial in  
 42 our case. Recall that  $W_{\mathbb{Q}_p}$  is the dense subgroup of the absolute Galois group

1  $G_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  generated by the inertia subgroup and the integral pow-  
 2 ers of a lift to  $G_p$  of the Frobenius automorphism  $x \rightarrow x^p$  of  $\overline{\mathbb{F}}_p$ . One knows  
 3 that  $\pi_p$  is in the principal series if and only if  $\sigma_p$  is reducible, necessarily of  
 4 the form  $\nu_p \oplus \nu_p^{-1}$ . We may write  $\nu_p$  as an unramified character times a finite  
 5 order character  $\lambda_p$ , since the inertia subgroup  $I_p$  acts by a finite quotient, and  
 6 so  $\sigma_p$  becomes unramified over the finite extension  $\mathbb{Q}_p(\lambda_p)$ , which is the cyclic  
 7 extension of  $\mathbb{Q}_p$  cut out by  $\lambda_p$  by class field theory. By looking at the  $\ell$ -adic  
 8 representation of  $G_p$  attached to  $\sigma_p$ , and using the Néron-Ogg-Shafarevich cri-  
 9 terion [Sil], we see that  $E$  also acquires good reduction over  $\mathbb{Q}_p(\lambda_p)$ . Suppose  
 10 now  $\sigma_p$  is irreducible, in which case  $\pi_p$  is supercuspidal. Again,  $\sigma_p$  is acted on  
 11 by  $I_p$  through a finite quotient  $H_p$ , because  $I_p$  is profinite and  $\sigma_p$  takes values  
 12 in  $\text{GL}(2, \mathbb{C})$ . Consequently, there exists a finite Galois extension  $F/\mathbb{Q}_p$  with Ga-  
 13 lois group  $H_p$  such that over  $F$ , the representation  $\sigma_p$  becomes unramified and  
 14  $E$  attains good reduction. Hence the claim.

15 We argue further that this line of reasoning shows that  $\pi_p$  is in the principal  
 16 series if and only if  $E$  acquires good reduction over an *abelian* extension of  $\mathbb{Q}_p$ .  
 17 Indeed, this is clear if  $\pi_p$  is in the principal series, so we may assume that  $\pi_p$   
 18 is not of this kind. Then  $\sigma_p$  is irreducible and becomes unramified only over a  
 19 more complicated extension  $F$  due to irreducibility;  $F$  is dihedral for  $p \geq 3$ , but  
 20 can be wilder for  $p = 2$ .

21 For  $p \geq 5$ , a theorem of Rohrlich [Roh] asserts that  $E$  acquires good reduction  
 22 over an abelian extension of  $\mathbb{Q}_p$  if and only if the following field is abelian over  
 23  $\mathbb{Q}_p$ :

$$F := \mathbb{Q}_p(\Delta^{1/e}),$$

24 where  $\Delta$  is the discriminant of  $E$  and

$$e = \frac{12}{(v_p(\Delta), 12)},$$

25 where the denominator on the right signifies the gcd of  $v_p(\Delta)$  and 12.

26 In the present case,  $p = 7$  and  $\Delta = -7^3$ , so that  $e = 12/3 = 4$ . Consequently,  
 27  $F$  is generated over  $\mathbb{Q}_7$  by a fourth root of 343. The only way  $F$  can be abelian  
 28 over  $\mathbb{Q}_7$  is for  $\mathbb{Q}_7$  to contain the fourth roots of unity, i.e., for  $-1$  to be a square  
 29 in  $\mathbb{Q}_7$ , and hence in  $\mathbb{F}_7$ . But this does not happen because the Legendre symbol  
 30  $(\frac{-1}{7})$  is  $(-1)^{(7-1)/2} = -1$ . Hence  $\pi_7$  is supercuspidal.

31 Finally, the conductor  $c(\pi)$  of  $\pi$  factors as

$$c(\pi) = \prod_p c(\pi_p),$$

32 and since  $\pi$  is associated to  $E$ , the conductor  $c(\pi)$  coincides with that of  $E$ ,  
 33 which is 49. Moreover, as  $\pi_p$  is unramified at every prime outside 7, we have  
 34  $c(\pi_p) = 1$  for every  $p \neq 7$ . It follows that  $c(\pi_7) = 49$ .  $\square$

35 **5.4. Local transfer to the quaternion algebra over  $\mathbb{Q}_7$ .** Denote by  $B$  the unique  
 36 quaternion division algebra over  $\mathbb{Q}_7$ . Then  $B^*$  is an inner form of  $\text{GL}(2)$  over  
 37  $\mathbb{Q}_7$ . By the local Jacquet-Langlands correspondence [JL, §16], there is a finite-  
 38 dimensional, irreducible representation  $\pi_7^B$  of  $B^*$  functorially associated to  $\pi_7$ .

1 Let  $\mathcal{O}_B$  denote the maximal order of  $B$ , and for any  $m \geq 0$ , let  $\Gamma_B(7^m)$  denote  
 2 the principal congruence subgroup of level  $7^m$ , with  $\Gamma_B(1)$  denoting  $\mathcal{O}_B^*$ .

3 **5.5. Lemma.** *The dimension of  $\pi_7^B$  is 2, and its conductor is 49. This representation*  
 4 *is non-trivial when restricted to  $\mathcal{O}_B^*$ , and the kernel contains  $\Gamma_B(7)$ .*

5 Some people would be tempted to say that  $\pi_7^B$  has conductor 7, but it would  
 6 not be in agreement with the convention used in the theory of automorphic  
 7 forms, where the trivial representation of  $B^*$  is said to have (normalized) con-  
 8 ductor 7 (see [Tun, §3]).

9 *Proof.* Two of the basic properties of the correspondence  $\pi_7 \rightarrow \pi_7^B$  are the fol-  
 10 lowing:

- 11 (1) The dimension of  $\pi_7^B$  is the formal dimension of  $\pi_7$   
 12 (2)  $c(\pi_7^B) = c(\pi_7)$

13 The second property is a consequence of the coincidence of the  $\varepsilon$ -factors of  $\pi_7^B$   
 14 and  $\pi_7$ .

15 Thanks to (1), it suffices to check that the formal dimension of  $\pi_7$  is 2. Let us  
 16 first make the following

17 **5.6. Claim.** *The 2-dimensional representation  $\sigma_7$  of  $W_{\mathbb{Q}_7}$  associated to  $\pi_7$  is induced*  
 18 *by a character  $\chi$  of  $W_F$ , where  $F$  is the unique unramified quadratic extension of  $\mathbb{Q}_7$ .*

19 To begin, as  $\pi$  is associated to a Hecke character  $\Psi$  of  $\mathbb{Q}[\sqrt{-7}]$ , the represen-  
 20 tation  $\sigma_7$  is induced by a character  $\psi$ , of  $W_k$ , where  $k$  is the ramified quadratic  
 21 extension of  $\mathbb{Q}_7$  obtained by completing  $L$  at the prime  $(\sqrt{-7})$ . (The character  
 22  $\psi$  is the local component at  $\sqrt{-7}$  of the global unitary idèle class character of  $L$   
 23 defined by  $\Psi$ .) Then we have

$$24 \quad c(\sigma_7) = \mathcal{N}_{k/\mathbb{Q}_7}(c(\psi)) \text{disc}_{k/\mathbb{Q}_7},$$

25 where  $\mathcal{N}$  denotes the norm and  $\text{disc}$  the discriminant. Since  $-7 \equiv 1 \pmod{4}$ ,  
 26  $\text{disc}_{k/\mathbb{Q}_7}$  is  $-7$ , which forces  $c(\psi)$  to be  $P := (\sqrt{-7})\mathcal{O}_k$ , because  $c(\sigma_7) = 49$  and  
 27  $\mathcal{N}_{k/\mathbb{Q}_7}(\sqrt{-7}) = 7$ . Moreover, we have

$$28 \quad \det(\sigma_7) = \text{Ver}_{k/\mathbb{Q}_7}(\psi)\eta_k,$$

29 where  $\text{Ver}$  denotes the *transfer map*, and  $\eta_k$  is the quadratic character  $W_{\mathbb{Q}_7}$  at-  
 30 tached to  $k/\mathbb{Q}_7$ . As the determinant of  $\sigma_7$  is 1, the restriction of  $\psi$  to  $\mathbb{Q}_7^*$  must be  
 31 the quadratic character  $\eta_k$ . If  $\tau$  denotes the non-trivial automorphism of  $k/\mathbb{Q}_7$ ,  
 32 we must have

$$33 \quad \psi^\tau = \psi\nu,$$

34 where  $\nu$  is the unique quadratic unramified character of  $W_k$ . (As usual,  $\psi^\tau(x) =$   
 35  $\psi(x^\tau)$ , for all  $x \in W_k$ .) Indeed, with  $\{U_k^j\}$  denoting the usual filtration of the unit  
 36 group  $U_k$ ,  $\psi$  and  $\psi^\tau$  are trivial on  $U_k^1$ , but not on  $U_k$ , since  $c(\psi) = P$ . Also,  $\psi/\psi^\tau$   
 37 is evidently trivial on  $\mathbb{Q}_7^*$ , implying that  $\psi/\psi^\tau$  is even trivial on the full unit  
 38 group  $U_k$ , and thus must be an unramified character  $\nu$ . Also, since  $(\psi^\tau)^\tau = \psi$ ,  
 39 we get  $\nu^2 = 1$ . On the other hand, since  $k = \mathbb{Q}_7(\sqrt{-7})$ , the automorphism  $\tau$   
 40 sends the uniformizer  $\sqrt{-7}$  to its negative, and so  $\psi^\tau$  differs from  $\psi$ ; in other

1 words,  $\nu$  must be the unique unramified quadratic character of  $W_k$ . Next ob-  
 2 serve that since  $k/\mathbb{Q}_7$  is ramified, the unique unramified quadratic extension of  
 3  $k$  is the compositum  $Fk$ , where  $F$  is the unique unramified quadratic extension  
 4 of  $\mathbb{Q}_7$ . This forces  $\nu$  to be of the form  $\eta_F \circ \mathcal{N}_{K/k}$ , and so we have

$$5 \quad \sigma_7 \otimes \eta_F \simeq I_k^{\mathbb{Q}_7}(\psi\nu) \simeq I_k^{\mathbb{Q}_7}(\psi^\tau) \simeq \sigma_7,$$

6 as  $\sigma_7$  is induced equally by  $\psi$  and  $\psi^\tau$ . Thus  $\sigma_7$  admits a self-twist under  $\eta_F$ , or  
 7 equivalently, that there is a  $W_{\mathbb{Q}_7}$ -homomorphism

$$8 \quad \eta_F \hookrightarrow \sigma_7 \otimes \sigma_7 \simeq 1 \oplus \text{sym}^2(\sigma_7).$$

9 Here we have used the triviality of  $\det(\sigma_7)$ , and  $\text{sym}^2(\sigma_7)$  denotes the (3-di-  
 10 mensional) symmetric square representation. Combining this with  $\sigma_7 \simeq \sigma_7^\vee$ ,  
 11 we get a  $W_{\mathbb{Q}_7}$ -isomorphism

$$12 \quad \text{End}(\sigma_7) \simeq \sigma_7^{\otimes 2} \simeq \eta_F \oplus \mathbf{1} \oplus \beta,$$

13 for a 2-dimensional representation  $\beta (\subset \text{sym}^2(\sigma_7))$ . Since  $\eta_F$  is trivial on  $W_F$ ,  
 14 we get

$$15 \quad \dim_{\mathbb{C}} \text{Hom}_{W_F}(\mathbf{1}, \text{End}(\sigma_7)) \geq 2.$$

16 So by Schur, the restriction of  $\sigma_7$  to  $W_F$  is reducible, and if  $\chi$  is a character of  $W_F$   
 17 appearing in  $\sigma_7|_{W_F}$ , Frobenius reciprocity implies that  $\sigma_7$  is induced by  $\chi$ . This  
 18 proves Claim 5.6.

19 By the local correspondence [JL], the representation  $\pi_7$  is associated to the  
 20 character  $\chi$  of  $F^*$  ( $\simeq W_F^{\text{ab}}$ ), with  $F/\mathbb{Q}_7$  unramified. Now we may apply Lemma  
 21 5 of [PR], which summarizes the results we want on the formal degree, etc. In  
 22 that lemma,  $\text{cond}(-)$  means the *exponent* of  $c(-)$ , and the quadratic extension  
 23  $K$  there corresponds to our (unramified)  $F$  here. (It should be noted that the  
 24  $K$  in that lemma is taken to be unramified if the representation is attached to  
 25 more than one quadratic extension.) Since  $\text{cond}(\chi) = 1$  for us, we see that  $\pi_7^B$   
 26 has dimension 2. Also,  $c(\pi_7^B) = 1$ . This implies the last part of the lemma  
 27 by the discussion in [Tun], cf. the two paragraphs in Section 3 before Theorem  
 28 3.6.  $\square$

29 **5.7. Base change to  $\mathbb{Q}[\sqrt{-3}]$  and the global transfer.** Put  $K = \mathbb{Q}[\sqrt{-3}]$ , in  
 30 which the prime 7 splits as follows:

$$31 \quad 7\mathcal{O}_K = Q\bar{Q}, \quad Q = \left(2 + (1 + i\sqrt{3})/2\right).$$

32 Let  $\pi_K = \otimes'_v \pi_{K,v}$  denote the base change [Lan] of  $\pi$  to  $K$ , which has conductor  
 33  $Q^2\bar{Q}^2$ , the reason being that 7 splits in  $K$  as above, and consequently,

$$34 \quad \pi_{K,Q} \simeq \pi_{K,\bar{Q}} \simeq \pi_7.$$

35 Also,  $\pi_K$  still has trivial central character. More importantly, since  $\pi_K$  has a su-  
 36 percuspidal component, it must be cuspidal. One can also see this from Deur-  
 37 ing's theory since  $K$  and  $L$  are disjoint.

1 Let  $D$  be, once again, the quaternion division algebra over  $K$ , which is rami-  
 2 fied only at  $Q$  and  $\bar{Q}$ . Then one has, by the Jacquet-Langlands correspondence,  
 3 a corresponding cuspidal automorphic representation  $\pi_K^D$  of  $D(\mathbb{A}_K)^*$  such that

$$4 \quad v \notin \{Q, \bar{Q}\} \implies \pi_{K,v}^D = \pi_{K,v},$$

5 which makes sense because  $D_V$  and  $\mathrm{GL}(2, K_v)$  are the same at any place  $v \neq$   
 6  $Q, \bar{Q}$ . More importantly,

$$7 \quad D_Q \simeq D_{\bar{Q}} \simeq B,$$

8 where  $B$  is the quaternion division algebra considered in the previous section,  
 9 and by the naturality of the Jacquet-Langlands correspondence,

$$10 \quad (5.8) \quad \pi_{K_Q}^D \simeq \pi_{K_{\bar{Q}}}^D \simeq \pi_7^B.$$

11 **5.9. Lemma.** *The conductor of  $\pi_K^D$  is 49. The corresponding principal congruence*  
 12 *subgroup  $U$  of  $D^*(\mathbb{A}_{K,f})$  is of level  $7\mathcal{O}_K = Q\bar{Q}$ . Moreover, the space of vectors fixed*  
 13 *by  $U$  is of dimension 4.*

14 *Proof.* If  $v$  is any finite place of  $K$  outside  $\{Q, \bar{Q}\}$ , the local representation  $\pi_{K,v}^D \simeq$   
 15  $\pi_{K,v}$  is unramified and hence has trivial conductor. Thus we have, by applying  
 16 Lemma 5.5,

$$17 \quad c\left(\pi_K^D\right) = c\left(\pi_{K,Q}^D\right) c\left(\pi_{K,\bar{Q}}^D\right) = c\left(\pi_7^B\right)^2 = 7^2 = (Q\bar{Q})^2.$$

18 By our convention on the conductor, it follows the corresponding principal  
 19 congruence subgroup is of level  $7\mathcal{O}_K$ . The final claim of the lemma follows  
 20 from Lemma 5.5, in view of (5.8) above.  $\square$

21 **5.10. The  $\pi^D$ -isotypic subspace.** Put

$$22 \quad \mathcal{V}(D) := L^2\left(Z(\mathbb{A}_K)D^* \backslash D(\mathbb{A}_K)^*\right),$$

23 where  $Z$  is the center ( $\simeq K^*$ ) of the algebraic group  $D^*$  over  $K$ . Then  $\mathcal{V}(D)$   
 24 is a unitary representation of  $D(\mathbb{A}_K)^*$  under the right action. The coset space  
 25  $Z(\mathbb{A}_K)D^* \backslash D(\mathbb{A}_K)^*$  is compact, which follows by Godement's compactness cri-  
 26 terion as  $D^*$  contains no unipotents  $\neq 1$ . Consequently, one has a (Hilbert)  
 27 direct sum decomposition (as unitary  $D(\mathbb{A}_K)^*$ -modules)

$$28 \quad \mathcal{V}(D) \simeq \bigoplus_{\eta} m(\eta) \mathcal{V}_{\eta},$$

29 where  $(\eta, \mathcal{V}_{\eta})$  runs over irreducible unitary, admissible representations of the  
 30 group  $D(\mathbb{A}_K)^*$ , with multiplicities  $m(\eta) \geq 0$ .

31 The Jacquet-Langlands correspondence which embeds  $\mathcal{V}(D)$  into the dis-  
 32 crete spectrum of the corresponding space for  $\mathrm{GL}(2, \mathbb{A}_K)$ , and since the multi-  
 33 plicities are 1 on the  $\mathrm{GL}(2)/K$ -side [JL], we get

$$34 \quad m(\eta) \leq 1, \quad \text{for all } \eta.$$

35 Since  $\pi_K^D = \pi_{K,\infty} \otimes \pi_{K,f}^D$  is by construction a cuspidal automorphic repre-  
 36 sentation of  $D(\mathbb{A}_K)^*$  of trivial central character, it gives rise to a non-trivial

1 summand in  $\mathcal{V}(D)$ . In particular,

$$2 \quad m(\pi_K^D) = 1.$$

3 The  $\pi_K^D$ -isotypic subspace of  $\mathcal{V}(D)$ , identifiable with  $\mathcal{V}_{\pi_K^D}$ , is infinite-dimen-  
4 sional. However, due to the admissibility of  $\pi_{K,f}^D$ , given any compact open  
5 subgroup  $U$  of  $D(\mathbb{A}_{K,f})^*$ , the space  $U$ -invariants in  $\pi_{K,f}^D$  is finite dimensional.  
6 Applying Lemma 5.9, we obtain the following

5.11. **Lemma.**

$$7 \quad \dim_{\mathbb{C}} \mathcal{V}_{\pi_{K,f}^D}^{U_D(7)} = 4.$$

8 5.12. **Connected components of the adèlic double coset space.** Now put  $G =$   
9  $D^*/K$  and consider the adèlic quotient space

$$10 \quad X_{\mathbb{A}} := G(K)Z(\mathbb{A}_K) \backslash \mathcal{H}^3 \times G(\mathbb{A}_{K,f}),$$

11 introduced in the middle of Section 3.7, which admits a right action by  $G(\mathbb{A}_{K,f})$ .  
12 Also set

$$13 \quad A = Q\bar{Q}, \quad \text{and} \quad X_{U_A} = X_{\mathbb{A}}/U_A,$$

14 where

$$15 \quad U_A = U_Q(Q) \times U_{\bar{Q}}(\bar{Q}) \times \left( \prod_{P \neq Q, \bar{Q}} G(\mathcal{O}_{K_P}) \right),$$

16 where  $U_Q(Q)$  is the principal congruence subgroup of  $G(K_Q)$  of level 1, i.e.,  
17 the kernel of the map to  $G(\mathcal{F}_7)$ , and the same for  $U_{\bar{Q}}(\bar{Q}) \leq G(K_{\bar{Q}})$ . We will  
18 write  $U_D(7)$  instead of  $U_A$  to refer to the specific choice of  $A$  here.

19 5.13. **Lemma.**  $X_{U_A}$  has two connected components, both diffeomorphic the manifold  
20  $M$  defined in Section 4.

21 *Proof.* The disconnectedness comes because we are working with the group  $D^*$   
22 and not its semisimple part

$$23 \quad D^1 := \text{Ker}(Nrd : D^* \rightarrow K^*),$$

24 where  $Nrd$  denotes the reduced norm. In our case, we have

$$25 \quad X_{U_A} = D^* \backslash \mathcal{H}^3 \times \left( Z(\mathbb{A}_{K,f}) \backslash D(\mathbb{A}_{K,f})^* / U_D(7) \right)$$

26 where  $D^*$  acts diagonally. Thus the number of connected components of  $M^{\mathbb{A}}$   
27 is

$$28 \quad (5.14) \quad \pi_0(M^{\mathbb{A}}) = D^* Z(\mathbb{A}_{K,f}) \backslash D(\mathbb{A}_{K,f})^* / U_D(7)$$

29 Were we working with  $D^1$ , we would have

$$30 \quad \pi_0 = D^1 \backslash D(\mathbb{A}_{K,f})^1 / U$$

31 where the right hand quotient is a single element as  $D^1$  is dense in  $D(\mathbb{A}_{K,f})^1$   
32 by strong approximation.

1 Returning to the case at hand, it is straightforward to check that we can eval-  
 2 uate the right hand side of (5.14) by taking its image under  $Nrd$ . Since  $Nrd$   
 3 surjects  $D^*$  onto  $K^*$ , as well as  $D(\mathbb{A}_K)^*$  onto  $\mathbb{A}_K^*$ , and  $Z(\mathbb{A}_K)$  onto  $\mathbb{A}_K^{*2}$ , we get

$$4 \quad \pi_0(M^{\mathbb{A}}) = \mathbb{A}_{K,f}^* \backslash {}^2K^* \backslash \mathbb{A}_{K,f}^* / Nrd(U_D(7)).$$

5 Note that since the class number of  $K = \mathbb{Q}[\sqrt{-3}]$  is 1, we have by strong  
 6 approximation,

$$7 \quad \mathbb{A}_K^* = K^* K_\infty^* \prod_P U_P,$$

8 where  $P$  runs over all the (finite) primes of  $\mathcal{O}_K$ , and  $U_P = \mathcal{O}_{K_P}^*$ . Moreover,

$$9 \quad Nrd(U_D(7)) = \left( \prod_{P \neq Q, \bar{Q}} U_P \right) \times U_Q^1 \times U_{\bar{Q}}^1,$$

10 where  $U_Q^1 = 1 + Q\mathcal{O}_{K_Q}$ .

11 As 7 splits into  $Q, \bar{Q}$  in  $K$ , we have  $\mathcal{O}_{K_Q} \simeq \mathcal{O}_{K_{\bar{Q}}} \simeq \mathbb{Z}_7$ , and moreover,  $\mathbb{Z}_7^* / \mathbb{Z}_7^{*2}$   
 12 identifies with  $\mathbb{F}_7^* / \mathbb{F}_7^{*2} \simeq \{\pm 1\}$ . It follows that

$$13 \quad \pi_0(M^{\mathbb{A}}) = \text{Coker} \left( \mathcal{O}_K^* \rightarrow \left( \mathcal{O}_{K_Q}^* / \mathcal{O}_{K_Q}^{*2} \right) \times \left( \mathcal{O}_{K_{\bar{Q}}}^* / \mathcal{O}_{K_{\bar{Q}}}^{*2} \right) \simeq \left( \mathbb{F}_7^* / \mathbb{F}_7^{*2} \right)^2 \right).$$

14 The map  $\mathcal{O}_K^* \rightarrow \mathbb{F}_7^* \times \mathbb{F}_7^*$  is onto the diagonal, and thus  $M^{\mathbb{A}}$  has two connected  
 15 components.

16 To complete the lemma, here is an explicit description of the two compo-  
 17 nents. Let  $x = (x_v)$  be an element of  $\mathbb{A}_K^*$  with  $x_v = 1$  for all  $v \neq Q$ , such that  
 18 the  $Q$ -component  $x_Q$  is an element of  $\mathcal{O}_{K_Q}^*$  ( $\simeq \mathbb{Z}_7^*$ ) mapping onto a non-square  
 19 of  $\mathbb{F}_7^*$ . By the preceding, we get an identification

$$20 \quad M^{\mathbb{A}} = M_\Gamma \cup M_{\Gamma'},$$

21 with connected components

$$22 \quad M_\Gamma = \Gamma \backslash \mathcal{H}^3, \quad M_{\Gamma'} = \Gamma' \backslash \mathcal{H}^3,$$

23 where

$$24 \quad \Gamma = \Gamma(Q\bar{Q}) = D^* \cap U_D(7), \quad \Gamma' = D^* \cap (xU_D(7)x^{-1}).$$

25 Now  $M_\Gamma$  is precisely the manifold described in Section 4, as claimed, and  $M_{\Gamma'}$   
 26 is diffeomorphic to  $M_\Gamma$  since  $U_D(7)$  is a normal subgroup of  $U_D$ .  $\square$

27 **5.15. Cohomology on the compact side.** Put  $G = D^*/Z$ . Then  $\Gamma = \Gamma(Q\bar{Q})$   
 28 defines a torsion-free lattice in  $G_\infty = \text{PGL}(2, \mathbb{C})$ . Since  $\mathcal{H}^3$  is contractible,  $M_\Gamma$   
 29 is an Eilenberg-MacLane space for  $\Gamma$ , and thus

$$30 \quad H^*(M_\Gamma, \mathbb{Q}) \simeq H^*(\Gamma, \mathbb{Q}).$$

31 Moreover, since  $\Gamma$  is cocompact, by a refinement of Shapiro's lemma [Bla], we  
 32 have

$$33 \quad H^*(\Gamma, \mathbb{C}) \simeq H_{\text{cont}}^* \left( G_\infty, L^2(\Gamma \backslash G_\infty)^\infty \right),$$

1 where  $H_{\text{cont}}^*$  denotes continuous cohomology, and the superscript  $\infty$  on (the  
 2 right regular representation)  $L^2(\Gamma \backslash G_\infty)$ , which is the unitary analog of the  
 3 group algebra of a finite group, denotes taking the smooth vectors. Conse-  
 4 quently, if  $L^2(\Gamma \backslash G_\infty)$  decomposes as a unitary  $G_\infty$ -module as  $\oplus m_\infty(\beta)W_\beta$ ,

$$5 \quad H^*(M_\Gamma, \mathbb{C}) \simeq H_{\text{cont}}^* \left( G_\infty, L^2(\Gamma \backslash G_\infty)^\infty \right) \simeq \oplus_{(\beta, W_\beta)} m_\infty(\beta) H_{\text{cont}}^*(G_\infty, W_\beta^\infty).$$

6 One knows that there is a unique irreducible, unitary representation  $(\beta_0, W_{\beta_0})$   
 7 of  $G_\infty$  for which

$$8 \quad H_{\text{cont}}^i(G_\infty, W_{\beta_0}^\infty) \neq 0, \quad \text{for } i = 1, 2,$$

9 and the dimension is 1. In fact (see [Clo], for example), the 2-dimensional rep-  
 10 resentation of  $W_{\mathbb{C}} = \mathbb{C}^*$  attached to this  $\beta_0$  (by the local archimedean corre-  
 11 spondence) is given by

$$12 \quad \sigma_\infty : z \rightarrow \left( \frac{z}{|z|} \right) \oplus \left( \frac{\bar{z}}{|z|} \right).$$

13 By construction, the parameter of our  $\pi_{K, \infty} \simeq \pi_{K, \infty}^D$  is also of this form. Hence  
 14 we have

$$15 \quad \pi_{K, \infty}^D \simeq W_{\beta_0}^\infty.$$

16 If  $L^2(G(K) \backslash G(\mathbb{A}_K) / U_D(7)) \simeq \oplus_{\Pi} m(\Pi) \Pi$  (as unitary  $G(\mathbb{A}_K)$ -modules), a  
 17 straightforward adèlic refinement of the above yields the following identifica-  
 18 tion:

$$\begin{aligned} H^*(M^{\mathbb{A}}, \mathbb{C}) &\simeq H_{\text{cont}}^* \left( G_\infty, L^2 \left( G(K) \backslash G(\mathbb{A}_K) / U_D(7) \right) \right) \\ &\simeq \oplus_{\Pi} H_{\text{cont}}^*(G_\infty, \mathcal{V}_{\Pi_\infty}) \otimes \Pi_f^{U_D(7)}, \end{aligned}$$

19 where  $\mathcal{V}_{\Pi} = \mathcal{V}_{\Pi_\infty} \otimes \mathcal{V}_{\Pi_f}$  denotes the admissible subspace of (the space of)  $\Pi =$   
 20  $\Pi_\infty \otimes \Pi_f$ .

21 Next recall that  $m_{\Pi}$  is 1, and that  $H_{\text{cont}}^i(G_\infty, W_{\beta_0}^\infty)$  is 1-dimensional for  $i = 1, 2$ .  
 22 So we get an isotypic decomposition (for  $i \in \{1, 2\}$ ),

$$23 \quad H^i(M^{\mathbb{A}}, \mathbb{C}) \simeq \oplus_{\{\Pi : \Pi_\infty \simeq \pi_{K, \infty}^D\}} H^i(M^{\mathbb{A}}, \Pi_f),$$

24 where

$$25 \quad \dim_{\mathbb{C}} H^i(M^{\mathbb{A}}, \Pi_f) = \dim_{\mathbb{C}} \mathcal{V}_{\Pi_f}^{U_D(7)}.$$

26 In particular, since our  $\pi_K^D$  is one such  $\Pi$ , and since the space of  $U_D(7)$ -invari-  
 27 ants of  $\pi_{K, f}^D$  is 4-dimensional, we obtain

28 **5.16. Lemma.** *For  $i = 1, 2$ , we have  $\dim_{\mathbb{C}} H^i(M^{\mathbb{A}}, \pi_{K, f}^D) = 4$ .*

29 Combining this with the discussion of the connected components of  $M^{\mathbb{A}}$  in  
 30 the previous section, since  $M^{\mathbb{A}}$  has two isomorphic components, we get the  
 31 following

32 **5.17. Theorem.** *For  $i = 1, 2$ , we have  $\dim_{\mathbb{C}} H^i(M_\Gamma; \mathbb{C})^{\text{new}} = 2$ .*

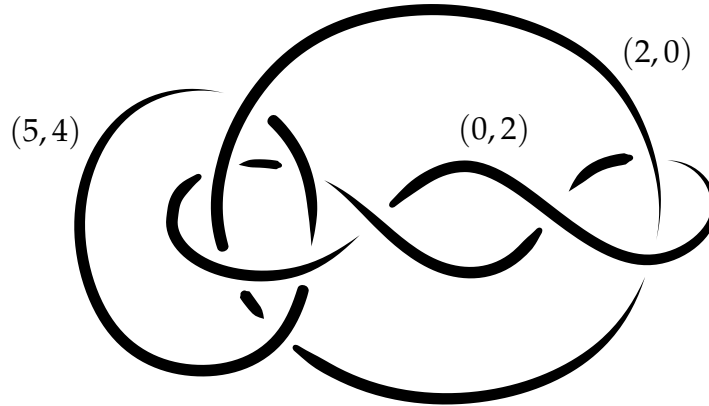


FIGURE 6.2. A Dehn surgery description of the hyperbolic 3-orbifold  $B$ . Our orientation conventions are given by  $\leftarrow \rightarrow$ , and match those of SnapPea [W].

6. THE THURSTON NORM AND FIBERED FACES OF  $M$

In this section, we study the topological properties of  $M$  and show

**6.1. Theorem.** *Let  $M = X(\overline{Q\overline{Q}})$  be the hyperbolic 3-manifold described in Section 4. Then  $H^1(M; \mathbb{Q})$  is 3-dimensional and*

- (1) *The Thurston norm ball  $B_T \subset H^1(M; \mathbb{R})$  is a parallelepiped, i.e. an affine cube.*
- (2) *Exactly four of the six faces of  $B_T$  are fibered.*
- (3) *The oldforms  $H^1(M; \mathbb{Q})^{\text{old}}$  consist of the line passing the barycenters of the non-fibered faces.*

Here, the *barycenter* of a face is the average of its vertices. We will now show that if Theorem 6.1 holds, then so does Theorem 4.1(2).

*Proof of Theorem 4.1(2) modulo Theorem 6.1.* Part (1) of Theorem 4.1 gives that the subspace  $V = H^1(M; \mathbb{Q})^{\text{new}}$  is 2-dimensional; we need to show that  $V$  intersects the interiors of one of the four fibered faces. We can change coordinates on  $H^1(M; \mathbb{Q})$  by an element of  $GL(3, \mathbb{Q})$  so that  $B_T$  is the standard cube with vertices  $(\pm 1, \pm 1, \pm 1)$ , and the non-fibered faces are the ones intersecting the  $z$ -axis. By part (3), the set of oldforms  $W$  is then the  $z$ -axis itself. Now  $\dim V = 2$ , and among all 2-dimensional subspaces, only two miss the interiors of the fibered faces, namely the ones containing a vertical edge of  $B_T$  where two fibered faces meet. But both of these subspaces contain the  $z$ -axis of oldforms  $W$  and so can't be  $V$ . So there is a fibered class in  $V$  as desired.  $\square$

The rest of this section is devoted to *proving* Theorem 6.1. We will produce an explicit triangulation of  $M$  and use a series of tricks to compute the Thurston norm. We believe these tricks are new; they are remarkably effective in many examples, and (we hope) are of independent interest. Our computations used an extensive variety of software: [W, G, BC, CD, SAGE]. Source code, data files, and complete computational details are available at [DR].

1 **6.3. Finding a topological description.** First, we give a concrete topological  
 2 description of  $M$ . Let's review the setup of Section 4. Let  $K = \mathbb{Q}(\sqrt{-3})$  and let  
 3  $Q$  and  $\bar{Q}$  be the prime ideals sitting over 7. Let  $D$  be the quaternion algebra over  
 4  $K$  ramified precisely at  $Q$  and  $\bar{Q}$ . Let  $\mathcal{O}_D$  be a maximal order of  $D$ . Let  $\phi$  be the  
 5 embedding of  $D^*/K^*$  into  $\mathrm{PGL}(2, \mathbb{C})$ , and set  $\Gamma = \phi(\mathcal{O}_D^*)$  and  $X = \Gamma \backslash \mathcal{H}^3$ . Also,  
 6 let  $\mathcal{O}_D^1$  denote the elements of norm 1, and set  $\Gamma' = \phi(\mathcal{O}_D^1)$  and  $X' = \Gamma' \backslash \mathcal{H}^3$ .  
 7 We are interested in the congruence covers  $X(Q)$ ,  $X(\bar{Q})$ , and  $M = X(Q\bar{Q})$ .

8 **6.4. The topology of  $X'$ .** It is often easier to find  $X'$  than  $X$  because of the  
 9 following characterization of the former. For a lattice  $\Lambda$  in  $\mathrm{PSL}(2, \mathbb{C})$ , let  $\tilde{\Lambda}$   
 10 denote the preimage lattice in  $\mathrm{SL}(2, \mathbb{C})$ . Let  $k\Lambda$  be the number field generated  
 11 by the traces of elements of  $\tilde{\Lambda}$ , and  $A\Lambda$  be the quaternion algebra generated by  
 12 the elements of  $\tilde{\Lambda}$  (see e.g. [MR, Ch 3] for details). Then Corollary 8.3.6 and  
 13 Theorem 11.1.3 of [MR] imply that  $\Lambda \backslash \mathcal{H}^3$  is  $X'$  if:

- 14 (1)  $k\Lambda = K$  and  $A\Lambda = D$ .
- 15 (2) The trace of every element in  $\tilde{\Lambda}$  is an algebraic integer.
- 16 (3) The volume of  $\Lambda \backslash \mathcal{H}^3$  is  $3^{3/2}36\zeta_K(2)/(4\pi^2) \approx 6.0896496384579219$ .

17 Here, the degree of  $K$  is small, as is the volume, so we can expect these condi-  
 18 tions to be verified by the program Snap [CGHN, G] if it is given a topological  
 19 description of the orbifold  $X'$ .

20 In our case, a brute-force search finds the following description of  $X'$ . Let  $B$   
 21 be the orbifold shown in Figure 6.2. From this topological description, SnapPea  
 22 [W] derives the following presentation for  $\pi_1(B)$ :

$$23 \quad \left\langle a, b \mid b^2 = aBABaba^4baBABabABA^4BA b = aBABaba^3baBABaba^3b = 1 \right\rangle,$$

24 where  $A = a^{-1}$  and  $B = b^{-1}$ . Hence  $H_1(B; \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/8$ , and so  $B$  has a  
 25 unique regular cover  $C$  with covering group  $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ . SnapPea can explic-  
 26 itly build<sup>1</sup>  $C$  as a cover of  $B$  given the action of  $\pi_1(B)$  on the cosets of  $\pi_1(C)$ .  
 27 Snap checks that  $C$  is  $X'$ . This is a little subtle as the reasonable spun ideal  
 28 triangulations of  $C$  have some flat ideal tetrahedra (though no negatively ori-  
 29 ented ones), and it is necessary to invoke [PW], in slightly modified form, to  
 30 justify that we have constructed the hyperbolic structure. For details, see [DR].

31 Though we did not check this rigorously, and will not use it in this paper,  
 32 the manifold  $X$  is almost certainly the 2-fold cover of  $B$  whose homology is  
 33  $\mathbb{Z}/8 \oplus \mathbb{Z}/8$ ; this cover is the SnapPea orbifold  $s883(2, -2)(2, -2)$ .

34 **6.5. Constructing  $X(Q)$ .** In order to build  $X(Q)$ , we begin by claiming that it  
 35 covers  $X'$  and that this cover  $X(Q) \rightarrow X'$  is a cyclic cover of degree 4. First,  
 36 we work out the structure of the local quaternion algebra  $D_Q = K_Q \otimes_K D$  by

<sup>1</sup>Prior to 2007/9/12, the version of the SnapPea kernel [W] that came with SnapPeaPython [WCD] contained a bug where it sometimes returned the wrong orbifold when the cover it was trying to construct was a proper orbifold. The underlying topology was correct but the orbifold loci are sometimes mislabeled by proper divisors of their real orders. This bug also affects the MacOS Classic versions of SnapPea, e.g. SnapPea 2.6 and the derived Windows port. The calculations here were, of course, done with a corrected version of the kernel.

1 following [MR, §6.4]. We have  $K_Q \cong \mathbf{Q}_7$  by Hensel’s lemma since  $-3$  has a  
 2 square-root mod 7. Let  $F$  be the unique unramified quadratic extension of  $K_Q$ ,  
 3 concretely  $F = \mathbf{Q}_7(i)$ . Then we have  $D_Q \cong F \oplus Fj$  where  $i$  and  $j$  anti-commute  
 4 and  $j^2 = 7$ . Then  $\mathcal{O}_{D_Q} \cong \mathcal{O}_F \oplus \mathcal{O}_{Fj}$ . Consider the homomorphism

5  $\mathcal{O}_{D_Q}^* \rightarrow \bar{F}^*$  defined by  $x + yj \mapsto \bar{x}$ , where  $\bar{F} \cong \mathbb{F}_{49}$  is the residue field of  $F$ .

6 If  $\Lambda$  is the kernel of the restricted homomorphism  $\mathcal{O}_D^* \rightarrow \bar{F}^*$ , then by definition  
 7  $X(Q) = \Lambda \backslash \mathcal{H}^3$ .

8 Now the norm map  $Nrd: D_Q \rightarrow K_Q$  is given by  $Nrd(x + yj) = Nrd(x) -$   
 9  $7Nrd(y)$ . Thus if  $\alpha$  is in the kernel of  $\mathcal{O}_D^* \rightarrow \bar{F}^*$ , we have  $Nrd(\alpha) = 1$  in  $\bar{K}_Q$ . As  
 10 the units of  $\mathcal{O}_K$ , the sixth roots of unity, all map to distinct elements in  $\bar{K}_Q^*$ , it  
 11 follows that  $\alpha$  is in fact in  $\mathcal{O}_D^1$ . Thus we have  $\Lambda \subset \mathcal{O}_D^1$  and so  $X'(Q) = X(Q)$ ;  
 12 henceforth we focus on the former description.

13 The image of  $\mathcal{O}_{D_Q}^1 \rightarrow \bar{F}^\times$  is just the elements of  $\mathbb{F}_{49}/\mathbb{F}_7$ -norm 1; these el-  
 14 ements form a (cyclic) subgroup  $C$  of order 8. By strong approximation, the  
 15 image of  $\mathcal{O}^1 \rightarrow \bar{F}^\times$  is the same. Now  $-1 \notin \Lambda$ , and so  $\Lambda \cong \pi_1(X'(Q))$ . On the  
 16 other hand,  $-1$  maps into  $C$  non-trivially, and so we conclude that  $\pi_1(X'(Q))$   
 17 is a normal subgroup of  $\pi_1(X')$  with quotient  $C/\{\pm 1\} \cong C_4$ .

18 Further, we can characterize  $\Lambda$  as exactly those  $\alpha \in \mathcal{O}_D^1$  such that  $\text{tr}(\alpha) \equiv 2$   
 19 mod  $Q$ . This is because if  $x \in \bar{F}$  has norm 1 and  $\text{tr}(x) = 2$  then  $x = 1$  because  
 20  $x$  satisfies  $x^2 - \text{tr}(x)x + Nrd(x)$ . Searching through the normal subgroups of  
 21  $\pi_1(X')$  with quotient  $C_4$ , one quickly finds a unique subgroup for which all  
 22 tested elements have trace which is  $\pm 2 \pmod{Q}$  in the topologically given  
 23  $\text{PSL}(2, \mathbf{C})$ -representation; this subgroup must be  $\Lambda$ . In fact,  $X(Q)$  and  $X(\bar{Q})$   
 24 are exactly those 4-fold cyclic covers of  $X'$  whose homology is  $\mathbf{Z}/28 \oplus \mathbf{Z}/28 \oplus$   
 25  $\mathbf{Z}$ . Repeating the same procedure for  $\bar{Q}$  builds  $\pi_1(X(\bar{Q}))$  as a subgroup of  
 26  $\pi_1(X') \leq \pi_1(B)$ . Taking their intersection yields  $\pi_1(X(Q\bar{Q}))$ . From the associ-  
 27 ated permutation action on  $\pi_1(B)/\pi_1(X(Q\bar{Q}))$ , SnapPea can build an explicit  
 28 triangulation for  $M = X(Q\bar{Q})$ .

29 **6.6. The Thurston norm of  $M = X(Q\bar{Q})$ .** For this manifold  $M$ , we calculate  
 30  $H_1(M; \mathbf{Z}) = \mathbf{Z}^3 \oplus (\mathbf{Z}/14)^4 \oplus (\mathbf{Z}/2)^2$ . Our next task is to compute the Thurston  
 31 norm ball  $B_T$  in  $H^1(M; \mathbf{R}) = \mathbf{R}^3$ . In general, there is an algorithm using normal  
 32 surfaces for computing the Thurston norm. Normal surfaces are those which  
 33 cut through each simplex in the triangulation like a union of affine planes, and  
 34 have been the bedrock algorithmic tool in 3-dimensional topology since Haken  
 35 first used them to detect the unknot [Hak]. For the Thurston norm, the running  
 36 time of this normal surface algorithm is “only” simply exponential in the size  
 37 of the triangulation [CT], but any triangulation of  $M$  must have at least 95  
 38 tetrahedra as its volume is  $\approx 97.434394$ , and the triangulations we constructed  
 39 had 130 or more. At that complexity, normal surface methods appear to be  
 40 hopeless. Instead, we take a different approach, which involves producing  
 41 normal surfaces representing certain classes in  $H_2(M)$  quite cheaply.

1 The other key tool will be the Alexander polynomial  $\Delta_M$  of  $M$  which lies in  
 2 the group ring  $\mathbb{Z}[H_1(M; \mathbb{Z})/(\text{torsion})]$ . Following McMullen [McM], if  $\Delta_M =$   
 3  $\sum a_i g_i$  for  $a_i \in \mathbb{Z}$  and  $g_i \in H_1(M; \mathbb{Z})$  then we define the *Alexander norm* on  
 4  $H^1(M; \mathbb{R})$  by

$$5 \quad \|\omega\|_A = \sup_{i,j} \omega(g_i - g_j).$$

6 The unit ball  $B_A$  of this norm is the dual polyhedron to the Newton polytope  
 7 of  $\Delta_M$  inside  $H_1(M; \mathbb{R})$ , which is the convex hull of the  $g_i$ . Since  $b_1(M) =$   
 8  $\dim H^1(M; \mathbb{R}) > 1$ , we have  $\|\omega\|_A \leq \|\omega\|_T$  for all  $\omega \in H^1(M; \mathbb{R})$ , or equiva-  
 9 lently  $B_T \subset B_A$  [McM]. For a general 3-manifold, these norms do not always  
 10 coincide, but it is actually quite common for them to do so, and we will show

11 **6.7. Lemma.** *The Alexander and Thurston norms agree for  $M = X(Q\bar{Q})$ .*

12 Computing directly from a presentation of  $\pi_1(M)$  yields:

$$13 \quad (6.8) \quad \Delta_M = (16xyz - xy - xz - y - z + 16)^4$$

14 Taking powers of a polynomial just dilates its Newton polytope, so the Newton  
 15 polytope of  $\Delta_M$  is, up the change of basis, an octahedron. Dually, this means  
 16 that  $B_A$  is a cube. Since  $B_T$  is a convex subset of  $B_A$ , to prove Lemma 6.7 it suf-  
 17 fices to check that the two norms agree on the vertices of the cube. Moreover,  
 18  $M$  is very symmetric as  $\pi_1(M) \triangleleft \pi_1(B)$ , and any symmetry of  $M$  preserves  
 19 both norms, as does the map  $\iota: \omega \rightarrow -\omega$  on  $H^1(M; \mathbb{R})$ . One calculates that the  
 20 image of

$$21 \quad \langle \iota, \pi_1(B) \rangle \rightarrow \text{Aut} \left( H^1(M; \mathbb{R}) \right)$$

22 has order 16. Now the full symmetry group of the cube has order 48, and  
 23 any subgroup of order 16 acts transitively on the vertices, since the full vertex  
 24 stabilizer has order 6. Thus, it suffices to check  $\|\omega\|_A = \|\omega\|_T$  for a single  
 25 vertex of the cube.

26 For computing the Thurston norm of a single element of  $\omega \in H^1(M; \mathbb{Z})$   
 27 we used the following method. Choose a (non-classical) triangulation  $\mathcal{T}$  of  
 28  $M$  which has only one vertex, à la Jaco-Rubinstein [JR]. Then there is a unique  
 29 simplicial 1-cocycle representing the given class  $\omega$ . As observed in [Cal], any 1-  
 30 cocycle gives a canonical map  $M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$  which is affine on each simplex  
 31 of  $\mathcal{T}$ . More precisely, the integer  $\omega$  assigns to an edge specifies how many  
 32 times to wrap it around the circle, and this extends over each simplex because  
 33 of the cocycle condition. The inverse image of a generic point in  $S^1$  is a normal  
 34 surface which is Poincaré dual to  $\omega$ . By randomizing the triangulation  $\mathcal{T}$  a few  
 35 times using Pachner moves, we were able to find a genus 3 surface representing  
 36 one of the vertices of  $B_A$ , which shows the two norms agree for that class. This  
 37 proves Lemma 6.7, and thus part (1) of Theorem 6.1.

38 **6.9. Remark.** In practice, this seems to be a very effective method for computing  
 39 the Thurston norm when the first betti number  $b_1$  is 1, even when the triangu-  
 40 lation has a couple hundred tetrahedra. When  $b_1 > 1$ , things are more difficult,  
 41 as it seems one cannot expect to find a triangulation where these special dual

1 surfaces realize the Thurston norm for all  $\omega$ . While here we evaded this is-  
 2 sue by using symmetry, a general approach would be to carry along cohomol-  
 3 ogy isomorphisms between the new triangulations and the original one while  
 4 changing the triangulation using Pachner moves. This should enable probing  
 5 the whole Thurston norm ball using this method.

6 **6.10. Fiberings of  $M$ .** Next, we determine which faces of  $B_T$  are fibered. Again,  
 7 there is a normal surface based algorithm to decide this [Sch, TW, JT]. How-  
 8 ever, instead of using it, we build on the method described above. First, we use  
 9 the symmetries of  $M$  to simplify the problem. We know each face of  $B_T = B_A$   
 10 is associated to a vertex of the Newton polytope of  $\Delta_M$ , and hence has a co-  
 11 efficient of  $\Delta_M$  associated to it. A generalization of the classical Alexander  
 12 polynomial obstruction to fibering says that if a face of  $B_T$  is fibered, then this  
 13 coefficient must be  $\pm 1$  (see e.g. [Dun, Thm. 5.1]). In our case, these vertex  
 14 coefficients are  $\{16^4, 16^4, 1, 1, 1, 1\}$ ; recall from (6.8) that  $\Delta_M$  is a power of a  
 15 simpler polynomial  $f$ , and so the coefficients are the corresponding powers of  
 16 the vertex coefficients of the Newton polytope of  $f$ . Thus there is one pair of  
 17 faces which are definitely not fibered (since their coefficients are  $16^4$ ), and two  
 18 pairs of faces which *likely* fiber. Recall we have a group  $S$  of order 16 acting  
 19 on  $H^1(M; \mathbb{R})$  which preserves  $B_T = B_A$ , including the coefficient labels on the  
 20 faces (up to sign). Thus  $S$  preserves the two faces with labels  $16^4$  and has a  
 21 subgroup of order 8 which leaves each of them invariant; this subgroup acts  
 22 there by the full symmetry group of the square, the dihedral group of order  
 23 8. In particular,  $S$  acts transitively on the remaining four faces of  $B_A$ . Thus to  
 24 show that all these faces fiber, and so prove part (2) of Theorem 6.1, it suffices  
 25 to show:

26 **6.11. Lemma.** *The manifold  $M = X(Q\bar{Q})$  fibers over the circle.*

27 To check this, we consider the 8-fold cover  $N$  of the orbifold  $B$  of Figure 6.2  
 28 given by

29  $\pi_1(B) \rightarrow S_8$  where  $a \mapsto (1, 2, 3, 6, 8, 5, 7, 4), b \mapsto (1, 3)(2, 5)(4, 6)(7, 8),$

30 which is a manifold. Now  $\pi_1(N)$  is not a subgroup of  $\pi_1(M)$ , but we can con-  
 31 sider their minimal common cover  $Y$  with fundamental group  $\pi_1(N) \cap \pi_1(M)$   
 32 as shown



34 A key fact is that  $\dim H^1(Y; \mathbb{R}) = 3$ . Thus we may apply the theorem of  
 35 Stallings mentioned in Section 2.1 and conclude, as  $Y$  and  $M$  have the same  
 36 first Betti number, that  $M$  fibers if and only if  $Y$  does. Thus if  $N$  fibers, so  
 37 does  $M$ . Now  $H_1(N; \mathbb{Z}) = \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}$ , and its Alexander polynomial  
 38 is  $\Delta_N = t^4 + 30t^2 + 1$ , suggesting that  $N$  is fibered by genus 2 surfaces. The

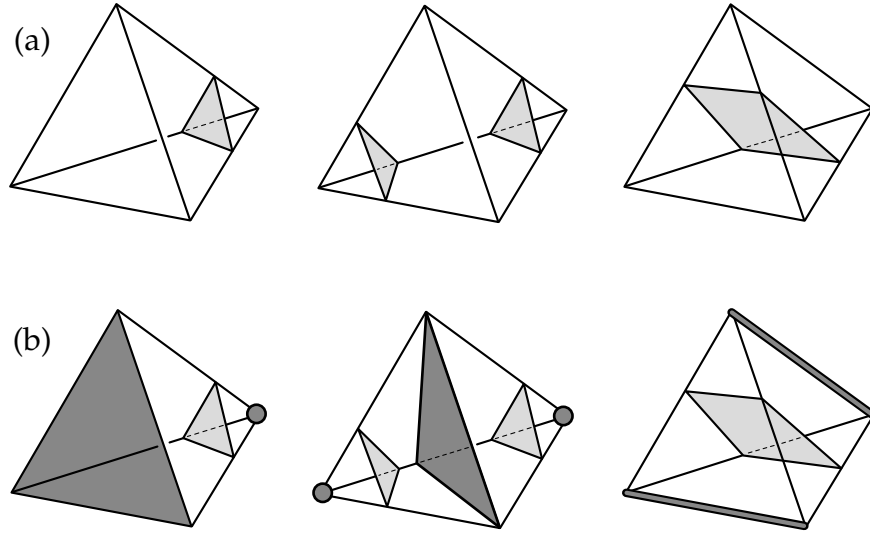


FIGURE 6.12. At the top, the three possibilities, besides the empty set, for how  $\Sigma$  intersects a tetrahedron of  $\mathcal{T}$ . The pictures at the bottom describe, in dark grey, a 2-complex  $P$  to which  $N \setminus \Sigma$  deformation retracts.

1 method of Section 6.6 finds a genus 2 normal surface  $\Sigma$  representing the gen-  
 2 erator of  $H_2(N; \mathbb{Z})$  (the particular triangulation of  $N$  we used, and hence an  
 3 explicit description of  $\Sigma$ , is available at [DR]). To show that  $N \setminus \Sigma$  is  $\Sigma \times I$  and  
 4 hence  $N$  fibers, it suffices to check that  $\pi_1(N \setminus \Sigma)$  is *abstractly* isomorphic to  
 5  $\pi_1(\Sigma)$  (see e.g. [Hem, Thm. 10.6]). To find an initial presentation of  $\pi_1(N \setminus \Sigma)$ ,  
 6 we can exploit the fact that in a suitable triangulation  $\mathcal{T}$  of  $N$ , the normal sur-  
 7 face  $\Sigma$  is very simple; in each tetrahedron it looks like one of the three possibili-  
 8 ties shown in Figure 6.12(a). Part (b) of the same figure describes a 2-complex  $P$   
 9 which is a spine for  $N \setminus \Sigma$ , i.e.  $N \setminus \Sigma$  deformation retracts to  $P$ . It is easy to read  
 10 off a presentation for  $\pi_1(N \setminus \Sigma)$  from this spine; simplifying this presentation  
 11 using Tietze transformations [MKS] and relabeling the generators yields

$$12 \quad \pi_1(N \setminus \Sigma) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$$

13 which is the standard presentation for the fundamental group of a genus 2  
 14 surface. So  $N$  fibers, and hence so does  $M$ .

15 6.13. *Remark.* This procedure for checking fibering is applicable much more  
 16 generally. For any normal surface, there is a natural spine for its comple-  
 17 ment, introduced by Casson in his study of 0-efficient triangulations (paral-  
 18 leling [JR]). Writing down a presentation of  $\pi_1(N \setminus \Sigma)$  from this is straightfor-  
 19 ward. In more complicated cases, simplifying the presentation of  $\pi_1(N \setminus \Sigma)$  in  
 20 the hopes of recognizing it as a surface group, it is important to employ not  
 21 just Tietze transformations, but Nielsen transformations as well, i.e. the appli-  
 22 cation of elements in  $\text{Aut}(F_n)$  to the generating set of the presentation in order  
 23 to shorten the relators. If one can reduce down to a presentation with only one  
 24 relator, then checking it is a surface group is often easy: just glue up the edges

1 of the relator polygon, and if there is only one vertex in what results, then the  
 2 group is indeed a surface group.

3 **6.14. Oldforms for  $M = X(\overline{Q})$ .** We turn now to the last assertion of Theo-  
 4 rem 6.1, namely that the space of oldforms in  $H^1(M; \mathbb{Q})$  is the line determined  
 5 by the barycenters of the non-fibered faces. By Theorem 4.1(1), we know that  
 6  $\dim H^1(M; \mathbb{Q})^{\text{new}} = 2$ , so as  $\dim H^1(M; \mathbb{Q}) = 3$  it follows that  $H^1(M; \mathbb{Q})^{\text{old}}$   
 7 has dimension 1 and is the image of  $\phi^*: H^1(X(Q); \mathbb{Q}) \rightarrow H^1(M; \mathbb{Q})$  where  
 8  $\phi = \phi_1: M \rightarrow X(Q)$  is the standard covering map. Since we have explicit pre-  
 9 sentations for both  $\pi_1(X(Q))$  and  $\pi_1(M)$  as subgroups of  $\pi_1(B)$ , computing  
 10 the image of  $\phi^*$  is straightforward; for details see [DR].

11 **7. PROOF OF THE MAIN RESULT**

12 **7.1. Proof of Theorem 1.4.** Continuing the notation of Section 4, let  $M$  be the  
 13 arithmetic hyperbolic 3-manifold described there, and  $\omega \in H^1(M; \mathbb{Z})$  be the  
 14 class given by part (2) of Theorem 4.1. Then  $\omega$  is fibered and for each prime  $P$  in  
 15 the special set  $\mathcal{P}_K$  we have  $T_P(\omega) = 0$ . Denote the primes of  $\mathcal{P}$  as  $p_1, p_2, p_3, \dots$   
 16 in increasing order, and fix an ordering, once and for all, of the primes of  $\mathcal{P}_K$   
 17 as  $P_1, P_2, \dots$  such that  $i \leq j \implies N(P_i) \leq N(P_j)$ .

18 Consider the congruence cover  $M_n = M(P_1 \cdots P_{2n})$ . By Theorem 3.12, the  
 19 number  $\nu_n$  of pairs of fibered faces of  $M_n$  is at least  $2^{2n}$ , and the degree  $d_n$  of  
 20  $M_n \rightarrow M$  is (see Section 3.6):

21 (7.2) 
$$d_n = \prod_{i=1}^{2n} (1 + \mathcal{N}_{K/\mathbb{Q}}(P_i)) = \left( \prod_{i=1}^n (1 + p_i) \right)^2$$

22 To prove Theorem 1.4 we will estimate  $d_n$  by using the fact that  $\mathcal{P}$  consists  
 23 of 1/4 of all rational primes. More precisely, recall that the *natural density* of  
 24 a set  $\mathcal{Y}$  of primes is  $\alpha \in [0, 1]$  if the number of primes in  $\mathcal{Y}$  which are  $\leq x$   
 25 is asymptotic (for  $x$  large) to  $\alpha$  times  $\pi(x) = |\{p \leq x\}|$ . In our case,  $\mathcal{P}$  is  
 26 the set of primes  $p$  which split in both  $\mathbb{Q}[\sqrt{-3}]$  and  $\mathbb{Q}[\sqrt{-7}]$ , i.e., those which  
 27 split completely in the biquadratic field  $\mathbb{Q}[\sqrt{-3}, \sqrt{-7}]$ , and so  $\mathcal{P}$  has natural  
 28 density 1/4 by the Tchebotarev density theorem. Theorem 1.4 now follows  
 29 directly from:

30 **7.3. Lemma.** *Let  $\mathcal{P} = \{p_1, p_2, p_3, \dots\}$  be a set of rational primes of natural density*  
 31 *1/4, and  $d_n$  the product defined in (7.2). Then*

32 
$$n = \left( \frac{1}{2} + o(1) \right) \frac{\log d_n}{\log \log d_n}.$$

33 *In particular,*

34 
$$2^{2n} = \exp \left( (\log 2 + o(1)) \frac{\log d_n}{\log \log d_n} \right) \leq c_t e^{(\log d_n)^t},$$

35 *for any  $t < 1$ , for a suitable constant  $c_t > 0$ .*

36 Some people prefer to work with weaker notions of density, like analytic  
 37 density, but in those cases this lemma may not hold.

1 *Proof.* If  $f$  and  $g$  are functions of  $x$  on a subset of  $\mathbb{R}_+$ , we will write  $f(x) \sim g(x)$   
 2 if their ratio tends to 1 as  $x$  goes to infinity. Now recall that the prime number  
 3 theorem asserts that  $\pi(x) \sim \frac{x}{\log x}$ , and consequently the  $n^{\text{th}}$  prime is  $\sim n \log n$ .

4 As the set  $\mathcal{P}$  has natural density  $1/4$ , the  $n^{\text{th}}$  prime  $p_n$  in  $\mathcal{P}$  satisfies

$$5 \quad (7.4) \quad p_n \sim 4n \log n.$$

6 Moreover, we claim that

$$7 \quad (7.5) \quad \log \left( \prod_{j=1}^n (1 + p_j) \right) \sim \frac{1}{4} p_n.$$

8 In fact, the Riemann hypothesis implies that this holds with an additive error  
 9 term  $\rho(n)$  of the order of  $n^{1/2+\epsilon}$ . Here, we only need that  $\rho(n)$  is  $o(n \log n)$ ,  
 10 which is known; in fact one has (cf. [MV, Ch. 6]):

$$11 \quad |\rho(n)| \leq n e^{-c\sqrt{\log n}} \quad \text{for some } c > 0.$$

12 Combining the asymptotic statements (7.4) and (7.5), and noting the formula  
 13 for  $d_n$ , we get

$$14 \quad 2n \log n \sim \log d_n.$$

15 This yields, upon taking logarithms,

$$16 \quad \log n + \log(2 \log n) \sim \log \log d_n,$$

17 and since the left hand side is  $(1 + o(1)) \log n$ , we obtain

$$18 \quad 2n \sim \frac{\log d_n}{\log n} = (1 + o(1)) \frac{\log d_n}{\log \log d_n}.$$

19 Since  $2^{2n} = e^{2n \log 2}$ , this implies

$$20 \quad (7.6) \quad 2^{2n} = \exp \left( (\log 2 + o(1)) \frac{\log d_n}{\log \log d_n} \right).$$

21 Moreover, for any real number  $t < 1$ , the quantity  $\frac{\log d_n}{\log \log d_n}$  dominates  $(\log d_n)^t$   
 22 as  $n \rightarrow \infty$ . So the right hand side of (7.6) is bounded below by a constant (de-  
 23 pending on  $t$ ) times  $\exp((\log d_n)^t)$ , as asserted by the lemma. This completes  
 24 the proof of Theorem 1.4.  $\square$

25 **7.7. A lower bound for  $b_1(M_n)$ .** Recall that the first Betti number of any of the  
 26 manifolds  $M_n$  is the same as the dimension of space of the cohomological cusp

27 forms on  $D^*$  of level  $I_n = Q\bar{Q} \prod_{j=1}^n P_j \bar{P}_j$ , with  $P_j, \bar{P}_j$  being the prime ideals in  $\mathcal{O}_K$

28 above the rational prime  $p_j \in \mathcal{P}$ . From the arithmetic point of view, the di-  
 29 mension of this space is a complete mystery. This is because the two standard  
 30 tools, namely the Riemann-Roch theorem and the Selberg trace formula, which  
 31 work so well in the case of Hilbert modular forms (of weight 2), are not appli-  
 32 cable here; the former because the  $M_n$  are not algebraic varieties, and the latter  
 33 because the relevant archimedean representation is not in the discrete series,

1 making it impossible to separate it from all the other archimedean representa-  
 2 tions. Nevertheless, we can get a lower bound as follows.

3 First recall that for any  $N \geq 1$ , the dimension of the space  $S_2(N)$  of clas-  
 4 sical, holomorphic cusp forms on the upper half-plane  $\mathcal{H}$  of level  $N$  is the  
 5 genus  $g_0(N)$  of the standard (cusp) compactification of the Riemann surface  
 6  $\Gamma_0(N) \backslash \mathcal{H}$ , where  $\Gamma_0(N)$  is the subgroup of  $\mathrm{SL}(2, \mathbb{Z})$  consisting of matrices  
 7  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $N \mid c$ . It is classical that for  $N$  square-free, one has

$$8 \quad g_0(N) = 1 + \frac{1}{12} \prod_{p|N} (p+1) - \frac{2^s}{4} - \frac{2^t}{3} - \frac{2^m}{2},$$

9 where  $m$  is the number of prime divisors of  $N$ , and  $s$  (resp.  $t$ ) the number of  
 10 such divisors which are 1 mod 4 (resp. 1 mod 3); the last three terms of the for-  
 11 mula are the error terms corresponding to the elliptic points  $i$  and  $(1 + \sqrt{-3})/2$   
 12 on  $\mathcal{H}$  and the cusp at infinity. Applying this with  $N = N_n = 7 \prod_{j=1}^n p_j$ , with  
 13  $p_j \neq 7$  being split in  $\mathbb{Q}[\sqrt{-3}]$  and inert in  $\mathbb{Q}[\sqrt{-7}]$ , we see by the formula for  
 14  $d_n$  that

$$15 \quad g_0(N_n) \geq \frac{\sqrt{d_n} - (13)2^n}{12}.$$

16 On the other hand, by Lemma 7.3, the term  $2^n$  is bounded from above by any  
 17 positive power of  $d_n$ . Moreover, the dimension of the subspace  $S_2(N_n)^{7\text{-new}}$   
 18 of  $S_2(N_n)$  consisting of cusp forms of level  $N$  which are new at 7 is  $\frac{5}{7}g_0(N)$ .  
 19 Putting these together, we obtain, for any  $\varepsilon > 0$ ,

$$20 \quad \dim_{\mathbb{C}} S_2(N_n)^{7\text{-new}} \geq \frac{5}{84} d_n^{\frac{1}{2}-\varepsilon},$$

21 for all large enough  $n$  (depending on  $\varepsilon$ ). It is an easy exercise to see that the  
 22 dimension on the left is derived purely from the knowledge of the dimensions  
 23 of the space  $S_2(N'_n)^{\text{new}}$  of *newforms* of level  $N'_n$  with  $7 \mid N'_n \mid N_n$ , together with  
 24 the factorization of  $N_n/N'_n$ .

25 Next consider the base change  $g \rightarrow g_K$  from  $\mathrm{GL}(2)/\mathbb{Q}$  to  $\mathrm{GL}(2)/K$  ([Lan]).  
 26 We claim that the image of each  $S_2(N'_n)^{\text{new}}$  under this base change has the same  
 27 dimension. Indeed, if  $g_K = h_K$  for two newforms  $g, h$  in this space, the new-  
 28 form  $h$  must be a twist of  $g$  by the quadratic Dirichlet character  $\delta$  corresponding  
 29 to  $K$ . But since  $g, h$  have odd levels, this would imply that  $h$  has level divisible  
 30 by 4, the level of  $\delta$ , which is impossible. Hence the claim. Now by construc-  
 31 tion, every newform  $g_k$ , or rather the cuspidal automorphic representation  $\beta_K$   
 32 of  $\mathrm{GL}(2, \mathbb{A}_K)$  attached to it, is Steinberg at each of the primes dividing 7, and  
 33 so transfers to our  $D^*$  and defines a cusp form  $\beta_K^D$  relative to  $\Gamma_D(N_n/7)$ ; it also  
 34 lifts to level  $\Gamma_D(N_n)$  by the natural map. The cusp forms on  $\mathrm{GL}(2)/K$  which  
 35 are Steinberg at  $\{Q, \overline{Q}\}$ , and which are old away from 7, are seen to correspond  
 36 in a one-to-one way to old forms on  $D^*$ , since  $D_v^* \simeq \mathrm{GL}(2, K_v)$  for any  $v \nmid 7$ .  
 37 Consequently, we get the following

1 **7.8. Proposition.** *Let  $\{M_n\}$  be the arithmetic tower of closed hyperbolic 3-manifolds*  
 2 *we have constructed above, with  $M_n$  a non-normal cover of  $M$  of degree  $d_n$ . Then for*  
 3 *every  $\varepsilon > 0$ , we can find an integer  $L_\varepsilon$  such that*

$$4 \quad n \geq L_\varepsilon \implies b_1(M_n) \geq \frac{5}{84} d_n^{\frac{1}{2}-\varepsilon}.$$

## 5 8. THE WHITEHEAD LINK COMPLEMENT

6 In this section, we give a very concrete example of covers of a finite volume  
 7 hyperbolic 3-manifold with cusps where the number of fibered faces increases  
 exponentially in the degree. Consider the Whitehead link shown in Figure 8.1,

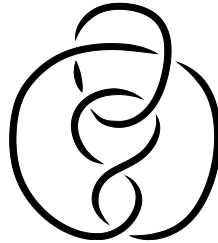


FIGURE 8.1. The Whitehead link.

8 and let  $W$  be its exterior. The interior of  $W$  admits a hyperbolic metric of finite  
 9 volume; indeed,  $W$  is arithmetic with  $\pi_1(W)$  a subgroup of  $\mathrm{PSL}(2, \mathbb{Z}[i])$  of in-  
 10 dex 12. We consider  $n$ -fold cyclic covers dual to the thrice punctured sphere  
 11 bounded by one of the components. More precisely, let  $\omega$  be an element of  
 12 the basis of  $H^1(W; \mathbb{Z})$  which is dual to a basis of  $H_1(W; \mathbb{Z}) \cong \mathbb{Z}^2$  consisting  
 13 of meridians. (This link is quite symmetric, so it doesn't matter which basis  
 14 element we pick.) Then let  $W_n$  be the cover corresponding to the kernel of the  
 15 composition  $\pi_1(W) \xrightarrow{\omega} \mathbb{Z} \rightarrow \mathbb{Z}/n$ . The manifold  $W_n$  is also the complement of  
 16 the link  $L_n$  pictured in Figure 8.2. Following that figure, let  $\mu_i \in H_1(W_n; \mathbb{Z})$  be a

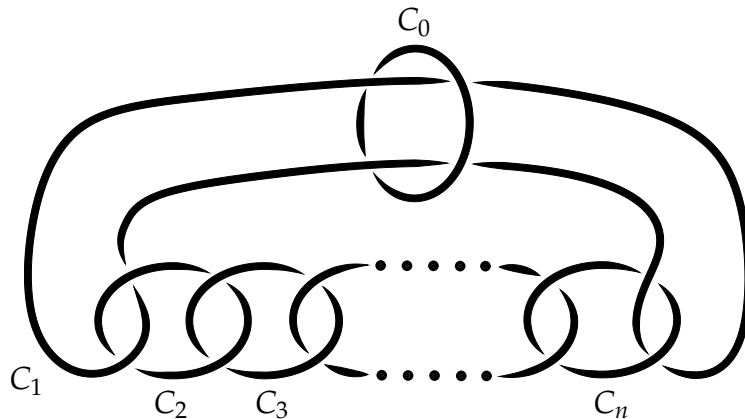


FIGURE 8.2. The covering manifold  $W_n$  is the exterior of the shown link  $L_n$ .

1 meridian for the component  $C_i$  of  $L_n$ . Let  $\{\omega_i\}$  be the dual basis for  $H^1(W_n; \mathbb{Z})$ .  
 2 This section is devoted to proving:

3 **8.3. Theorem.** *Let  $W_n$  be the  $n$ -fold cyclic cover of the exterior of the Whitehead link*  
 4 *described above. Then the number of fibered faces of the Thurston norm ball  $B_T$  of  $W_n$*   
 5 *is  $2^{n+1}$ . More precisely,  $B_T$  is the convex hull of the  $\omega_i$  and all of its top-dimensional*  
 6 *faces are fibered.*

7 The polytope  $B_T$  is called an  $n$ -orthoplex or a cross polytope; it is dual to  
 8 the unit cube in  $H_1(W_n; \mathbb{Z})$  spanned by the  $\mu_i$ . Note that this form for  $B_T$  is  
 9 plausible as each component of  $L_n$  bounds an obvious twice-punctured disc,  
 10 and hence  $\|\omega_i\|_T = 1$  for every  $i$  as hyperbolic manifolds contain no simpler  
 11 essential surfaces. The theorem will follow easily from

12 **8.4. Lemma.** *Any class of  $H^1(W_n; \mathbb{Z})$  of the form*

13 
$$\omega = \epsilon_0\omega_0 + \epsilon_1\omega_1 + \cdots + \epsilon_n\omega_n \quad \text{for } \epsilon_i \in \{-1, 1\}$$

14 *fibers and  $\|\omega\|_T = n + 1$ .*

15 Let us first derive the theorem from the lemma.

16 *Proof of Theorem 8.3.* We need to show that for any choices of  $\epsilon_i \in \{-1, 1\}$ , the  
 17 simplex  $\Delta$  spanned by  $\eta_i = \epsilon_i\omega_i$  is a face of  $B_T$ . We know  $\|\eta_i\|_T = 1$ , and so  
 18  $\Delta \subset B_T$ . By Lemma 8.4, the class  $\eta = \frac{1}{n+1} \sum \eta_i$  has norm 1; this gives another  
 19 point in  $\Delta$  which we know lies in  $\partial B_T$ . Now consider  $\alpha \in \Delta$ , which necessarily  
 20 has the form

21 
$$\alpha = \sum_{i=0}^n a_i \eta_i \quad \text{for } a_i \geq 0 \text{ and } \sum a_i = 1.$$

22 We need to show  $1 \leq \|\alpha\|_T$ , as then  $\alpha$  lies in  $\partial B_T$ ; knowing this for all  $\alpha$  implies  
 23  $\Delta$  is a face of  $B_T$ .

24 After permuting the coordinates, we can assume  $a_0 \geq a_i$  for all  $i$ . As a con-  
 25 sequence,  $a_0 \geq 1/(n+1)$  and

26 
$$\eta = b_0\alpha + b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n \quad \text{where } b_i \geq 0 \text{ and } \sum_{i=0}^n b_i = 1.$$

27 Now the sublinearity of the Thurston norm gives

28 
$$\|\eta\|_T \leq b_0\|\alpha\|_T + b_1\|\eta_1\|_T + b_2\|\eta_2\|_T + \cdots + b_n\|\eta_n\|_T.$$

29 Evaluating the norms we know gives:

30 
$$1 \leq b_0\|\alpha\|_T + b_1 + \cdots + b_n$$

31 Since  $\sum b_i = 1$ , this implies  $1 \leq \|\alpha\|_T$ , as required to prove the theorem. □

32 We conclude this section with

33 *Proof of Lemma 8.4.* Since the Thurston norm is invariant under  $\omega \mapsto -\omega$ , we  
 34 can assume  $\epsilon_0 = 1$ . We analyze this class using Murasugi sum, closely fol-  
 35 lowing the approach used in [Lei] to analyze chain links. Given  $a_i \in \{-1, 1\}$ ,  
 36 consider the link  $L(a_1, a_2, \dots, a_n)$  shown in Figure 8.6, which is described as  
 37 the boundary of a surface which is the Murasugi sum of the surface in Fig-  
 38 ure 8.5 with  $n$  Hopf bands, where the twist on the  $i^{\text{th}}$  Hopf band is right- or

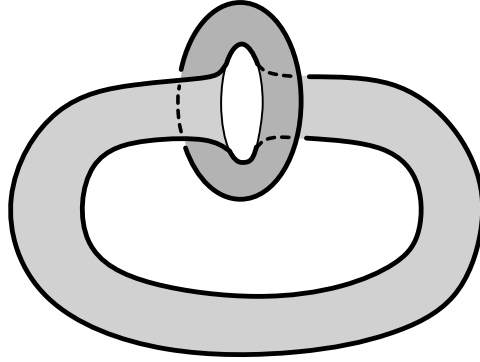


FIGURE 8.5. The base link/surface for the Murasugi sum shown in Figure 8.6.

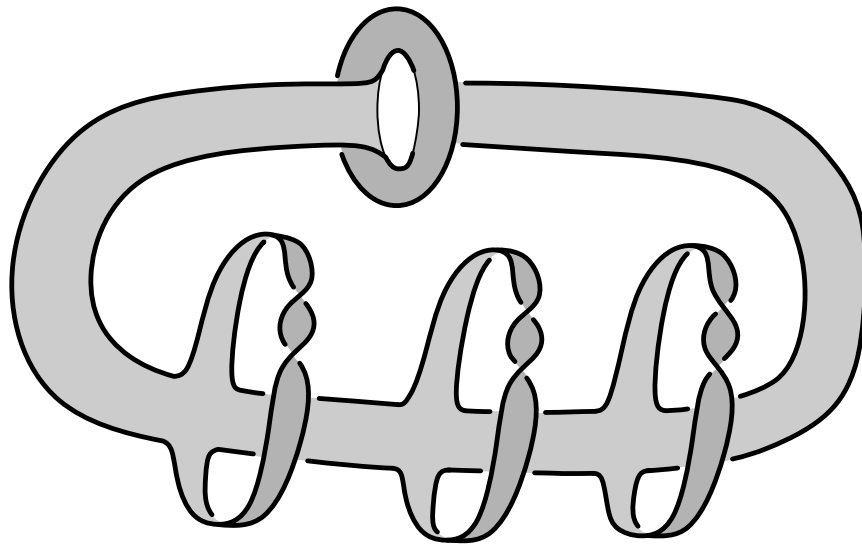


FIGURE 8.6. The link  $L(a_1, a_2, \dots, a_n)$  is the Murasugi sum of the link in Figure 8.5 with  $n$  Hopf bands with appropriately oriented twists. Shown is  $L(1, 1, -1)$ .

- 1 left-handed depending on  $a_i$ . Let  $M(a_1, \dots, a_n)$  be the exterior of  $L(a_1, \dots, a_n)$ .
- 2 Since all the surfaces in the Murasugi sum are fibers, the surface  $\Sigma$  pictured in
- 3 Figure 8.5 is a fiber of a fibration of  $M(a_1, \dots, a_n)$  over the circle [Sta2, Gab2].
- 4 Orient the components of  $L(a_1, \dots, a_n)$  as the boundary of the surface shown in
- 5 Figure 8.6, and let  $\omega_i$  be the corresponding dual basis of  $H^1(M(a_1, \dots, a_n); \mathbb{Z})$ .
- 6 Thus  $\Sigma$  is Poincaré dual to  $\omega = \omega_1 + \dots + \omega_n$  and  $\|\omega\|_T = n + 1$ . The key to
- 7 the lemma is to show:

**8.7. Claim.** Fix  $a_1, \dots, a_n$  in  $\{-1, 1\}$ , and pick  $k \in \{1, 2, \dots, n\}$ . Suppose  $b_i = a_i$  except  $b_{k-1} = -a_{k-1}$  and  $b_k = -a_k$ . (If  $k = 1$  then here  $k - 1$  should be interpreted as  $n$ .) Then  $M(a_1, \dots, a_n) \cong M(b_1, \dots, b_n)$  via a homeomorphism that acts on  $H^1$  by

$$\begin{aligned} \omega_i &\mapsto \omega_i \quad \text{for } i \neq k \text{ and} \\ \omega_k &\mapsto -\omega_k \end{aligned}$$

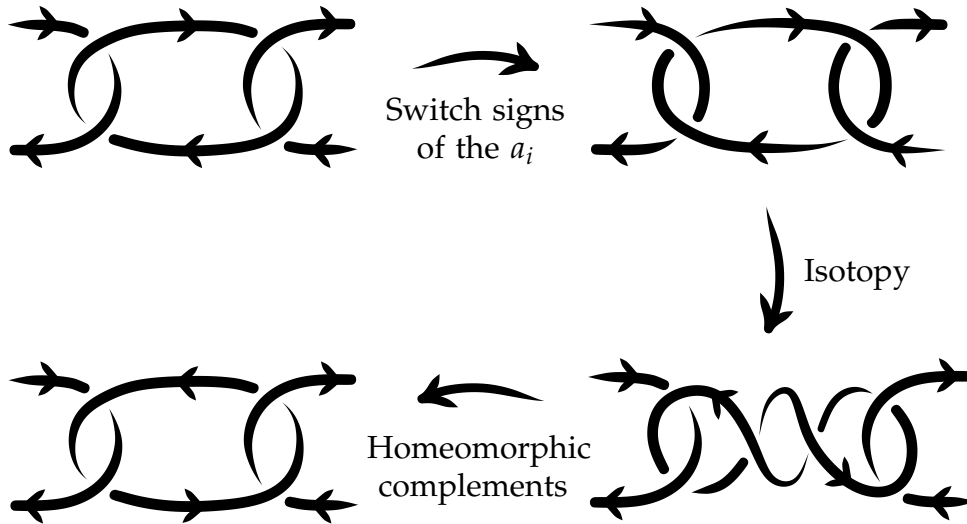


FIGURE 8.8. Switching the signs of two consecutive  $a_i$  can be partially reversed at the cost of adding a twist as shown in the bottom right picture. (Here both  $a_i$  are 1, but the other cases are similar.) We can undo said twist by a homeomorphism of the link complements by cutting along a twice punctured disc bounded by component  $C_0$  of the link and doing a full twist. To prove Claim 8.7, just note the different orientations of the central loop in the two pictures at left.

1 The proof of this claim is given in Figure 8.8. Now to prove the lemma,  
 2 note that  $L(1, 1, \dots, 1)$  is  $L_n$  and applying the claim repeatedly to a suitable  
 3 starting  $L(a_1, \dots, a_n)$  allows us to see any that class of the form  $\omega_0 + \epsilon_1\omega_1 +$   
 4  $\dots + \epsilon_n\omega_n$  is represented by a fiber surface with Euler characteristic  $-(n + 1)$   
 5 as required. □

6 9. WORK OF LONG AND REID

7 In this section, we give a proof of Theorem 1.10 which is simpler than the  
 8 original one in [LR]. A different simplification was given by Agol in [Ago1].  
 9 We begin by introducing an invariant of a fibered cohomology class  $\omega$  of a  
 10 hyperbolic manifold  $M$ . Let  $\mathcal{F}_\omega$  be the associated transverse pseudo-Anosov  
 11 flow described in Section 2.2. As discussed, this flow has a positive finite  
 12 number of singular orbits, each of which is closed. From the closed geodesics  
 13  $\gamma_1, \gamma_2, \dots, \gamma_n$  homotopic to these orbits, define  $c(\omega)$  to be the subset of the uni-  
 14 versal cover  $\mathcal{H}^3$  of  $M$  consisting of the inverse images of the  $\gamma_i$ . By Theo-  
 15 rem 2.3, any other class  $\eta$  in the fibered face of  $\omega$  has isotopic  $\mathcal{F}_\eta$ , and hence  $c$   
 16 is really an invariant of a fibered face. In fact, we regard the geodesics in  $c(\omega)$   
 17 as unoriented, so that  $c$  is an invariant of a fibered face pair. With this invariant  
 18 in hand, we turn to the proof itself.

1 *Proof of Theorem 1.10.* Let  $\omega$  be a fibered class for our arithmetic 3-manifold  
 2  $M = \Gamma \backslash \mathcal{H}^3$ . We will first find a cover of  $M$  with two pairs of fibered faces.  
 3 Following Section 3.5, given  $g \in \text{Comm}(\Gamma)$  we have the associated manifold  
 4  $M_g$  with two covering maps  $p_g, q_g: M_g \rightarrow M$ . Then  $M_g$  has two fibered classes  
 5  $p_g^*(\omega)$  and  $q_g^*(\omega)$ , with associated  $c$  invariants equal to  $c(\omega)$  and  $g^{-1} \cdot c(\omega)$  re-  
 6 spectively. Thus these two fibered classes of  $M_g$  lie in genuinely distinct faces  
 7 provided  $c(\omega)$  is not setwise invariant under  $g$ . Now the setwise stabilizer of  
 8  $c(\omega)$  in  $\text{PSL}(2, \mathbb{C})$  is a discrete subgroup which contains  $\Gamma$  as a subgroup of fi-  
 9 nite index; as  $\text{Comm}(\Gamma)$  is dense in  $\text{PSL}(2, \mathbb{C})$ , there are plenty of  $g \in \text{Comm}(\Gamma)$   
 10 which do not fix  $c(\omega)$ , as desired.

11 To create any number of fibered faces, we simply repeat this process induc-  
 12 tively, assuming at each stage we have constructed a cover  $M_n$  with fibered  
 13 classes  $\omega_1, \dots, \omega_n$  where the  $c(\omega_i)$  are all distinct and then building a cover of  
 14 using a  $g \in \text{Comm}(\pi_1(M_n))$  so that all  $c(\omega_i)$  and  $g \cdot c(\omega_k)$  are distinct.  $\square$

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