Existence of Ramanujan primes for GL(3)

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To Joe Shalika with admiration

Introduction

Let π be a cusp form on $\mathrm{GL}(n)/\mathbb{Q}$, i.e., a cuspidal automophic representation of $\mathrm{GL}(n, \mathbb{A})$, where $\mathbb{A}$ denotes the adele ring of $\mathbb{Q}$. We will say that a prime p is a **Ramanujan prime for** π iff the corresponding π_p is *tempered*. The local component π_p will necessarily be *unramified* for almost all p, determined by an unordered n-tuple $\{\alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{n,p}\}$ of non-zero complex numbers, often represented by the corresponding diagonal matrix $A_p(\pi)$ in $\mathrm{GL}(n, \mathbb{C})$, unique up to permutation of the diagonal entries. The L-factor of π at p is given by

$$L(s, \pi_p) = \det(I - A_p(\pi)p^{-s})^{-1} = \prod_{j=1}^{n}(1 - \alpha_{j,p}p^{-s})^{-1}.$$

As π is unitary, π_p is tempered (in the unramified case) iff each $\alpha_{j,p}$ is of absolute value 1. It was shown in [Ra] that for $\mathbf{n = 2}$, the set $\mathcal{R}(\pi)$ of Ramanujan primes for π is of lower density at least 9/10. When one applies in addition the deep recent results of H. Kim and F. Shahidi ([KSh]) on the symmetric cube and the symmetric fourth power liftings for GL(2), the lower bound improves from 9/10 to 34/35 (*loc. cit.*), which is $0.971428\ldots$.

For $n > 2$ there is a dearth of results for general π, though for cusp forms of *regular infinity type*, assumed for $n > 3$ to have a discrete series component at a finite place, one knows by [Pic] for $n = 3$, which relies on the works of J. Rogawski, et al, and by the work of Clozel ([Cℓ]) for $n > 3$

(see also the non-trivial refinement due to Harris and Taylor ([HaT])), that *all* the unramified primes are Ramanujan primes for π. (Of course for $n = 2$, a regular cusp form is necessarily holomorphic of weight $k \geq 2$, and it is known, by Eichler-Shimura for $k = 2$ and by Deligne for $k > 2$, that every prime is a Ramanujan prime in this case; ditto for $k = 1$ by the work of Deligne and Serre.) One is interested in knowing whether there exists even one Ramanujan prime for general π on $\mathrm{GL}(n)$. Thanks to the work of Kim and Shahidi, one sees the importance of knowing a positive answer to such a question.

The main result of this Note is the following:

Theorem A *Let π be a cusp form on $GL(3)/\mathbb{Q}$. Then there exist infinitely many Ramanujan primes for π.*

Let us now explain the main issues behind its proof. One can show (see section 1) that at any prime p where π is unramified, *if* the coefficient $a_p(\pi)$ is bounded in absolute value by 1, *then* π_p is tempered. A general result proved in [Ra] for $\mathrm{GL}(n)$ implies that for any real number $b > 1$, the set of p where $|a_p(\pi)| \leq b$ is infinite, even of lower Dirichlet density $\geq 1 - \frac{1}{b^2}$. But this gives us nothing for $b = 1$. Our aim here is to show that for infinitely many primes p, $a_p(\pi)$ is indeed bounded in absolute value by 1. The key idea is to exploit the adjoint L-function (whose definition makes sense for π on $\mathrm{GL}(n)$ for any n):

$$L(s, \pi; Ad) = \frac{L(s, \pi \times \overline{\pi})}{\zeta(s)}, \tag{0.1}$$

where $L(s, \pi \times \overline{\pi})$ is the Rankin-Selberg L-function of the pair $(\pi, \overline{\pi})$. (As usual, $\overline{\pi}$ signifies the complex conjugate of π, which, by the unitarity of π, is the same as the contragredient of π.) One knows (see [HRa], Lemma a of section 2) that $L(s, \pi \times \overline{\pi})$ is of *positive type*, i.e., the Dirichlet series defined by its logarithm has non-negative coefficients. The proof of Theorem A relies on the following

Proposition B *Let π be a cusp form on $GL(n)/\mathbb{Q}$. Then for any finite set S of primes containing infinity, the incomplete adjoint L-function $L^S(s, \pi, Ad)$ is not of positive type.*

The proof given here of this Proposition, and hence of Theorem A, will work over any number field F having no real zeros in the interval $(0, 1)$. In the case of real zeros one has to proceed differently.

When I was at Hopkins as an Assistant Professor during 1983-85, I learnt a lot from Joe Shalika about the L-functions of $\mathrm{GL}(n)$. It is a pleasure to dedicate this article to him. Thanks are due to Jeff Lagarias and Freydoon Shahidi for making comments on an earlier version of this article, which led to an improvement of the exposition, and to the NSF for financial support through the grant DMS–0100372.

1 Why Proposition B implies Theorem A

Let $\pi \simeq \pi_\infty \otimes (\otimes'_p \pi_p)$ be a cuspidal automorphic representation of $\mathrm{GL}(3, \mathbb{A}) = \mathrm{GL}(3, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{A}_f)$. Let S_0 be the set of primes p where π_p is unramified and tempered, and let S_1 be the finite set of primes where π_p is ramified. Put

$$S = S_0 \cup S_1 \cup \{\infty\}. \tag{1.1}$$

For any L-function with an Euler product $\prod_v L_v(s)$ over $\mathbb{Q}$, put

$$L^S(s) = \prod_{p \notin S} L_p(s), \tag{1.2}$$

which we call the *incomplete Euler product* relative to, or outside, S.

Pick any p outside S and consider the Langlands class

$$A_p = A_p(\pi) = \{\alpha_{1,p}, \alpha_{2,p}, \alpha_{3,p}\}. \tag{1.3}$$

As π_p is by assumption non-tempered, there is a non-zero real number t and a complex number u of absolute value 1 such that, after possibly renumbering the $\alpha_{j,p}$,

$$\alpha_{1,p} = up^t.$$

On the other hand, by the unitarity of π_p, we must have

$$\{\overline{\alpha}_{1,p}, \overline{\alpha}_{2,p}, \overline{\alpha}_{3,p}\} = \{\alpha_{1,p}^{-1}, \alpha_{2,p}^{-1}, \alpha_{3,p}^{-1}\}.$$

This then implies that

$$A_p = \{up^t, up^{-t}, w\}, \tag{1.4}$$

for some complex number w of absolute value 1. We may, and we will, assume that t is positive. Put

$$u^{-1}w = e^{i\theta}, \tag{1.6}$$

for some $\theta \in [0, 2\pi) \subset \mathbb{R}$.

So we have

$$|a_p|^2 = (p^t + p^{-t} + \cos\theta)^2 + \sin^2\theta = 3 + p^{2t} + p^{-2t} + 2\cos\theta(p^t + p^{-t}). \tag{1.6}$$

Now let us look at the adjoint L-function. By definition,

$$L^S(s, \pi; Ad) = \frac{\prod_{p \notin S} L(s, \pi_p \overline{\pi}_p)}{\prod_{p \notin S}(1 - p^{-s})^{-1}}. \tag{1.7}$$

So for any p outside S, the Langlands class of the Adjoint L-function is

$$A_p(\pi; Ad) = A_p \otimes \overline{A}_p - \{1\}.$$

Applying (1.4) and (1.5), we get

$$A_p(\pi; Ad) = \{p^{2t}, p^{-2t}, 1, 1, u\overline{w}p^t, u\overline{w}p^{-t}, \overline{u}wp^t, \overline{u}wp^{-t}\}. \tag{1.8}$$

and

$$a_p(\pi; Ad) = \mathrm{tr}(A_p(\pi; Ad)) = 2 + p^{2t} + p^{-2t} + 2\cos\theta(p^t + p^{-t}).$$

Consequently,

$$\log L^S(s, \pi; Ad) = \sum_{p \notin S} \sum_{m \geq 1} \frac{a_{p^m}(\pi; Ad)}{p^{ms}}, \tag{1.9}$$

where (by (1.8) and (1.6))

$$a_{p^m}(\pi; Ad) = 2 + p^{2mt} + p^{-2mt} + 2\cos m\theta(p^{mt} + p^{-mt}). \tag{1.10.}$$

Since

$$p^{mt} + p^{-mt} \geq 2,$$

and since

$$a_{p^m}(\pi; Ad) = (p^{mt} + p^{-mt})((p^{mt} + p^{-mt}) + 2\cos m\theta)),$$

we get the following

Lemma 1.11 *Let π be a cusp form on $GL(3)/\mathbb{Q}$ and S the set of primes containing ∞, the primes where π is ramified and the Ramanujan primes for π. Then $L^S(s, \pi; Ad)$ is of positive type.*

But if S_0, and hence S, is finite, this Lemma contradicts the conclusion of Proposition B. Hence the set of Ramanujan primes for π must be infinite, once we accept Proposition B.

2 Proof of Proposition B

In this section π will be a unitary, cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A})$. At each place v, the local factor of $L(s, \pi; Ad)$ is given by

$$L(s, \pi_v; Ad) \,=\, \frac{L(s, \pi_v \times \overline{\pi}_v)}{\zeta_v(s)}, \tag{2.1}$$

where $\zeta_v(s)$ is $(1 - p^{-s})^{-1}$ if v is a finite place defined by a prime p, and it equals $\pi^{s/2}\Gamma(s/2)$ if v is the archimedean place. By convention, $\zeta(s)$ is the product of $\zeta_v(s)$ over all the *finite* v, while all the other automorphic L-functions occurring in this paper will also involve the archimedean factors.

The L-group of $\mathrm{GL}(n)$ is $\mathrm{GL}(n, \mathbb{C})$, and the Euler factor $L(s, \pi; Ad)$ is associated to the representation

$$Ad : \, \mathrm{GL}(n, \mathbb{C}) \,\to\, \mathrm{GL}(n^2 - 1, \mathbb{C}), \tag{2.2}$$

given by composing the natural projection of $\mathrm{GL}(n, \mathbb{C})$ onto $\mathrm{PGL}(n, \mathbb{C})$ with the $(n^2 - 1)$-dimensional *Adjoint representation* of $\mathrm{PGL}(n, \mathbb{C})$, whence the notation Ad. In any case, we have for *every* v:

$$L(s, \pi_v; Ad) \,\neq\, 0 \;\forall\, s \in \mathbb{C}. \tag{2.3}$$

One way to see this will be to use the local Langlands correspondence, established recently by Harris-Taylor ([HaT]) and Henniart ([He]), associating to each π_v an n-dimensional representation σ_v of $W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$ (resp. W_{F_v}) for v finite (resp. infinite). (Here W_{F_v} denotes as usual the Weil group of F_v.) Since this correspondence preserves the local factors of pairs and matches identifie the central character of π_v with the determinant of σ_v, one gets in particular,

$$L(s, \pi_v; Ad) \,=\, L(s, Ad(\sigma_v)),$$

where $Ad(\sigma_v)$ denotes $\sigma_v \otimes \sigma_v^\vee \ominus 1$, which is a *genuine* representation because the trivial representation occurs in $\sigma_v \otimes \sigma_v^\vee \simeq \mathrm{End}(\sigma_v)$. It is well known that for any representation τ_v of $W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$, such as $Ad(\sigma_v)$, the associated L-factor has no zeros.

Now let S be any finite set of primes containing ∞ and the primes where π is ramified. Put

$$L^S(s, \pi; Ad) \;=\; \prod_{v \notin S} L(s, \pi_v; Ad). \tag{2.4}$$

Suppose $L^S(s, \pi; Ad)$ is of positive type. Then by definition, its logarithm defines a Dirichlet series with positive coefficients, absolutely convergent in a right half plane. By the theory of Landau, this Dirichlet series converges on (β, ∞), where β is the largest real number where $\log L^S(s, \pi; Ad)$ diverges. But such a point of divergence must be a pole, and not a zero, of $L^S(s, \pi; Ad)$ because its logarithm is positive in (β, ∞).

Lemma 2.5 *Let β be the largest real number such that $L^S(s, \pi; Ad)$ converges for all real $s > \beta$. Then*

$$\beta \;<\; 0.$$

Proof of Lemma 2.5. By the standard properties of the Rankin-Selberg L-functions ([JPSS], [JS], [Sh1-3], [MW] – see also [BRa]), $L^S(s, \pi \times \overline{\pi})$ is invertible for $\Re(s) > 1$ and admits a meromorphic continuation to the whole s-plane with a unique *simple* pole at $s = 1$. The same properties hold of course for $\zeta^S(s)$; so $L^S(s, \pi; Ad)$ has no pole in $\Re(s) \geq 1$. In other words we have $\beta < 1$. Moreover, one knows that $\zeta^S(s)$ is **non-zero** on $(0, 1) \subset \mathbb{R}$. Hence we have

$$\beta \;\leq\; 0.$$

Now let us look at the point $s = 0$. By definition,

$$L^S(s, \pi; Ad) \;=\; \frac{L^\infty(s, \pi \times \overline{\pi})}{\zeta(s) \prod_{v \in S - \{\infty\}} L(s, \pi_v; Ad)}.$$

The numerator on the right has no pole at $s = 0$, and $\zeta(s)$ does not vanish at $s = 0$. Moreover, as we noted above, the local factors $L(s, \pi_v; Ad)$ have no zeros. Consequently, $L^S(s, \pi; Ad)$ has no pole at $s = 0$, and this proves the Lemma.

Lemma 2.6 *$L(s, \pi_\infty; Ad)$ has a pole at $s = 0$.*

Proof of Lemma 2.6. There exist complex numbers z_1, z_2, z_3 such that

$$L(s, \pi_\infty) \;=\; \prod_{j=1}^{3} \Gamma_{\mathbb{R}}(s + z_j + \delta_j), \tag{2.7}$$

with $\delta_j \in \{\pm 1\}$, $\forall j$, and

$$\Gamma_{\mathbb{R}}(s) \;=\; \pi^{-s/2}\Gamma(s/2).$$

By the unitarity of π_∞, we see that either all the z_j have absolute value 1, in which case π_∞ is tempered, or exactly one of the z_j, say z_1, has absolute value 1, and moreover,

$$z_2 = u + t, \; z_3 = u - t,$$

for some positive real number t and a complex number u of absolute value 1. In either case we see that the set

$$B(\pi_\infty; Ad) := \{z_1, z_2, z_3\} \cup \{\overline{z}_1, \overline{z}_2, \overline{z}_3\} - \{0\} \tag{2.8}$$

contains 0. The standard yoga of Langlands L-functions furnishes the identity

$$L(s, \pi_\infty; Ad) \;=\; \prod_{z \in B(\pi_\infty; Ad)} \Gamma_{\mathbb{R}}(s + z). \tag{2.9}$$

Recall that $\Gamma_{\mathbb{R}}(s)$ never vanishes and has simple poles at the even negative integers, in particular at $s = 0$. Since $B(\pi_\infty; Ad)$ contains 0, $\Gamma_{\mathbb{R}}(s)$ is a factor of $L(s, \pi_\infty; Ad)$. We must then have

$$-\mathrm{ord}_{s=0} L(s, \pi_\infty; Ad) \;\geq\; 1,$$

as asserted.

Proof of Proposition B (contd.) As the local factors $L(s, \pi_v; Ad)$ never vanish, and since S is finite by assumption, the function

$$L_{S-\{\infty\}}(\pi; Ad) := \prod_{p \in S-\{\infty\}} L(s, \pi_p; Ad)$$

is non-zero at $s = 0$. Hence by Lemma 2.6,

$$-\mathrm{ord}_{s=0} L_S(s, \pi; Ad) \geq 1. \tag{2.10}$$

But we know that the full adjoint L-function $L(s, \pi; Ad)$ has no pole at $s = 1$, and hence at $s = 0$ by the functional equation. So all this forces the following:

$$L^S(0, \pi; Ad) = 0,$$

which contradicts Lemma 2.5. The only unsupported assumption we made was that $L^S(s, \pi; Ad)$ is of positive type, which must be wrong if S is finite. We are done.
QED

Note that in the proof uses the base field $\mathbb{Q}$ in order to use the crucial property of the Riemann zeta function that it does not vanish in the real interval $(0, 1)$. For general number fields F, the Dedekind zeta function $\zeta_F(s)$ should not have any such real zero either, save possibly at $s = 1/2$. Clearly, Theorem A will follow for any F for which Proposition B can be established. One way to get around the difficulty for general F would be to prove *a priori* that the adjoint L-function, which has been studied from different points of view by D. Ginzburg, Y. Flicker, H. Jacquet, S. Rallis, F. Shahidi and D. Zagier, has no pole in $(0, 1)$, which is, to our knowledge, unknown. To elaborate a little, a particular version of the trace formula due to H. Jacquet and D. Zagier ([JZ]) suggests that the divisibility of $L(s, \pi \times \overline{\pi})$ by $\zeta_F(s)$ for all cuspidal automorphic representations π of $\mathrm{GL}(n, \mathbb{A}_F)$ with trivial central character is equivalent to the divisibility of $\zeta_K(s)$ by $\zeta_F(s)$ for all commutative cubic algebras K over F. Since the latter is known for $n = 3$, one hopes that the former holds. This divisibility has been investigated by relating it to an Eisenstein series on G_2 by D. Jiang and S. Rallis ([JiR]), and the desired result could be close to being established in the $n = 3$ case.

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