AN EXERCISE CONCERNING THE SELFDUAL CUSP FORMS ON GL(3)

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The object of this Note is to supply a proof of the following result, which is in the folklore, and deduce a Corollary. There is no pretension to anything creative here, and all that is involved is a synthesis of results due to various people.

Theorem A *Let F be a number field, and Π a cuspidal, selfdual automorphic representation of $GL_3(\mathbb{A}_F)$. Then there exists a non-dihedral cusp form π on $GL(2)/F$, and an idele class character ν of F with $\nu^2 = 1$, such that*

$$(1) \qquad \Pi \simeq Ad(\pi) \otimes \nu.$$

The form π is unique up to a character twist, while ν is simply the central character of Π. The central character ω of π may be chosen to be of finite order. Moreover, we may choose π such that, for any finite place v, π_v is unramified, resp. Steinberg, when $\Pi_v \otimes \nu_v$ is unramified, resp. Steinberg.

Here $\mathrm{Ad}(\pi)$ denotes the Adjoint of π, a selfdual automorphic form on $\mathrm{GL}(3)/F$, defined to be $\mathrm{sym}^2(\pi) \otimes \omega^{-1}$, where $\mathrm{sym}^2(\pi)$ is the symmetric square of π, defined by Gelbart and Jacquet in [GJ]. As π is non-dihedral, $\mathrm{Ad}(\pi)$ is cuspidal.

Note that Theorem A remains valid for any cusp form Π on $\mathrm{GL}(3)/F$ which satisfies $\Pi^\vee \simeq \Pi \otimes |\cdot|^t$ for some t, the reason being that we may replace Π by $\Pi \otimes |\cdot|^{t/2}$, which is selfdual.

Theorem A has been known to experts for a while. It is a consequence of a comparison of the stable trace formula for $\mathrm{SL}(2)/F$ with the twisted trace formula for $\mathrm{PGL}(3)/F$ (relative to transpose inverse); this fundamental idea of Langlands has been carried out in detail by Flicker in a series of papers. It will also be a special case of Arthur's forthcoming major work relating selfdual automorphic representations of $\mathrm{GL}(n)$ with those of suitable classical groups, again comparing appropriate trace formulae. In this Note we deduce Theorem A in a different way, via L-functions, by appealing to the backwards lifting ("descent") of Ginzburg, Rallis and Soudry, as well as the forward transfer, for generic cusp forms, from odd orthogonal groups to $\mathrm{GL}(n)$, due to Cogdell, Kim, Piatetski-Shapiro, and Shahidi.

Corollary B *Let F, Π be as in Theorem, with associated (π, ν). Then at any archimedean place w, Π_w has regular parameter iff π_w does. Moreover, if Π is algebraic and F totally real, then π can be chosen to be algebraic as* *well. Consequently, over totally real fields, π can be chosen to be regular algebraic, hence cohomological, when Π has that property.*

Proof of Theorem A Fix a finite set S of places of F containing the ramified and archimedean places, and write $L^S(s)$, given any Euler product $L(s) = \prod_v L_v(s)$, for $\prod_{v \notin S} L_v(s)$. Then, Π being selfdual implies that its central character ν, say, must also be selfdual, hence either trivial or quadratic. The selfduality of Π also results in a pole at $s = 1$ of the Rankin-Selberg L-function (on the left hand side of the following factorization):

$$L^S(s, \Pi \times \Pi) \,=\, L^S(s, \Pi; \mathrm{sym}^2) L^S(s, \Pi; \Lambda^2), \tag{2}$$

where the right hand side factors are the (incomplete) symmetric and exterior square L-functions of Π. Moreover, one has the identity

$$L^S(s, \Pi; \Lambda^2) \,=\, L^S(s, \Pi \otimes \nu), \tag{3}$$

which can be checked factor by factor explicitly. Indeed, at any $v \notin S$, if the unordered triple ("Langlands class") associated to Π_v is $\{\alpha_v, \beta_v, \gamma_v\}$, then we have

$$\Lambda^2\{\alpha_v, \beta_v, \gamma_v\} \,=\, \frac{1}{\alpha_v \beta_v \gamma_v}\{\alpha_v^{-1}, \beta_v^{-1}, \gamma_v^{-1}\},$$

and so the v-part of (2) follows by noting that $\nu_v(\varpi_v) = \alpha_v \beta_v \gamma_v$, where ϖ_v is a uniformizer at v, and $\Pi_v^\vee \simeq \Pi_v$ has the Langlands class $\{\alpha_v^{-1}, \beta_v^{-1}, \gamma_v^{-1}\}$.

The utility of (3) is that it shows that $L^S(s, \Pi, \Lambda^2)$ is, being the L-function (outside S) of the cusp form $\Pi \otimes \nu$, invertible at $s = 1$. So we have, thanks to (2) and (3),

$$-\mathrm{ord}_{s=1} L^S(s, \Pi; \mathrm{sym}^2) \,=\, 1. \tag{4}$$

Consequently, the parameter $\phi = \phi_\Pi$ of Π lands in $O(3, \mathbb{C})$. It lands in $\mathrm{SO}(3, \mathbb{C})$ iff $\nu = 1$. Note that if we put

$$\Pi_1 := \Pi \otimes \nu,$$

then Π_1 is still selfdual. Moreover, its central character $\nu_1 = \omega_{\Pi_1}$ satisfies

$$\nu_1 \,=\, \nu \cdot (\nu)^3 \,=\, 1,$$

implying that the parameter of Π_1 lands in $\mathrm{SO}(3, \mathbb{C})$.

Thus, after replacing Π by $\Pi \otimes \nu$, we may assume that it has trivial central character, and consequently has parameter in $\mathrm{SO}(3, \mathbb{C})$. Now applying the *descent theorem* of Ginzburg, Rallis and Soudry ([GRS], [Sou]), we can find a cuspidal, globally generic automorphic representation π_0 of $\mathrm{Sp}_2(\mathbb{A}_F)$, which is the same as $\mathrm{SL}_2(\mathbb{A}_F)$. Furthermore, if r denotes the standard (3-dimensional) representation of the dual group of SL(2), which is $\mathrm{PGL}_2(\mathbb{C})$, we have the following:

Proposition C *The descent $\Pi \mapsto \pi_0$ satisfies the following:*

(a) *If v is a non-archimedean place of F, then*

$$L(s, \Pi) \,=\, L^S(s, \pi_0; r).$$

(b) *If w is an archimedean place of F,*

$$\sigma_w(\Pi) \simeq r(\sigma_w(\pi_0)),$$

where $\sigma_w(\Pi)$, resp. $\sigma_w(\pi_0)$, denotes the parameter of Π_w, resp. $\pi_{0,w}$, i.e., the associated representation of the Weil group W_w into $GL(3,\mathbb{C})$, resp. $PGL(2,\mathbb{C})$.

Proof of Proposition C. By the work of Cogdell, Kim, Piatetski-Shapiro ad Shahidi ([CKPSS]), we can transfer π_0 back to a cusp form Π' on $\mathrm{GL}(3)/F$ such that the arrow $\pi_{0,v} \mapsto \Pi'_v$ is compatible, at every unramified or archimedean place v of F, with the descent of [GRS], [Sou]. So Π' and Π are equivalent almost everywhere, hence isomorphic by the strong multiplicity one theorem. So the composition of the parameters of π_0 with the natural embedding

$$\mathrm{PGL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C}) \hookrightarrow \mathrm{GL}(3,\mathbb{C}) \tag{5}$$

are the same as the parameters of Π at the various places v. The assertions of the Proposition now follow.

□

Proof of Theorem A (contd.)

The next object is to find a generic cuspidal representation of $\mathrm{GL}(2,\mathbb{A}_F)$ whose restriction to $\mathrm{SL}(2,\mathbb{A}_F)$ contains π_0. This can be done by appealing to Labesse and Langlands ([LL]). But we want to refine their construction in such a way that we keep track of what happens at the finite primes in order that we do not introduce new ramification. Here is what we do.

First choose a character ω_1 of $Z(\mathbb{A}_F)$, where Z denotes the center of $\mathrm{GL}(2)$, such that ω_1 is trivial on $Z_\infty^+ Z(F)$ and agrees with the restriction of π_0 to $Z(\mathbb{A}_F)\cap \mathrm{SL}(2,\mathbb{A}_F)$. The pair (π_0,ω_1) defines a representation π_1 of the group $H := \mathrm{SL}(2,\mathbb{A}_F)Z(\mathbb{A}_F)$, such that the central character ω_1 of π_1 is trivial on Z_∞^+ and $Z(F)$. If Π has conductor $\mathcal{N}$, then π_1 is also unramified outside $\mathcal{N}$. Moreover, by the Proposition, the transfer at the ramified primes is still functorial and respects the level.

Note that $H(\mathbb{A}_F)$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{A}_F)$ with a countable quotient group. Now induce π_1 to $\mathrm{GL}(2,\mathbb{A}_F)$, and choose (as follows) a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ occurring in the induced representation, which is necessarily globally generic. Denote by ω the central character of π, which, by virtue of being trivial on Z_∞^+, is of finite order.

Let $K(\mathcal{M})$ denote a principal congruence subgroup of $\mathrm{GL}(2,\mathbb{A}_{F,f})$ such that π_1 has a fixed vector under $K_1(\mathcal{M}) := K(\mathcal{M})\cap \mathrm{SL}(2,\mathbb{A}_{F,f})$. Then the induced representation will, by Frobenius reciprocity, have at least one constituent which will have a vector fixed under $K(\mathcal{M})$, and such a π is what we choose. In particular, thanks to Proposition C, at any finite place v, π_v is unramified whenever Π_v is. Suppose next that v divides $\mathcal{M}$. Then it is not hard to see that π_1, and hence π, is not unramified, the reason being that the descent of [GRS] is compatible with the transfer of [CKPSS], which preserves the epsilon factors.

Note that by construction, the adjoint representation of the parameter of π is just the symmetric square of that of π_0. It follows that

$$L^S(s,\Pi) = L^S(s, Ad(\pi)), \tag{6a}$$

and at each infinite place w,

$$\sigma_w(\Pi) = Ad(\sigma_w(\pi)). \tag{6b}$$

In other words,

$$\Pi \simeq Ad(\pi), \tag{6c}$$

and if π' is another candidate, then by the multiplicity one theorem for SL(2) [Ram], we must have

$$\pi' \simeq \pi \otimes \mu,$$

for an idele class character μ of F.

As noted earlier, we can take π to be unramified at a finite place v when Π is so. Similarly, when Π_v has an Iwahori fixed vector at a finite place v, we may choose π_v to also have an Iwahori fixed vector. Since π_v is ramified (as seen above) when Π_v is, π_v cannot be fixed by the full maximal compact subgroup at v. So π_v is Steinberg when Π_v is.

Finally, note that all this applies when the central character ν of Π is trivial central character. But when ν is not trivial, as we observed earlier, $\Pi \otimes \nu$ has trivial central character and we can apply the construction above to deduce that

$$\Pi \otimes \nu \simeq \mathrm{Ad}(\pi), \tag{7}$$

for a suitable cusp form π on $\mathrm{GL}(2)/F$. As $\nu^2 = 1$, (7) is equivalent to (1). □

Proof of Corollary B

Evidently, if $\sigma_w(\Pi)$ is regular at an archimedean place w, i.e., has $\mathbb{C}^*$ acting on it with multiplicity one, then the same necessarily holds, thanks to (1), for $\sigma_w(\pi)$, for any choice of π.

Next recall that the algebraicity of any cusp form η of $\mathrm{GL}(n, \mathbb{A}_F)$ implies by definition (cf. [Clo], Section 1.2.3) that for any $w \mid \infty$, and for any character χ of $\mathbb{C}^*$ appearing in the restriction of $\sigma_w(\eta)$ to $\mathbb{C}^*$, we have, for suitable integers p_w, q_w, $\chi(z) = z^{p_w+(n-1)/2}\overline{z}^{q_w+(n-1)/2}$, for all $z \in \mathbb{C}^*$. Since the central character ω of π is of finite order by construction, the restriction of $\sigma_w(\pi)$ to $\mathbb{C}^*$ is, for any $w \mid \infty$, of the form $\mu \oplus \mu^{-1}$. Then we have

$$\sigma_w(\mathrm{Ad}(\pi))_{|\mathbb{C}^*} = \sigma_w(\Pi)_{|\mathbb{C}^*} = \mu^2 \oplus 1 \oplus \mu^{-2}. \tag{8}$$

As Π is algebraic, we have

$$\mu^2(z) = z^{p_w+1}\overline{z}^{q_w+1}, \ \forall\, z \in \mathbb{C}^*, \tag{9}$$

for some $p_w, q_w \in \mathbb{Z}$.

By the *archimedean purity theorem* for algebraic cusp forms on $\mathrm{GL}(n)$ (cf. [Clo], p. 112), we see that $p_w + q_w$ is constant for all the characters of $\mathbb{C}^*$ appearing in $\sigma_w(\Pi)$, and it is also independent of $w \mid \infty$. Since the trivial character also occurs by (8), we must have

$$p_w + q_w \;=\; 0, \;\; \forall\, w \mid \infty. \tag{10}$$

In other words, $\sigma_w(\Pi) \otimes |\cdot|^{-1}$ is tempered at each $w \mid \infty$.

Now let F be totally real. If Π_w is not regular, then $p_w = 0$ or $p_w = -p_w$, and in either case p_w is even and π_w is algebraic. So let Π be regular. Then Π_w must be an isobaric sum $\mathcal{D}_{k_w} \boxplus 1$, with $\mathcal{D}_{k_w}$ a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$. We get

$$\sigma_w(\Pi) \otimes |\cdot|^{-1} \;\simeq\; \mathrm{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}} \left(\left(\frac{z}{\sqrt{z\overline{z}}} \right)^{k_w - 1} \right) \oplus 1, \tag{11}$$

with $k \geq 2$. (Here $W_{\mathbb{R}}$ denotes the real Weil group.) Comparing (8) with the restriction to $\mathbb{C}^*$ of (11), we see that $k_w - 1$ must be even, hence of the form $2(m-1)$, for an integer m. Since $k_w \geq 2$, $m = (k_w + 1)/2 \in \mathbb{Z}$ is also ≥ 2. It follows that π_w is the discrete series representation $\mathcal{D}_m$, showing that it is algebraic. Since this holds at every archimedean place w, π is algebraic.

In sum, when F is totally real, π can be chosen to be regular algebraic, hence cohomological, when Π has that property. This completes the proof of Corollary B.

□

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