EULER CHARACTERISTICS IN RELATIVE K-GROUPS

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1. Introduction

Suppose M is a finite module under the Galois group of a local or global field. Ever since Tate's papers [17], [18] one has a simple and explicit formula for the Euler-Poincaré characteristic of the cohomology of M. In this note we are interested in a refinement of this formula when M also carries an action of some algebra $\mathcal{A}$, commuting with the Galois action (Prop. 5.2 and Theorem 5.1 below). This refinement naturally takes the shape of an identity in a relative K-group attached to $\mathcal{A}$ (see section 2 below). We shall deduce such an identity whenever one has a formula for the ordinary Euler characteristic, the key step in the proof being the representability of certain functors by perfect complexes (see section 3). This representability may be of independent interest in other contexts.

Our formula for the equivariant Euler characteristic over $\mathcal{A}$ implies the "isogeny invariance" of the equivariant conjectures on special values of L-function put forward in [3], and this was our motivation to write this note. Incidentally, isogeny invariance (of the conjectures of Birch and Swinnerton-Dyer) was also a motivation for Tate's original paper [18]. I am very grateful to J.P. Serre for illuminating discussions on the subject of this note, in particular for suggesting that I consider representability. I'd also like to thank D. Burns for insisting on a most general version of the results in this paper.

2. The relative K_0

We fix a prime number p and a $\mathbb{Z}_p$-algebra $\mathcal{A}$ (associative with unit) which is either finite or finitely generated and free as a $\mathbb{Z}_p$-module. Let $K_0(\mathcal{A})$ (resp. $K_0(\mathcal{A}, \mathbb{Q}_p)$) be the Grothendieck group of the exact category $H(\mathcal{A})$ (resp. $HF(\mathcal{A})$)

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of finitely generated (resp. finite) $\mathcal{A}$-modules of finite projective dimension. For an object M of $HF(\mathcal{A})$ we denote by $[M]_{\mathcal{A}} \in K_0(\mathcal{A}, \mathbb{Q}_p)$ its class and for a bounded complex $M^{\cdot}$ of objects of $HF(\mathcal{A})$ we put $[M]_{\mathcal{A}} = \sum (-1)^i [M^i]_{\mathcal{A}}$. One has a long exact localization sequence

$$K_1(\mathcal{A}) \to K_1(\mathcal{A}_{\mathbb{Q}}) \to K_0(\mathcal{A}, \mathbb{Q}_p) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}_{\mathbb{Q}}) \tag{1}$$

where $\mathcal{A}_{\mathbb{Q}} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This is [1][Th. IX.6.3] if $\mathcal{A}$ is a free $\mathbb{Z}_p$-module whilst if $\mathcal{A}$ is Artinian then obviously $K_0(\mathcal{A}, \mathbb{Q}_p) = K_0(\mathcal{A})$ and $K_i(\mathcal{A}_{\mathbb{Q}}) = 0$.

2.1. **Examples.**

2.1.1. If $\mathcal{A} = \mathbb{Z}_p$ then $HF(\mathcal{A})$ consists of of all finite $\mathbb{Z}_p$-modules. We write $[M]$ for $[M]_{\mathbb{Z}_p}$. There is an isomorphism $K_0(\mathcal{A}, \mathbb{Q}_p) \cong \mathbb{Z}$ which can be chosen so that $\#M = p^{[M]}$ for any finite $\mathbb{Z}_p$-module M. More generally, if $\mathcal{A}$ is Dedekind, $K_0(\mathcal{A}, \mathbb{Q}_p)$ is the group of divisors on $\mathrm{Spec}(\mathcal{A})$ or, equivalently, of fractional $\mathcal{A}$-ideals and $[M]_{\mathcal{A}}$ is the Fitting ideal of (any finite $\mathcal{A}$-module) M.

2.1.2. If $\mathcal{A}_{\mathbb{Q}}$ is a semisimple $\mathbb{Q}_p$-algebra it is shown in [3][§2.2] how to construct elements in $K_0(\mathcal{A}, \mathbb{Q}_p)$ from a perfect complex $P^{\cdot}$ of $\mathcal{A}$-modules together with a *trivialization*, i.e. an isomorphism $\bigoplus_{i \text{ even}} H^i(P^{\cdot}_{\mathbb{Q}}) \xrightarrow{\sim} \bigoplus_{i \text{ odd}} H^i(P^{\cdot}_{\mathbb{Q}})$ where $P^{\cdot}_{\mathbb{Q}} = P^{\cdot} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This construction is crucial in formulating conjectures on special values of equivariant L-functions of motives whose p-adic realisation carries an action of $\mathcal{A}_{\mathbb{Q}}$. The isogeny invariance of these conjectures will follow from Theorem 5.1 below.

2.1.3. Let N be a finite abelian p-group, say of order p, and $\mathcal{A} = \mathbb{Z}_p \oplus N$ where $nm = 0$ for all $n, m \in N$. This algebra does not satisfy our running assumptions, and it is indeed not hard to verify that $K_0(\mathcal{A}, \mathbb{Q}_p) = 0$ (with our above definition) whereas any group $K_0(\mathcal{A}, \mathbb{Q}_p)$ fitting into a long exact sequence (1) has to be nonzero.

Remark 1. Consider triples (X, g, Y) where X and Y are finitely generated projective $\mathcal{A}$-modules and $g : X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to Y \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an $\mathcal{A}_{\mathbb{Q}}$-isomorphism. There is a rather different description of $K_0(\mathcal{A}, \mathbb{Q}_p)$ as the group generated by such triples together with certain relations [1]. If one defines $K_0(\mathcal{A}, \mathbb{Q}_p)$ this way the sequence (1) holds true in example 2.1.3, and in fact for an arbitrary ring homomorphism in place of $\mathcal{A} \to \mathcal{A}_{\mathbb{Q}}$. This description of $K_0(\mathcal{A}, \mathbb{Q}_p)$ is also crucial for the constructions of example 2.1.2 but it is of little use for the purposes of this paper.

Remark 2. Let $D^{perf}(\mathcal{A})$ (resp. $D^{fperf}(\mathcal{A})$) be the full triangulated subcategory of the derived category of (left) $\mathcal{A}$-modules consisting of complexes quasi-isomorphic to a bounded complex of finitely generated projective $\mathcal{A}$-modules (and with finite cohomology). Any bounded complex of objects of $H(\mathcal{A})$ (resp. $HF(\mathcal{A})$) is an object of $D^{perf}(\mathcal{A})$ (resp. $D^{fperf}(\mathcal{A})$). There is a notion of K_0 of a triangulated category [10][Exposé VIII] and one can show that $K_0(\mathcal{A}) \cong K_0(D^{perf}(\mathcal{A}))$ (loc.cit, section 7). Using for example the results of [19] one can also show that $K_0(\mathcal{A}, \mathbb{Q}_p) \cong K_0(D^{fperf}(\mathcal{A}))$. In some sense this is the most natural point of view on $K_0(\mathcal{A}, \mathbb{Q}_p)$ for the purposes of this paper because we shall be interested in the classes of cohomology complexes which happen to be quasi-isomorphic to bounded complexes of objects of $HF(\mathcal{A})$ but which are naturally only determined up to quasi-isomorphism in the derived category of all $\mathcal{A}$-modules.

3. Representability

Let $\mathcal{B}$ be a profinite $\mathbb{Z}_p$-algebra and $\mathcal{C} = \mathcal{C}(\mathcal{B})$ (resp. $\mathcal{D} = \mathcal{D}(\mathcal{B})$) the category of profinite (resp. discrete) continuous $\mathcal{B}$-modules with continuous homomorphisms. For M, N objects of either $\mathcal{C}$ or $\mathcal{D}$ we denote by $\mathrm{Hom}^c_{\mathcal{B}}(M, N) \subseteq \mathrm{Hom}_{\mathcal{B}}(M, N)$ the set of continuous homomorphisms. $\mathcal{C}$ is an abelian category with enough projectives [2][Lemma 1.6] and contains all finitely generated continuous $\mathcal{B}$-modules. $\mathcal{C} \cap \mathcal{D}$ is the category of finite continuous $\mathcal{B}$-modules. There might exist finite, hence also finitely generated $\mathcal{B}$-modules on which the action of $\mathcal{B}$ is not continuous but we shall never consider these. From now on we assume all $\mathcal{B}$-modules continuous without further mention.

A well known representability theorem of Grothendieck asserts that any left exact functor $F : \mathcal{C} \cap \mathcal{D} \to \mathrm{Ab}$ is isomorphic to the functor $X \mapsto \mathrm{Hom}^c_{\mathcal{B}}(M, X)$ for some object M of $\mathcal{C}$ [7][CH. V,§2 Th. 3.1], [9]. M is projective in $\mathcal{C}$ (resp. finitely generated) if and only if F is exact [2][Prop. 3.1] (resp. satisfies a certain growth condition, see Lemma 3 below). Our aim in this section is to establish the following representability theorem for complexes.

Proposition 3.1. *Assume $M \mapsto C^{\cdot}(M)$ is a functor from $\mathcal{C} \cap \mathcal{D}$ to (cochain) complexes of $\mathbb{Z}_p$-modules such that*

i) $\exists\, n, m \in \mathbb{Z} \quad \forall M \in Ob(\mathcal{C} \cap \mathcal{D}) \quad H^i(C^{\cdot}(M)) = 0$ *for* $i \notin [n, m]$

ii) $\exists\, a_i \geq 0 \quad \forall M \in Ob(\mathcal{C} \cap \mathcal{D}) \qquad \# H^i(C^{\cdot}(M)) \leq (\# M)^{a_i}$.

iii) *Each C^i is an exact functor in M.*

Then there exists a bounded (chain) complex $P.$ of finitely generated, projective $\mathcal{B}$-modules and a natural isomorphism in the derived category of $\mathbb{Z}_p$-modules

$$Hom_{\mathcal{B}}(P., M) \xrightarrow{\sim} C^{\cdot}(M).$$

Note that $\mathrm{Hom}^c_{\mathcal{B}}(P_i, M) = \mathrm{Hom}_{\mathcal{B}}(P_i, M)$ *since P_i is finitely generated. Conversely, if $P.$ is such a complex then the functor* $\mathrm{Hom}_{\mathcal{B}}(P., -)$ *satisfies i), ii) and iii).*

Remark 3. If there are only finitely many simple $\mathcal{B}$-modules up to isomorphism then by an easy dévissage argument conditions i) and ii) can be relaxed to

i) $\forall M \in \mathrm{Ob}(\mathcal{C} \cap \mathcal{D}) \quad \exists n, m \in \mathbb{Z} \quad H^i(C^{\cdot}(M)) = 0$ for $i \notin [n, m]$

ii) $\# H^i(C^{\cdot}(M))$ is finite for all $M \in \mathrm{Ob}(\mathcal{C} \cap \mathcal{D})$.

We shall be mostly interested in the case where $\mathcal{B} = \mathbb{Z}_p[[G]]$ is the profinite group algebra of a profinite group G. In this case $\mathcal{B}$ will have only finitely many simple modules if G contains a pro-p group of finite index.

Proof of Proposition. Let n, m be as in i). Then the truncated complex

$$0 \to \ker(\delta^{n-1}) \to C^{n-1}(M) \xrightarrow{\delta^{n-1}} C^n(M) \to \cdots$$

is quasi-isomorphic to $C^{\cdot}(M)$ and still satisfies iii) since $\ker(\delta^{n-1}) = \mathrm{coker}(\delta^{n-3})$ is both a left and a right exact functor in M. So from now on we assume that $C^{\cdot}(M)$ is bounded below.

Let Q_i be the (projective) object of $\mathcal{C}$ which represents the functor C^i and which exists by Grothendieck's theorem. Using a Yoneda-Lemma type argument we obtain a bounded below (chain) complex $\cdots \to Q_i \to Q_{i-1} \to \cdots \to 0$ in $\mathcal{C}$ and an isomorphism of complexes

$$\mathrm{Hom}^c_{\mathcal{B}}(Q., M) \cong C^{\cdot}(M) \tag{2}$$

for all finite M. We shall now successively replace the terms in $Q_\cdot$ by finitely generated projective $\mathcal{B}$-modules.

Lemma 1. *Suppose $Q_\cdot^{(n)} \xrightarrow{\sim} Q_\cdot$ is a quasi-isomorphism (qis), all $Q_i^{(n)}$ are projective, and $Q_i^{(n)}$ is finitely generated for $i < n$. Then there is a qis $Q_\cdot^{(n+1)} \xrightarrow{\sim} Q_\cdot^{(n)}$ with $Q_i^{(n+1)} = Q_i^{(n)}$ for $i < n$, $Q_i^{(n+1)}$ finitely generated for $i < n+1$ and all $Q_i^{(n+1)}$ projective.*

Proof. We first remark that since any quasi-isomorphism between bounded below complexes of projective objects is a homotopy equivalence, all complexes $\mathrm{Hom}^c_{\mathcal{B}}(Q_\cdot^{(k)}, M)$ are quasi-isomorphic to $C^\cdot(M)$. Denoting differentials in $Q_\cdot^{(n)}$ by d we find an isomorphism

$$\ker\Big(\mathrm{Hom}^c_{\mathcal{B}}(Q_n^{(n)}, M) \to \mathrm{Hom}^c_{\mathcal{B}}(Q_{n+1}^{(n)}, M)\Big) \cong \mathrm{Hom}^c_{\mathcal{B}}(Q_n^{(n)}/\operatorname{im}(d_{n+1}), M)$$

and an exact sequence

$$\mathrm{Hom}^c_{\mathcal{B}}(Q_{n-1}^{(n)}, M) \xrightarrow{\kappa} \mathrm{Hom}^c_{\mathcal{B}}(Q_n^{(n)}/\operatorname{im}(d_{n+1}), M) \to H^n(\mathrm{Hom}^c_{\mathcal{B}}(Q_\cdot^{(n)}, M)) \to 0. \tag{3}$$

Since $Q_{n-1}^{(n)}$ is finitely generated Lemma 3 below implies that there is a constant β such that $\#\,\mathrm{Hom}^c_{\mathcal{B}}(Q_{n-1}^{(n)}, M) \le (\#M)^\beta$ and by assumption ii) in the proposition we have

$$\#H^n(\mathrm{Hom}^c_{\mathcal{B}}(Q_\cdot^{(n)}, M)) = \#H^n(C^\cdot(M)) \le (\#M)^{a_n}.$$

From (3) we deduce $\#\,\mathrm{Hom}^c_{\mathcal{B}}(Q_n^{(n)}/\operatorname{im}(d_{n+1}), M) \le (\#M)^{\beta + a_n}$ and hence by Lemma 3 that $Q_n^{(n)}/\operatorname{im}(d_{n+1})$ is finitely generated over $\mathcal{B}$. We pick a surjection $Q_n^{(n+1)} \twoheadrightarrow Q_n^{(n)}/\operatorname{im}(d_{n+1})$ where $Q_n^{(n+1)}$ is finitely generated and projective. This surjection can be lifted to a map $Q_n^{(n+1)} \twoheadrightarrow Q_n^{(n)}$ and we arrive at a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Q_{n+1}^{(n)} \arrow[r] & Q_n^{(n)} \arrow[r] & Q_{n-1}^{(n)} \arrow[r] & Q_{n-2}^{(n)} \arrow[r] & \cdots \\
 & & Q_n^{(n+1)} \arrow[u] \arrow[r, "d'_n"] & Q_{n-1}^{(n+1)} \arrow[r] \arrow[u, equal] & Q_{n-2}^{(n+1)} \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd} \tag{4}$$

such that (I) $Q_i^{(n+1)}$ is projective for $i \le n$ (II) the vertical map induces an isomorphism in homology for $i < n$ and (III) the vertical map induces a surjection $\ker(d'_n) \twoheadrightarrow H_n(Q_\cdot^{(n)})$. The conditions (I)-(III) are the inductive assumptions in the proof of [8][Prop. 11.9.1] with K'' (resp. K') the class of projective (resp. all) objects in $\mathcal{C}$. Applying this proof we can inductively complete (4) to a qis $Q_\cdot^{(n+1)} \xrightarrow{\sim} Q_\cdot^{(n)}$ as in the Lemma (where however $Q_i^{(n+1)}$ need not be finitely generated for $i > n$). $\square$

Lemma 2. *If $H^n(C^\cdot(M)) = 0$ for all finite $\mathcal{B}$-modules M then $H_n(Q_\cdot) = 0$.*

Proof. The exact sequence

$$0 \to \frac{\ker d_n}{\operatorname{im} d_{n+1}} \to \frac{Q_n}{\operatorname{im} d_{n+1}} \to \frac{Q_n}{\ker d_n} \to 0$$

induces an exact sequence

$$0 \to \mathrm{Hom}^c_{\mathcal{B}}(\frac{Q_n}{\ker d_n}, M) \xrightarrow{\iota} \mathrm{Hom}^c_{\mathcal{B}}(\frac{Q_n}{\operatorname{im} d_{n+1}}, M) \to \mathrm{Hom}^c_{\mathcal{B}}(\frac{\ker d_n}{\operatorname{im} d_{n+1}}, M)$$

and the map κ in (3) factors through ι. So if $H^n(C^\cdot(M)) = H^n(\mathrm{Hom}^c_{\mathcal{B}}(Q_\cdot^{(n)}, M)) = 0$ we must have $\operatorname{im}(\kappa) = \operatorname{im}(\iota) = \mathrm{Hom}^c_{\mathcal{B}}(\frac{Q_n}{\operatorname{im} d_{n+1}}, M)$ or, in other words, that any $\phi \in \mathrm{Hom}^c_{\mathcal{B}}(\frac{Q_n}{\operatorname{im} d_{n+1}}, M)$ restricts to zero in $\mathrm{Hom}^c_{\mathcal{B}}(\frac{\ker d_n}{\operatorname{im} d_{n+1}}, M)$. Taking ϕ to be the natural projection onto $M := \frac{Q_n}{\operatorname{im} d_{n+1}}/U$ where U runs through a fundamental system of neighborhoods of 0 in $\frac{Q_n}{\operatorname{im} d_{n+1}}$ we deduce

$$H_n(Q_\cdot) = \frac{\ker d_n}{\operatorname{im} d_{n+1}} \subseteq \bigcap U = 0.$$

$\square$

Remark. Unless $\mathcal{B}$ is Noetherian, $H_n(Q_\cdot)$ need not be finitely generated over $\mathcal{B}$.

We continue with the proof of Prop. 3.1. Since $Q_\cdot$ is bounded below we may put $Q_\cdot^{(n)} := Q_\cdot$ for some $n << 0$ and apply Lemma 1 inductively to arrive at a complex $Q_\cdot^{(\infty)} \xrightarrow{\sim} Q_\cdot$ consisting of finitely generated projective $\mathcal{B}$-modules and quasi-isomorphic to $Q_\cdot$. In fact it suffices to stop at $Q_\cdot^{(m+2)}$ where m is as in assumption i) in Prop. 3.1. By Lemma 2 the complex $Q_\cdot$ (and hence the complex $Q_\cdot^{(k)}$ for any k) is then acyclic in degrees greater than m. Define $P_\cdot$ to be the truncated complex

$$0 \to \operatorname{im}(d_{m+1}) \to Q_m^{(\infty)} \to Q_{m-1}^{(\infty)} \to \cdots \to 0. \tag{5}$$

We have a projective resolution

$$\cdots \to Q^{(\infty)}_{m+2} \to Q^{(\infty)}_{m+1} \to \mathrm{im}(d_{m+1}) \to 0$$

of $\mathrm{im}(d_{m+1})$. Since

$$\mathrm{Ext}^i_{\mathcal{C}}(\mathrm{im}(d_{m+1}), M) = H^i(\mathrm{Hom}^c_{\mathcal{B}}(Q^{(\infty)}_{\cdot+m+1}, M)) \cong H^{i+m+1}(C^{\cdot}(M)) = 0$$

for $i > 0$ and all finite M we find that $\mathrm{im}(d_{m+1})$ is projective in $\mathcal{C}$, again using [2][Prop. 3.1]. Since $Q^{(\infty)}_{m+1}$ is finitely generated over $\mathcal{B}$, so is $\mathrm{im}(d_{m+1})$. The natural quasi-isomorphism $Q^{(\infty)}_{\cdot} \xrightarrow{\sim} P_{\cdot}$ is a homotopy equivalence so that we have natural quasi-isomorphisms

$$\mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M) \to \mathrm{Hom}^c_{\mathcal{B}}(Q^{(\infty)}_{\cdot}, M) \leftarrow \mathrm{Hom}^c_{\mathcal{B}}(Q_{\cdot}, M) \to C^{\cdot}(M). \tag{6}$$

The complex $P_{\cdot}$ therefore satisfies all requirements. The converse statement in Prop. 3.1 follows easily from the fact that $P_{\cdot}$ is bounded together with Lemma 3 below. □

In the remainder of this section we briefly discuss the extent to which the representability result of Proposition 3.1 is valid on larger categories.

Proposition 3.2. *Suppose $C^{\cdot}$ in Prop. 3.1 is the restriction of a functor on $\mathcal{C}$ (resp. $\mathcal{D}$) with values in the category of complexes of $\mathbb{Z}_p$-modules. Assume also that $C^{\cdot}$ commutes with filtered inverse (resp. direct) limits. Then there is an isomorphism of functors from $\mathcal{C}$ (resp. $\mathcal{D}$) into the derived category of $\mathbb{Z}_p$-modules*

$$Hom_{\mathcal{B}}(P_{\cdot}, -) \xrightarrow{\sim} C^{\cdot}(-)$$

where $P_{\cdot}$ is as in Prop. 3.1.

Proof. The first two maps in (6) are quasi-isomorphisms for any object M of either $\mathcal{C}$ or $\mathcal{D}$ because they are induced by homotopy-equivalences of complexes. To show that the third map is a quasi-isomorphism represent M as a filtered inverse (resp. direct) limit

$$M = \varprojlim M_\iota \qquad (\text{resp. } M = \varinjlim M_\iota) \tag{7}$$

of objects M_ι of $\mathcal{C} \cap \mathcal{D}$ and use the fact that both $\mathrm{Hom}^c_{\mathcal{B}}(Q_i, -)$ and $C^i(-)$ commute with these limits. In the case of $\mathrm{Hom}^c_{\mathcal{B}}(Q_i, -)$ this is by definition of an inverse limit, resp. by [2][Lemma A.3] for the direct limit. □

In the situation of Prop. 3.2 the functor $\mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, -)$ on $\mathcal{C}$ (resp. $\mathcal{D}$) actually takes values in the bounded derived category of the abelian category $\mathcal{C}(\mathbb{Z}_p)$ (resp. $\mathcal{D}(\mathbb{Z}_p)$) as is easily seen by representing M as in (7). Let D be the derived category of $\mathcal{C}$ (resp. $\mathcal{D}$) and $D(\mathbb{Z}_p)$ the derived category of $\mathcal{C}(\mathbb{Z}_p)$ (resp. $\mathcal{D}(\mathbb{Z}_p)$). A functor $C^{\cdot}$ as in Prop. 3.2 naturally extends to a functor $F : D \to D(\mathbb{Z}_p)$ by sending $M^{\cdot}$ to the total complex of $C^{\cdot}(M^{\cdot})$ or, equivalently, of $\mathrm{Hom}^c_{\mathcal{B}}(P_{\cdot}, M^{\cdot})$. F will satisfy properties i), ii) in Proposition 3.1 when considered on $\mathcal{C} \cap \mathcal{D}$ via the natural full embedding $\mathcal{C} \cap \mathcal{D} \to D$. F will also be exact in the sense of being a *triangulated* functor.

Question. Conversely, is every triangulated functor $F : D \to D(\mathbb{Z}_p)$ which satisfies i) and ii) of Proposition 3.1 of the form $M^{\cdot} \mapsto \mathrm{Hom}^{\cdot}(P_{\cdot}, M^{\cdot})$ for some bounded complex $P_{\cdot}$ of finitely generated projective $\mathcal{B}$-modules? A positive answer to this question would be important in situations where one doesn't have "standard resolutions".

Lemma 3. *The following are equivalent for an object M of $\mathcal{C}$.*

a) *M is finitely generated over $\mathcal{B}$.*

b) *There is a constant $a \geq 0$ such that $\# \mathrm{Hom}^c_{\mathcal{B}}(M, X) \leq (\#X)^a$ for all finite $\mathcal{B}$-modules X.*

c) *There is $a \geq 0$ such that $\dim_D \mathrm{Hom}^c_{\mathcal{B}}(M, X) \leq a \cdot \dim_D X$ for all* simple *$\mathcal{B}$-modules X. Here D is the finite field $End_{\mathcal{B}}X$.*

Proof. a) $\Rightarrow$ b). If M is finitely generated by $m_1, ..., m_r$ as a $\mathcal{B}$-module, any $\phi \in \mathrm{Hom}^c_{\mathcal{B}}(M, X)$ is uniquely determined by $(\phi(m_1), ..., \phi(m_r)) \in X^r$. Hence $\# \mathrm{Hom}^c_{\mathcal{B}}(M, X) \leq \#X^r = (\#X)^r$.

b) $\Rightarrow$ c). Choosing X to be simple and taking logarithms to the base $\#D$ on both sides of b) gives $\dim_D \mathrm{Hom}^c_{\mathcal{B}}(M, X) \leq a \cdot \dim_D X$.

c) $\Rightarrow$ a). First we make use of the fact that any object M of $\mathcal{C}$ has a *projective hull* that is a homomorphism $\phi : P \to M$ with P projective, $\phi(P) = M$, and $\phi(N) \neq M$ for all proper closed submodules N of P. For Artinian $\mathcal{B}$ this is proved in [16][Prop.41] and the general case follows by noting that for any surjection $B \twoheadrightarrow B'$ of Artinian rings and projective hull $P \to M$, $P \otimes_B B' \to M \otimes_B B'$ is a projective hull. Using then the fact that projective hulls are unique up to isomorphism we get, for general profinite $\mathcal{B}$, an inverse system of projective hulls over the finite quotients of $\mathcal{B}$ whose inverse limit is projective in $\mathcal{C}$ [2][Cor. 3.3]. If X is simple, it follows easily from the definition that a projective hull ϕ induces a bijection

$$\mathrm{Hom}^c_{\mathcal{B}}(M,X) \xrightarrow{\sim} \mathrm{Hom}^c_{\mathcal{B}}(P,X). \tag{8}$$

Now we make use of the structure theorem for projective objects in $\mathcal{C}$ as given in [7][Ch V, §2, Th. 4.5 & Example 4.6 b)] which also follows from the corresponding theorem over Artinian rings [16][Cor. to Prop. 41]. Let Σ be a set of representatives for the isomorphism classes of simple $\mathcal{B}$-modules and for each $S \in \Sigma$ choose a projective hull $P_S \to S$. For any projective object P of $\mathcal{C}$ there exist index sets $I_S(P)$, of cardinality uniquely determined by P, and an isomorphism of objects in $\mathcal{C}$

$$P \cong \prod_{S\in\Sigma} \prod_{I_S(P)} P_S$$

where the right hand side carries the product topology. In particular if $X \in \Sigma$, $D := \mathrm{End}_{\mathcal{B}} X$, we find using (8) with $M = S \in \Sigma$

$$\mathrm{Hom}^c_{\mathcal{B}}(P,X) = \bigoplus_{S\in\Sigma} \bigoplus_{I_S(P)} \mathrm{Hom}^c_{\mathcal{B}}(P_S,X) = \bigoplus_{S\in\Sigma} \bigoplus_{I_S(P)} \mathrm{Hom}^c_{\mathcal{B}}(S,X) = \bigoplus_{I_X(P)} D. \tag{9}$$

This formula applied to $P = \mathcal{B}$ gives

$$X \cong \mathrm{Hom}^c_{\mathcal{B}}(\mathcal{B},X) = \bigoplus_{I_X(\mathcal{B})} D \tag{10}$$

and we find that the set $I_X(\mathcal{B})$ is finite (for any $X \in \Sigma$) because X is finite. Using (8), (9), (10) and the assumption in c) we obtain

$$\#I_X(P) = \dim_D \mathrm{Hom}^c_{\mathcal{B}}(P,X) = \dim_D \mathrm{Hom}^c_{\mathcal{B}}(M,X) \leq a \cdot \dim_D X = a \cdot \#I_X(\mathcal{B}).$$

Hence we can choose an injection of sets $I_X(P) \hookrightarrow \coprod_{a \text{ copies}} I_X(\mathcal{B})$ and obtain a surjection in $\mathcal{C}$

$$\left(\prod_{I_X(\mathcal{B})} P_X\right)^a \twoheadrightarrow \prod_{I_X(P)} P_X.$$

After taking the product over all $X \in \Sigma$ we find a surjection

$$\mathcal{B}^a = \left(\prod_{S\in\Sigma} \prod_{I_S(\mathcal{B})} P_S\right)^a \twoheadrightarrow \prod_{S\in\Sigma} \prod_{I_S(P)} P_S = P \xrightarrow{\phi} M$$

which shows that M is finitely generated. □

Remark 4. As in Remark 3 above, if Σ is finite, conditions b) and c) in Lemma 3 can be relaxed to

b') $\mathrm{Hom}^c_{\mathcal{B}}(M, X)$ is finite for all finite $\mathcal{B}$-modules X.

c') $\dim_D \mathrm{Hom}^c_{\mathcal{B}}(M, X)$ is finite for all simple $\mathcal{B}$-modules X where $D = \mathrm{End}_{\mathcal{B}} X$.

See also [13] for a proof of Lemma 3 in this case.

4. Computing the equivariant Euler characteristic

Let $\mathcal{A}$ be as in section 2 and $\mathcal{B}$ as in section 3. In this section we shall consider objects M of $\mathcal{C}$ which also carry an $\mathcal{A}$-action commuting with the $\mathcal{B}$-action, in other words $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-modules. We shall assume throughout that M is finitely generated over $\mathcal{A}$ (hence over $\mathbb{Z}_p$).

Proposition 4.1. *Assume $M \mapsto C^\cdot(M)$ is a functor as in Prop. 3.1, induced from a functor on $\mathcal{C}$ as in Prop. 3.2.*

a) *$C^\cdot$ maps $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-modules of finite projective $\mathcal{A}$-dimension to perfect complexes of $\mathcal{A}$-modules, i.e. objects of $D^{perf}(\mathcal{A})$.*

b) *$C^\cdot$ maps finite $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-modules of finite projective $\mathcal{A}$-dimension to objects of $D^{fperf}(\mathcal{A})$.*

c) *Suppose in addition that $[C^\cdot(M)] = d[M]$ in $K_0(\mathbb{Z}_p, \mathbb{Q}_p)$ for some integer d and all finite M. Then $[C^\cdot(M)]_{\mathcal{A}} = d[M]_{\mathcal{A}}$ in $K_0(\mathcal{A}, \mathbb{Q}_p)$ for all finite $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-modules M of finite projective $\mathcal{A}$-dimension.*

Proof. We use the notation of the proof of Lemma 3 and the structure theorem for projective objects of $\mathcal{C}$ mentioned there. If M is a (left) $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-module and P an object of $\mathcal{C}$ then $\mathrm{Hom}_{\mathcal{C}}(P, M) = \mathrm{Hom}^c_{\mathcal{B}}(P, M)$ retains a (left) $\mathcal{A}$-action. For any simple module $S \in \Sigma$ put $M_S = \mathrm{Hom}^c_{\mathcal{B}}(P_S, M)$.

Lemma 4. *We have $M_S = 0$ for all but finitely many $S \in \Sigma$.*

Proof. The action of $\mathcal{B}$ on M factors through the algebra $B' = \mathrm{im}\,(B \to \mathrm{End}_{\mathcal{A}} M)$ which is finitely generated as a $\mathbb{Z}_p$-module and hence has only finitely many simple modules up to isomorphism. If $S \in \Sigma$ is not one of those simple B'-modules we have

$$M_S = \mathrm{Hom}^c_{\mathcal{B}}(P_S, M) = \varprojlim \mathrm{Hom}^c_{\mathcal{B}}(P_S, M \otimes_B B'')$$

where B'' runs through the finite quotients of B'. Choosing a composition series of the finite B'-module $M \otimes_B B''$ none of its simple subquotients will be isomorphic to S, hence $\mathrm{Hom}^c_{\mathcal{B}}(P_S, M \otimes_B B'') = 0$ by an easy inductive argument using (8). $\square$

Similarly to (10) there is a direct sum decomposition of $\mathcal{A}$-modules

$$M \cong \mathrm{Hom}^c_{\mathcal{B}}(\mathcal{B}, M) \cong \bigoplus_{S\in\Sigma} \bigoplus_{I_S(\mathcal{B})} M_S \tag{11}$$

where the right hand direct sum is in fact *finite* by Lemma 4. If M has finite projective dimension over $\mathcal{A}$, the direct summand M_S of M will also have finite projective dimension. Now consider the complex $P.$ of Proposition 3.1. Since

$$\mathrm{Hom}^c_{\mathcal{B}}(P_i, M) = \bigoplus_{S\in\Sigma} \bigoplus_{I_S(P_i)} M_S \tag{12}$$

the complex $\mathrm{Hom}^c_{\mathcal{B}}(P., M) \cong C^{\cdot}(M)$ is a bounded complex of finitely generated $\mathcal{A}$-modules of finite projective $\mathcal{A}$-dimension, hence perfect. This gives a). If M is finite, so are the M_S which gives b). To prove c) note that

$$[C^{\cdot}(M)]_{\mathcal{A}} = \sum_{i\in\mathbb{Z}} (-1)^i \sum_{S\in\Sigma} \#I_S(P_i)[M_S]_{\mathcal{A}} = \sum_{S\in\Sigma} \left(\sum_{i\in\mathbb{Z}} (-1)^i \#I_S(P_i) \right) [M_S]_{\mathcal{A}}. \tag{13}$$

If $M \in \Sigma$ we have $M_S = 0$ for $S \neq M$ by (8). Taking $\mathcal{A} = \mathbb{Z}_p$ and $M \in \Sigma$ in (13) the assumption in part c) gives

$$d[M] = \sum_{i\in\mathbb{Z}} (-1)^i \#I_M(P_i)[M_M] \tag{14}$$

and equation (11) yields $[M] = (\#I_M(\mathcal{B}))[M_M]$ for $M \in \Sigma$. We conclude that

$$\sum_{i\in\mathbb{Z}} (-1)^i \#I_M(P_i) = d \cdot \#I_M(\mathcal{B}) \tag{15}$$

for all $M \in \Sigma$ since $[M_M] \neq 0$ in $K_0(\mathbb{Z}_p, \mathbb{Q}_p) \cong \mathbb{Z}$. Together with (11) and (13) this yields part c). $\square$

Remark. It is clear from the proof of Proposition 4.1 that any other "explicit" formula for the Euler characteristic of $C^{\cdot}(M)$ would determine the integers $\sum_{i\in\mathbb{Z}}(-1)^i \#I_S(P_i)$ and hence the equivariant Euler characteristic $[C^{\cdot}(M)]_{\mathcal{A}}$.

We conclude this section with another typical application of Proposition 3.1.

Proposition 4.2. *Assume $M \mapsto C^{\cdot}(M)$ is a functor as in Prop. 3.1, induced from a functor on $\mathcal{C}$ as in Prop. 3.2. Let $\mathcal{A} \to \mathcal{A}'$ be a homomorphism of algebras satisfying the assumptions of section 2 and M a $\mathcal{A} \otimes_{\mathbb{Z}_p} \mathcal{B}$-module, finitely generated and projective over $\mathcal{A}$. Then the natural map*

$$C^{\cdot}(M) \otimes^L_{\mathcal{A}} \mathcal{A}' \to C^{\cdot}(M \otimes_{\mathcal{A}} \mathcal{A}')$$

is a quasi-isomorphism.

Proof. By (11) and (12) the complex $\mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M)$ is a bounded complex of finitely generated projective $\mathcal{A}$-modules. Hence the quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M) \cong C^{\cdot}(M)$$

of Prop. 3.1 is a flat resolution of $C^{\cdot}(M)$ and by definition $C^{\cdot}(M) \otimes^L_{\mathcal{A}} \mathcal{A}' \cong \mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M) \otimes_{\mathcal{A}} \mathcal{A}'$. But the natural map

$$\mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M) \otimes_{\mathcal{A}} \mathcal{A}' \to \mathrm{Hom}_{\mathcal{B}}(P_{\cdot}, M \otimes_{\mathcal{A}} \mathcal{A}') \cong C^{\cdot}(M \otimes_{\mathcal{A}} \mathcal{A}')$$

is clearly a quasi-isomorphism: One reduces to P a free $\mathcal{B}$-module and then to $P = \mathcal{B}$ in which case the map $\mathrm{Hom}_{\mathcal{B}}(P, M) \otimes_{\mathcal{A}} \mathcal{A}' \to \mathrm{Hom}_{\mathcal{B}}(P, M \otimes_{\mathcal{A}} \mathcal{A}')$ is simply the identity of $M \otimes_{\mathcal{A}} \mathcal{A}'$. □

5. Examples

5.1. **Profinite groups.** Suppose Γ is a profinite group such that

a) $cd_p(\Gamma) < \infty$

b) $H^i(\Gamma, M)$ is finite for all finite $\mathbb{Z}_p$-modules M with continuous Γ-action.

Assumption b) is too weak to apply Prop. 3.1 directly to the standard continuous cochain complex $C^{\cdot}(\Gamma, M)$ with $\mathcal{B} = \mathbb{Z}_p[[\Gamma]]$ the profinite group algebra of Γ. However, one has

Proposition 5.1. *(cf.*[15]*[Remark after Prop 3.5.2]) Assume Γ satisfies a) and b) and G is a quotient of Γ which contains a pro-p subgroup of finite index. Then there exists a bounded complex $P_{\cdot}$ of finitely generated projective $\mathbb{Z}_p[[G]]$-modules such that $R\Gamma(\Gamma, M) \cong \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P_{\cdot}, M)$ for all continuous, profinite or discrete, $\mathbb{Z}_p[[G]]$-modules M.*

Proof. For finite M this follows from Prop. 3.1 and Remark 3 applied to $\mathcal{B} = \mathbb{Z}_p[[G]]$. If one defines the group cohomology $R\Gamma(\Gamma, M)$ for all profinite or discrete $\mathcal{B}$-modules via the standard continuous cochain complex $C^{\cdot}(\Gamma, M)$ the assumptions of Prop.3.2 are satisfied which gives the proposition. □

Remark 5. If $\mathcal{C}$ denotes the abelian category of profinite $\mathcal{B} = \mathbb{Z}_p[[\Gamma]]$-modules considered in section 3 then $C^{\cdot}(\Gamma, M)$ is quasi-isomorphic to $R\operatorname{Hom}_{\mathcal{C}}(\mathbb{Z}_p, M)$ for any object M of $\mathcal{C}$. Indeed, giving a continuous Γ-equivariant map $\Gamma \times \cdots \times \Gamma \to M$ is equivalent to giving a continuous $\mathbb{Z}_p[[\Gamma]]$-homomorphism $F_n \to M$ where $F_n := \varprojlim_U \mathbb{Z}_p[\Gamma/U \times \cdots \times \Gamma/U]$ (the limit taken over all open subgroups U of Γ). By [2][Cor. 3.3] F_n is projective in $\mathcal{C}$ because $\mathbb{Z}_p[\Gamma/U \times \cdots \times \Gamma/U]$ is a projective (indeed free) $\mathbb{Z}_p[\Gamma/U]$-module. Moreover, the standard boundary maps give a projective resolution $F. \twoheadrightarrow \mathbb{Z}_p$ in $\mathcal{C}$ so that $\operatorname{Hom}_{\mathcal{C}}(F., M) = C^{\cdot}(\Gamma, M)$.

Proposition 5.2. *Assume Γ satisfies a) and b) and M is a continuous $\mathcal{A}[[\Gamma]]$-module, finitely generated over $\mathcal{A}$. If M has finite projective dimension over $\mathcal{A}$ (and is finite) then $R\Gamma(\Gamma, M)$ is an object of $D^{perf}(\mathcal{A})$ ($D^{fperf}(\mathcal{A})$). If $[R\Gamma(\Gamma, M)] = d[M]$ for all finite M then $[R\Gamma(\Gamma, M)]_{\mathcal{A}} = d[M]_{\mathcal{A}}$ for all finite M of finite projective $\mathcal{A}$-dimension.*

Proof. This follows by applying Prop. 4.1 to $\mathcal{B} := \mathbb{Z}_p[[G]]$ and $C^{\cdot}(-) := C^{\cdot}(\Gamma, -)$, where $G = \operatorname{im}(\Gamma \to \operatorname{Aut}_{\mathcal{A}}(M))$. Since M is finitely generated over $\mathbb{Z}_p$, G contains a pro-p subgroup of finite index and Remark 3 applies. □

5.2. **Local fields.** Let Γ be the absolute Galois group of a finite extension K of $\mathbb{Q}_l$. Then a) and b) hold for Γ and one also knows that

$$[R\Gamma(\Gamma, M)] = -[K : \mathbb{Q}_l]\delta_{l,p}[M] \tag{16}$$

for finite continuous $\mathbb{Z}_p[[\Gamma]]$-modules M, where $\delta_{l,p}$ is the Kronecker delta [14][Th. II.5]. Hence Prop. 5.1 and Prop. 5.2 apply. In fact one can show slightly more.

Proposition 5.3. *Let Γ be the absolute Galois group of a finite extension K of $\mathbb{Q}_l$. Then there exists a bounded complex $P.$ of finitely generated, projective $\mathcal{B} :=$*

$\mathbb{Z}_p[[\Gamma]]$*-modules such that* $R\Gamma(\Gamma, M) = \mathrm{Hom}^c_{\mathcal{B}}(P., M)$ *for all continuous, profinite or discrete,* $\mathcal{B}$*-modules* M*.*

Proof. One has the estimates $\#H^0(\Gamma, M) \leq \#M$,

$$\#H^2(\Gamma, M) = \#H^0(\Gamma, M^*(1)) \leq \#M^*(1) = \#M$$

by local duality [14][Th. II.2], and

$$\#H^1(\Gamma, M) = \#H^0(\Gamma, M)\#H^2(\Gamma, M)\#M^{[K:\mathbb{Q}_l]\delta_{l,p}} \leq \#M^{[K:\mathbb{Q}_l]\delta_{l,p}+2}$$

by (16). Hence Prop. 3.1 applies directly to $\mathcal{B} = \mathbb{Z}_p[[\Gamma]]$ and $C^\cdot(M) = C^\cdot(\Gamma, M)$. □

For more examples see [14],[15].

5.3. **Étale Cohomology.** Let X be a scheme such that

a) $cd_p(X_{et}) < \infty$

b) $H^i(X_{et}, \mathcal{F})$ is finite for all constructible sheaves of $\mathbb{Z}_p$-modules $\mathcal{F}$ on X.

If R is a separably closed or local field of characteristic different from p, or $R = \mathbb{Z}[p^{-1}]$ and $p \neq 2$ then any scheme $X \xrightarrow{\pi} \mathrm{Spec}(R)$ of finite type over $\mathrm{Spec}(R)$ satisfies a) and b). This is an immediate consequence of the Leray spectral sequence for π together with the following facts

- A constructible sheaf of $\mathbb{Z}_p$-modules $\mathcal{F}$ on $\mathrm{Spec}(R)$ has finite cohomology and $cd_p(\mathrm{Spec}(R)) < \infty$ [12][II, Th. 3.1].
- If $\mathcal{F}$ is constructible on X then the sheaves $R^i\pi_*(\mathcal{F})$ are constructible on $\mathrm{Spec}(R)$ [6][Finitude, Th.1.1]
- $R^i\pi_*(\mathcal{F}) = 0$ for constructible $\mathcal{F}$ and $i > N$ (an integer depending on X but not $\mathcal{F}$)[6][Finitude, Rem. before 1.4].

Let $Y \to X$ be a (profinite) Galois cover of schemes with group G and put $\mathcal{B} = \mathbb{Z}_p[[G]]$. Any profinite continuous $\mathcal{B}$-module M, finitely generated over $\mathbb{Z}_p$, gives rise to a locally constant $\mathbb{Z}_p$-adic sheaf on X which we denote by the same letter M (this functor from modules to sheaves is exact).

Proposition 5.4. *Suppose* X *is a scheme satisfying a) and b) and* $Y \to X$ *is a Galois cover with group* G *such that* G *contains a pro-*p *subgroup of finite index.*

Then there exists a bounded complex of finitely generated $\mathcal{B} = \mathbb{Z}_p[[G]]$-modules $P.$ and a natural quasi-isomorphism

$$R\Gamma(X_{et}, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(P., M)$$

for all continuous $\mathcal{B}$-modules M, finitely generated over $\mathbb{Z}_p$.

Proof. Let $M \to \mathcal{G}^{\cdot}(M)$ be the Godement resolution of M [11][Remark III.1.20 (c)]. Then the complex $C^{\cdot}(M) = H^0(X, \mathcal{G}^{\cdot}(M))$ is functorial and exact in M and computes $R\Gamma(X_{et}, M)$. The proposition then follows from Prop. 3.1 and Remark 3 applied to $\mathcal{B}$ and $C^{\cdot}(M)$. □

Proposition 5.5. *Assume X satisfies a) and b) and M is a locally constant sheaf of $\mathcal{A}$-modules on X, finitely generated over $\mathcal{A}$. If M has finite projective dimension over $\mathcal{A}$ (and is finite) then $R\Gamma(X_{et}, M)$ is an object of $D^{perf}(\mathcal{A})$ ($D^{fperf}(\mathcal{A})$). If $[R\Gamma(X_{et}, M)] = d[M]$ for all finite M then $[R\Gamma(X_{et}, M)]_{\mathcal{A}} = d[M]_{\mathcal{A}}$ for all finite M of finite projective $\mathcal{A}$-dimension.*

Proof. Results of this type are more or less well known, at least when $\mathcal{A}$ is Artinian [6][Finitude, Rem. 1.7 and Rapport, Lemme 4.5.1]. The proof of this proposition is the same as that of Prop. 5.2, using the Godement resolution as in Prop. 5.4. □

There are variants of these statements for the cohomology with compact support $R\Gamma_c(X_{et}, \mathcal{F})$.

5.4. **Global fields.** This is the case which gave rise to this note. Suppose K is a global field of characteristic different from p and S a finite set of places of K including the archimedean ones and those dividing p. Denote by G_S the Galois group of the maximal algebraic extension of K unramified at places not in S and by G_v the absolute Galois group of the complete local field K_v for $v \in S$. If M is a continuous (profinite or discrete) G_S-module put

$$C^{\cdot}(M) = \mathrm{Cone}\big(C^{\cdot}(G_S, M) \to \prod_{v \in S} C^{\cdot}(G_v, M)\big)[-1] \tag{17}$$

where $C^{\cdot}(-,-)$ is the standard continuous cochain complex as in example 5.1. Then $C^{\cdot}(M)$ is in fact quasi-isomorphic to the étale cohomology with compact support

$R\Gamma_c(U,M)$ considered in [4][1.9] where U is the spectrum of the ring of S-integers in K.

Theorem 5.1. *Suppose $\mathcal{A}$ is as in section 2, U as above and M is a continuous $\mathcal{A}[[G_S]]$-module, finitely generated over $\mathcal{A}$ (resp. finite) and of finite projective dimension over $\mathcal{A}$. Then $R\Gamma_c(U,M)$ is a perfect complex of $\mathcal{A}$-modules (resp. an object of $D^{fperf}(\mathcal{A})$ and*

$$[R\Gamma_c(U,M)]_{\mathcal{A}} = 0$$

in $K_0(\mathcal{A},\mathbb{Q}_p)$).

Proof. If we put $\mathcal{B} = \mathbb{Z}_p[[G]]$ where $G = \mathrm{im}(G_S \to \mathrm{Aut}_{\mathcal{A}}(M))$, the complex $C^\cdot(-)$ defined in (17) satisfies all assumptions of Prop. 4.1. One also knows that $[C^\cdot(M)] = 0$ in $K_0(\mathbb{Z}_p,\mathbb{Q}_p)$ (this is an immediate reformulation of Tate's formula for the global Euler characteristic [12][I, Th.5.1]). □

Remark 6. If $\mathcal{A}$ is commutative, using Remark 1 and the determinant functor as in [5], one can show that $K_0(\mathcal{A},\mathbb{Q}_p)$ is generated by all triples $(X,g,\mathcal{A})$ where X is an invertible $\mathcal{A}$-module and $g: X\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \cong \mathcal{A}_{\mathbb{Q}}$ an isomorphism, i.e. $K_0(\mathcal{A},\mathbb{Q}_p)$ appears as the group of line bundles on $\mathrm{Spec}(\mathcal{A})$ together with a trivialization on $\mathrm{Spec}(\mathcal{A}_{\mathbb{Q}})$. From this point of view the statement of [4][Prop. 1.20b)] amounts to an identity in $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)$ and Theorem 5.1 is a direct generalization of [4][Prop. 1.20b)] to any algebra $\mathcal{A}$ as in section 2.

If $\mathcal{A}$ is Artinian and the Cartan map $K_0(\mathcal{A}[G]) \to G_0(\mathcal{A}[G])$ is injective for finite groups G, Theorem 5.1 can be proved along the lines of the proof of [4][Prop. 1.20]. However, in [3] one needs the case where $\mathcal{A}$ is an order in a finite dimensional semi-simple $\mathbb{Q}_p$-algebra. There does not seem to be a straightforward way to deduce the general case from the Artinian case. Also, there are Artinian rings for which the Cartan map is not injective.

Question. Do the hypotheses of Prop. 3.1 hold for $\mathcal{B} = \mathbb{Z}_p[[G_S]]$ and $C^\cdot(M)$ as in (17) (or $C^\cdot(M) = C^\cdot(G_S,M)$ if $p \neq 2$)? This question was raised by J.P. Serre in discussions with the author. He also expressed his belief that the answer to the question is in fact negative.

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