ON THE NUMBER OF BOUND STATES FOR SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS

DIRK HUNDERTMARK

ABSTRACT. Cwikel's bound is extended to an operator-valued setting. One application of this result is a semi-classical bound for the number of negative bound states for Schrödinger operators with operator-valued potentials. We recover Cwikel's bound for the Lieb–Thirring constant $L_{0,3}$ which is far worse than the best available by Lieb (for scalar potentials). However, it leads to a uniform bound (in the dimension $d \geq 3$) for the quotient $L_{0,d}/L^{\mathrm{cl}}_{0,d}$, where $L^{\mathrm{cl}}_{0,d}$ is the so-called classical constant. This gives some improvement in large dimensions.

1. Introduction

The Lieb-Thirring inequalities bound certain moments of the negative eigenvalues of a one-particle Schrödinger operator by the corresponding classical phase space moment. More precisely, for "nice enough" potentials one has

$$\operatorname{tr}_{L^2(\mathbb{R}^d)}(-\Delta + V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \iint\limits_{\mathbb{R}^d\mathbb{R}^d} d\xi dx\, (\xi^2 + V(x))_-^\gamma. \tag{1}$$

Here and in the following, $(x)_- = \frac{1}{2}(|x| - x)$ is the negative part of a real number or a self-adjoint operator. Doing the ξ integration explicitly with the help of scaling the above inequality is equivalent to its more often used form

$$\operatorname{tr}_{L^2(\mathbb{R}^d)}(-\Delta + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} dx\, V(x)_-^{\gamma+d/2}, \tag{2}$$

where the Lieb-Thirring constant $L_{\gamma,d}$ is given by $L_{\gamma,d} = C_{\gamma,d} L^{\mathrm{cl}}_{\gamma,d}$ with the classical Lieb-Thirring constant

$$L^{\mathrm{cl}}_{\gamma,d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp (1 - p^2)_+^\gamma. \tag{3}$$

This integral is, of course, explicitly given by a quotient of Gamma functions, but we will have no need for this. The Lieb-Thirring inequalities are valid as soon as the potential V is in $L^{\gamma+d/2}(\mathbb{R}^d)$.

Department of Mathematics 253–37, California Institute of Technology, Pasadena, CA 91125, U.S.A.; E-mail: dirkh@caltech.edu.

2000 Mathematics subject classification. Primary: 35P15, 47B10; Secondary: 81Q10, 47L20. Key words: CLR estimate, weak type estimates, singular values.

To appear in Arkiv för matematik.

These inequalities are important tools in the spectral theory of Schrödinger operators and they are known to hold if and only if $\gamma \geq \frac{1}{2}$ if $d = 1$, $\gamma > 0$ if $d = 2$, and $\gamma \geq 0$ if $d \geq 3$. The bound for the critical case $\gamma = 0$, that is, the bound for the number of negative eigenvalues of a Schrödinger operator in three or more dimensions is the celebrated Cwikel-Lieb-Rozenblum bound [6, 17, 21]. Later, different proofs for this were given by Conlon and Li and Yau [5, 16]. The remaining case $\gamma = \frac{1}{2}$ in $d = 1$ was settled in [29]. The well-known Weyl asymptotic formula

$$\lim_{\lambda\to\infty} \operatorname{tr}(-\Delta + \lambda V)_-^\gamma = L^{\mathrm{cl}}_{\gamma,d} \int dx\, V(x)_-^{\gamma+d/2}$$

immediately gives the lower bound $C_{\gamma,d} \geq 1$. There are certain refined lower bounds [20, 9] for small values of γ. In particular, one always has $C_{\gamma,d} > 1$ for $\gamma < 1$; see [9]. In one dimension this even happens for $\gamma < 3/2$, and in two dimensions, one always has $C_{1,2} > 1$ [20].

Depending on the dimension there are certain conjectures for the optimal value of the constants in these inequalities [19, 20]. One part of the conjectures on the Lieb-Thirring constants is that, indeed, $C_{\gamma,d} = 1$ for $d \geq 3$ and moments $\gamma \geq 1$. For the physically most important case $\gamma = 1$, $d = 3$ this would imply, via a duality argument, that the kinetic energy of fermions is bounded below by the Thomas-Fermi ansatz for the kinetic energy, which in turn has certain consequences for the energy of large quantum Coulomb systems [17, 19].

Laptev and Weidl [14] realized that a, at first glance, purely technical extension of the Lieb-Thirring inequality from scalar to operator-valued potentials already suggested in [12] is a key in proving at least a part of the Lieb-Thirring conjecture. It allowed them to show that $C_{\gamma,d} = 1$ for all $d \in \mathbb{N}$ as long as $\gamma \geq 3/2$. To prove this they considered Schrödinger operators of the form $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ on the Hilbert space $L^2(\mathbb{R}^d, \mathcal{G})$ where V now is an operator-valued potential with values $V(x)$ in the bounded self-adjoint operators on the auxiliary Hilbert space $\mathcal{G}$. In this case the Lieb-Thirring inequalities (1) and (2) are modified to

$$\operatorname{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \iint\limits_{\mathbb{R}^d\mathbb{R}^d} dp dx\, \operatorname{tr}_{\mathcal{G}}(p^2 + V(x))_-^\gamma, \tag{4}$$

or, again doing the ξ integral explicitly with the help of the spectral theorem and scaling

$$\operatorname{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} dx\, \operatorname{tr}_{\mathcal{G}}(V(x)_-^{\gamma+d/2}). \tag{5}$$

Here we abused the notation slightly in using the same symbol for the constants as in the scalar case. But in the following, we will only consider the operator-valued case anyway. Laptev and Weidl realized that this extension of the Lieb-Thirring inequality gives rise to the possibility of an inductive proof for $C_{3/2,d} = 1$ as long as one has the a priori information $C_{3/2,1} = 1$ for operator-valued potentials. This idea together with ideas in [11] was then later used in [10] to prove improved bounds on $C_{\gamma,d}$ in the range $1/2 \leq \gamma \leq 3/2$; in particular, it was shown that $C_{1,d} \leq 2$ uniformly in $d \in \mathbb{N}$.

Unlike the scalar case, however, the range of parameters γ and d for which (4) or equivalently (5) holds is not known. The results in [10] only show that these inequalities are true for $\gamma \geq 1/2$ and all $d \in \mathbb{N}$. This shortcoming has to do with the way the Lieb-Thirring estimates are proven for operator-valued potentials: First, the estimate is shown to hold in one dimension. Then a suitable induction proof, using the one-dimensional result, is set up to prove the full result in all dimensions. This turns out to give good estimates for the coefficients $C_{\gamma,d}$ in the Lieb-Thirring inequality, for example, they are independend of the dimension. However, moments below $1/2$ cannot be addressed with this method, since the a priori estimate fails already for scalar potentials.

This led Ari Laptev [13], see also [15], to ask the question whether, in particular, the Cwikel-Lieb-Rozenblum estimate holds for Schrödinger operators with operator-valued potentials. In this note we answer his question affirmatively, that is, the Lieb-Thirring inequalities for operator-valued potentials are shown to hold also for $\gamma = 0$ as long as $d \geq 3$ and then, by a monotonicity argument also for all $\gamma \geq 0$. More precisely, we want to show that Cwikel's proof of the Cwikel-Lieb-Rozenblum bound can be adapted to the operator-valued setting. However, the bound for $C_{0,d}$ is far from being optimal since we use Cwikel's approach. But, nevertheless, reasoning similar to Laptev and Weidl, any a priori bound on $C_{0,3}$ implies the bound $C_{0,d} \leq C_{0,3}$ for $d \geq 3$, thus giving a *uniform* bound in the dimension, whereas the best available bound in the scalar case due to Lieb [17] grows like $\sqrt{\pi d}$, see [20].

2. Statement of the results

Let $\mathcal{G}$ be a (separable) Hilbert space with norm $\|.\|_{\mathcal{G}}$, scalar product $\langle .,. \rangle_{\mathcal{G}}$, and let $\mathbf{1}_{\mathcal{G}}$ be the identity operator on $\mathcal{G}$. We follow the convention that scalar products are linear in the second component. Furthermore, $\mathcal{B}(\mathcal{G})$ is the Banach space of bounded operators equipped with the operator norm $\|.\|_{\mathcal{B}(\mathcal{G})}$ and $\mathcal{K}(\mathcal{G})$ the (separable) ideal of the compact operators on $\mathcal{G}$. For a compact operator $A \in \mathcal{K}(\mathcal{G})$, the singular values $\mu_n(A)$, $n \in \mathbb{N}$ are the eigenvalues of $|A| := (A^*A)^{1/2}$ arranged in decreasing order counting multiplicity. A^* is the adjoint of A. $\mathcal{S}^q(\mathcal{G})$ denotes the ideal of compact operators $A \in \mathcal{K}(\mathcal{G})$ whose singular values are q-summable, that is, $\sum_n \mu_n(A)^q < \infty$. In particular, $\mathcal{S}^1(\mathcal{G})$ and $\mathcal{S}^2(\mathcal{G})$ are the trace class and Hilbert-Schmidt operators on $\mathcal{G}$. We will often write $\mathcal{B}$, $\mathcal{K}$, and $\mathcal{S}^q$ if there is no ambiguity. Of course, $A \in \mathcal{S}^q$ if and only if $\mathrm{tr}_{\mathcal{G}}(|A|^q) = \mathrm{tr}_{\mathcal{G}}((A^*A)^{q/2}) < \infty$, where $\mathrm{tr}_{\mathcal{G}}$ is the trace on $\mathcal{G}$.

The Hilbert space $L^2(\mathbb{R}^d, \mathcal{G})$ is the space of all measurable functions $\phi : \mathbb{R}^d \to \mathcal{G}$ such that

$$\|\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})} := \int_{\mathbb{R}^d} dx \ \|\psi(x)\|^2_{\mathcal{G}} < \infty$$

and the Sobolev space $H^1(\mathbb{R}^d,\mathcal{G})$ consists of all functions $\psi\in L^2(\mathbb{R}^d,\mathcal{G})$ with finite norm

$$\|\psi\|^2_{H^1(\mathbb{R}^d,\mathcal{G})} := \sum_{l=1}^d \|\partial_l\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})} + \|\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})}\,.$$

As in the scalar case, the quadratic form

$$h_0(\psi,\psi) := \sum_{l=1}^d \|\partial_l\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})}$$

is closed in $L^2(\mathbb{R}^d,\mathcal{G})$ on the domain $H^1(\mathbb{R}^d,\mathcal{G})$. Naturally, this form corresponds to the Laplacian $-\Delta\otimes\mathbf{1}_{\mathcal{G}}$ on $L^2(\mathbb{R}^d,\mathcal{G})$.

$L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ is the space of operator-valued functions $f\colon\mathbb{R}^d\to\mathcal{B}(\mathcal{G})$ with finite norm

$$\|f\|_q^q = \|f\|^q_{L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))} := \int_{\mathbb{R}^d} dx\ \|f(x)\|^q_{\mathcal{B}(\mathcal{G})}$$

and $L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))$ the space of operator-valued functions f whose norm

$$\|f\|^q_{q,r} = \|f\|^q_{L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))} := \int_{\mathbb{R}^d} dx\, \mathrm{tr}_{\mathcal{G}}(|f(x)|^r)^{q/r}$$

is finite. A potential is a function $V\in L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ such that $V(x)$ is a symmetric operator for almost every $x\in\mathbb{R}^d$. If

$$q\geq 1 \text{ for } d=1,\quad q>1 \text{ for } d=2,\quad \text{and } q\geq d/2 \text{ for } d\geq 3 \tag{6}$$

one sees, using Sobolev embedding theorems as in the scalar case, that the real-valued quadratic form

$$v[\psi,\psi] := \int_{\mathbb{R}^d} dx\, \langle\psi(x),V(x)\psi(x)\rangle_{\mathcal{G}}$$

is infinitesimally form-bounded with respect to h_0. Hence the form sum

$$h[\psi,\psi] := h_0[\psi,\psi] + v[\psi,\psi]$$

is closed and semi-bounded from below on $H^1(\mathbb{R}^d,\mathcal{G})$ and thus generates the self-adjoint operator

$$H = -\Delta\otimes\mathbf{1}_{\mathcal{G}} + V$$

on $L^2(\mathbb{R}^d,\mathcal{G})$ by the KLMN theorem [23]. It is easy to see that any potential $V\in L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ satisfying (6) for which $V(x)\in\mathcal{K}(\mathcal{G})$ for almost every $x\in\mathbb{R}^d$ is relatively form compact with respect to h_0. Hence by Weyl's theorem for such potentials, the negative eigenvalues $E_0\leq E_1\leq E_3\leq\cdots\leq 0$ are at most a countable set with accumulation point zero and their eigenspaces are finite-dimensional. In particular, this is the case for potentials $V\in L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))$.

Our first result is a generalized version of a basic observation of Laptev and Weidl: The two versions (4) and (5) of the Lieb-Thirring inequality give rise to two different monotonicity properties of $C_{\gamma,d}$ in d.

Theorem 2.1 (Sub-multiplicativity of $\mathbf{C}_{\gamma,\mathbf{d}}$). *If, for dimensions n and $d-n$, the Lieb-Thirring inequality holds for operator-valued potentials then it also holds in dimension d. Moreover,*

$$\begin{aligned} C_{\gamma,d} &\leq C_{\gamma,n} C_{\gamma,d-n} \qquad \textit{and} &(7)\\ C_{\gamma,d} &\leq C_{\gamma,n} C_{\gamma+n/2,d-n}. &(8)\end{aligned}$$

Remarks 2.2. i) In the scalar case Aizenman and Lieb [1] showed that the map $\gamma \to C_{\gamma,d} = L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is decreasing. This monotonicity holds also in the general case, so, in fact, (8) implies (7). The monotonicity in γ is most easily seen in the phase space picture: By scaling one has, for $\gamma > \gamma_0 \geq 0$,

$$\int_0^\infty (s+t)_-^{\gamma_0}\, t^{\gamma-\gamma_0-1} dt = (s)_-^{\gamma}\, B(\gamma-\gamma_0, \gamma_0+1),$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ is the Beta function. In other words, for each choice of $\gamma > \gamma_0 \geq 0$ there exists a positive measure μ on $\mathbb{R}_+$ with $(s)_-^\gamma = \int_{\mathbb{R}_+} (s+t)_-^{\gamma_0}\, d\mu(t)$. Using this, the functional calculus, and the Fubini-Tonelli theorem, we immediately get

$$\begin{aligned} \mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(\Delta+V)_-^\gamma &= \int_0^\infty \mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(\Delta+V+t)_-^{\gamma_0}\, d\mu(t)\\ &\leq \frac{C_{\gamma_0,d}}{(2\pi)^d} \int_0^\infty d\mu(t) \iint d\xi dx\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x)+t)_-^{\gamma_0}\\ &= \frac{C_{\gamma_0,d}}{(2\pi)^d} \iint d\xi dx \int_0^\infty d\mu(t)\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x)+t)_-^{\gamma_0}\\ &= \frac{C_{\gamma_0,d}}{(2\pi)^d} \iint d\xi dx\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x))_-^{\gamma}. \end{aligned}$$

ii) Theorem 2.1 is a slight extension of a very nice observation of Laptev and Weidl [12, 14]. They used it to show $C_{\gamma,d} = 1$ as long as $\gamma \geq 3/2$. Basically this follows immediately by induction and the above monotonicity from (7) for $n = 1$ once one knows that $C_{3/2,1} = 1$. The beauty of this observation is that this bound is well-known in the scalar case [20] and Laptev and Weidl gave a proof for it in the general case. See also [2] for an elegant alternative proof which avoids the proof of Buslaev-Fadeev-Zhakarov type sum rules for matrix-valued potentials.

iii) Using $C_{\gamma,d} = 1$ for $\gamma \geq 3/2$ and (8), we get the bound

$$C_{\gamma,d} \leq C_{\gamma,3}$$

in $d \geq 3$ for all $\gamma \geq 0$. In particular, this implies a uniform bound (in d) for the constant in the Cwikel-Lieb-Rozenblum bound as soon as such an estimate is established in dimension three for operator-valued potentials. Below we will recover Cwikel's bound $C_{0,3} \leq 3^4 = 81$, see Corollary 2.4. It is, already for scalar potentials, known, that $C_{0,3} \geq 8/\sqrt{3} > 4.6188$, [7] [20, eq. (4.24)] (see also the discussion in [28, page 96–97]); in fact, it is conjectured to be the correct value [7, 20, 27]. In the *scalar case* Lieb's proof [17] of the CLR-bound gives by far the best estimate, $C_{0,3}^{\text{scalar}} \le 6.87$. However, Lieb's estimate grows like $\sqrt{\pi d}$ for large dimensions [20, eq. (5.5)]. While we get a quite large bound on $C_{0,3}$ this at least furnishes the uniform bound $C_{0,d} \le 81$ for all $d \ge 3$. It would be nice to extend Lieb's or even Conlon's proof [5] of the CLR-bound to operator-valued potentials.

To state our second result, Cwikel's bound in the operator-valued case, we need some more notation: $L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$, the analog of the weak L^q-space $L^q_w(\mathbb{R}^d)$, is given by all operator-valued functions $g\colon \mathbb{R}^d \to \mathcal{B}(\mathcal{G})$ for which

$$\|g\|^*_{q,w} = \|g\|^*_{L^q_w(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))} := \sup_{t>0} \big(t\, |\{\|g(\cdot)\|_{\mathcal{B}(\mathcal{G})} > t\}|^{1/q}\big) < \infty.$$

Here $|B|$ is the d-dimensional Lebesgue measure of a Borel set $B \subset \mathbb{R}^d$. Note that $\|\cdot\|^*_{q,w}$ is not a norm since it fails to obey the triangle inequality already for scalar g. But, as in the scalar case, one can give a norm on $L^q_w(\mathbb{R}^d; \mathcal{B}(\mathcal{G}))$ which is equivalent to $\|\cdot\|^*_{L^q(\mathbb{R}^d;\mathcal{B}(\mathcal{G}))}$. However, we will not need this.

With p we abbreviate the operator $-i\nabla$ and similarly to the scalar case we define the operator $f(x)g(p)$ to be

$$\psi \to f(x)g(p)\psi(x) = f(x)\frac{1}{(2\pi)^{d/2}} \int e^{ix\zeta} g(\zeta)\hat{\psi}(\zeta)\, d\zeta,$$

that is, $f(x)g(p) = M_f \mathcal{F}^{-1} M_g \mathcal{F}$ with M_f, M_g the "multiplication" operators by $f(x)$ and $g(\xi)$ and $\mathcal{F}$ the Fourier transform. A priori, $f(x)g(p)$ is well-defined only for simple functions, but it will turn out to be a compact operator for rather general "functions" f and g. The extension of Cwikel's bound to the operator-valued case is

Theorem 2.3 (Cwikel's bound, operator-valued case)**.** *Let f and g be operator-valued functions on an auxiliary Hilbert space $\mathcal{G}$. Assume that $f \in L^q(\mathbb{R}^d, \mathcal{S}^q(\mathcal{G}))$ and $g \in L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ for some $q > 2$. Then $f(x)g(p)$ is a compact operator on $L^2(\mathbb{R}^d, \mathcal{G})$. In fact, it is in the weak operator ideal $\mathcal{S}^q_w(L^2(\mathbb{R}^d, \mathcal{G}))$ and, moreover,*

$$\|f(x)g(p)\|^*_{q,w} := \sup_{n\ge1} n^{1/q}\mu_n\big(f(x)g(p)\big) \le K_q\, \|f\|_{q,q}\, \|g\|^*_{q,w} \tag{9}$$

where the constant K_q is given by

$$K_q = (2\pi)^{-d/q}\frac{q}{2}\left(\frac{8}{q-2}\right)^{1-2/q}\left(1+\frac{2}{q-2}\right)^{1/q}.$$

As in the scalar case Theorem 2.3 gives a bound for the number of negative eigenvalues of Schrödinger operators with operator-valued potentials.

Corollary 2.4. *Let $\mathcal{G}$ be some auxiliary Hilbert space and V a potential in $L^{d/2}(\mathbb{R}^d, \mathcal{S}^{d/2}(\mathcal{G}))$. Then the operator $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ has a finite number N of negative eigenvalues. Furthermore, we have the bound*

$$N \le L_{0,d} \int_{\mathbb{R}}^{d} \operatorname{tr}_{\mathcal{G}}(V(x)_-^{d/2})\, dx$$

with

$$L_{0,d} \leq (2\pi\, K_d)^d L^{\mathrm{cl}}_{0,d},$$

that is, $C_{0,d} \leq (2\pi\, K_d)^d$.

Proof. For completeness we explicitly derive the estimate for the number of negative eigenvalues of $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ from Theorem 2.3. Replacing V with $-(V)_-$ if necessary and using the min–max principle, we can assume V to be non-positive. Let N be the number of negative eigenvalues of $-\Delta \otimes +V$ and put $Y := |V|^{1/2}\,(|p|^{-1} \otimes \mathbf{1}_{\mathcal{G}})$. By the Birman-Schwinger principle [3, 25, 4, 26, 24] one has

$$1 \leq \mu_N(Y).$$

But $\xi \to |\xi|^{-1} \otimes \mathbf{1}_{\mathcal{G}}$ has weak $L^d(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$-norm $\tau_d^{1/d}$, τ_d being the volume of the unit ball in $\mathbb{R}^d$. With Theorem 2.3 we arrive at

$$1 \leq K_d \tau_d^{1/d} \| |V|^{1/2} \|_{d,d} N^{-1/d},$$

that is,

$$N \leq K_d^d \tau_d \| |V|^{1/2} \|_{d,d}^d = (2\pi K_d)^d L^{\mathrm{cl}}_{0,d} \int \mathrm{tr}_{\mathcal{G}}(|V(x)|^{d/2})\, dx,$$

since $L^{\mathrm{cl}}_{0,d} = \tau_d/(2\pi)^d$. ∎

Remark 2.5. Corollary 2.4 gives the a priori bound $C_{0,d} \leq (2\pi\, K_d)^d$ for $d \geq 3$. Using Theorem 2.1 and the fact that $C_{\gamma,d} = 1$ if $\gamma \geq 3/2$, [14], we know that $C_{0,d} \leq \min_{n=3,\dots,d} C_{0,n}$. Since the a priori bound given in Corollary 2.4 increases rather fast in the dimension, the best we can conclude is $C_{0,d} \leq (2\pi\, K_3)^3 = 3^4 = 81$.

3. Proof of the sub-multiplicativity of the Lieb-Thirring constants

We proceed very similarly to [14], but freeze the first $n < d$ variables. Let $x_< = (x_1, \dots, x_n)$, $x_> = (x_{n+1}, \dots, x_d)$ and $\xi_<$, $\xi_>$ similarly defined. Put

$$W(x_<) := (-\Delta_> + V(x_<, .))_-,$$

where $\Delta_>$ is the Laplacian in the $x_>$ variables. Clearly, by assumption on V, W is a non-negative compact operator on $L^2(\mathbb{R}^{d-n}, \mathcal{G})$ for almost all $x_< \in \mathbb{R}^n$ and, moreover,

$$\begin{aligned}\mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta + V)_-^\gamma &\leq \mathrm{tr}_{L^2(\mathbb{R}^n, L^2(\mathbb{R}^{d-n},\mathcal{G}))}(-\Delta_< - W)_-^\gamma \\ &\leq \frac{C_{\gamma,n}}{(2\pi)^n} \iint\limits_{\mathbb{R}^n \mathbb{R}^n} d\xi_< dx_<\, \mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - W(x_<))_-^\gamma. \end{aligned} \tag{10}$$

Since $(t - (s)_-)_- = (t+s)_-$ for $t \geq 0$, $s \in \mathbb{R}$, the spectral theorem gives

$$\begin{aligned}\mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - W(x_<))_-^\gamma &= \mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - \Delta_> + V(x_<, .))_-^\gamma \\ &\leq \frac{C_{\gamma,d-n}}{(2\pi)^{d-n}} \iint\limits_{\mathbb{R}^{d-n}\mathbb{R}^{d-n}} d\xi_> dx_>\, \mathrm{tr}_{\mathcal{G}}(\xi_<^2 + \xi_>^2 + V(x_<, x_>))_-^\gamma.\end{aligned}$$

This together with (10) and the Fubini-Tonelli theorem shows (7).
For the other inequality we use the more usual form (5) of the Lieb-Thirring inequality. Again, freezing the first n coordinates and proceeding as before, we immediately get

$$L_{\gamma,d} \leq L_{\gamma,n} L_{\gamma+n/2,d-n}, \tag{11}$$

where $L_{\gamma+n/2,d-n}$ enters now because in the first application of the Lieb-Thirring inequality (5) the exponent is raised from γ to $\gamma + n/2$. Using the definition (3) for the classical Lieb-Thirring constant together with the Fubini-Tonelli theorem and scaling, one easily sees

$$\begin{aligned} L^{\mathrm{cl}}_{\gamma,d} &= \int_{\mathbb{R}^d} dp\,(|p|^2 - 1)^{\gamma}_{-} \\ &= \int_{\mathbb{R}^n} dp_<\,(|p_<|^2 - 1)^{\gamma}_{-} \int_{\mathbb{R}^{(d-n)}} dp_>\,(|p_>|^2 - 1)^{\gamma+n/2}_{-} \\ &= L^{\mathrm{cl}}_{\gamma,n} L^{\mathrm{cl}}_{\gamma+n/2,d-n}. \end{aligned}$$

This together with (11) proves (8) and thus Theorem 2.1.

4. Proof of Cwikel's bound

The proof of Theorem 2.3 follows closely Cwikel's original proof. We first need a criterion for $f(x)g(p)$ to be a Hilbert-Schmidt operator.

Lemma 4.1. *Let $f \in L^2(\mathbb{R}^d, \mathcal{S}^2(\mathcal{G}))$ and assume g obeys $\|g(.)\|_{\mathcal{B}(\mathcal{G})} \in L^2(\mathbb{R}^d)$. Then the operator $f(x)g(p)$ is Hilbert-Schmidt and we have the estimate*

$$\begin{aligned} \|f(x)g(p)\|^2_{HS} &= (2\pi)^{-d} \iint \mathrm{tr}_{\mathcal{G}}[g^*(\xi) f(x)^* f(x) g(\xi)]\, dx d\xi \\ &\leq (2\pi)^{-d} \int \mathrm{tr}_{\mathcal{G}}[|f(x)|^2]\, dx \int \|g(\xi)\|^2_{\mathcal{G}}\, d\xi. \end{aligned}$$

Proof. In the scalar case this is well-known and is usually shown by noting that in this case $f(x)g(p)$ is a convolution operator. Another proof is by changing the basis: Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^d, \mathcal{G})$, that is,

$$\mathcal{F}u(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi\cdot x} u(x)\, dx.$$

Then the Hilbert-Schmidt norms of $f(x)g(p)$ and $M_f\mathcal{F}^{-1}M_g$ are equal. The operator $M_f\mathcal{F}^{-1}M_g$ has "kernel" $(2\pi)^{-d/2}\mathrm{e}^{ix\cdot\xi}f(x)g(\xi)$ and thus by [22, Theorem VI.23] or [8, Section III.9]

$$\begin{aligned}
\|f(x)g(p)\|_{HS}^2 &= \left\|f(x)\mathcal{F}^{-1}g(\xi)\right\|_{HS}^2 \\
&= (2\pi)^{-d}\iint \operatorname{tr}_{\mathcal{G}}(g(\xi)^*f(x)^*f(x)g(\xi))\,dxd\xi \\
&= (2\pi)^{-d}\iint \operatorname{tr}_{\mathcal{G}}(|f(x)|^2|g(\xi)^*|^2)\,dxd\xi \\
&\le (2\pi)^{-d}\iint \operatorname{tr}_{\mathcal{G}}(|f(x)|^2)\,\|g(\xi)\|_{\mathcal{G}}^2)\,dxd\xi \\
&= (2\pi)^{-d}\int \operatorname{tr}_{\mathcal{G}}(|f(x)|^2)\,dx\int \|g(\xi)\|_{\mathcal{G}}^2)\,d\xi.
\end{aligned}$$

■

The first step rests on splitting the operator $f(x)g(p)$ (which is a priori only defined on simple functions) into manageable pieces. Fix $t>0$, $r>1$ and assume that f and g are non-negative, in particular, self-adjoint, operator-valued functions. For a Borel subset B of $\mathbb{R}$ let $\chi_B(f(x))$ and $\chi_B(g(\xi))$ be the spectral projection operators of $f(x)$ and $g(\xi)$, respectively. By the functional calculus we have

$$\begin{aligned}
f(x) &= \sum_{l\in\mathbb{Z}} f(x)\chi_{t(r^{l-1},r^l]}(f(x)) = \sum_{l\in\mathbb{Z}} f_l(x) \\
g(\xi) &= \sum_{l\in\mathbb{Z}} g(\xi)\chi_{(r^{l-1},r^l]}(g(\xi)) = \sum_{l\in\mathbb{Z}} g_l(\xi),
\end{aligned} \tag{12}$$

where f_l (resp., g_m) are mutually orthogonal operators. We use this decomposition of f and g to split the operator $f(x)g(p)$ into

$$f(X)g(P) = B_t + H_t \tag{13}$$

with $B_t := \sum_{l+m\le 1} f_l(x)g_m(p)$, $H_t := \sum_{l+m>1} f_l(x)g_m(p)$. Note that this decomposition of $f(x)g(p)$ is slightly different from the one used by Cwikel. We have

Lemma 4.2. *Let f and g be non-negative operator-valued functions. If $q>2$ and $f\in L^q(\mathbb{R}^d,\mathcal{S}^q(\mathcal{G}))$, $g\in L^q_w(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ with $\|f\|_{q,q}=1$ and $\|g\|_{q,w}^*=1$ then*
a) *B_t is a bounded operator with operator norm bounded by*

$$\|B_t\|_{L^2(\mathbb{R}^d,\mathcal{G})} \le t\frac{r}{1-r^{-1}}.$$

b) *H_t is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by*

$$\|H_t\|_{HS}^2 \le (2\pi)^{-d}t^{-(q-2)}\Big(1+\frac{2}{q-2}\Big).$$

Remarks 4.3. i) Due to our choice of B_t, H_t the bound in Lemma 4.2.b) is *independent* of r and in a) it easy to see that the choice $r=2$ is optimal.

ii) This lemma also shows that $f(x)g(p)$ is a compact operator since it is the norm limit for $t \to 0$ of the Hilbert-Schmidt operators H_t.

Proof. Part a) follows completely Cwikel's original proof: Since the f_l (resp., g_m) are orthogonal operators for different indices we get, for simple functions ψ and ϕ, say,

$$
\begin{aligned}
|\langle \psi, B_t \phi\rangle| &\le \sum_{l+m\le 1} r^{l+m} \big\| r^{-l} f_l(x)\psi \big\|_2 \, \big\| r^{-m} g_m(p)\phi \big\|_2 \\
&\le \sum_{s\le 1} r^s \sum_{m\in\mathbb{N}} \big\| r^{-(s-m)} f_{s-m}(x)\psi \big\|_2 \big\| r^{-m} g_m(p)\phi \big\|_2 \\
&\le \sum_{s\le 1} r^s \left(\sum_{m\in\mathbb{N}} \big\| r^{-(s-m)} f_{s-m}(x)\psi \big\|_2^2 \right)^{1/2} \left(\sum_{m\in\mathbb{N}} \big\| r^{-m} g_m(p)\phi \big\|_2^2 \right)^{1/2} \\
&= \sum_{s\le 1} r^s \Big\| \sum_{m\in\mathbb{N}} r^{-(s-m)} f_{s-m}(x)\psi \Big\|_2 \Big\| \sum_{m\in\mathbb{N}} r^{-m} g_m(p)\phi \Big\|_2 \\
&\le r(1-r^{-1})^{-1} t \, \|\psi\| \, \|\phi\| ,
\end{aligned}
$$

since $\sum_l r^{-l} f_l(x) \le t\mathbf{1}_{\mathcal{G}}$ and $\sum_m r^{-m} g_m(\xi) \le \mathbf{1}_{\mathcal{G}}$. Thus B_t extends to a bounded operator on $L^2(\mathbb{R}^d, \mathcal{G})$ with the given bound for its norm.

To prove part b) observe that by Lemma 4.1 and the cyclicity of the trace, we have

$$
\|H_t\|_{\mathrm{HS}}^2 = \sum_{l+m>1} \iint \mathrm{tr}_{\mathcal{G}}[f_l(x) g_m(\xi)^2 f_l(x)] \, dx d\xi .
$$

Assume for $x, \xi \in \mathbb{R}^d$ the operator inequality

$$
\begin{aligned}
\sum_{l+m>1} f_l(x) g_m(\xi)^2 f_l(x) &\le \Big(\|g(\xi)\| \, f(x) \chi_{(t,\infty)}(\|g(\xi)\| \, f(x)) \Big)^2 \\
&=: h(x,\xi)^2
\end{aligned}
\tag{14}
$$

on the Hilbert space $\mathcal{G}$. Note that the projection operator $\chi_{(t,\infty)}(\|g(\xi)\| \, f(x))$ (on $\mathcal{G}$) commutes with $f(x)$ for all $x, \xi \in \mathbb{R}^d$. Let $\lambda_j(x)$ be the j^{th} ordered eigenvalue of $f(x)$, and $E_j(\alpha) := \{\|g(.)\| \, \lambda_j(.) > \alpha\}$. Each E_j has $2d$ dimensional Lebesgue measure

$$
|E_j(\alpha)|_{2d} = \int |\{\|g(.)\| > \alpha/\lambda_j(x)\}|_d \, dx \le \alpha^{-q} \int \lambda_j(x)^q \, dx,
$$

since $\|g\|^*_{q,w} = 1$ by assumption. Thus we see

$$\begin{aligned}
\|H_t\|^2_{HS} &\le (2\pi)^{-d} \iint \mathrm{tr}_{\mathcal{G}}[h(x,\xi)^2]\, dx d\xi \\
&= (2\pi)^{-d} \sum_j 2 \int_0^\infty |E_j(\max(\alpha, t))|_{2d} \alpha\, d\alpha \\
&= (2\pi)^{-d} \left(\sum_j 2 \int_0^t |E_j(t)|_{2d} \alpha\, d\alpha + \sum_j 2 \int_t^\infty |E_j(\alpha)|_{2d} \alpha\, d\alpha \right) \\
&\le (2\pi)^{-d} t^{-(q-2)} \big(1 + 2/(q-2)\big) \sum_j \int \lambda_j(x)^q\, dx \\
&= (2\pi)^{-d} t^{-(q-2)} \big(1 + 2/(q-2)\big),
\end{aligned}$$

since $\sum_j \int \lambda_j(x)^q\, dx = \|f\|^q_{q,q} = 1$ by assumption. It remains to prove (14): Again, let $s = l + m$ and note that the $g_m(\xi) = g(\xi)\chi_{(r^{m-1}, r^m]}(g(\xi))$ are orthogonal operators for different indices. As operators on $\mathcal{G}$,

$$\begin{aligned}
\sum_{l+m>1} f_l(x) g_m(\xi)^2 f_l(x) &= \sum_{l\in\mathbb{Z}} \sum_{s\ge 2} f_l(x) g_{s-l}(\xi)^2 f_l(x) \\
&= \sum_{l\in\mathbb{Z}} f_l(x) \big(\sum_{s\ge 2} g_{s-l}(\xi)^2 \big) f_l(x) = \sum_{l\in\mathbb{Z}} f_l(x) g(\xi)^2 \chi_{(r^{1-l},\infty)}(g(\xi)) f_l(x) \\
&\le \sum_{l\in\mathbb{Z}} f_l(x)\, \|g(\xi)\|^2_{\mathcal{G}}\, \chi_{(r^{1-l},\infty)}(\|g(\xi)\|) f_l(x) \\
&= f(x)^2\, \|g(\xi)\|^2_{\mathcal{G}} \sum_{l\in\mathbb{Z}} \underbrace{\chi_{(r^{1-l},\infty)}(\|g(\xi)\|) \chi_{t(r^{l-1}, r^l]}(f(x))}_{\le \chi_{(t,\infty)}(\|g(\xi)\| f(x)) \chi_{t(r^{l-1},r^l]}(f(x))} \\
&\le f(x)^2\, \|g(\xi)\|^2_{\mathcal{G}}\, \chi_{(t,\infty)}(\|g(\xi)\|\, f(x)) \sum_{l\in\mathbb{Z}} \chi_{t(r^{l-1},r^l]}(f(x)) \\
&= f(x)^2\, \|g(\xi)\|^2_{\mathcal{G}}\, \chi_{(t,\infty)}(\|g(\xi)\|\, f(x)),
\end{aligned}$$

which proves (14) and hence the lemma. ∎

Given the above bounds the proof of Theorem 2.3 is by now a standard interpolation argument. We give this argument for the sake of completeness:

Proof of Theorem 2.3. First, without loss of generality assume that f and g are non-negative operator-valued functions. Indeed, let $\mathcal{F}$ be the Fourier transform and M_f and M_g the operators of "multiplication" by f and g and note that $f(x)g(p)$ and $M_f \mathcal{F}^{-1} M_g$ have the same singular values. With the polar decompositions $f(x) = U_1(x)|f(x)|$ and $g(\xi) = |g^*(\xi)|U_2^*(\xi)$ in the Hilbert space $\mathcal{G}$ we have

$$M_f \mathcal{F}^{-1} M_g = U_1 M_{|f|} \mathcal{F}^{-1} M_{|g^*|} U_2^*,$$

where U_j, $j \in \{1,2\}$ are fibered partial isometries in the space $L^2(\mathbb{R}^d, \mathcal{G})$, for example, $(U_1\psi)(x) = U_1(x)\psi(x)$. Hence the singular values of $f(x)g(p)$ are bounded by the singular values of $M_{|f|}\mathcal{F}^{-1}M_{|g^*|}$ and $\||g^*|\|^*_{q,w} = \|g\|^*_{q,w}$.

By one of the consequences of Ky Fan's inequality [8] we have

$$\begin{aligned}\mu_n(f(x)g(p)) = \mu_n(B_t + H_t) \leq \mu_1(B_t) + \mu_n(H_t) \leq \|B_t\| + \frac{1}{\sqrt{n}}\|H_t\|_{HS} \\ \leq t\frac{r}{1-r^{-1}} + (2\pi)^{-d/2}t^{-(q-2)/2}\big(1+\frac{2}{q-2}\big)^{1/2}\frac{1}{\sqrt{n}}\end{aligned}$$

using Lemma 4.2. Choosing t and r $(=2)$ optimal gives

$$\mu_n(f(x)g(p)) \leq (2\pi)^{-d/q}\frac{q}{2}\left(\frac{8}{q-2}\right)^{1-2/q}\left(1+\frac{2}{q-2}\right)^{1/q} n^{-1/q}$$

which proves Theorem 2.3. ∎

Acknowledgment: It is a pleasure to thank Ari Laptev and Timo Weidl for stimulating discussions and Michael Solomyak for comments and discussions on an earlier version of the paper.

References

[1] Aizenmann, M. and Lieb, E. H., *On semi-classical bounds for eigenvalues of Schrödinger operators.* Phys. Lett. **66A** (1978), 427–429.
[2] Benguria, R. and Loss, M., *On a theorem of Laptev and Weidl.* Math. Research Letters, **7** (2000), 195–203.
[3] Birman, M. S., *The spectrum of singular boundary problems.* Mat. Sb. **55** no. 2 (1961), 125–174, translated in Amer. Math. Soc. Trans. (2), **53** (1966), 23–80.
[4] Birman, M. S. and Solomyak, M. Z., *Spectral theory of selfadjoint operators in Hilbert space.* Translated from the 1980 Russian original by S. Khrushchv and V. Peller. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987. xv+301 pp.
[5] Conlon, J. G., *A new proof of the Cwikel-Lieb-Rosenbljum bound.* Rocky Mountain J. Math. **15** no. 1 (1985), 117–122.
[6] Cwikel, M., *Weak type estimates for singular values and the number of bound states of Schrödinger operators.* Ann. Math. **106** no. 1 (1977), 93–100.
[7] Glaser, V., Grosse, H., Martin, A., and Thirring, W., *A family of optimal conditions for the absence of bound states in a potential.* Studies in Math. Phys., Essays in Honor of Valentine Bargmann, Princeton University Press, New Jersey (1976).
[8] Gohberg, I. C. and Krein, M. C., *Introduction to the theory of linear non-selfadjoint operators.* Trans. Math. Monographs vol. **18**, AMS (1969).
[9] Helffer, B.,and Robert, D., *Riesz means of bound states and semi-classical limit connected with a Lieb-Thirring conjecture. I, II.* Asymptotic Anal. **3** (1990), no. 2, 91–103. Ann. Inst. H. Poincaré Phys. Théor. **53** no. 2 (1990), 139–147.
[10] Hundertmark, D., Laptev, A., and Weidl, T., *New bounds on the Lieb-Thirring constant.* Invent. Math. **140** no. 3 (2000), 693–704
[11] Hundertmark, D., Lieb, E. H., and Thomas, L. E., *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator.* Adv. Theor. Math. Phys. **2**, no. 4 (1998), 719–731.

[12] Laptev, A., *Dirichlet and Neumann eigenvalue problems on domains in Euclidian spaces.* J. Funct. Anal. **151** (1997), 531–545.

[13] Laptev, A. Private communication.

[14] Laptev, A. and Weidl, T., *Sharp Lieb-Thirring inequalities in high dimensions.* Acta Math. **184** no. 1 (2000), 87–111.

[15] Laptev, A. and Weidl, T., *Recent results on Lieb-Thirring inequalities.* To appear in Journées Équation aux dérivées partielles.

[16] Li, P. and S.-T. Yau, S. T., *On the Schrödinger equation and the eigenvalue problem.* Comm. Math. Phys. **88** (1983), 309–318.

[17] Lieb, E. H., *Bounds on the eigenvalues of the Laplace and Schrödinger operators.* Bull. Amer. Math. Soc., **82** (1976), 751–753. See also Proc. A.M.S. Symp. Pure Math. **36** (1980), 241–252.

[18] Lieb, E. H., *Thomas-Fermi and related theories of atoms and molecules.* Rev. Modern Phys. **53**, no. 4 (1981), 603–641.

[19] Lieb, E. H. and W. Thirring, W., *Bound for the kinetic energy of fermions which proves the stability of matter.* Phys. Rev. Lett. **35** (1975), 687–689. Errata **35** (1975), 1116.

[20] Lieb, E. H. and W. Thirring, W., *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities.* Studies in Math. Phys., Essays in Honor of Valentine Bargmann, Princeton University Press, New Jersey (1976),

[21] Rozenblum, G. V., *Distribution of the discrete spectrum of singular differential operators.* Dokl. Akad. Nauk SSSR, **202**, **N 5** (1972), 1012-1015. Translated in Soviet Math. Dok. **13** (1972), 245–249. See also Izv. VUZov, Matematika, **N.1**(1976), 75–86.

[22] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis.* Revised and enlarged edition. Academic Press, New York (1980).

[23] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics II: Fourier Analysis, Self–Adjointness.* Academic Press, New York (1975).

[24] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics IV: Analysis of Operators.* Academic Press, New York (1978).

[25] Schwinger, J., *On the bound states of a given potential.* Proc. Nat. Acad. Sci. U.S.A. **47**, (1961), 122–129.

[26] Simon, B., *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms.* Princeton Series in Physics, Princeton University press, New Jersey (1971).

[27] Simon, B., *Bound states of two-body Schrödinger operators – A review.* Studies in Math. Phys., Essays in Honor of Valentine Bargmann, Princeton University Press, New Jersey (1976).

[28] Simon, B., *Functional Integration and Quantum Physics.* Academic Press, New York, (1977).

[29] Weidl, T., *On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$.* Comm. Math. Phys. **178** no. 1 (1996), 135–146.