A DESCENT CRITERION FOR ISOBARIC REPRESENTATIONS

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This will appear as an appendix to the paper *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* ([K]) by Henry Kim.

The object here is to prove the following extension (from cuspidal) to *isobaric* automorphic representations of Proposition 3.6.1 of [Ra], which was itself an extension to $\mathrm{GL}(n)$ of the Proposition 4.2 (for $\mathrm{GL}(2)$) in [BR]. The argument is essentially the same as in [Ra], but requires some delicate bookkeeping.

Proposition. *Fix $n, p \in \mathbb{N}$ with p prime. Let F a number field, $\{K_j \mid j \in \mathbb{N}\}$ an infinite family of cyclic extensions of F with $[K_j : F] = p$, and for each $j \in \mathbb{N}$, π_j an isobaric automorphic representation of $GL(n, \mathbb{A}_{K_j})$. Suppose that, for all $j, r \in \mathbb{N}$, the base changes of π_j, π_r to the compositum K_jK_r satisfy*

$$(DC) \qquad (\pi_j)_{K_jK_r} \simeq (\pi_r)_{K_jK_r}.$$

Then there exists a unique isobaric automorphic representation π of $GL(n, \mathbb{A}_F)$ such that

$$(\pi)_{K_j} \simeq \pi_j,$$

for all but a finite number of j.

Proof. Recall that the set *Isob* of isobaric automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$ for all $n \geq 1$ admits a sum operation $\boxplus$, called the *isobaric sum*, such that

$$L(s, \pi \boxplus \pi') = L(s, \pi)L(s, \pi'), \ \forall\, \pi, \pi' \in Isob.$$

Moreover, given any isobaric automorphic representation π of $\mathrm{GL}(n, \mathbb{A}_F)$ there exist cuspidal automorphic representations $\pi^1, \dots, \pi^d$ of $\mathrm{GL}(n_1, \mathbb{A}_F), \dots, \mathrm{GL}(n_d, \mathbb{A}_F)$, with $n = n_1 + \dots + n_d$, such that

$$(1) \qquad \pi \simeq \pi^1 \boxplus \dots \boxplus \pi^d.$$

Here the cuspidal datum $(\pi^1, \dots, \pi^d)$ is unique up to (isomorphism and) permutation. We will say that π is of *width* d. For the basic properties of isobaric representations see [La] and [JS].

Given any isobaric automorphic representation π of width d in the form (1), and any d-tuple $\chi := (\chi^1, \dots, \chi^d)$ of idele clas characters of F, we define the χ-twist of π to be

$$\pi[\chi] := (\pi^1 \otimes \chi^1) \boxplus \dots \boxplus (\pi^d \otimes \chi^d). \tag{2}$$

If an isobaric automorphic representation π' is isomorphic to $\pi[\chi]$ for some χ, we will say that π' is a twist of π. And if μ is an idele class character of F and $m = (m(1), \dots, m(d))$ a d-tuple of integers, we will set

$$\mu^m := (\mu^{m(1)}, \dots, \mu^{m(d)}).$$

Now we need the following

Lemma. *Let $\pi = \pi^1 \boxplus \dots \boxplus \pi^d$ be an isobaric automorphic representation of $GL(n, \mathbb{A}_F)$, with $\pi^1, \dots, \pi^d$ being cuspidal automorphic representations of $GL(n_1, \mathbb{A}_F)$, $\dots$, $GL(n_d, \mathbb{A}_F)$, $n = n_1 + \dots + n_d$. Then there exist at most a finite number of d-tuples $\chi = (\chi^1, \dots, \chi^d)$ of idele class characters such that*

$$\pi \simeq \pi[\chi].$$

Proof of Lemma. By the uniqueness of the isobaric sum decompostition of π into cuspidals, there must be a permutation σ in S_d such that we have, for each $i \leq d$, an isomorphism

$$\pi^i \simeq \pi^{\sigma(i)} \otimes \chi_{\sigma(i)}.$$

We must necessarily have $n_i = n_{\sigma(i)}$ for each i. So the Lemma is a consequence of the following

Sublemma. *Let η, η' be cuspidal automorphic representation of $GL(m, \mathbb{A}_F)$. Then the set X of idele class characters μ such that*

$$\eta \simeq \eta' \otimes \mu$$

is finite.

Proof of Sublemma. We may assume that X is non-empty, as there is nothing to prove otherwise. Pick, and fix, any member, call it ν, of X. Put

$$Y = \{\mu\nu^{-1} \mid \mu \in X\}.$$

Since X and Y have the same cardinality, it suffices to prove that Y is finite. We *claim* that for any χ in Y,

$$\eta \simeq \eta \otimes \chi.$$

Indeed, if $\chi = \mu\nu^{-1}$ with $\mu \in X$, we have

$$\eta \simeq \eta' \otimes \mu \simeq (\eta' \otimes \nu) \otimes (\mu\nu^{-1}) \simeq \eta \otimes \chi,$$

whence the claim.

Now the set Y, which parametrizes the self-twists of η, is finite by Lemma 3.6.2 of [Ra], and hence the Sublemma is proved; so is the Lemma. $\square$

Proof of Proposition (contd.)

For each j, let θ_j be a generator of $\mathrm{Gal}(K_j/F)$, and δ_j a character of F cutting out K_j (by class field theory). Note that, for each $i \geq 1$, the pull back to K_i of δ_j by the norm map N_i from K_i to F cuts out the compositum K_iK_j.

We will write, for each j,

$$\pi_j \simeq \boxplus_{k=1}^{d(j)} \pi_j^k, \tag{3}$$

with each π_j^k a cuspidal automorphic representation of $\mathrm{GL}(n_k(j), \mathbb{A}_F)$, with $n = \sum_{k=1}^{d(j)} n_k(j)$.

We claim that

$$\pi_j \circ \theta_j \simeq \pi_j \quad (\forall j). \tag{4}$$

For all $j, r \geq 1$, let $\theta_{j,r}$ denote the automorphism of K_jK_r such that (i) $\theta_{j,r}|_{K_j} = \theta_j$, and (ii) $\theta_{j,r}|_{K_r} = 1$ (where 1 denotes the identity automorphism). It is easy to see that the base change of $\pi_j \circ \theta_j$ to K_jK_r is simply $(\pi_j)_{K_jK_r} \circ \theta_{j,r}$. (For the basic results on base change, see [AC]; for a quick summary see Prop. 2.3.1 of [Ra].) Applying (DC), we then have

$$(\pi_j \circ \theta_j)_{K_jK_r} \simeq (\pi_r)_{K_jK_r} \circ \theta_{j,r} \simeq (\pi_r)_{K_jK_r} \simeq (\pi_j)_{K_jK_r},$$

since $\theta_{j,r}$ is trivial on K_r. Since K_jK_r is a cyclic extension of K_j of prime degree, we must have by Arthur-Clozel,

$$\pi_j \circ \theta_j \simeq \pi_j[(\delta_r \circ N_j)^{m_r}], \tag{5}$$

for some $d(j)$-tuple $m_r = (m_r(1), \ldots, m_r(d(j)))$ of integers in $\{0, 1, \ldots, p-1\}$. For every fixed $r \geq 1$, and for all $k \neq r$, we then have the self-twist identity

$$\pi_j \simeq \pi_j[(\delta_r \circ N_j)^{m_r}][(\delta_k \circ N_j)^{-m_k}].$$

Note that $\delta_r \circ N_j$ and $\delta_k \circ N_j$ must be distinct unless their ratio is a power of δ_j. So the Lemma above forces m_r to be the zero vector for all but a finite number of r. The claimed identity now follows by taking r to be outside this exceptional finite set.

As a result, by applying base change ([AC]; Prop. 2.3.1 of [Ra]) once again, we see that there exists, for each $j \geq 1$, an isobaric automorphic representation

$$\pi(j) = \boxplus_{k=1}^{b(j)} \pi(j)^k$$

of $\mathrm{GL}(n, \mathbb{A}_F)$, with each $\pi(j)^k$ a cuspidal automorphic representation of $\mathrm{GL}(N_k(j), \mathbb{A}_F)$ and

$$n = \sum_{k=1}^{b(j)} N_k(j),$$

such that

$$\pi_j \simeq (\pi(j))_{K_j}. \tag{6}$$

Such a $\pi(j)$ is of course unique only up to replacing it by $\pi(j)[\delta_j^a]$ for some $d(j)$-tuple $a = (a_1, \ldots, a_{b(j)})$ of integers in $\{0, 1, \ldots, p-1\}$. Clearly we have

$$b(j) \leq d(j),$$

but equality need not hold.

It is important to note that, for any $r \neq j$, we have the following compatibility for base change in (cyclic) stages:

$$((\pi(j))_{K_j})_{K_j K_r} \simeq ((\pi(j))_{K_r})_{K_j K_r}. \tag{7}$$

We see this as follows. Let v be a finite place of $K_j K_r$ which is unramified for the data. Denote by u (resp. w, resp. w') the place of F (resp. K_j, resp. K_r) below v. If σ_u denotes the representation of W'_{F_u} associated to $\pi(j)_u$, then

$$\mathrm{res}^{(K_j)_w}_{(K_j K_r)_v}(\mathrm{res}^{F_u}_{(K_j)_w}(\sigma_u)) \simeq \mathrm{res}^{(K_r)_{w'}}_{(K_j K_r)_v}(\mathrm{res}^{F_u}_{(K_r)'_w}(\sigma_u)).$$

Then (2.3.0) of [Ra] implies the local identity (for all such v)

$$((\pi(j)_u)_{(K_j)_w})_{(K_j K_r)_v} \simeq ((\pi(j)_u)_{(K_r)'_w})_{(K_j K_r)_v}.$$

The global isomorphism (7) follows by the strong multiplicity one theorem for isobaric automorphic representations ([JS]).

We can then rewrite (DC) as saying, for all $j, r \geq 1$,

$$((\pi(j))_{K_j})_{K_j K_r} \simeq ((\pi(r))_{K_j})_{K_j K_r}. \tag{8}$$

Consequently we must have, after renumbering, an equality of partitions $(\forall\,(r, j))$:

$$(N_1(j), \ldots, N_{b(j)}(j)) = (N_1(r), \ldots, N_{b(r)}(r))$$

of n. In particular, we have

$$b := b(j) = b(r) \quad \text{and} \quad N_k := N_k(j) = N_k(r). \tag{9}$$

Moreover,

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j}[(\delta_r \circ N_j)^{m(r,j)}], \tag{10}$$

for some b-tuple $m(r,j) = (m(r,j)_1, \ldots, m(r,j)_b)$ of integers. We can replace $\pi(r)$ by $\pi(r)[\delta_r^{-m(r,j)}]$ and get

$$(\pi(j))_{K_j} \simeq (\pi(r))_{K_j}. \tag{11}$$

Then, by replacing $\pi(j)$ by a twist by δ_j^a for a b-tuple a of integers, we can arrange to have $\pi(j)$ and $\pi(r)$ be isomorphic. In sum, we have produced, for *every pair* (j,r), a *common descent*, say $\pi(j,r)$, of π_j, π_r, i.e., with

$$\pi(j,r)_{K_j} \simeq \pi_j \quad \text{and} \quad \pi(j,r)_{K_r} \simeq \pi_r. \tag{12}$$

Fix non-zero vectors a, c in $(\mathbb{Z}/p)^b$, and consider the possible isomorphism

$$\pi(j,r) \simeq \pi(j,r)[\delta_j^a][\delta_r^{-c}]. \tag{13}$$

We claim that this cannot happen outside a finite set $S_{a,c}$ of pairs (j,r). To see this fix a pair (i,ℓ) and consider the relationship of $\pi(i,\ell)$ to $\pi(j,r)$. Since $\pi(i,\ell)$ and $\pi(j,\ell)$ have the same base change to K_ℓ, they must differ by twisting by a b-tuple power of δ_ℓ. Similarly, $\pi(j,\ell)$ and $\pi(j,r)$ differ by a twist as they have the same base change to K_r. Put together, this shows that $\pi(i,\ell)$ and $\pi(j,r)$ are twists of each other. Then (13) would imply that

$$\pi(i,\ell) \simeq \pi(i,\ell)[\delta_j^a][\delta_r^{-c}] \simeq \pi(i,\ell)[\chi_{a,-c}], \tag{14}$$

where

$$\chi_{a,-c} = (\delta_j^{a_1}\delta_r^{-c_1}, \ldots, \delta_j^{a_b}\delta_r^{-c_b}).$$

The claim now follows since, by the Lemma above, $\pi(i,\ell)$ admits only a finite number of self-twists, and since the b-tuples $\chi_{a,-c}$ are all distinct for distinct pairs (j,r) (as a, c are fixed).

Now choose a pair (j,r) *not belonging* to $S_{a,c}$ for *any* pair (a,c) of non-zero vectors in $(\mathbb{Z}/p)^b$, and set

$$\pi = \pi(j,r). \tag{15}$$

We assert that for all but a finite number of indices m,

$$\pi_{K_m} \simeq \pi_m. \tag{16}$$

It suffices to show that, for any large enough m, $\pi = \pi(j,r)$ is isomorphic to either $\pi(j,m)$ or $\pi(m,r)$. Suppose neither is satisfied. Then there exist non-zero vectors a, c in $(\mathbb{Z}/p)^b$ such that

$$\pi(j,m) \simeq \pi(j,r)[\delta_j^a] \quad \text{and} \quad \pi(m,r) \simeq \pi(j,r)[\delta_r^c].$$

We also have $\pi(j,m) \simeq \pi(m,r)[\delta_m^e]$, for some vector e in $(\mathbb{Z}/p)^b$. Putting these together, we get the self-twisting identity

$$\pi(j,r) \simeq \pi(j,r)[\delta_j^a][\delta_r^{-c}][\delta_m^{-e}]. \tag{17}$$

By our choice of (j,r), e cannot be the zero vector. But for each non-zero e, the set of indices m for which such an identity can hold is finite, again by the Lemma. Hence we get a contradiction for large enough m, which implies that a or c should be 0, giving the requisite contradiction. Thus $\pi = \pi(j,r)$ must be isomorphic to either $\pi(j,m)$ or $\pi(m,r)$ for large enough m. Since we have, by (12),

$$\pi(j,m)_{K_m} \simeq \pi_m \simeq \pi(m,r)_{K_m},$$

the Proposition is now proved. □

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