

**REMARKS ON THE SYMMETRIC POWERS OF CUSP
FORMS ON $GL(2)$**

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*To Steve Gelbart
On the occasion of his sixtieth birthday*

Introduction

Let F be a number field, and π a cuspidal automorphic representation of $GL(2, \mathbb{A}_F)$ of conductor \mathcal{N} . For every $m \geq 1$ one has its *symmetric m -th power L -function* $L(s, \pi; \text{sym}^m)$, which is an Euler product over the places v of F , with the v -factors (for finite $v \nmid \mathcal{N}$ of norm q_v) being given by

$$L_v(s, \pi; \text{sym}^m) = \prod_{j=0}^m (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1},$$

where the unordered pair $\{\alpha_v, \beta_v\}$ defines the diagonal conjugacy class in $GL_2(\mathbb{C})$ attached to π_v . Even at a ramified (resp. archimedean) place v , one has by the local Langlands correspondence a 2-dimensional representation σ_v of the extended Weil group $W_{F_v} \times SL(2, \mathbb{C})$ (resp. of the Weil group W_{F_v}), and the v -factor of the symmetric m -th power L -function is associated to $\text{sym}^m(\sigma_v)$. A basic conjecture of Langlands asserts that there is, for each m , an (isobaric) automorphic representation $\text{sym}^m(\pi)$ of $GL(m+1, \mathbb{A})$ whose standard (degree $m+1$) L -function $L(s, \text{sym}^m(\pi))$ agrees, at least at the primes not dividing \mathcal{N} , with $L(s, \pi; \text{sym}^m)$. It is well known that such a result will have very strong consequences, such as the *Ramanujan conjecture* and the *Sato-Tate conjecture* for π . The *modularity*, also called *automorphy*, has long been known for $m=2$ by the pioneering work of Gelbart and Jacquet ([GJ]); we will write $\text{Ad}(\pi)$ for the selfdual representation $\text{sym}^2(\pi) \otimes \omega^{-1}$, ω being the central character of π . A *major breakthrough*, due to *Kim and Shahidi* ([KS2, KS1, Kim]), has established the modularity of $\text{sym}^m(\pi)$ for $m=3, 4$, along with a useful cuspidality criterion (for $m \leq 4$). Furthermore, when $F = \mathbb{Q}$ and π is defined by a holomorphic newform f of weight 2, \mathbb{Q} -coefficients and level N , such that at some prime p , the component π_p is Steinberg, a recent *dramatic theorem of Taylor* ([Tay3]), which depends on earlier works of his with Clozel, Harris and Shepherd-Baron, furnishes the *potential modularity* of $\text{sym}^{2m}(\pi)$ (for every $m \geq 1$), i.e., its modularity over a number field K , thereby finessing the Sato-Tate

conjecture in this case. It should however be noted that such a beautiful result is not (yet) available for π defined by newforms φ of higher weight, for instance for the ubiquitous cusp form $\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n)q^n$, where $z \in \mathcal{H}$ and $q = e^{2\pi iz}$, which is holomorphic of weight 12, level 1 and trivial character.

In this Note we consider the following more modest, but nevertheless basic, question:

Suppose $\text{sym}^m(\pi)$ is an automorphic representation of $GL_{m+1}(\mathbb{A}_F)$. When is it cuspidal?

If $\text{sym}^m(\pi_v)$ is, for some finite place v , in the discrete series, which happens for example when π_v is Steinberg, it is well known that the global representation $\text{sym}^m(\pi)$ will necessarily be cuspidal (once it is automorphic). On the other hand, one knows that the answer to the question above is negative already for $m = 2$, as shown by Gelbart and Jacquet ([GJ]), if π is dihedral, i.e., associated to an idele class character χ of a quadratic extension K of F ; indeed this is *necessary and sufficient* for $\text{sym}^2(\pi)$ to be non-cuspidal. There is a non-trivial extension of such a criterion in the work of Kim and Shahidi ([KS1]), who show that for a non-dihedral π , $\text{sym}^3(\pi)$ is Eisensteinian iff π is *tetrahedral*, while $\text{sym}^4(\pi)$ is cuspidal iff π is not tetrahedral or octahedral. We will say that π is *solvable polyhedral* iff it is dihedral, tetrahedral or octahedral. Finally, if π is associated to an irreducible 2-dimensional Galois representation ρ which is icosahedral, i.e., with projective image isomorphic to the alternating group A_5 , one knows that $\text{sym}^6(\rho)$ is reducible, suggesting that $\text{sym}^6(\pi)$ is not cuspidal. However, as noted by Song Wang ([Wan]), $\text{sym}^5(\rho)$ is, in the icosahedral case, necessarily a tensor product $\text{sym}^2(\rho') \otimes \rho$, where ρ' is the Galois conjugate representation of ρ (which is defined over $\mathbb{Q}[\sqrt{5}]$). This allowed Wang to prove that $\text{sym}^5(\pi)$ is cuspidal by making use of the construction (cf [KS2]) of the functorial product $\Pi \boxtimes \pi'$ (in $GL(6)/F$), for Π (resp. π') a cusp form on $GL(3)/F$ (resp. $GL(2)/F$).

The following result was suggested by the philosophy of Langlands ([Lan4]) which predicts that any cuspidal π on $GL(2)/F$ should be naturally associated to a reductive subgroup $H(\pi)$ of $GL_2(\mathbb{C})$, as well as the results of [Wan].

Theorem A *Let π a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$, which is not solvable polyhedral, of central character ω . Suppose $\text{sym}^m(\pi)$ is modular for all m . Then we have*

- (a) $\text{sym}^5(\pi)$ is cuspidal.
- (b) $\text{sym}^6(\pi)$ is non-cuspidal iff we have

$$\text{sym}^5(\pi) \simeq \text{Ad}(\pi') \boxtimes \pi \otimes \omega^{-4},$$

for a cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$.

- (c) *If $\text{sym}^6(\pi)$ is cuspidal, then so is $\text{sym}^m(\pi)$ for all $m \geq 1$.*

- (d) If $F = \mathbb{Q}$ and π is defined by a non-CM, holomorphic newform φ of weight $k \geq 2$, then $\text{sym}^m(\pi)$ is cuspidal for all m .

One can do a bit better than this in that for a given symmetric power, one does not need information on all the $\text{sym}^m(\pi)$. See Theorem A' in section 2 for a precise statement, as well as Theorem B, a variant, given in that section. The proofs are then given in sections 3, 4 and 5.

When π is associated to an icosahedral Galois representation ρ , $\text{sym}^6(\pi)$ is well known to be non-cuspidal (if automorphic), but with $\text{sym}^m(\pi)$ cuspidal for $m \leq 4$. In fact, such a ρ is defined over $F = \mathbb{Q}[\sqrt{5}]$, and if the conjugate ρ' of ρ - by the non-trivial automorphism of F - is also modular, then $\text{sym}^5(\pi)$ is cuspidal as well ([Wan]). So the result above is consistent with (and motivated by) this Galois picture.

The results of this paper were essentially established some time ago, but the questions raised to me in the past two years by some colleagues have led me to believe in the possible usefulness of their being in print. While the inspiration for the results here came from Langlands and the paper of Wang, the proofs depend, at least partly, on the beautiful constructions [KS2, KS1, Kim] of Kim and Shahidi. Use is also made of the papers [Ram2, Ram6].

Acknowledgement: Like so many others interested in Automorphic Forms, I was decidedly influenced during my graduate student years (in the late seventies), and later, by Steve Gelbart's book, *Automorphic Forms on adèle groups*, and his expository papers, *Automorphic forms and Artin's conjecture* and *Elliptic curves and automorphic representations*, as well as his seminal work with Jacquet, *A relation between automorphic forms on $GL(2)$ and $GL(3)$* . His later works have also been influential. Furthermore, Steve has been incredibly friendly and generous over the years, and it is a great pleasure to dedicate this paper to him. To end, I would be remiss if I do not acknowledge support from the NSF through the grant DMS0402044.

1. PRELIMINARIES

1.1. The standard L -function of $GL(n)$. Let F be a number field with adèle ring \mathbb{A}_F . Fix a non-trivial character $\psi : \mathbb{A}_F \rightarrow S^1$, which is trivial on the discrete, cocompact subgroup F . For each place v , denote by ψ_v the v -component of Ψ , which is a character of F_v . Let $\text{ram}(\Psi)$ denote the finite set of places where ψ_v is ramified. Denote by $dx = (dx_v)$ the Haar measure on \mathbb{A}_F which is self-dual relative to Ψ . The ε -factors will depend on these choices, which will suppress in our notation. We will also simply write dx to denote the induced measure on the quotient group \mathbb{A}_F/F . We will take the Haar measure on I_F , resp. F_v^* (for any place v), to be $dx^* = dx/|x|$, resp. $dx_v^* = dx_v/|x_v|$, where $|\cdot|$ is the normalized absolute value on \mathbb{A}_F , resp. F_v .

For every algebraic group G over F , let $G(\mathbb{A}_F)$ denote the restricted direct product $\prod'_v G(F_v)$, endowed with the usual locally compact topology.

For any $m \geq 1$ write

$$G_m = \mathrm{GL}(m).$$

Write Z_m for its center consisting of scalar matrices, A_m for the subgroup of diagonal matrices, and B_m for the Borel subgroup of upper triangular matrices. Then $B_m = A_m N_m$ (semi-direct product), where N_m denotes the upper triangular matrices with 1s on the diagonal. Let dz_m^* , da_m^* , and dn_m , be the respective Haar measures on $Z_m(\mathbb{A}_F)$, $A_m(\mathbb{A}_F)$, and $N_m(\mathbb{A}_F)$, induced by dx^* , resp. dx . Let $Z_{m,v}, A_{m,v}, N_{m,v}$ denote the F_v -values points of Z_m, A_m, N_m respectively, and let $dz_{m,v}^*, da_{m,v}^*, dn_{m,v}$ denote the corresponding local Haar measures. Let $K_{m,v}$ denote, for each finite (resp. real, resp. complex) place v , the maximal compact subgroup $G_m(\mathfrak{O}_v)$ (resp. $O(m)$, resp. $U(m)$). Choose a Haar measure $dk_{m,v}$ on $K_{m,v}$, for each v . By the Iwasawa decomposition, we have $G_{m,v} = A_{m,v} N_{m,v} K_{m,v}$. Take the measure $dg_{m,v} = da_{m,v} dn_{m,v} dk_{m,v}$ on $G_{m,v}$ and take the product measure $dg_m = \prod_v dg_{m,v}$ on $G_m(\mathbb{A}_F)$. One knows that under this measure, the volume of $Z_m(\mathbb{A}_F) G_m(f) \backslash G_m(\mathbb{A}_F)$ is finite.

By a *unitary cuspidal* representation of $G_m(\mathbb{A}_F) = G_m(F_\infty) \times G_m(\mathbb{A}_{F,f})$, we will always mean an irreducible, automorphic representation occurring in the space of cusp forms in $L^2(Z_m(\mathbb{A}_F) G_m(F) \backslash G_m(\mathbb{A}_F), \omega)$ relative to a character ω of $Z_m(\mathbb{A}_F)$. By a *cuspidal representation*, we will mean an irreducible admissible representation of $G_m(\mathbb{A}_F)$ for which there exists a real number, called the *weight* of π such that $\pi \otimes |\cdot|^{w/2}$ is a unitary cuspidal representation. Such a representation is in particular a restricted tensor product $\pi = \otimes'_v \pi_v = \pi_\infty \otimes \pi_f$, where each π_v is an (irreducible) admissible representation of $G(F_v)$ for v finite, and an admissible (Lie G_v, K_v)–module for v archimedean, with K_v denoting a compact modulo center subgroup of $G(F_v)$; π_f (resp. π_∞) is the restricted tensor product of π_v over all finite (resp. archimedean) places v . By definition, π_v must be unramified at almost all v .

For any irreducible, automorphic representation π of $GL(n, \mathbb{A}_F)$, let $L(s, \pi) = L(s, \pi_\infty) L(s, \pi_f)$ denote the associated *standard* L –function ([Jac]) of π ; it has an Euler product expansion

$$(1.1.1) \quad L(s, \pi) = \prod_v L(s, \pi_v),$$

convergent in a right-half plane. If v is an archimedean place, then one knows (cf. [Lan3]) how to associate a semisimple n –dimensional \mathbb{C} –representation $\sigma(\pi_v)$ of the Weil group W_{F_v} , and $L(\pi_v, s)$ identifies with $L(\sigma_v, s)$. On the other hand, if v is a finite place where π_v is unramified, there is a corresponding semisimple (Langlands) conjugacy class $A_v(\pi)$ (or $A(\pi_v)$) in $\mathrm{GL}(n, \mathbb{C})$ such that

$$(1.1.2) \quad L(s, \pi_v) = \det(1 - A_v(\pi)T)^{-1} |_{T=q_v^{-s}}.$$

We may find a diagonal representative $\mathrm{diag}(\alpha_{1,v}(\pi), \dots, \alpha_{n,v}(\pi))$ for $A_v(\pi)$, which is unique up to permutation of the diagonal entries. Let $[\alpha_{1,v}(\pi), \dots, \alpha_{n,v}(\pi)]$

denote the resulting unordered n -tuple. Since $W_{F,v}^{\text{ab}} \simeq F_v^*$, $A_v(\pi)$ clearly defines an abelian n -dimensional representation $\sigma(\pi_v)$ of $W_{F,v}$. One has

Theorem 1.1.3 ([GJ, Jac]) *Let $n \geq 1$, and π a non-trivial cuspidal representation of $GL(n, \mathbb{A}_F)$. Then $L(s, \pi)$ is entire. Moreover, for any finite set S of places of F , the incomplete L -function $L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v)$ is holomorphic in $\Re(s) > 0$.*

When $n = 1$, such a π is simply a unitary idele class character and this result is due to Hecke. Also, when π is trivial, $L(s, \pi) = \zeta_F(s)$.

1.2. Isobaric automorphic representations. By the theory of Eisenstein series, one has a sum operation \boxplus ([Lan2]), which results in the following

Theorem 1.2.1 ([JS]) *Given any m -tuple of cuspidal representations π_1, \dots, π_m of $GL(n_1, \mathbb{A}_F), \dots, GL(n_m, \mathbb{A}_F)$ respectively, there exists a unitary, irreducible, automorphic representation $\pi_1 \boxplus \dots \boxplus \pi_m$ of GLn, \mathbb{A}_F , $n = n_1 + \dots + n_m$, which is unique up to equivalence, such that for any finite set S of places,*

$$L^S(s, \boxplus_{j=1}^m \pi_j) = \prod_{j=1}^m L^S(s, \pi_j).$$

Call such a (Langlands) sum $\pi \simeq \boxplus_{j=1}^m \pi_j$, with each π_j cuspidal, an *isobaric* representation. Denote by $\text{ram}(\pi)$ the finite set of finite places where π is ramified, and let $\mathfrak{N}(\pi)$ be its conductor ([JPSS1]).

For every integer $n \geq 1$, set:

$$(1.2.2) \quad \mathcal{A}(n, F) = \{\pi : \text{isobaric representation of } GL(n, \mathbb{A}_F)\} / \simeq,$$

and

$$\mathcal{A}_0(n, F) = \{\pi \in \mathcal{A}(n, F) \mid \pi \text{ cuspidal}\}.$$

Put $\mathcal{A}(F) = \cup_{n \geq 1} \mathcal{A}(n, F)$ and $\mathcal{A}_0(F) = \cup_{n \geq 1} \mathcal{A}_0(n, F)$.

Definition 1.2.3 *Given $\pi, \eta \in \mathcal{A}(F)$, if we can find an $\eta' \in \mathcal{A}(F)$ such that $\pi \simeq \eta \boxplus \eta'$, we will call η an isobaric summand of π and write*

$$[\pi : \eta] > 0.$$

Remark. One can also define the analogs of $\mathcal{A}(n, F)$ for local fields F , where the ‘‘cuspidal’’ subset $\mathcal{A}_0(n, F)$ consists of essentially square-integrable representations of $GL(n, F)$. See [Lan2] and [Ram1] for details.

1.3. Symmetric powers of $\mathrm{GL}(2)$. Since the L -group of $\mathrm{GL}(2)$ is $\mathrm{GL}(2, \mathbb{C}) \times W_F$, the principle of functoriality of Langlands ([Lan1]) predicts that for any algebraic representation

$$(1.3.1) \quad r : \mathrm{GL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(N, \mathbb{C}),$$

and any number field F , there should be a map

$$(1.3.2) \quad \mathcal{A}(2, F) \rightarrow \mathcal{A}(N, F), \pi \rightarrow r(\pi),$$

with compatible local maps, such that for all finite unramified places v (for π), we have the equality of Langlands classes

$$r(A(\pi_v)) = A(r(\pi)_v).$$

More precisely, one expects *exact equality* at every place of the (formally defined) L -function $L(s, \pi, r)$ and the standard L -function $L(s, r(\pi))$.

It suffices to establish this for irreducible representations r , which are all of the form $\mathrm{sym}^n(\rho) \otimes L^{\otimes k}$, with $n, k \in \mathbb{Z}, n \geq 0$. As in chapter 1, ρ denotes the standard representation of $\mathrm{GL}(2, \mathbb{C})$ with determinant L , and $\mathrm{sym}^n(\rho)$ denotes the symmetric n -th power representation of ρ . Also, $L^{\otimes k}$ corresponds to ω^{bk} .

It is enough to construct the $\mathrm{sym}^n(\pi)$ for π cuspidal. When it exists, by which we mean it exists in $\mathcal{A}(F)$, we will write (for $\pi \in \mathcal{A}(2, F)$)

$$\mathrm{sym}^n(\pi) = \mathrm{sym}^n(\rho)(\pi).$$

It may be useful to recall that if

$$L(s, \pi_v) = [(1 - \alpha_v q_v^{-s})(1 - \beta_v q_v^{-s})]^{-1}$$

at any unramified finite place v with norm q_v , with $A(\pi_v)$ being represented by the diagonal matrix with entries α_v, β_v , then for every $n \geq 1$,

$$(1.3.3) \quad L(s, \pi_v, \mathrm{sym}^n) = \left[\prod_{j=0}^n (1 - \alpha_v^j \beta_v^{n-j} q_v^{-s}) \right]^{-1}.$$

It is well known that when $r = L$, $r(\pi) \in \mathcal{A}(1, F)$ is given by the central character $\omega = \omega_\pi$ of π . (Of course, $\rho(\pi) = \pi$.) Consequently, if one can establish the lifting for $r = \mathrm{sym}^n(\rho)$, then one can also achieve it for $r = \mathrm{sym}^n(\rho) \otimes L^{\otimes k}$ by twisting by ω^k , i.e., by setting

$$\left(\mathrm{sym}^n(\rho) \otimes L^{\otimes k} \right) (\pi) = \mathrm{sym}^n(\pi) \otimes \omega^k.$$

So it suffices to establish the transfer $\pi \rightarrow r(\pi)$ for $\mathrm{sym}^n(\rho)$ for all n . Clearly, $\mathrm{sym}^1(\pi) = \pi$.

Proposition 1.3.4 *Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ which is associated to a two-dimensional, continuous \mathbb{C} -representation ρ of $\mathrm{Gal}(\overline{F}/F)$ so that $L(s, \rho) = L(s, \pi)$. Suppose $\mathrm{sym}^m(\pi)$ exists in $\mathcal{A}(F)$ for every $m \geq 1$. It is then cuspidal iff $\mathrm{sym}^m(\rho)$ is irreducible.*

One expects the same when ρ is an ℓ -adic Galois representation (attached to π), but this is unknown except for small m (cf. [Ram6, Ram5]).

It is a well known, classical result of Gelbart and Jacquet ([GJ]) that $\text{sym}^2(\pi)$ exists for any $\pi \in \mathcal{A}_0(2, F)$. It is cuspidal iff π is not *dihedral*, i.e., π is not automorphically induced by an idele class character of a quadratic field.

When π is dihedral, it is easy to see that $\text{sym}^m(\pi)$ exists for all m , and that it is an isobaric sum of elements of $\mathcal{A}(1, F)$ and $\mathcal{A}_0(2, F)$. So we may, and we will, henceforth restrict our attention to non-dihedral forms π .

Here is a ground-breaking result due to Kim and Shahidi which we will need:

Theorem 1.3.5 (Kim-Shahidi [KS2], [KS1], Kim [Kim]) *Let $\pi \in \mathcal{A}_0(2, F)$ be non-dihedral. Then $\text{sym}^n(\pi)$ exists in $\mathcal{A}(F)$ for all $n \leq 4$. Moreover, $\text{sym}^3(\pi)$ (resp. $\text{sym}^4(\pi)$) is cuspidal iff π is not tetrahedral (resp. octahedral).*

A non-dihedral π is *tetrahedral* iff $\text{sym}^2(\pi)$ is monomial, while π is *octahedral* if it is not dihedral or tetrahedral but whose symmetric cube is not cuspidal upon base change to some quadratic extension K of F . We will say that π is *solvable polyhedral* if it is either dihedral, or tetrahedral, or octahedral.

1.4. Rankin-Selberg L -functions. Let π, π' be isobaric automorphic representations in $\mathcal{A}(n, F), \mathcal{A}(n', F)$ respectively. Then there exist an associated Euler product $L(s, \pi \times \pi')$ ([JPSS2], [JS, JPSS2, Sha2, Sha1, MW]), which converges in $\{\Re(s) > 1\}$, and admits a meromorphic continuation to the whole s -plane and satisfies the functional equation

$$(1.4.1) \quad L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1-s, \pi^\vee \times \pi'^\vee),$$

with

$$\varepsilon(s, \pi \times \pi') = W(\pi \times \pi') N(\pi \times \pi')^{\frac{1}{2}-s},$$

where $N(\pi \times \pi')$ is a positive integer not divisible by any rational prime not intersecting the ramification loci of F/\mathbb{Q} , π and π' , while $W(\pi \times \pi')$ is a non-zero complex number, called the *root number of the pair* (π, π') . As in the Galois case, $W(\pi \times \pi') W(\pi^\vee \times \pi'^\vee) = 1$, so that $W(\pi \times \pi') = \pm 1$ when π, π' are self-dual.

When v is archimedean or a finite place unramified for π, π' ,

$$(1.4.2) \quad L_v(s, \pi \times \pi') = L(s, \sigma(\pi_v) \otimes \sigma(\pi'_v)).$$

In the archimedean situation, $\pi_v \rightarrow \sigma(\pi_v)$ is the arrow to the representations of the Weil group W_{F_v} given by [La1]. When v is an unramified finite place, $\sigma(\pi_v)$ is defined in the obvious way as the sum of one dimensional representations defined by the Langlands class $A(\pi_v)$.

When $n = 1$, $L(s, \pi \times \pi') = L(s, \pi\pi')$, and when $n = 2$ and $F = \mathbb{Q}$, this function is the usual Rankin-Selberg L -function, extended to arbitrary global fields by Jacquet.

Theorem 1.4.3 [JS, JPSS2]) *Let $\pi \in \mathfrak{A}_0(n, F)$, $\pi' \in \mathfrak{A}_0(n', F)$, and S a finite set of places. Then $L^S(s, \pi \times \pi')$ is entire unless π is of the form $\pi'^{\vee} \otimes |\cdot|^w$, in which case it is holomorphic outside $s = w, 1 - w$, where it has simple poles.*

1.5. The (conjectural) automorphic tensor product. The *Principle of Functoriality* implies that given isobaric automorphic representations $\pi, \pi' \in \text{GL}_n(\mathbb{A}_F), \text{GL}_{n'}(\mathbb{A}_F)$ respectively, there should exist an isobaric automorphic representation $\pi \boxtimes \pi'$, called the *automorphic tensor product*, or the *functorial product*, of $\text{GL}(nn', \mathbb{A}_F)$ such that

$$(1.5.1) \quad L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi').$$

We will say that an automorphic $\pi \boxtimes \pi'$ is a *weak automorphic tensor product* of π, π' if the identity (1.5.1) of Euler products holds outside a finite set S of places, i.e, if $L^S(s, \pi \boxtimes \pi')$ equals $L^S(s, \pi \times \pi')$.

The (conjectural) functorial product \boxtimes is the automorphic analogue of the usual tensor product of Galois representations. For the importance of this product, see [Ram1], for example.

One can always construct $\pi \boxtimes \pi'$ as an *admissible* representation of $\text{GL}(nn', \mathbb{A}_F)$, but the subtlety lies in showing that this product is automorphic. Also, if one knows how to construct it for cuspidal π, π' , then one can do it in general.

The automorphy of \boxtimes is known in the following cases, which will be useful to us:

$$(1.5.2) \quad \begin{aligned} (\mathbf{n}, \mathbf{n}') = (\mathbf{2}, \mathbf{2}): & \quad \text{[Ram2]} \\ (\mathbf{n}, \mathbf{n}') = (\mathbf{2}, \mathbf{3}): & \quad \text{Kim-Shahidi ([KS2]} \end{aligned}$$

The reader is also referred to section 11 of [Ram4], which contains some refinements, explanations, and (minor) errata for [Ram2]. Furthermore, it may be worthwhile remarking that Kim and Shahidi effectively use their construction of the functorial product on $\text{GL}(2) \times \text{GL}(3)$ to prove the automorphy of *symmetric cube* transfer from $\text{GL}(2)$ to $\text{GL}(4)$, mentioned in section 1.3. A *cuspidality criterion* for the image under this transfer is proved in [RW], with an application to the cuspidal cohomology of congruence subgroups of $\text{SL}(6, \mathbb{Z})$.

2. STATEMENTS OF RESULTS

Here is a precise, though a bit more cumbersome, version of Theorem A, which was stated in the Introduction.

Theorem A' *Let π a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ of central character ω . Assume, for the first three parts that π is not solvable polyhedral. Then we have the following:*

- (a) *If $\text{sym}^5(\pi)$ is modular, then it is cuspidal.*

- (b) If $\text{sym}^5(\pi)$ and $\text{sym}^6(\pi)$ are both modular, $\text{sym}^6(\pi)$ is non-cuspidal iff we have

$$\text{sym}^5(\pi) \simeq \text{Ad}(\pi') \boxtimes \pi \otimes \omega^2,$$

for a cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$; in this case, $\text{Ad}(\pi')$ and $\text{Ad}(\pi)$ are not twist equivalent.

- (c) Let $m \geq 6$, with $\text{sym}^j(\pi)$ modular for every $j \leq 2m$. If $\text{sym}^6(\pi)$ is cuspidal, then so is $\text{sym}^m(\pi)$.
- (d) If $F = \mathbb{Q}$ and π is defined by a non-CM, holomorphic newform φ of weight $k \geq 2$, then $\text{sym}^m(\pi)$ is cuspidal whenever it is modular.

Here is a variant of (the first three parts) of this result, where the hypothesis is not modularity of the appropriate symmetric powers, but rather the existence of the automorphic tensor product on $GL(2) \times GL(r)$ for suitable r .

Theorem B *Let $m \geq 5$ and π a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ of central character ω , which is not solvable polyhedral. Assume that (i) $\text{sym}^j(\pi)$ is modular for all $j < m$, and (ii) $\text{sym}^i(\pi) \boxtimes \text{sym}^{m-i}(\pi)$ is modular for some positive integer i with $i \leq m$. Then we have the following:*

- (a) $\text{sym}^m(\pi)$ is modular, and even cuspidal if $m = 5$.
- (b) When $m = 6$, $\text{sym}^6(\pi)$ is non-cuspidal iff we have

$$\text{sym}^5(\pi) \simeq \text{Ad}(\pi') \boxtimes \pi \otimes \omega^2,$$

for a cuspidal automorphic representation π' of $GL_2(\mathbb{A}_F)$; in this case, $\text{Ad}(\pi')$ and $\text{Ad}(\pi)$ are not twist equivalent.

- (c) When $m \geq 6$, suppose that $\pi \boxtimes \tau$ is modular for any cusp form τ on $GL(r)/F$, with $r \leq \lfloor \frac{m}{2} + 1 \rfloor$. Then $\text{sym}^m(\pi)$ is cuspidal if $\text{sym}^6(\pi)$ is cuspidal.

Clearly, $i = m - 1$ is the most interesting case, and such an argument was already used in [KS2] for showing the automorphy of the symmetric cube of π .

3. Proof of Theorem A', parts (a)–(c)

3.1. A simple lemma. In this and the following sections, S will always denote a finite set of places of F containing the archimedean and finite ramified (for π) places of F .

Lemma 3.1 *Suppose $\text{sym}^r(\pi)$ is modular for all $r < m$. Pick any positive integer $i \leq m$. Then $\text{sym}^m(\pi)$ is modular iff $\text{sym}^i(\pi) \boxtimes \text{sym}^{m-i}(\pi)$ is modular.*

Proof. Since \boxtimes is commutative, we may assume that $i \leq m/2$. By the Clebsch-Gordon identities, if r_0 denotes the standard 2-dimensional representation of $GL(2, \mathbb{C})$, we have

$$\text{sym}^i(r_0) \times \text{sym}^{m-i}(r_0) \simeq \bigoplus_{j=0}^i \text{sym}^{m-2j}(r_0) \otimes \det^j.$$

It follows that

$$(3.2) \quad L^S(s, \text{sym}^i(\pi) \times \text{sym}^{m-i}(\pi)) = \prod_{j=0}^i L^S(s, \text{sym}^{m-2j}(\pi) \otimes \omega^j).$$

By hypothesis, $\text{sym}^j(\pi)$ is modular for all $j < m$. If $\text{sym}^m(\pi)$ is also modular, we may set

$$\text{sym}^i(\pi) \boxtimes \text{sym}^{m-i}(\pi) := \boxplus_{j=0}^i \text{sym}^{m-2j}(\pi) \otimes \omega^j,$$

which defines the desired automorphic form on $\text{GL}((i+1)(m-i+1))/F$, uniquely defined by the strong multiplicity one theorem. Conversely, if $\text{sym}^i(\pi) \boxtimes \text{sym}^{m-i}(\pi)$ is modular, then by (3.2), it must have a unique isobaric summand Π , with

$$\text{sym}^i(\pi) \boxtimes \text{sym}^{m-i}(\pi) := \Pi \boxplus (\boxplus_{j=1}^i \text{sym}^{m-2j}(\pi) \otimes \omega^j).$$

It follows that at any place v one has, for every integer $k \leq m$ and for every irreducible admissible representation η of $\text{GL}_k(F_v)$, identities of the Rankin-Selberg local factors:

$$L(s, \Pi_v \times \eta) = L(s, \text{sym}^m(\pi) \times \eta),$$

and

$$\varepsilon(s, \Pi_v \times \eta) = \varepsilon(s, \text{sym}^m(\pi_v) \times \eta).$$

From the local converse theorem, one gets an isomorphism of Π_v with $\text{sym}^m(\pi_v)$. Hence $\text{sym}^m(\pi)$ is modular. \square

3.2. Proof of part (a) of Theorem A'. By the work of Kim and Shahidi (see section 1), we know that for all $j \leq 4$, $\text{sym}^j(\pi)$ is modular, even cuspidal since π is not solvable polyhedral. By hypothesis, $\text{sym}^5(\pi)$ is modular. Applying Lemma 3.1 above with $i = 4$, we get the modularity of $\text{sym}^4(\pi) \boxtimes \pi$. Suppose $\text{sym}^5(\pi)$ is Eisensteinian. Then it must have an isobaric summand τ , say, which is cuspidal on $\text{GL}(r)/F$ for some $r \leq 3$. We know (see section 1) that $\pi \boxtimes \tau^\vee$ is automorphic on $\text{GL}(2r)/F$. Using (3.2) we get the identity $L^S(s, \text{sym}^4(\pi) \times (\pi \boxtimes \tau^\vee)) = L^S(s, \text{sym}^5(\pi) \times \tau^\vee) L^S(s, \text{sym}^3(\pi) \otimes \omega \times \tau^\vee)$.

As τ is an isobaric summand of $\text{sym}^5(\pi)$, the first L -function on the right has a pole at $s = 1$. And by the Rankin-Selberg theory, the second L -function on the right has no zero at $s = 1$. It follows that

$$-\text{ord}_{s=1} L^S(s, \text{sym}^4(\pi) \times (\pi \boxtimes \tau^\vee)) \geq 1.$$

Since $\text{sym}^4(\pi)$ is a cusp form on $\text{GL}(5)/F$, we are forced to have $r = 3$. Comparing dimensions, we must then have an isobaric sum decomposition

$$\pi^\vee \boxtimes \tau \simeq \text{sym}^4(\pi) \boxplus \nu,$$

where ν is an idele class character of F . This implies that

$$-\text{ord}_{s=1} L^S(s, \pi^\vee \boxtimes \tau \otimes \nu^{-1}) \geq 1,$$

which is impossible unless $r = 2$ and $\tau \simeq \pi \otimes \nu$. But we have $r = 3$, furnishing the desired contradiction. hence $\text{sym}^5(\pi)$ must be cuspidal. \square

3.3. Proof of part (b) of Theorem A'. By hypothesis, $\text{sym}^j(\pi)$ is modular for all $j \leq 6$, even cuspidal for $j \leq 5$ by part (a). By Lemma 1, $\text{sym}^j(\pi) \boxtimes \pi$ is also modular for each $j \leq 5$.

First suppose we have an isomorphism

$$\text{sym}^5(\pi) \simeq \text{sym}^2(\pi') \boxtimes \pi \otimes \nu,$$

for a cusp form π' on $GL(2)/F$ and idele class character ν of F . This results in the identity:

$$(3.3.1) \quad L^S(s, \text{sym}^5(\pi) \boxtimes \pi) = L^S(s, (\text{sym}^2(\pi') \boxtimes \pi) \times \pi \otimes \nu).$$

The L -function on the right is the same as

$$(3.3.2) \quad L^S(s, \text{sym}^2(\pi') \times \text{sym}^2(\pi) \otimes \nu) L^S(s, \text{sym}^2(\pi') \otimes \omega \nu).$$

As $\text{sym}^2(\pi')^\vee \otimes (\omega \nu)^{-1}$ is equivalent to $\text{sym}^2(\pi') \otimes \omega \nu^{-1}$, we see that by Lemma 3.1, $\Pi' := \text{sym}^2(\pi') \boxtimes \text{sym}^2(\pi')^\vee \otimes (\omega \nu)^{-1}$ makes sense as an automorphic form on $GL(6)/F$. And since $\text{sym}^5(\pi) \boxtimes \pi$ is isomorphic to $\text{sym}^6(\pi) \boxplus (\text{sym}^4(\pi) \otimes \omega)$, we obtain by using (3.3.1) and (3.3.2):

$$(3.3.3 - a) \quad L^S(s, \text{sym}^6(\pi) \times \text{sym}^2(\pi')^\vee (\omega \nu)^{-1}) L^S(s, \text{sym}^4(\pi) \times \text{sym}^2(\pi')^\vee \otimes \nu^{-1})$$

equals

$$(3.3.3 - b) \quad L^S(s, \Pi' \times \text{sym}^2(\pi')) L^S(s, \text{sym}^2(\pi') \boxtimes \text{sym}^2(\pi')^\vee).$$

The second L -function of (3.3.3-b) has a pole at $s = 1$. And since $\text{sym}^4(\pi)$ is a cusp form on $GL(5)/F$, the second L -function of (3.3.3-a) has no pole at $s = 1$, and the first L -function of (3.3.3-b) has no zero at $s = 1$. Consequently,

$$-\text{ord}_{s=1} L^S(s, \text{sym}^6(\pi) \times \text{sym}^2(\pi')^\vee \otimes (\omega \nu)^{-1}) \geq 1.$$

As $\text{sym}^2(\pi')^\vee$ is automorphic on $GL(3)/F$, this cannot be unless $\text{sym}^6(\pi)$ is not cuspidal. We are done in this direction.

Now let us prove the *converse*, by supposing that $\text{sym}^6(\pi)$ is Eisensteinian. In this case it must admit an isobaric summand τ which is cuspidal on $GL(k)/F$ with $k \leq 3$. Since we have

$$\text{sym}^6(\pi) \boxplus \text{sym}^4(\pi) \simeq \text{sym}^5(\pi) \boxtimes \pi,$$

τ must be an isobaric summand of $\text{sym}^5(\pi) \boxtimes \pi$. It follows that

$$-\text{ord}_{s=1} L^S(s, \text{sym}^5(\pi) \times (\pi \boxtimes \tau^\vee)) \geq 1,$$

where $\pi \boxtimes \tau^\vee$ is modular since $k \leq 3$. Since $\text{sym}^5(\pi)$ is a cusp form on $GL(6)/F$, we are forced to have $k = 3$, and moreover,

$$(3.3.4) \quad \text{sym}^5(\pi) \simeq \pi^\vee \boxtimes \tau.$$

As $\text{sym}^6(\pi)$ cannot have a $\text{GL}(1)$ isobaric summand, no twist of τ can be an isobaric summand either. On the other hand, since the dual of $\text{sym}^6(\pi)$ is its twist by ω^{-6} , τ^\vee is an isobaric summand of $\text{sym}^6(\pi) \otimes \omega^{-6}$. So we must have

$$(3.3.5) \quad \tau^\vee \simeq \tau \otimes \omega^{-6},$$

showing τ is essentially selfdual. In fact, if we put

$$(3.3.6) \quad \eta := \tau \otimes \omega^{-3},$$

it is immediate that η is even *selfdual*. It follows that

$$L^S(s, \eta, \text{sym}^2)L^S(s, \eta, \Lambda^2) = L^S(s, \eta \times \eta^\vee),$$

showing that the left hand side has a pole at $s = 1$. Since η is a cusp form on $\text{GL}(3)/F$, the second L -function cannot have a pole at $s = 1$ (see [JS]). Hence

$$(3.3.7) \quad -\text{ord}_{s=1} L^S(s, \eta, \text{sym}^2) \geq 1.$$

By the backwards lifting results of Ginzburg, Rallis and Soudry ([GRS]), we then have a functorially associated cuspidal, necessarily generic, automorphic representation π'_0 of $\text{SL}(2, \mathbb{A}_F)$ ($= \text{Sp}(2, \mathbb{A}_F)$) of trivial central character. We may extend it (see [RS], for example) to an irreducible cusp form π' of $\text{GL}(2)/F$, which is only unique up to twisting by a character, such that

$$(3.3.7) \quad L^S(s, \text{Ad}(\pi')) = L^S(s, \eta).$$

By the strong multiplicity one theorem, η is isomorphic to $\text{Ad}(\pi')$. An alternate way to find such a π' is to use descend using the comparison between the twisted trace formula on $\text{GL}(3)$, relative to $g \rightarrow {}^t g^{-1}$, and the stable trace formula on $\text{SL}(2)$, which has been carried out by Flicker.

Combining with (3.3.4) and (3.3.26), we get

$$\text{sym}^5(\pi) \simeq \text{Ad}(\pi') \boxtimes \pi \otimes \omega^2,$$

as asserted in part (b) of Theorem A'.

Finally suppose $\text{Ad}(\pi)$ and $\text{Ad}(\pi')$ are twist equivalent. Then $\text{sym}^5(\pi)$ would need to be twist equivalent to $\text{sym}^2(\pi) \boxtimes \pi$, which is Eisensteinian of the form $\text{sym}^3(\pi) \boxplus \pi \otimes \omega$. This contradicts the cuspidality of $\text{sym}^5(\pi)$, and we are done. □

3.4. Proof of part (c) of Theorem A'. There is nothing to prove if $m = 6$, so let $m \geq 7$, and assume by induction that the conclusion holds for all $n \leq m - 1$. In particular, $\text{sym}^n(\pi)$ is cuspidal for every $n < m$. Moreover, by hypothesis, $\text{sym}^j(\pi)$ is modular for all $j \leq 2m$, and this implies, by Lemma 3.1, that $\text{sym}^m(\pi) \boxtimes \text{sym}^m(\pi)$ is modular.

Suppose $\text{sym}^m(\pi)$ is not cuspidal. Then by [JS],

$$(3.4.1) \quad -\text{ord}_{s=1} L^S(s, \text{sym}^m(\pi) \times \text{sym}^m(\pi)^\vee) \geq 2.$$

We have by Clebsch-Gordon,

$$\mathrm{sym}^m(\pi) \boxtimes \mathrm{sym}^m(\pi)^\vee \simeq \boxplus_{j=0}^m \mathrm{sym}^{2j}(\pi) \otimes \omega^{-j},$$

and of course we have a similar formula for $\mathrm{sym}^{m-1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)^\vee$, where the sum goes from $j = 0$ to $j = m - 1$. Consequently,

$$(3.4.2) \quad \mathrm{sym}^m(\pi) \boxtimes \mathrm{sym}^m(\pi)^\vee \simeq (\mathrm{sym}^{m-1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)^\vee) \boxplus (\mathrm{sym}^{2m}(\pi) \otimes \omega^{-m}).$$

Since $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, $L^S(s, \mathrm{sym}^{m-1}(\pi) \times \mathrm{sym}^{m-1}(\pi)^\vee)$ has a simple pole at $s = 1$ (cf. [JS]). Combining this with (3.4.1) and (3.4.2), we obtain

$$(3.4.3) \quad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{2m}(\pi) \otimes \omega^{-m}) \geq 1.$$

Since $\mathrm{sym}^{2m}(\pi)$ is automorphic, it must admit ω^m as an isobaric summand.

On the other hand, we have (by Clebsch-Gordon)

$$(3.4.4) \quad \mathrm{sym}^{m+1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi) \simeq \boxplus_{j=0}^{m-1} \mathrm{sym}^{2(m-j)}(\pi) \otimes \omega^j.$$

It follows that ω^m must be an isobaric summand of $\mathrm{sym}^{m+1}(\pi) \boxtimes \mathrm{sym}^{m-1}(\pi)$, implying

$$(3.4.5) \quad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m+1}(\pi) \times (\mathrm{sym}^{m-1}(\pi) \otimes \omega^{-m})) \geq 1.$$

Since $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, this can only happen (cf. [JS]) if $\mathrm{sym}^{m-1}(\pi)^\vee \otimes \omega^m$ is an isobaric summand of $\mathrm{sym}^{m+1}(\pi)$. Therefore

$$\mathrm{sym}^{m+1}(\pi) \simeq (\mathrm{sym}^{m-1}(\pi)^\vee \otimes \omega^m) \boxplus \tau,$$

where τ is an (isobaric) automorphic form on $GL(2)/F$.

Hence τ is an isobaric summand of $\mathrm{sym}^m(\pi) \boxtimes \pi$, which is isomorphic to $\mathrm{sym}^{m+1}(\pi) \boxplus (\mathrm{sym}^{m-1}(\pi) \otimes \omega)$. Recall that $\pi^\vee \boxtimes \tau$ is modular. Then there is an isobaric summand β of $\pi^\vee \boxtimes \tau$, which is cuspidal on $GL(r)/F$ with $r \leq 4$, such that

$$-\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^m(\pi) \times \beta^\vee) \geq 1.$$

In other words, β is an isobaric summand of $\mathrm{sym}^m(\pi)$, and hence of $\mathrm{sym}^{m-1}(\pi) \boxtimes \pi$. Consequently,

$$(3.4.6) \quad -\mathrm{ord}_{s=1} L^S(s, (\mathrm{sym}^{m-1}(\pi) \boxtimes \pi) \times \beta^\vee) \geq 1.$$

First suppose $r \leq 3$. Then we know that $\pi \boxtimes \beta^\vee$ is modular on $GL(2r)$ (by [Ram2] for $r=2$, and [KS2] for $r = 3$). As $\mathrm{sym}^{m-1}(\pi)$ is by induction cuspidal, (3.4.6) forces the bound

$$(3.4.7) \quad m \leq 2r \leq 6.$$

So we are done in this case.

Next suppose that $r = 4$, which means $\beta = \pi^\vee \boxtimes \tau$ is cuspidal. Since $\pi \boxtimes \pi^\vee \simeq \mathrm{sym}^2(\pi) \boxplus \omega$, it follows that $\pi \boxtimes \beta^\vee$ is modular, with

$$\pi \boxtimes \beta^\vee \simeq (\mathrm{sym}^2(\pi) \boxtimes \tau^\vee) \boxplus (\omega \otimes \tau^\vee),$$

where the first summand is on $\mathrm{GL}(6)/F$ and the second on $\mathrm{GL}(4)$. As a result, we have from (3.4.6),

$$(3.4.8) \quad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m-1}(\pi) \times \delta) \geq 1,$$

for an isobaric summand δ of $\pi \boxtimes \beta^\vee$, which is a cusp form on $\mathrm{GL}(n)$, for some $n \leq 6$. So, once again, the inequality (3.4.7) holds. So we are done - again. \square

4. PROOF OF THEOREM B

The modularity of $\mathrm{sym}^m(\pi)$ follows from Lemma 3.1, thanks to the hypotheses. Then $\mathrm{sym}^{m-1}(\pi) \boxtimes \pi$ is also modular. Suppose $\mathrm{sym}^5(\pi)$ is not cuspidal. Then it will admit an isobaric component τ , which is cuspidal on $\mathrm{GL}(r)/F$, for some $r \leq 3$. We get a contradiction by proceeding exactly as in the proof of part (a) of Theorem A'. Similarly, the proof of part (b) is very similar to the corresponding part of Theorem A'.

The proof of part (c) is a bit different, however, because we are not assuming (for this Theorem) good properties of $\mathrm{sym}^j(\pi)$ for j all the way up to $2m$.

We may take $m > 6$ and assume by induction that $\mathrm{sym}^j(\pi)$ is cuspidal for all $j \leq m-1$. Suppose $\mathrm{sym}^m(\pi)$ is Eisensteinian. Then it must have an isobaric summand η , which is cuspidal on $\mathrm{GL}(r)/F$ with $r \leq \lceil \frac{m+1}{2} \rceil$, where $\lceil x \rceil$ denotes the integral part of x . Then η must be an isobaric summand of $\mathrm{sym}^{m-1}(\pi) \boxtimes \pi$, because of the decomposition

$$\mathrm{sym}^{m-1}(\pi) \boxtimes \pi \simeq \mathrm{sym}^m(\pi) \boxplus (\mathrm{sym}^{m-2}(\pi) \otimes \omega).$$

By our hypothesis, $\pi \boxtimes \eta^\vee$ is modular on $\mathrm{GL}(2r)/F$. So we get

$$(4.1) \quad -\mathrm{ord}_{s=1} L^S(s, \mathrm{sym}^{m-1}(\pi) \times (\pi \boxtimes \eta^\vee)) \geq 1.$$

As $\mathrm{sym}^{m-1}(\pi)$ is cuspidal, we are forced to have $m \leq 2r$. Combining this with the upper bound for r , we get

$$(4.2) \quad \frac{m}{2} \leq r \leq m+1.$$

So the only possible (isobaric) decomposition of $\mathrm{sym}^m(\pi)$ we can have, up to renaming η , is

$$(4.3) \quad \mathrm{sym}^m(\pi) \simeq \eta \boxplus \eta',$$

with

$$\eta \in \mathcal{A}_0(\lceil (m+1)/2 \rceil, F) \quad \text{and} \quad \eta' \in \mathcal{A}_0(m+1 - \lceil (m+1)/2 \rceil, F).$$

And by our hypothesis, $\eta \boxtimes \pi^\vee$ and $\eta' \boxtimes \pi^\vee$ are modular. We deduce that

$$(4.4) \quad [\mathrm{sym}^{m-1}(\pi), \eta \boxtimes \pi^\vee] > 0, \quad \text{and} \quad [\mathrm{sym}^{m-1}(\pi), \eta' \boxtimes \pi^\vee] > 0.$$

First consider the case when m is odd. (This is similar to the argument above for $m = 5$.) Then $r = [(m + 1)/2] = m + 1 - [(m + 1)/2]$, and since $\text{sym}^{m-1}(\pi) \in \mathcal{A}_0(m, F)$, we must have

$$\eta \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi) \boxplus \mu$$

and

$$\eta' \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi) \boxplus \mu',$$

with μ, μ' in $\mathcal{A}(1, F)$. Then it follows that the Rankin-Selberg L -functions $L^S(s, \eta \times (\pi^\vee \otimes \mu^{-1}))$ and $L^S(s, \eta' \times (\pi^\vee \otimes \mu'^{-1}))$ both have poles at $s = 1$. This forces the following:

$$m = 3, \quad \eta \simeq \pi \otimes \mu, \quad \text{and} \quad \eta' \simeq \pi \otimes \mu'.$$

So this cannot happen for $m \neq 3$.

Next consider the case when m is even. Then $\eta \in \mathcal{A}_0(m/2, F)$ and $\eta' \in \mathcal{A}_0(m/2 + 1, F)$. We get

$$\eta \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi)$$

and

$$\eta' \boxtimes \pi^\vee \simeq \text{sym}^{m-1}(\pi) \boxplus \tau,$$

with τ in $\mathcal{A}_0(2, F)$. Then η' must occur in $\pi \boxtimes \tau$, which is in $\mathcal{A}(4, F)$. So we must have

$$m/2 + 1 \leq 4.$$

In other words, m must be less than or equal to 6, which is not the case.

Thus we get a contradiction in either case. The only possibility is for $\text{sym}^m(\pi)$ to be cuspidal. Done proving part (c), and hence all of Theorem B. □

5. PROOF OF THEOREM A', PART (D)

Finally, we want to restrict to $F = \mathbb{Q}$ and analyze the case of holomorphic newforms f of weight ≥ 2 . One knows that the level N of f is the same as the conductor of the associated cuspidal automorphic representation π of $GL(2, \mathbb{A}_{\mathbb{Q}})$. Moreover, as f is not of CM type, π is not dihedral.

Fix a prime ℓ not dividing N and consider the *cyclotomic character*

$$(5.1) \quad \chi_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_\ell^*,$$

defined by the Galois action on the projective system $\{\mu_{\ell^r} \mid r \geq 1\}$, where μ_{ℓ^r} denotes the group of ℓ^r -th roots of unity in $\overline{\mathbb{Q}}$. Then by a theorem of Deligne, one has at our disposal an irreducible, continuous representation

$$(5.2) \quad \rho_\ell(\pi) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL(2, \overline{\mathbb{Q}}_\ell),$$

unramified outside $N\ell$, such that for every prime p not dividing $N\ell$,

$$(5.3) \quad \text{Tr}(\rho_\ell(\pi)(Fr_p)) = a_p,$$

where Fr_p denotes the Frobenius at p and a_p the p -th Hecke eigenvalue of f . Moreover,

$$(5.4) \quad \det(\rho_\ell(\pi)) = \omega \chi_\ell^{k-1}.$$

When f is of CM-type, there exists an imaginary quadratic field K , and an algebraic Hecke character Ψ of K such that

$$(5.5) \quad \rho_\ell(\pi) \simeq \text{Ind}_{\text{Gal}(\overline{\mathbb{Q}}/K)}^{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\Psi_\ell),$$

where Ψ_ℓ is the ℓ -adic character associated to Ψ ([Ser]). Let θ denote the non-trivial automorphism of $\text{Gal}(K/\mathbb{Q})$. Then it is an immediate exercise to check that for any $m \geq 1$, $\text{sym}^m(\rho_\ell)$ is of the form $\bigoplus_j \beta_{j,\ell}$, where each $\beta_{j,\ell}$ is either one-dimensional defined by an idele class character of \mathbb{Q} or a two-dimensional induced by $\Psi_\ell^a(\Psi_\ell^\theta)^{m-a}$ for some $a \geq 0$, with Ψ_ℓ^θ denoting the conjugate of Ψ_ℓ under θ . It is clear this is modular, but not cuspidal for any $m \geq 2$.

Let us assume henceforth that f is not of CM-type. Denote by G_ℓ the Zariski closure of the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under $\rho_\ell(\pi)$; it is an ℓ -adic Lie group. Since f is of weight ≥ 2 and not of CM-type, a theorem of K. Ribet ([Rib]) asserts that for large enough ℓ ,

$$(5.6) \quad G_\ell = \text{GL}(2, \overline{\mathbb{Q}}_\ell).$$

We will from now on consider only those ℓ large enough for this to hold. Since the symmetric power representations of the algebraic group $\text{GL}(2)$ are irreducible, we get the following

Lemma 5.7 *For any non-CM newform f of weight $k \geq 2$ and for any $m \geq 1$ and large enough ℓ , the representation $\text{sym}^m(\rho_\ell)$ is irreducible, and it remains so under restriction to $\text{Gal}(\overline{\mathbb{Q}}/E)$ for any finite extension E of \mathbb{Q} .*

Since f is not of CM-type, $\text{sym}^2(\pi)$ is cuspidal. In view of parts (a)–(c) (of Theorem A'), we need only prove the following to deduce part (d):

Proposition 5.8 *For any non-CM newform f of weight $k \geq 2$ and level N , with associated cuspidal automorphic representation π of $\text{GL}(2, \mathbb{A}_\mathbb{Q})$, assume that $\text{sym}^m(\pi)$ is modular for all $m \geq 2$. Then the following hold:*

- (i) *For any quadratic field K , the base change $\text{sym}^3(\pi)_K$ to $\text{GL}(4)/K$ is cuspidal*
- (ii) *$\text{sym}^6(\pi)$ is cuspidal*

This Proposition suffices, because (i) implies that π is not solvable polyhedral, and (ii) implies what we want by part (c) of Theorem A'.

Let f be as in the Proposition. Suppose $m \geq 1$ is such that $\text{sym}^j(\pi)$ is cuspidal for all $j < m$, but Eisensteinian for $j = m$. Then we have, as in the proof of the earlier parts of Theorem A', a decomposition

$$(5.9) \quad \text{sym}^m(\pi) \simeq \eta \boxplus \eta',$$

with

$$\eta \in \mathcal{A}_0([(m+1)/2], \mathbb{Q}) \quad \text{and} \quad \eta' \in \mathcal{A}_0(m+1 - [(m+1)/2], \mathbb{Q}),$$

with η, η' are essentially self-dual. Moreover, we have

Lemma 5.10 *The infinity types of η, η' are both algebraic and regular.*

Some explanation of the terminology is called for at this point. Recall that $W_{\mathbb{R}}$ is the unique non-split extension of $\text{Gal}(\mathbb{C}/\mathbb{R})$ by \mathbb{C}^* , which is concretely described as $\mathbb{C}^* \cup j\mathbb{C}^*$, with $jzj^{-1} = \bar{z}$, for all $z \in \mathbb{C}^*$. Let Π be an irreducible automorphic representation of $GL(n, \mathbb{A}_F)$. Since the restriction of $\sigma_{\infty}(\Pi)$ is semisimple and since \mathbb{C}^* is abelian, we get a decomposition

$$\sigma_{\infty}(\Pi)|_{\mathbb{C}^*} \simeq \bigoplus_{i \in J} \chi_i,$$

where each χ_i is in $\text{Hom}_{\text{cont}}(\mathbb{C}^*, \mathbb{C}^*)$. Π_{∞} is said to be **regular** iff this decomposition is multiplicity-free, i.e., iff $\chi_i \neq \chi_r$ for $i \neq r$. It is **algebraic** ($[\text{Clo}]$) iff each $\chi_i| \cdot |^{(m-1)/2}$ is of the form $z \rightarrow z^{-a_i} \bar{z}^{-b_i}$, for some integers a_i, b_i . An algebraic Π is said to be **pure** if there is an integer w , called the **weight** of Π , such that $w = a_i + b_i$ for each $i \in J$.

It is well known that, since π is defined by a holomorphic newforms f of weight $k \geq 2$,

$$(5.11) \quad \sigma_{\infty}(\pi) \otimes | \cdot |^{-1/2} \simeq \text{Ind}(W_{\mathbb{R}}, \mathbb{C}^*; z_{1-k}),$$

where z_n denotes, for each integer n , the continuous homomorphism $\mathbb{C}^* \rightarrow \mathbb{C}^*$ given by $z \rightarrow z^n$. Note that π_{∞} is regular (as $k > 1$) and algebraic of weight $k - 1$. From here on to the end of this chapter, we will simply write $I(-)$ for $\text{Ind}(W_{\mathbb{R}}, \mathbb{C}^*; -)$. Set

$$\nu_{1-k} = z_{1-k}|_{\mathbb{R}^*}.$$

Then we have

$$(5.12) \quad \omega_{\infty} = \text{sgn} \nu_{1-k},$$

where sgn denotes the sign character of \mathbb{R}^* . Clearly, $\omega_{\infty} = \text{sgn}^{1-k} \nu_{1-k}$. But as f has trivial character, k is forced to be even, so $\text{sgn}^{1-k} = \text{sgn}$. (Here we have identified, as we may, ω_{∞} with $\sigma_{\infty}(\omega)$.)

SubLemma 5.13 *For each $j \leq [m/2]$,*

$$(i) \quad \sigma_{\infty}(\text{sym}^{2j+1}(\pi)) \simeq I(z_{1-k}^{2j+1}) \oplus (I(z_{1-k}^{2j-1}) \otimes | \cdot |^{1-k}) \oplus \dots \oplus (I(z_{1-k}) \otimes | \cdot |^{(1-k)j}),$$

and

$$(ii) \quad \sigma_{\infty}(\text{sym}^{2j}(\pi)) \simeq I(z_{1-k}^{2j}) \oplus (I(z_{1-k}^{2j-2}) \otimes | \cdot |^{1-k}) \oplus \dots \oplus (I(z_{1-k}^2) \otimes | \cdot |^{(1-k)(j-1)}) \oplus \nu_{1-k}^j.$$

Proof of SubLemma. Everything is fine for $j = 0$. So we may let $j > 0$ and assume by induction that the identities hold for all $r < j$. Applying (i) for $j - 1$ together with (5.3)_{2j}, (5.11) and (3.19), we see that

$$\sigma_{\infty}(\text{sym}^{2j}(\pi)) \oplus (\sigma_{\infty}(\text{sym}^{2j-2}(\pi)) \otimes | \cdot |^{1-k})$$

is isomorphic to

$$(I(z_{1-k}^{2j-1}) \oplus (I(z_{1-k}^{2j-3}) \otimes |\cdot|^{1-k}) \oplus \dots \oplus (I(z_{1-k}) \otimes |\cdot|^{(1-k)(j-1)}) \otimes I(z_{1-k})).$$

By Mackey theory, we have for all $a \geq b$,

$$I(z_{1-k}^a) \otimes I(z_{1-k}^b) \simeq I(z_{1-k}^{a+b}) \oplus I(z_{1-k}^a \bar{z}_{1-k}^b) \simeq I(z_{1-k}^{a+b}) \oplus (I(z_{1-k}^{a-b}) \otimes |\cdot|^{1-k}).$$

Since $I(-) \otimes \text{sgn} \simeq I(-)$, $I(-) \otimes |\cdot|^{1-k}$ is isomorphic to $I(-) \otimes \nu_{1-k}$. Combining these and using the inductive assumption for $\sigma(\text{sym}^{2j-2}(\pi))$, we get (ii) for j . The proof of (ii) is similar and left to the reader. \square

Now Lemma 5.10 follows easily from the SubLemma and the definition of regular algebraicity.

Proof of Proposition (contd.) We need only examine $\text{sym}^m(\pi)$ for $m = 3$ and $m = 6$.

First suppose $m = 3$. Let K be any quadratic field. Then η_K and η'_K are both essentially self-dual forms on $\text{GL}(2)/K$ with algebraic, regular infinity types. Consequently, one knows that for $\beta \in \{\eta, \eta'\}$, there exists a semisimple representation

$$\rho_\ell(\beta) : \text{Gal}(\bar{\mathbb{Q}}/K) \rightarrow \text{GL}(2, \bar{\mathbb{Q}}_\ell)$$

such that for primes P in a set of Dirichlet density 1, we have

$$(5.14) \quad L(s, \beta_P) = \det(1 - \text{Fr}_P(NP)^{-s} | \rho_\ell(\beta))^{-1}.$$

If β is Eisensteinian, which in fact cannot happen, this is easy to establish. Ditto if it is dihedral. So we may take β to be cuspidal and non-dihedral. If K is totally real, the existence of $\rho_\ell(\beta)$ is a well known result, due independently to R. Taylor ([Tay1]) and to Blasius-Rogawski ([BR]); in fact a stronger assertion holds in that case. In this case, β corresponds to a Hilbert modular form, either one of weight $3k-2$ or to a twist of one of weight $3k-4$. If K is imaginary, the existence of $\rho_\ell(\beta)$ is a theorem of R. Taylor ([Tay2]), partly based on his joint work with M. Harris and D. Soudry. (Note that here, the central character of the unitary version of β is trivial.)

By part (a) of the Lemma, we then get the following at all primes P in a set of density 1:

$$(5.15) \quad L(s, \text{sym}^3(\pi_K)_P) = \det(1 - \text{Fr}_P(NP)^{-s} | \rho_\ell(\eta) \oplus \rho_\ell(\eta'))^{-1}.$$

But by construction,

$$(5.16) \quad L(s, \text{sym}^3(\pi_K)_P) = \det(1 - \text{Fr}_P(NP)^{-s} | \text{sym}^3(\rho_\ell(\pi)_K))^{-1}.$$

Thus we have, by the Tchebotarev density theorem,

$$\text{sym}^3(\rho_\ell(\pi)_K) \simeq \rho_\ell(\eta) \oplus \rho_\ell(\eta').$$

We get a contradiction as we know (cf. Lemma 5.7) that $\text{sym}^3(\rho_\ell(\pi)_K)$ is an irreducible representation.

Thus $\text{sym}^3(\pi_K)$ is cuspidal. This proves part (i) of the Proposition, and implies that π is not solvable polyhedral.

Next we turn to the question of cuspidality of $\text{sym}^6(\pi)$. Again, thanks to the hypothesis of modularity $\text{sym}^6(\pi)$, $\text{sym}^j(\pi)$ is cuspidal for all $j \leq 5$.

Suppose $\text{sym}^6(\pi)$ is not cuspidal. Let η, η' be as in the decomposition $\text{sym}^m(\pi)$ given by (5.9). Since $m = 6$, $\eta \in \mathcal{A}_0(3, \mathbb{Q})$ and $\eta' \in \mathcal{A}_0(4, \mathbb{Q})$. Specializing Lemma 3.1 to $(i, m) = (5, 6)$, we get

$$(5.17) \quad \text{sym}^5(\pi) \boxtimes \pi \simeq \eta \boxplus \eta' \boxplus (\text{sym}^4(\pi) \otimes |\cdot|^{1-k}).$$

Lemma 5.18 Let $\beta \in \{\eta, \eta'\}$. Take $m = 3$ if $\beta = \eta$ and $m = 4$ if $\beta = \eta'$. Then for any prime ℓ away from the ramification locus of β , there exists a semisimple ℓ -adic representation

$$\rho_\ell(\beta) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL(m, \overline{\mathbb{Q}}_\ell)$$

such that for almost all primes p , we have

$$(5.19) \quad L(s, \beta_p) = \det(1 - \text{Fr}_p p^{-s} | \rho_\ell(\beta))^{-1}.$$

Proof of Lemma. First Note that since the dual of $\text{sym}^6(\pi)$ is $\text{sym}^6(\pi) \otimes \omega^{-6}$, the twisted representation $\text{sym}^6(\pi) \otimes \omega^{-3}$ is selfdual. So, we may, after replacing $\text{sym}^6(\pi)$, η and η' by their respective twists by ω^3 , assume that they are all selfdual. (Since η, η' are irreducible representations of unequal dimensions, they cannot be contragredients of each other, and so are forced to be selfdual themselves.) As we have seen, they are also regular and algebraic. Now the discussion in [Ram6] explains how to deduce the existence of the desired Galois representations attached to η, η' (see also [RS, Ram3, Lau, Wei]).

□

Proof of Proposition 5.8 (contd.). Applying Lemma 5.18 we get for almost all primes p ,

$$L(s, \text{sym}^6(\pi)_p) = \det(1 - \text{Fr}_p p^{-s} | \rho_\ell(\eta) \oplus \rho_\ell(\eta'))^{-1}.$$

By the Tchebotarev density theorem,

$$\text{sym}^6(\rho_\ell(\pi)) \simeq \rho_\ell(\eta) \oplus \rho_\ell(\eta').$$

Again we get a contradiction since by Lemma 5.7, $\text{sym}^6(\rho_\ell(\pi))$ is an irreducible representation.

Thus $\text{sym}^6(\pi)$ is cuspidal.

□

We have now completely proved Theorem A', which implies Theorem A of the Introduction.

□

REFERENCES

- [BR] D. Blasius and J. D. Rogawski. Motives for Hilbert modular forms. *Invent. Math.* **114** (1993), 55–87.
- [Clo] L. Clozel. Motifs et formes automorphes: applications du principe de functorialité. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990.
- [GJ] S. Gelbart and H. Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. École Norm. Sup. (4)* **11** (1978), 471–542.
- [GRS] D. Ginzburg, S. Rallis, and D. Soudry. On explicit lifts of cusp forms from GL_m to classical groups. *Ann. of Math. (2)* **150** (1999), 807–866.
- [GJ] R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 260.
- [JPSS1] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika. Conducteur des représentations du groupe linéaire. *Math. Ann.* **256** (1981), 199–214.
- [JPSS2] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.* **105** (1983), 367–464.
- [JS] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.* **103** (1981), 777–815.
- [Jac] H. Jacquet. Principal L -functions of the linear group. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 63–86. Amer. Math. Soc., Providence, R.I., 1979.
- [JS] H. Jacquet and J. Shalika. Exterior square L -functions. In *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 143–226. Academic Press, Boston, MA, 1990.
- [Kim] H. H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2 . *J. Amer. Math. Soc.* **16** (2003), 139–183 (electronic). With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.
- [KS1] H. H. Kim and F. Shahidi. Cuspidality of symmetric powers with applications. *Duke Math. J.* **112** (2002), 177–197.
- [KS2] H. H. Kim and F. Shahidi. Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2 . *Ann. of Math. (2)* **155** (2002), 837–893. With an appendix by Colin J. Bushnell and Guy Henniart.
- [Lan1] R. P. Langlands. Automorphic representations, Shimura varieties, and motives. Ein Märchen. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 205–246. Amer. Math. Soc., Providence, R.I., 1979.
- [Lan2] R. P. Langlands. On the notion of an automorphic representation. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 189–217. Amer. Math. Soc., Providence, R.I., 1979.
- [Lan3] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation theory and harmonic analysis on semisimple Lie groups*, volume 31 of *Math. Surveys Monogr.*, pages 101–170. Amer. Math. Soc., Providence, RI, 1989.
- [Lan4] R. P. Langlands. Beyond endoscopy. In *Contributions to automorphic forms, geometry, and number theory*, pages 611–697. Johns Hopkins Univ. Press, Baltimore, MD, 2004.
- [Lau] G. Laumon. Fonctions zêtas des variétés de Siegel de dimension trois. *Astérisque* (2005), 1–66. Formes automorphes. II. Le cas du groupe $GSp(4)$.

- [MW] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $GL(n)$. *Ann. Sci. École Norm. Sup. (4)* **22** (1989), 605–674.
- [Ram1] D. Ramakrishnan. Pure motives and automorphic forms. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 411–446. Amer. Math. Soc., Providence, RI, 1994.
- [Ram2] D. Ramakrishnan. Modularity of the Rankin-Selberg L -series, and multiplicity one for $SL(2)$. *Ann. of Math. (2)* **152** (2000), 45–111.
- [Ram3] D. Ramakrishnan. Modularity of solvable Artin representations of $GO(4)$ -type. *Int. Math. Res. Not.* (2002), 1–54.
- [Ram4] D. Ramakrishnan. Algebraic cycles on Hilbert modular fourfolds and poles of L -functions. In *Algebraic groups and arithmetic*, pages 221–274. Tata Inst. Fund. Res., Mumbai, 2004.
- [Ram5] D. Ramakrishnan. Irreducibility of ℓ -adic associated to regular cusp forms on $GL(4)/\mathbb{Q}$, preprint, 2004.
- [Ram6] D. Ramakrishnan. Irreducibility and Cuspidality. In *Representation Theory and Automorphic Functions*.
- [RS] D. Ramakrishnan and F. Shahidi. Siegel modular forms of genus 2 attached to elliptic curves. *Math. Res. Lett.* **14** (2007), 315–332.
- [RW] D. Ramakrishnan and S. Wang. A cuspidality criterion for the functorial product on $GL(2) \times GL(3)$ with a cohomological application. *Int. Math. Res. Not.* (2004), 1355–1394.
- [Rib] K. A. Ribet. On l -adic representations attached to modular forms. *Invent. Math.* **28** (1975), 245–275.
- [Ser] J.-P. Serre. *Abelian l -adic representations and elliptic curves*. Advanced Book Classics. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989. With the collaboration of Willem Kuyk and John Labute.
- [Sha1] F. Shahidi. On certain L -functions. *Amer. J. Math.* **103** (1981), 297–355.
- [Sha2] F. Shahidi. On the Ramanujan conjecture and finiteness of poles for certain L -functions. *Ann. of Math. (2)* **127** (1988), 547–584.
- [Tay1] R. Taylor. On Galois representations associated to Hilbert modular forms. *Invent. Math.* **98** (1989), 265–280.
- [Tay2] R. Taylor. l -adic representations associated to modular forms over imaginary quadratic fields. II. *Invent. Math.* **116** (1994), 619–643.
- [Tay3] R. Taylor. Automorphy for some l -adic lifts of automorphic mod l representations II, preprint, 2006.
- [Wan] S. Wang. On the symmetric powers of cusp forms on $GL(2)$ of icosahedral type. *Int. Math. Res. Not.* (2003), 2373–2390.
- [Wei] R. Weissauer. Four dimensional Galois representations. *Astérisque* (2005), 67–150. Formes automorphes. II. Le cas du groupe $GSp(4)$.

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