Measurable chromatic and independence numbers for ergodic graphs and group actions

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0 Introduction

We study in this paper some combinatorial invariants associated with ergodic actions of infinite, countable (discrete) groups.

Let (X, μ) be a standard probability space and Γ an infinite, countable group with a set of generators $1 \notin S \subseteq \Gamma$. Given a free, measure-preserving action a of Γ on (X, μ), we consider the graph $G(S, a) = (X, E(S, a))$, whose vertices are the points in X and where $x \neq y \in X$ are adjacent if there is a generator $s \in S$ taking one to the other. It is clear that the connected components of this graph are isomorphic to the Cayley graph $\mathrm{Cay}(\Gamma, S)$ and thus parameters such as the chromatic number of $G(S, a)$ are identical to those of $\mathrm{Cay}(\Gamma, S)$. This however requires selecting an element from each connected component and thus essentially depends upon a use of the Axiom of Choice. However, the situation is vastly different when one considers instead measurable colorings and the associated measurable chromatic numbers.

Let us introduce first the combinatorial invariants we will be interested in.

Consider a locally countable, Borel graph $G = (X, E)$ on a standard probability space (X, μ). We denote by E^* the associated Borel equivalence relation whose classes are the connected components of G. Given a property of equivalence relations $\mathcal{P}$, we say that G has property $\mathcal{P}$ if E^* has property $\mathcal{P}$. This explains what it means to say that G is (μ-)measure preserving, ergodic, hyperfinite, smooth, etc. In particular, the graphs $G(S, a)$ discussed before are measure preserving, and they are ergodic iff the action a is ergodic.

Given such a graph $G = (X, E)$ its (μ-)*measurable chromatic number*, $\chi_\mu(G)$, is the smallest cardinality of a standard Borel space Y for which there is a (μ-)measurable coloring $c : X \to Y$ (i.e., $xEy \Rightarrow c(x) \neq c(y)$). Clearly $\chi_\mu(G) \in \{1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$. It is well known (see, e.g., [26]) that there are acyclic such graphs G, for which of course the usual chromatic number $\chi(G)$ is equal to 2, with $\chi_\mu(G) = 2^{\aleph_0}$.

In addition, we consider the *approximate* (μ-)*measurable chromatic number*, $\chi_\mu^{\mathrm{ap}}(G)$, which is defined as the smallest cardinality of a standard Borel space Y such that for each $\varepsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a measurable coloring $c : A \to Y$ of the induced subgraph $G|A$. Clearly $\chi_\mu^{\mathrm{ap}}(G) \leq \chi_\mu(G)$.

Finally, the (μ-)*independence number* of G, $i_\mu(G)$, is the supremum of the measures of Borel independent sets ($A \subseteq X$ is *independent* if no two elements of A are adjacent in G). Clearly $i_\mu(G) \in [0, 1]$. It is easy to check that $\chi_\mu^{\mathrm{ap}}(G) \geq \frac{1}{i_\mu(G)}$, so graphs with small independence number have large (approximate) measurable chromatic number.

We discuss in §2 various examples of measure-preserving, ergodic graphs G with small chromatic number $\chi(G)$ (e.g., acyclic) but for which $\chi_\mu^{\mathrm{ap}}(G)$ or $\chi_\mu(G)$ take various finite or infinite values, and others in which $i_\mu(G)$ takes any value in $[0, 1)$ (the value 1 can be easily seen to be impossible to realize in such a graph).

However for graphs of bounded degree, there are further restrictions (see 2.19, 2.20), which are analogs of the classical Brooks' Theorem in finite graph theory (see, e.g., Diestel [10]), which asserts that for finite graphs G the chromatic number is bounded by the maximum degree d of the graph, unless $d = 2$ and G contains an odd cycle or $d \geq 3$ and G contains a complete graph of size $d + 1$.

Theorem 0.1. *Let (X, μ) be a standard probability space and $G = (X, E)$ a Borel graph with degree bounded by $d \geq 2$. Then*

(i) [26] $\chi_\mu(G) \leq d + 1$ *and thus* $i_\mu(G) \geq 1/(d + 1)$.

(ii) If $d = 2$ and G has no odd cycles (i.e., G is bipartite) or else $d \geq 3$ and G has no cliques (i.e., complete subgraphs) of size $d+1$, then $\chi_\mu^{\mathrm{ap}}(G) \leq d$ and thus $i_\mu(G) \geq 1/d$.

In §3, we consider the case of hyperfinite graphs. Denote below by $\chi^*(G)$ the smallest chromatic number of an induced subgraph $G|A$, where A is an

E^*-invariant Borel set of measure 1. In particular, $\chi^*(G) \leq \chi(G)$. Using some techniques of Miller [39], we show (see 3.1, 3.8):

Theorem 0.2. *Let (X, μ) be a standard probability space and G a locally countable, acyclic, (μ-)hyperfinite graph. Then $\chi_\mu^{\mathrm{ap}}(G) \leq 2$ and thus $i_\mu(G) \geq 1/2$. If moreover G is locally finite, (μ-)hyperfinite, but not necessarily acyclic, then $\chi_\mu^{\mathrm{ap}}(G) \leq \chi^*(G)$ and thus $i_\mu(G) \geq 1/\chi^*(G)$, and if G is also measure preserving, then $\chi_\mu^{\mathrm{ap}}(G) = \chi^*(G)$.*

In §4 we consider the graphs associated with group actions, as discussed in the beginning of this introduction. Let $\chi_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$, $i_\mu(S, a)$ be the parameters associated with $G(S, a)$. It is easy to see that $i_\mu(S, a) \leq 1/2$.

Let $a \prec b$ be the relation of weak containment among measure preserving actions of Γ on (X, μ); see [24]. We have $a \prec b$ iff a is in the closure, in the weak topology, of the conjugacy class of b. We now have the following monotonicity properties (see 4.2, 4.3).

Theorem 0.3. *Let Γ be a countable group and S a finite set of generators. Then*

$$a \prec b \Rightarrow i_\mu(S, a) \leq i_\mu(S, b), \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, b).$$

It follows that $i_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$ are invariants of weak equivalence, $a \sim b$, where $a \sim b \Leftrightarrow a \prec b$ and $b \prec a$.

Note that $\mathrm{Cay}(\Gamma, S)$ is bipartite iff there is no odd length word in $S \cup S^{-1}$ equal to the identity in Γ. We show in 4.5 that (for any Γ, S) if $\mathrm{Cay}(\Gamma, S)$ is not bipartite, then $i_\mu(S, a) < 1/2$ and $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$. In fact in this case $i_\mu(S, a) \leq 1/2 - 1/(2g)$, where g is the odd girth (= length of shortest odd cycle) in $\mathrm{Cay}(\Gamma, S)$. From this and 0.1 it follows that for $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \langle s, t | s^2 = 1, t^3 = 1\rangle$ and $S = \{s, t\}$, we have $i_\mu(S, a) = 1/3$ for every free, measure-preserving action a of Γ.

It is known, see, e.g., [24], 13.2, that any two free, measure-preserving, ergodic actions of an amenable group Γ are weakly equivalent, thus for any finite generating set $S \subseteq \Gamma$, we have that

$$i_\mu(\Gamma, S) = i_\mu(S, a), \chi_\mu^{\mathrm{ap}}(\Gamma, S) = \chi_\mu^{\mathrm{ap}}(S, a)$$

are independent of the action a. We can identify $i_\mu(\Gamma, S)$, $\chi_\mu^{\mathrm{ap}}(\Gamma, S)$ in terms of $\mathrm{Cay}(\Gamma, S)$. For a *finite* graph $G = (X, E)$, we define the *independence ratio* $i(G)$ to be $i_\mu(G)$, where μ is the normalized counting measure on X (so $i(G) = \frac{\alpha(G)}{|X|}$, where $\alpha(G)$ is the maximum cardinality of an independent subset of X). For (Γ, S) as above and finite $F \subseteq \Gamma$, let $i(F, S)$ be the the independence ratio of the induced subgraph $\text{Cay}(\Gamma, S)|F$. Let (F_n) be a Følner sequence for Γ, i.e., $F_n \subseteq \Gamma$ are finite, non-empty, and $\forall \gamma \in \Gamma$, $\frac{|\gamma F_n \triangle F_n|}{|F_n|} \to 0$. Using a result that can be proved easily using the quasi-tiling machinery of Ornstein-Weiss [43] (see Gromov [14], 1.3 and Lindenstrauss-Weiss [31], Appendix), one can show that for any Følner sequence (F_n),

$$\lim_{n\to\infty} i(F_n, S) = i(\Gamma, S)$$

exists (and is of course independent of (F_n)). We call this the *independence number* of $\text{Cay}(\Gamma, S)$. We now have (see 4.7, 4.10)

Theorem 0.4. *Let Γ be a countable, amenable group and S a finite set of generators. Then:*

(i) $\chi^{\text{ap}}_\mu(\Gamma, S) = \chi(\text{Cay}(\Gamma, S))$,

(ii) $i_\mu(\Gamma, S) = i(\Gamma, S)$.

For non-amenable Γ, $i_\mu(S, a)$ and $\chi^{\text{ap}}_\mu(S, a)$ are not necessarily constant. In fact for any Γ and finite set of generators S with $\text{Cay}(\Gamma, S)$ bipartite, we have the following characterization of amenability (see 4.13, 4.14).

Theorem 0.5. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators with* $\text{Cay}(\Gamma, S)$ *bipartite. Then the following are equivalent:*

(i) Γ *is amenable,*

(ii) $i_\mu(S, a)$ *is constant for any free, measure-preserving action a of Γ,*

(iii) $i_\mu(S, a) = 1/2$*, for any free, measure-preserving action a of Γ,*

(iv) $\chi^{\text{ap}}_\mu(S, a)$ *is constant for any free, measure-preserving action a of Γ,*

(v) $\chi^{\text{ap}}_\mu(S, a) = 2$*, for every free, measure-preserving action a of Γ.*

Subsequently an alternative proof of this result was given in Abért-Elek [1].

We also have an analogous characterization of groups that have property (T) and the Haagerup Approximation Property HAP (see 4.15).

Theorem 0.6. *Let Γ be an infinite, countable group and $S \subseteq \Gamma$ a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *has property* (T),

(ii) $i_\mu(S, a) < 1/2$*, for every free, measure-preserving, weakly mixing action* a *of* Γ,

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$*, for every free, measure-preserving, weakly mixing action* a *of* Γ.

Also the following are equivalent:

(i*) Γ *does not have the* HAP,

(ii*) $i_\mu(S, a) < 1/2$*, for every free, measure-preserving, mixing action* a *of* Γ,

(iii*) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$*, for every free, measure-preserving, mixing action* a *of* Γ.

We next consider the shift action s_Γ of the free group $\Gamma = \mathbb{F}_m$, with m free generators $S = \{a_1, \ldots, a_m\}$, on 2^Γ with the product measure. Using a result of Kesten [28] for the norm of averaging operators, we show the following result (see 4.17).

Theorem 0.7. *Let $\Gamma = \mathbb{F}_m$ be the free group with a free set of generators S and let s_Γ be its shift action on 2^Γ. Then*

$$\frac{1}{2m} \leq i_\mu(S, s_\Gamma) \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}},$$

and

$$2m \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m + \sqrt{2m-1}}{\sqrt{2m-1}}.$$

This has the following consequence (see 2.5), answering a question in [39].

Corollary 0.8. *For each $2 \leq n < \aleph_0$, there is an acyclic, bounded degree, measure-preserving, ergodic Borel graph G with $\chi_\mu(G) = n$.*

Without the requirements of having bounded degree or preserving measure, such examples were first found by Laczkovich (see [26], Appendix).

The exact values of $i_\mu(S, s_\Gamma)$, $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma)$ in 0.7 are unknown. It should be noted that there is no known example of (Γ, S), with Γ amenable and S finite, for which $\chi_\mu(S, s_\Gamma) > \chi(\mathrm{Cay}(\Gamma, S)) + 1$. For instance, Gao-Jackson-Miller (unpublished) and recently Adam Timar (private communication) have shown that for $\Gamma = \mathbb{Z}^m$, with S the usual set of generators (for which of course $\chi(\mathrm{Cay}(\Gamma, S)) = 2$), $\chi_\mu(S, s_\Gamma) = 3$.

We also present in §4 other examples of free, measure-preserving, ergodic actions of $\mathbb{F}_m$ that satisfy the bounds in 0.7.

Finally in §4 we discuss, for any (Γ, S), canonical finite graphs that "approximate" the infinite graph $G(S, s_\Gamma)$ associated with the shift of Γ on 2^Γ. Labeled and "weighted" versions of these graphs were also used in Bowen [7]. Applying this to $\Gamma = \mathbb{F}_m$ and a free set of generators S produces a natural explicit family of finite graphs $G_{n,m,k}$ $(n, m, k \geq 1)$ which simultaneously have arbitrarily large odd girth $g_{\mathrm{odd}}(G)$ and arbitrarily small independence ratio $i(G)$, thus arbitrarily large chromatic numbers. Such explicit families appear to be of interest in finite graph theory. More precisely, we have (see 4.22):

Theorem 0.9. *There is an explicit family of finite graphs $G_{m,k,n}$ $(m, k, n \geq 1)$ and a map $m, k \mapsto N(m, k)$ such that for any m, k, if $n > N(m, k)$, then*

$$\begin{aligned} g_{\mathrm{odd}}(G_{m,k,n}) &> k, \\ i(G_{m,k,n}) &\leq \frac{2\sqrt{2m-1}}{m + \sqrt{2m-1}}, \end{aligned}$$

and thus

$$\chi(G_{m,k,n}) \geq \frac{m + \sqrt{2m-1}}{2\sqrt{2m-1}}.$$

In §5 we return to Brooks' Theorem and study Borel analogs of it, especially in the context of graphs generated by group actions.

Let Γ be an infinite countable group and $1 \notin S$ a finite set of generators for Γ. Let $d = |S^{\pm 1}|$, where $S^{\pm 1} = S \cup S^{-1}$, be the degree of the Cayley graph of Γ, S. Let A be a free Borel action of Γ on a standard Borel space X (we do not assume here that a measure is necessarily present on X). We can define as before the graph $G(S, A)$ associated with this action and the set of generators S. We denote be $\chi_B(S, A)$ the *Borel chromatic number* of this graph, i.e., the smallest cardinality of a standard Borel space Y for which there is a Borel coloring $c : X \to Y$ for $G(S, A)$. It follows from results of [26] that $\chi_B(S, A) \leq d + 1$. We examine under what circumstances this can be improved to $\chi_B(S, A) \leq d$ as in Brooks' Theorem. We show the following (see 5.12)

Theorem 0.10. *Suppose that Γ is a countable infinite group isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Suppose further that Γ has finitely many ends. Let S be a finite set of generators for Γ and put $d = |S^{\pm 1}|$. Then for any free Borel action A of Γ on a standard Borel space X, we have $\chi_B(S, A) \leq d$.*

The requirement that Γ is not isomorphic to $\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is necessary as the free part of the shift action of these groups Γ on 2^Γ has Borel chromatic number equal to 3 (with respect to the usual set of generators). Groups that satisfy the hypotheses of the preceding theorem include: groups with property (T), direct products of two infinite groups, amenable groups, and, more generally, groups of cost 1, etc. In particular, if Γ is a 2-generated such group, then for any free Borel action of Γ that admits an invariant Borel probability measure with respect to which it is weakly mixing, the corresponding Borel chromatic number is either 3 or 4. This extends a result of Gao-Jackson [12] and Miller, who proved this for the free part of the shift action of $\mathbb{Z}^2$ on $2^{\mathbb{Z}^2}$ (see §4).

In the last section §6 we discuss a matching problem in the Borel and measurable contexts related to earlier work of Laczkovich [30] and Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [29].

Addendum. After the first version of this paper has been completed, we received a preliminary draft of a paper by Lyons and Nazarov [35], with subject matter closely related to this paper. In particular, it contains a version of Proposition 4.16 below. Also, its main result, which is that the graph associated with the shift action of a non-amenable group Γ, and a finite set of generators $S \subseteq \Gamma$, on $[0, 1]^\Gamma$ admits a measurable matching, provides the solution to a problem that we discussed in §6 of the original version. Moreover, Lyons (private communication) mentioned that they have also considered the finite graphs approximating the graphs $G(S, s_\Gamma)$, discussed in §4, although these do not appear in [35]. Finally in [35] the authors mention that earlier results of Frieze-Luczak on random graphs imply that *for large values of m, $i_\mu(S, s_\Gamma) \leq \frac{\log 2m}{m}$* and thus $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m}{\log 2m}$.

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1 Preliminaries

(**A**) A *graph* is a pair $G = (X, E)$, where X is a set whose elements we call *vertices* of G and $E \subseteq X^2$ satisfies: $(x, x) \notin E$ (i.e., there are no loops) and $(x, y) \in E \Leftrightarrow (y, x) \in E$ (i.e., the graph is symmetric). We often write xEy to denote $(x, y) \in E$, and we identify E with the set of unordered pairs $\{\{x, y\} : xEy\}$, which we call the *edges* of G. If xEy we say that x, y are *adjacent*.

Occasionally we will also consider *graphs with* (possible) *loops*, those in which $(x, x) \in E$ is allowed, for some $x \in X$, and *directed graphs*, those in which $E \subseteq X^2$ is not necessarily symmetric.

A *path* in G is a sequence $x_0, x_1, \ldots, x_n$, $n \geq 1$, of distinct vertices such that $x_i E x_{i+1}$, $0 \leq i < n$. The *length* of such a path is the number of edges it uses, so the length of $x_0, x_1, \ldots, x_n$ is n. Such a path is a *cycle* if $n \geq 2$ and, moreover, $x_n E x_0$; its length is $n + 1$.

We denote by E^* the smallest equivalence relation containing E. Its equivalence classes are the *connected components* of G, and thus two vertices x, y are *connected* exactly when $x = y$ or there is a path $x = x_0, x_1, \ldots, x_n = y$. If x, y are connected, we set $\rho_G(x, y)$ equal to the length of the shortest path from x to y, and call it the (G)-*distance* from x to y. The graph is *connected* if it has a unique connected component.

A graph is *acyclic* if it contains no cycles; we sometimes call such graphs *forests* and their connected components *trees*. One easily sees that if x, y are distinct vertices of a tree, then there is a unique path from x to y.

The *girth* of G, in symbols $g(G)$, is the length of the smallest cycle in G. By convention, we set $g(G) = \infty$ when G is a forest. The *odd girth* of G,

$g_{\text{odd}}(G)$, is the length of the smallest odd cycle in G. Again $g_{\text{odd}}(G) = \infty$ if there are no odd cycles.

The *degree* of a vertex x, denoted $d_G(x)$ or just $d(x)$, if there is no danger of confusion, is the cardinality of the set $E_x = \{y \in X : xEy\}$. We also let $\Delta(G) = \sup\{d_G(x) : x \in X\}$. If $\Delta(G) \leq \aleph_0$, we say that G is *locally countable*, and if $d_G(x) < \aleph_0$ for all x, we say that G is *locally finite*. If $\Delta(G) < \aleph_0$, we say that G has *bounded degree*.

Given $A \subseteq X$, we define the *induced subgraph* on A, written $G|A$, to be $(A, E \cap A^2)$. We say that $A \subseteq X$ is *independent* for G if $G|A$ is trivial, i.e., no two vertices in A are adjacent.

A graph $G = (X, E)$ is *bipartite* if there is a partition $X = X_1 \sqcup X_2$, with each X_i $(i = 1, 2)$ independent. It is well known that a graph is bipartite if it has no odd length cycles. Any acyclic graph is therefore bipartite.

The *chromatic number* of a graph G, in symbols $\chi(G)$, is the smallest cardinality of a set Y for which there is a map $c : X \to Y$ (a vertex coloring) such that $xEy \Rightarrow c(x) \neq c(y)$. Thus the graph is bipartite iff $\chi(G) \leq 2$.

(**B**) Let now X be a standard Borel space. By a *measure* on X we mean a finite Borel measure. If μ is a measure on X with $0 < \mu(X) < \infty$ we call the pair (X, μ) a *standard measure space*. If $\mu(X) = 1$, we call (X, μ) a *standard probability space* and μ a *probability measure*. *Unless otherwise indicated or clear from context (e.g., when X is finite), measures will be assumed to be non-atomic.*

(**C**) When $G = (X, E)$ is a graph and X a standard Borel space, we say that G is *Borel* if $E \subseteq X^2$ is Borel. In this situation we have that E^* is an analytic equivalence relation, but if we assume in addition that G is locally countable, then E^* is a countable (i.e., having all its classes countable) Borel equivalence relation. We will be primarily interested in locally countable, Borel graphs, and will thus borrow from the theory of countable Borel equivalence relations. For more details see, e.g., [25].

A countable Borel equivalence relation R on a standard measure space (X, μ) is *measure preserving*, abbreviated *m.p.*, if whenever $f : X \to X$ is a Borel automorphism with graph contained in R, $f_*\mu = \mu$ (where, as usual, $f_*\mu(A) = \mu(f^{-1}(A))$). For $A \subseteq X$, we define the *R-saturation* of A, written $[A]_R$, to be the set $\{x \in X : \exists y \in A(xRy)\}$. We say that R is *ergodic* (relative to μ) if for all Borel $A \subseteq X$, $[A]_R$ is either μ-null or μ-conull. When R can be written as an increasing union of finite (i.e., having all its classes finite) Borel equivalence relations on X we call it *hyperfinite*, and when R admits a

Borel transversal (i.e., a set meeting each R-class in exactly one point), we say it is *smooth*. Similarly, we call R *μ-hyperfinite* (resp., *μ-smooth*) if its restriction to a conull R-invariant Borel set is hyperfinite (resp., smooth). We say $G = (X, E)$ is *m.p.* if E^* is; likewise we say that G is *ergodic*, *hyperfinite*, or *smooth*, if E^* is.

2 Chromatic and independence numbers

(**A**) A *coloring* of a locally countable, Borel graph $G = (X, E)$ on a standard Borel space X is a map $c : X \to Y$, where Y is a standard Borel space, such that

$$xEy \Rightarrow c(x) \neq c(y),$$

i.e., $\forall y \in Y(c^{-1}(\{y\})$ is independent). The *chromatic number* of G, in symbols

$$\chi(G),$$

is the smallest cardinality of a space Y as above for which there is a coloring $c : X \to Y$. Clearly $\chi(G) \in \{1, 2, \dots, n, \dots, \aleph_0\}$ (and $\chi(G) \geq 2$ unless $E = \emptyset$).

(**B**) If the coloring c as in (**A**) is Borel as a map from X into Y, we call c a *Borel coloring*. We define the *Borel chromatic number* of G, in symbols

$$\chi_B(G),$$

to be the smallest cardinality of a standard Borel space Y for which there is a Borel coloring $c : X \to Y$. Clearly $\chi_B(G) \in \{1, 2, 3, \dots, n, \dots, \aleph_0, 2^{\aleph_0}\}$ and

$$\chi(G) \leq \chi_B(G).$$

Example 2.1. In [26] and Miller [39], various examples of non-smooth $G = (X, E)$ are discussed, all of which are acyclic (so $\chi(G) = 2$), but $\chi_B(G)$ ranges over all values in $\{2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$. It follows that for any $m, n \in \{2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ with $m \leq n$ and $m < 2^{\aleph_0}$, there is such a G with $\chi(G) = m$ and $\chi_B(G) = n$ (just add a single connected component of chromatic number m to a graph G that has $\chi(G) = 2$ and $\chi_B(G) = n$).

(**C**) Suppose now that (X, μ) is a standard measure space (perhaps with atoms) and $G = (X, E)$ is a locally countable, Borel graph on X. We define the (μ-)*measurable chromatic number* of G, in symbols

$$\chi_\mu(G),$$

as the smallest cardinality of a standard Borel space Y for which there is a (μ-)measurable coloring $c : X \to Y$. Again,

$$\chi(G) \leq \chi_\mu(G) \leq \chi_B(G).$$

Example 2.2. a) There is an acyclic, locally countable, Borel graph G on a standard measure space (X, μ) with G m.p., ergodic and

$$2 = \chi(G) < \chi_\mu(G) = \chi_B(G) = 2^{\aleph_0}.$$

To see this, take a compact, metrizable group X that contains a dense subset $\{a_n\}_{n\in\mathbb{N}}$ which generates freely a free subgroup (e.g., $X = \mathrm{SO}_3(\mathbb{R})$) and let μ be the Haar measure on X. Consider the graph $G = (X, E)$, where $xEy \Leftrightarrow \exists n(x = a_n^{\pm 1} y)$. Then G is acyclic and m.p., ergodic. So $\chi(G) = 2$ but if $c : X \to \mathbb{N}$ is a measurable coloring, then for some n, $Y = c^{-1}(\{n\})$ has positive measure, so YY^{-1} contains an open neighborhood of 1. Then there are $x, y \in Y$, $k \in \mathbb{N}$ with $xy^{-1} = a_k$, thus xEy, a contradiction.

b) Another family of examples that have the above properties are the following: Take an infinite countable group Γ and an infinite set of generators S. Let a be a free, mixing, measure preserving action of Γ on (X, μ) and let $G(S, a)$ be the associated graph. Then, by the mixing property, if $A \subseteq X$ has positive measure, then for some $s \in S$, $s \cdot A \cap A$ has also positive measure, thus A is not independent and therefore $\chi_\mu(G(S, a)) = 2^{\aleph_0}$. If we take Γ to be the free subgroup on infinitely many (free) generators S, then $G(S, a)$ is also acyclic.

Example 2.3. There is an acyclic, locally countable, Borel graph G on a standard measure space (X, μ) with G m.p., ergodic and

$$2 = \chi(G) < 3 = \chi_\mu(G) < \chi_B(G) = 2^{\aleph_0}.$$

Take, for instance, the graph G_0 on $X = 2^{\mathbb{N}}$ defined in [26], where it is shown that G_0 is acyclic (thus $\chi(G_0) = 2$), but $\chi_B(G_0) = 2^{\aleph_0}$. Miller [39] showed that $\chi_\mu(G_0) = 3$, where μ is the usual product measure on $2^{\mathbb{N}}$, for which G_0 is m.p., ergodic.

Example 2.4. It is easy to construct an example of a locally countable, Borel graph G on a standard measure space (X, μ) with G m.p., ergodic, for which $\chi(G) = \chi_\mu(G) = \chi_B(G) = 2$. Take a countable, measure-preserving, ergodic equivalence relation R on (X, μ) and let $X = A \sqcup B$ be a Borel partition of X with A, B meeting each R-class. Let E be the bipartite graph with edges between all pairs of R-related points, one in A and the other in B. Clearly $\chi(G) = \chi_\mu(G) = \chi_B(G) = 2$ and G generates R.

Example 2.5. We will see in Section 4 examples of acyclic, bounded degree, Borel graphs $G = (X, E)$ on standard (X, μ) which are m.p., ergodic, and $\chi_\mu(G)$ is finite but arbitrarily large (although of course $\chi(G) = 2$). From this it follows that for each $2 \leq n < \aleph_0$, there is an acyclic, bounded degree $G = (X, E)$ on a standard measure space (X, μ) with G m.p., ergodic such that $\chi_\mu(G) = n$. To go from such a G that has $\chi_\mu(G) = k + 1 > 3$ to a $\overline{G}, \overline{\mu}$ that has $\chi_{\overline{\mu}}(\overline{G}) = k$, take a partition $A_0 \sqcup \cdots \sqcup A_k = X$ given by a measurable coloring of G, and assume without loss of generality that $\mu(A_0) < 1$. Let $X' = A_1 \sqcup \cdots \sqcup A_k$, $G' = G|X' = (X', E')$ be the induced subgraph on X' and let $\mu' = \mu|X'$. Clearly G' is acyclic, $\chi_{\mu'}(G') = k$, and G' is m.p. (for μ'). Consider then the ergodic decomposition associated with $(E')^*$ (on X'). If all the pieces of this decomposition have (for the induced subgraph G') measurable chromatic number $\leq k-1$ then, by measurable selection, we can find a μ'-conull set on which G' admits a $k-1$ measurable coloring, and since G' has chromatic number ≤ 2, it follows that $\chi_\mu(G') \leq k-1$, a contradiction. So there is a piece of the ergodic decomposition $(\overline{X}, \overline{\mu})$, so that if $\overline{G} = G'|\overline{X}$ is the induced subgraph, then $\overline{G}$ is acyclic, $\chi_{\overline{\mu}}(\overline{G}) = k$, and $\overline{G}$ is m.p., ergodic. Finally, $\overline{\mu}$ is non-atomic, else $\overline{X}$ would have to be finite, and so $\overline{G}$ would have chromatic (and so $\overline{\mu}$-measurable chromatic) number at most $2 < k$, a contradiction.

An analogous argument produces, for each $2 \leq n < \aleph_0$, an acyclic, bounded degree, Borel graph G with $\chi_B(G) = n$, answering a question in [39].

Example 2.6. There is an example of a locally countable, Borel graph G on a standard measure space (X, μ) with G invariant, ergodic for which $\chi(G) \leq 3$ and $\chi_\mu(G) = \aleph_0$. To see this, take, for each n, as in Example 2.5, an acyclic, locally countable, Borel graph $G_n = (X_n, E_n)$ on a standard probability space (X_n, μ_n) with G_n invariant, ergodic, and $\chi_{\mu_n}(G_n) > n$. Fix a standard probability space (X, μ) and a Borel partition $X = \bigsqcup_{n=1}^\infty A_n$ with $\mu(A_n) = 1/2^n$. Fix a Borel bijection $\varphi_n : X_n \to A_n$ sending μ_n to

$\mu_{A_n} = \frac{\mu|A_n}{\mu(A_n)}$ and let $G'_n = (A_n, E'_n)$ be the image of G_n under this bijection. Find, for each n, two disjoint, Borel sets $C_n, D_n \subseteq A_n$ of positive measure which are independent for G'_n, and such that there are measure-preserving Borel isomorphisms $\omega_n : C_n \to C_{n+1}$, for $n \geq 1$, n odd, and $\psi_n : D_n \to D_{n+1}$ for $n \geq 2$, n even. Let $G = (X, E)$ be the graph on X whose edges are those in $\bigcup_n E'_n$ together with the graphs of ω_n, ψ_n, and their inverses. Then G is m.p., ergodic and $\chi_\mu(G) = \aleph_0$. Finally, $\chi(G) \leq 3$. Indeed, fix the same colors a, b witnessing the 2-colorability of each G'_n. Then change the color of each element of $\bigcup_{n,\text{ odd}} C_n \cup \bigcup_{n,\text{ even}} D_n$ to some third color c. This gives a 3-coloring of G.

We do not know an example of G, X, μ as above for which G is acyclic (or even has $\chi(G) = 2$) but $\chi_\mu(G) = \aleph_0$. (**Addendum.** Recently Conley-Miller [8] constructed an example of an acyclic such G with $\chi_B(G) = \chi_\mu(G) = \aleph_0$.) More generally, we do not know what are the possible values of $k, l, m \in \{2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ with $k \leq l \leq m$ such that there is a locally countable, Borel graph G on a standard measure space (X, μ) which is m.p., ergodic, and

$$\chi(G) = k, \chi_\mu(G) = l, \chi_B(G) = m.$$

Remark 2.7. One can also define the (μ-)*almost everywhere measurable chromatic number* of G, in symbols $\chi_\mu^{\text{ae}}(G)$, as the smallest cardinality of a standard Borel space Y for which there is a Borel set $A \subseteq X$ with $\mu(A) = 1$ and a Borel coloring $c : A \to Y$ of the induced subgraph $G|A = (A, E \cap A^2)$. Clearly, $\chi_\mu^{\text{ae}}(G) \leq \chi_\mu(G)$. However, if $\chi_\mu^{\text{ae}}(G) \geq \chi(G)$, which will be the case for most graphs that we will be interested in, then $\chi_\mu^{\text{ae}}(G) = \chi_\mu(G)$.

(**D**) Finally, when (X, μ) is a standard measure space and G is a locally countable, Borel graph on X, we define the *approximate* (μ-)*measurable chromatic number* of G, in symbols

$$\chi_\mu^{\text{ap}}(G),$$

to be the smallest cardinality of a standard Borel space Y such that for every $\varepsilon > 0$ there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a measurable coloring $c : A \to Y$ of the induced subgraph $G|A$. Again,

$$\chi_\mu^{\text{ap}}(G) \leq \chi_\mu(G),$$

but clearly $\chi(G) \leq \chi^{\text{ap}}_\mu(G)$ may fail, since $\chi(G)$ can be altered arbitrarily on a single connected component without affecting $\chi^{\text{ap}}_\mu(G)$. On the other hand, let

$$\chi^*(G)$$

be the minimum of all $\chi(G|A)$, where A is an E^*-invariant Borel set of measure 1. Clearly $\chi^*(G) \leq \chi(G)$. Then it is easy to see that if G is m.p., then

$$\chi^*(G) \leq \chi^{\text{ap}}_\mu(G).$$

This is clear if $\chi^{\text{ap}}_\mu(G) \geq \aleph_0$, so assume that $\chi^{\text{ap}}_\mu(G) = k < \aleph_0$. Let Y_n be a Borel set of measure at least $1 - 2^{-n}$ such that $G|Y_n$ is k-colorable. Let $Z_n = \bigcap_{m \geq n} Y_m$, so that $Z_n \subseteq Z_{n+1}$ and $\mu(Z_n) \geq 1 - \sum_{m \geq n} 2^{-m} \to 1$, as $n \to \infty$. Then $Z = \bigcup_n Z_n$ has measure 1, thus contains an E^*-invariant Borel set $W \subseteq Z$ of measure 1. To show that $G|W$ is k-colorable, it suffices to show that $G|F$ is k-colorable for every finite $F \subseteq W$, but this is clear since there must exist some n such that $F \subseteq Z_n$.

Example 2.8. There is an acyclic, bounded degree, Borel graph $G = (X, E)$ on a standard measure space (X, μ) which is m.p., ergodic, and $\chi^{\text{ap}}_\mu(G) < \chi_\mu(G)$. For instance, consider the shift S on $2^{\mathbb{Z}}$ and let $X \subseteq 2^{\mathbb{Z}}$ be its aperiodic part. Let μ be the restriction of the usual product measure to X (note that $\mu(X) = 1$). Let for $x, y \in X$, $xEy \Leftrightarrow x = S^{\pm 1}(y)$. Then by Rokhlin's Lemma (see, e.g., [25] 7, 7.5), $\chi^{\text{ap}}_\mu(G) = 2$. On the other hand, $\chi_\mu(G) = 3$. Otherwise, there is a measurable partition $X = A \sqcup B$ into independent sets. Then $S(A) = B$ and $S(B) = A$, and so $\mu(A) = \mu(B) = 1/2$ and both A, B are S^2-invariant, which is impossible as S^2 is ergodic.

We have seen in Example 2.2 examples of acyclic, locally countable, Borel G on standard (X, μ) which are m.p., ergodic and have no independent sets of positive measure, thus $\chi^{\text{ap}}_\mu(G) = 2^{\aleph_0}$. It is also easy to see that there is no such G, X, μ with $\chi^{\text{ap}}_\mu(G) = 1$ (i.e., there cannot exist Borel independent sets whose measure is arbitrarily close to 1). Indeed, if $G = (X, E)$ is such that G is m.p., ergodic, then by the uniformization theorem for Borel sets with countable sections, there is a measure-preserving Borel bijection $\varphi : A \to B$ between Borel sets of positive measure such that $(x, \varphi(x)) \in E$, for all $x \in A$. If $\mu(A) = \mu(B) = \delta$ and $\varepsilon < \delta/2$, there can be no Borel independent set of measure bigger than $1 - \varepsilon$.

Example 2.9. There is an example of G,X,μ as in Example 2.6 with $\chi(G) \leq 3$ and $\chi^{\text{ap}}_\mu(G) = \aleph_0$. The graphs $G = (X, E)$ on (X, μ) from Section 4 (mentioned previously in Example 2.5) which have arbitrarily large finite χ_μ actually have arbitrary large finite $\chi_\mu^{\rm ap}$. Then, as in Example 2.6, this gives examples of G with $\chi(G) \leq 3$ and $\chi_\mu^{\rm ap}(G) = \aleph_0$.

Again, we do not know examples of such G with $\chi(G) = 2$ and $\chi_\mu^{\rm ap}(G) = \aleph_0$. Also, we do not know if there are such examples for which $\chi_\mu^{\rm ap}(G)$ takes an *arbitrary* value $3 \leq k < \aleph_0$.

The more general problem is again whether there is any other relationship between $\chi(G)$, $\chi^*(G)$, $\chi_\mu(G)$, $\chi_\mu^{\rm ap}(G)$, beyond the obvious $\chi(G) \leq \chi_\mu(G)$, $\chi^*(G) \leq \chi_\mu^{\rm ap}(G) \leq \chi_\mu(G)$ for locally countable (or locally finite), Borel graphs G on standard measure spaces (X, μ) which are m.p., ergodic.

(**E**) Let finally G be a locally countable, Borel graph on a standard probability space (X, μ). We define the *independence number* of G, in symbols

$$i_\mu(G),$$

by

$$i_\mu(G) = \sup\{\mu(Y) : Y \subseteq X \text{ is a Borel independent set}\}.$$

Clearly, we can replace "Borel" by "(μ-)measurable" in this definition. (If μ is not a probability measure we replace $\mu(Y)$ by $\mu(Y)/\mu(X)$ in the definition above.)

Obviously, $0 \leq i_\mu(G) \leq 1$ and $i_\mu(G) = 0$ means that there is no positive measure independent set. We have seen in 2.2 examples of such graphs (they clearly have $\chi_\mu(G) = 2^{\aleph_0}$). If $G = (X, E)$ is a locally countable, Borel graph on a standard probability space (X, μ) with G m.p., ergodic, then we have seen in (**D**) that $i_\mu(G) < 1$ (otherwise $\chi_\mu^{\rm ap}(G) = 1$).

Example 2.10. For each $0 < a < 1$ there is an acyclic, locally countable Borel graph G on a standard probability space (X, μ) which is m.p., ergodic and $i_\mu(G) = a$ with the supremum being attained.

To see this, first fix an acyclic $G_1 = (X_1, E_1)$ on (X_1, μ_1) with G_1 invariant, ergodic, $\mu_1(X_1) = 1 - a$, and $i_{\mu_1}(G_1) = 0$. Also fix $k > \frac{a}{1-a}$. Let X_2 be an uncountable standard Borel space disjoint from X_1 and partition it into k uncountable Borel sets: $X_2 = A_1 \sqcup \cdots \sqcup A_k$. Fix a Borel subset Y_1 of X_1, meeting each E_1^*-class, such that $\mu_1(Y_1) = \frac{a}{k} < 1 - a$. For each $1 \leq i \leq k$, let $f_i : Y_1 \to A_i$ be a Borel bijection. Use f_i to copy the measure $\mu_1 | Y_1$ to A_i, say ν_i, and let $\mu_2 = \sum_{i=1}^n \nu_i$. Then $\mu_2(X_2) = k \cdot \frac{a}{k} = a$. Let $X = X_1 \sqcup X_2$, $\mu = \mu_1 + \mu_2$. Define the graph $G = (X, E)$ as follows: the edges of G are those in E_1 together with the graph of each f_i and its inverse. Clearly it is acyclic and it is easy to see that G is m.p., ergodic. Finally, X_2 is independent for G and if $A \subseteq X$ is Borel independent, then clearly $\mu(A \cap X_1) = 0$, so $\mu(A) \leq \mu(X_2) = a$. So $i_\mu(G) = a$ and the sup is attained.

Remark 2.11. When X is a finite set and μ is normalized counting measure, $i_\mu(G) = i(G)$ is usually called the *independence ratio* and $\alpha(G) =$ (the maximum cardinality of an independent subset of X) is called the independence number (thus $i(G) = \frac{\alpha(G)}{|X|}$). We will not use $\alpha(G)$ in this paper, so this should not cause any confusion.

We now have the following simple inequality:

Proposition 2.12. *Let G be a locally countable Borel graph on a standard probability space (X, μ) (perhaps with atoms). Then*

$$\chi_\mu^{\mathrm{ap}}(G) \geq \frac{1}{i_\mu(G)}.$$

Proof. This is clear if $i_\mu(G) = 0$. If $\chi_\mu^{\mathrm{ap}}(G) = k \in \mathbb{N}$, fix $\varepsilon > 0$ and independent, pairwise disjoint, Borel sets $A_1, \dots, A_k$ with $\mu\left(\bigcup_{i=1}^k A_i\right) > 1 - \varepsilon$. Then

$$k \cdot i_\mu(G) \geq \mu\left(\bigcup_{i=1}^k A_i\right) > 1 - \varepsilon,$$

so $k > \frac{1-\varepsilon}{i_\mu(G)}$, and we are done. □

(**F**) We will often be interested in locally finite graphs and, in particular, those of bounded degree, where recall that G has *bounded degree* if

$$\Delta(G) = \sup\{d_G(x) : x \in X\} < \aleph_0.$$

Proposition 2.13 ([26] 4.5, 4.6)**.** *If G is a locally finite, Borel graph, then $\chi_B(G) \leq \aleph_0$. If G is of bounded degree, then $\chi_B(G) \leq \Delta(G) + 1$.*

Corollary 2.14. *Let (X, μ) be a standard probability space and $G = (X, E)$ a locally finite, Borel graph. Then $i_\mu(G) > 0$.*

Example 2.15. For each $0 < a < 1$, there is a bounded degree $G = (X, E)$ on a standard probability space (X, μ) which is m.p., ergodic, such that $i_\mu(G) = a$ and the supremum is attained.

To see this, let S be the shift on $2^{\mathbb{Z}}$ and let $X_1 \subset 2^{\mathbb{Z}}$ be its aperiodic part. Let μ_1 be the restriction to X_1 of the product measure on $2^{\mathbb{Z}}$. Let E_1 be the union of the graph of $S|X_1$ and its inverse. This is invariant, ergodic on (X_1, μ_1). Let $n > 3$ be such that $a \in \left[\frac{1}{n}, \frac{n-1}{n}\right]$, so that $a = \frac{1}{n}\alpha + \frac{n-1}{n}\beta$ for some $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Let $A \sqcup B = X_1$ be a Borel partition with $\mu_1(A) = \alpha$, $\mu_1(B) = \beta$. Let $X = X_1 \times \{1, \ldots, n\}$ and give X the product measure $\mu = \mu_1 \times \nu$, where ν is the normalized counting measure on $\{1, \ldots, n\}$. Let $G = (X, E)$ be the following graph on X:

$$\begin{aligned} E &= \{((x,1),(y,1)) : xE_1y\} \cup \\ &\quad \{((x,i),(x,j)) : x \in A, 1 \leq i \neq j \leq n\} \cup \\ &\quad \{((x,1),(x,j)) : x \in B, 2 \leq j \leq n\} \cup \\ &\quad \{((x,j),(x,1)) : x \in B, 2 \leq j \leq n\}. \end{aligned}$$

Clearly, $d(G) \leq n+1$. It is easy to see that $(x,i)E^*(y,j) \Leftrightarrow xE_1^*y$, and thus G is m.p., ergodic. We claim that $i_\mu(G) = a$ and the supremum is attained. First note that if $Y \subseteq X$ is independent, then for each $x \in A$ there is at most one $1 \leq i \leq n$ with $(x,i) \in Y$ and for each $x \in B$ there are at most $n-1$ many $1 \leq j \leq n$ with $(x,j) \in Y$. Thus $\mu(Y) \leq \frac{1}{n}\mu(A) + \frac{n-1}{n}\mu(B) = a$. On the other hand, $Y = \{(x,2) : x \in A\} \cup \{(x,j) : x \in B \text{ and } j \in \{2, \ldots, n\}\}$ is independent and $\mu(Y) = \frac{1}{n}\mu(A) + \frac{n-1}{n}\mu(B) = a$, so $i_\mu(G) = a$ and the supremum is attained.

We do not know however if examples as in 2.15 with arbitrary $i_\mu(G) = a \in (0,1)$ can be found which are acyclic, even if we replace "bounded degree" by "locally finite." The referee pointed out that for any given integer $d > 2$, there is an upper bound $f(d) = \frac{d}{d+1} < 1$ for the independence number $i_\mu(G)$ of every m.p., ergodic G with $\Delta(G) \leq d$. This is because if A is independent and B is its complement, then the measure of A is bounded by the integral of $d(x)$ over B. We will see in Section 3 that actually for $d = 2$, $f(2) = 1/2$ works. Note that in the examples of 2.15, to achieve $i_\mu(G) = a < 1$, we needed a graph G of degree $n+1$, where $n \geq \frac{1}{1-a}$.

We now prove a strengthening of 2.13 with applications in computing bounds for approximate chromatic numbers (and thus independence numbers). Recall that we say a graph is *aperiodic* if all of its connected components are infinite.

Proposition 2.16. *Suppose that G is an aperiodic Borel graph on X with degree bounded by d. Then there is a decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of subsets of X with $\bigcap_n A_n = \emptyset$ and Borel $(d+1)$-colorings $c_n : X \to \{0, 1, \ldots, d\}$ of G with $c_n^{-1}(0) \subseteq A_n$.*

Proof. Recall that a *complete section* or *marker set* for an equivalence relation is one meeting each equivalence class (see, e.g., [25] 6.7). We now say that a subset A of X is a *strong marker set* for G if it meets every connected component, is independent, and there is some natural number k such that every point of X is connected to a point in A via a path in G of length less than k. The next lemma is a variation of Lemma 3.14 in [20].

Lemma 2.17. *Suppose that G is a locally finite, aperiodic Borel graph on X. Then there is a decreasing sequence of Borel strong marker sets $A_1 \supseteq A_2 \supseteq \cdots$ with $\bigcap_n A_n = \emptyset$.*

Proof. We let B_1 be a maximal independent Borel set (see [26] 4.2, 4.5). We then let B_2 be a maximal Borel subset of B_1 subject to the constraint that no two points of B_2 are within distance two in the graph metric of G (the existence of such B_2 follows from the fact that the distance two graph is locally finite). We continue in this fashion, letting B_{n+1} be a maximal Borel subset of B_n with no two points within distance $n+1$.

We claim that every point of X is within distance n^2 of B_n. This is a simple induction. If every point x is within distance n^2 of B_n, then, by maximality of B_{n+1}, every point of B_n is within distance $n+1$ of B_{n+1} and thus x is within distance $n^2 + n + 1 < (n+1)^2$ of B_n.

It may not be the case that $\bigcap_n B_n = \emptyset$, but this intersection meets each connected component of G it at most one point. Since G is aperiodic, each set $A_n = B_n \setminus \bigcap_n B_n$ is still a strong marker set. The sequence (A_n) is as desired. □

Lemma 2.18. *Suppose that G is a Borel graph on X and that $A \subseteq X$ is a Borel set such that every point in $X \setminus A$ has degree less than d. Then any Borel d-coloring $c : A \to d$ of $G|A$ may be extended to a Borel d-coloring $c' : X \to d$ of G.*

Proof. Partition $X \setminus A = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_d$ into Borel G-independent sets. Extend c to $c_1 : A \cup B_1 \to d$ by following a greedy algorithm, i.e., for each $x \in B_1$ set $c_1(x)$ to be the least color not used by a neighbor of x. Similarly extend to $B_2, \ldots, B_d$. □

To finish the proof of the proposition, we take the vanishing sequence of strong markers $A_1 \supseteq A_2 \supseteq \cdots$ granted by Lemma 2.17 as the sequence in the statement of the lemma, and describe how to Borel $(d+1)$-color G with one color contained in A_n. Fix k such that every point of X is within distance k of A_n, and partition X into $X_0 = A_n \sqcup X_1 \sqcup \cdots \sqcup X_k$, where $X_i = \{x \in X : \rho_G(x, A_n) = i\}$.

Now $G|X_k$ has degree bounded by $d-1$ (since everything is connected to at least one point in X_{k-1}), and thus admits a Borel d-coloring. In the graph $G|(X_k \cup X_{k-1})$, points in X_{k-1} have degree bounded by $d-1$ and so Lemma 2.18 allows us to extend the d-coloring on X_k to one on $X_k \cup X_{k-1}$. Continuing in this fashion, we obtain a Borel coloring of $X \setminus A_n$ with d colors. Using the remaining color on A_n itself completes the proof. $\square$

And now, for the promised application, an analogue of Brooks' theorem. Recall that the *clique number* of a graph, $\mathrm{clq}(G)$, is the largest cardinality of a complete (induced) subgraph of G.

Theorem 2.19. *Let (X, μ) be a standard probability space, $G = (X, E)$ a Borel graph on X with degree bounded by d, where $d \geq 3$. Suppose further that $\mathrm{clq}(G) \leq d$. Then $\chi_\mu^{\mathrm{ap}}(G) \leq d$ and thus $i_\mu(G) \geq 1/d$.*

Proof. Brooks' theorem allows us to d-color the finite components of G in a Borel fashion: indeed, whenever $G = (X, E)$ is a Borel graph with finite connected components, then $\chi(G) = \chi_B(G)$. To see this, simply choose a Borel transversal T of E^*. Since each $x \in T$ sees only finitely many ways (but at least one) of coloring its connected component using the colors $\{1, 2, \ldots, \chi(G)\}$, the required coloring is granted by the uniformization theorem for Borel sets with countable sections.

So, it remains to handle the infinite connected components of G. Fix $\varepsilon > 0$. Since the sequence $A_1 \supseteq A_2 \supseteq \cdots$ granted by Proposition 2.16 vanishes, we may fix n such that $\mu(A_n) < \varepsilon$. But then 2.16 provides us with the required d-coloring of $X \setminus A_n$. $\square$

The analogy extends also to the case $d = 2$:

Theorem 2.20. *Let (X, μ) be a standard probability space, $G = (X, E)$ a bipartite Borel graph on X with degree bounded by 2. Then $\chi_\mu^{\mathrm{ap}}(G) \leq 2$ and thus $i_\mu(G) \geq 1/2$.*

Proof. Fix $\varepsilon > 0$. Then by the marker lemma (see, e.g., [25] 6.7) there is a Borel set A with $\mu(A) < \varepsilon$ meeting every infinite E^*-class. The connected components of $G|(X \setminus A)$ are either finite or infinite with a single endpoint of degree one. This easily implies that there is a Borel 2-coloring of $G|(X \setminus A)$. □

In some cases, the conclusion of Theorem 2.19 can be slightly improved.

Proposition 2.21. *Let (X, μ) be a standard probability space, $G = (X, E)$ a bipartite Borel graph on X with degree bounded by d, where $d \geq 3$. Then there is an independent Borel set of measure $\geq 1/d$.*

Proof. Let A be a maximal independent Borel set. If $\mu(A) \geq 1/d$ we are done. Else, $1 - \mu(A) > 1 - 1/d$ and so we can choose $\varepsilon > 0$ small enough so that $(1 - \mu(A)) \left(\frac{1}{d-1} - \varepsilon\right) \geq \frac{1}{d}$. Since $G|(X \setminus A)$ has degree bounded by $d - 1 \geq 2$, by 2.19 or 2.20 there is an independent subset of $X \setminus A$ with measure at least

$$\mu(X \setminus A) \left(\frac{1}{d-1} - \varepsilon \right),$$

so of measure at least $1/d$. □

Proposition 2.22. *Let (X, μ) be a standard probability space, $G = (X, E)$ a Borel graph on X with degree bounded by d, where $d \geq 4$. Suppose further that* $\mathrm{clq}(G) \leq d - 1$. *Then there is an independent Borel set of measure $\geq 1/d$.*

Proof. Essentially the same argument as 2.21, noting that the assumption of small clique number allows us to always apply 2.19 in finding the large independent subset of $X \setminus A$. □

3 Hyperfinite graphs

(**A**) Recall that a countable Borel equivalence relation R on a standard Borel space X is called *hyperfinite* if it can be written as an increasing union $\bigcup_{n=1}^{\infty} F_n$, with each F_n a finite Borel equivalence relation. If instead R is on a standard measure space (X, μ), we say that R is *μ-hyperfinite* if there is a conull Borel set $A \subseteq X$ such that $R|A$ is hyperfinite. By Connes-Feldman-Weiss (see, e.g., [25], 10.1), measure-preserving actions of amenable groups give rise to μ-hyperfinite orbit equivalence relations. We will examine such actions in Section 4.

(**B**) We say that a locally countable, Borel graph $G = (X, E)$ on a standard measure space (X, μ) is *μ-hyperfinite* if the equivalence relation E^* is μ-hyperfinite. In Miller [39] it is shown that if G is μ-hyperfinite and acyclic, then $\chi_\mu(G) \leq 3$. A slight modification of these techniques allows us to compute $\chi_\mu^{\text{ap}}(G)$ for such graphs.

We say that a locally countable, Borel graph $G = (X, E)$ is *smooth* if E^* admits a Borel selector. Such a graph G is *directable* if there exists a Borel function $f : X \to X$ such that $xEy \Leftrightarrow y = f(x)$ or $x = f(y)$. Finally, such a graph is *essentially linear* if there is a Borel set $B \subseteq X$ such that every connected component of G contains exactly one connected component of $G|B$ and, moreover, $G|B$ is an acyclic graph which is regular of degree two (i.e., it is a forest of lines).

Theorem 3.1. *Let G be a locally countable, acyclic, μ-hyperfinite, Borel graph on a standard probability space (X, μ). Then $\chi_\mu^{\text{ap}}(G) \leq 2$, and thus $i_\mu(G) \geq 1/2$.*

Proof. Following [39] 3.1 and [20] 3.19, we may find pairwise disjoint, E^*-invariant Borel sets X_0, X_1, X_2 such that $\mu(X_0 \cup X_1 \cup X_2) = 1$, $G|X_0$ is smooth, $G|X_1$ is directable, and $G|X_2$ is essentially linear. We handle these three parts separately.

Fix a Borel transversal A of $E^*|X_0$, and color each point $x \in X_0$ by the parity of $\rho_G(x, A)$. Thus, $\chi_B(G|X_0) \leq 2$, and consequently $\chi_\mu^{\text{ap}}(G|X_0) \leq 2$.

We next handle X_2. Fix $\varepsilon > 0$ and a Borel set B witnessing the essential linearity of $G|X_2$. By Theorem 2.20, we may find a Borel partition $B = B_0 \sqcup B_1 \sqcup B_2$, with $\mu(B_2) < \varepsilon$ and B_0,B_1 forming a 2-coloring of $G|(B \setminus B_2)$ (if $\mu(B) = 0$ we may take $B_2 = B$). We may extend this to a 2-coloring of $G|(X_2 \setminus B_2)$ in the obvious way: for each $x \in X_2$ set $b(x)$ to be the closest element of B to x, then color x by the parity of $\rho_G(x, b(x))$ if $b(x) \in B_0$ and by the parity of $\rho_G(x, b(x)) + 1$ if $b(x) \in B_1 \cup B_2$. Thus, $\chi_\mu^{\text{ap}}(G|X_2) \leq 2$.

Finally, we handle X_1. Fix $\varepsilon > 0$ and a Borel function $f : X_1 \to X_1$ witnessing that $G|X_1$ is directable. Define a partial order on X_1 by $x \leq y \Leftrightarrow \exists n(y = f^n(x))$. The following generalization of the marker lemma ensures that we may find small sets cofinal in this partial order. For a relation R on X, we say that $A \subseteq X$ is an *R-complete section* if A meets every vertical section of R, i.e., for all x in X, $\exists y \in A$ (xRy).

Lemma 3.2 (Miller)**.** *Suppose that X is a Polish space and R is a transitive, reflexive Borel binary relation on X whose vertical sections are all countably infinite. Then there are Borel R-complete sections $A_0 \supseteq A_1 \supseteq \cdots$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.*

Using the above lemma, we may find a Borel set $C \subseteq X_1$ with $\mu(C) < \varepsilon$ so that for all $x \in X_1$ there exists $y \in C$ with $x \leq y$. We may then color each $x \in X_1 \setminus C$ by the parity of the least n such that $f^n(x) \in C$, so $\chi_\mu^{\mathrm{ap}}(G|X_1) \leq 2$. □

Proof of Lemma 3.2. Fix an enumeration $B_0, B_1, \ldots$ of a countable family of Borel subsets of X which separates points, and for each $s \in 2^{<\mathbb{N}}$, define $B_s \subseteq X$ by

$$B_s = \left(\bigcap_{s(i)=0} X \setminus B_i \right) \cap \left(\bigcap_{s(i)=1} B_i \right).$$

For each $n \in \mathbb{N}$, define $S_n : X \to \mathcal{P}(2^n)$ by

$$S_n(x) = \{ s \in 2^n : \forall y \in R_x \ (|B_s \cap R_y| = \aleph_0) \}.$$

Sublemma 3.3. *Suppose that $x, y \in X$, $n \in \mathbb{N}$, $s \in 2^n$, and $i \in \{0, 1\}$. Then:*

(1) $(x, y) \in R \Rightarrow S_n(x) \subseteq S_n(y)$;

(2) $si \in S_{n+1}(x) \Rightarrow s \in S_n(x)$.

Proof. The first claim is a consequence of the transitivity of R, and the second is a trivial consequence of the definition of S_n. □

For each $s \in 2^n$, define $C_s \subseteq X$ by

$$C_s = \{ x \in X : \forall y \in R_x \ (s = \min_{\mathrm{lex}} S_n(y)) \},$$

and for each $n \in \mathbb{N}$, define $D_n \subseteq X$ by

$$D_n = \bigcup_{s \in 2^n} B_s \cap C_s.$$

We will show that the sets $D_0, D_1, \ldots$ are nearly as desired.

Sublemma 3.4. $\forall n \in \mathbb{N} \ (D_{n+1} \subseteq D_n)$.

Proof. Fix $n \in \mathbb{N}$ and suppose that $x \in D_{n+1}$. Then there exists $s \in 2^n$ and $i \in \{0,1\}$ such that $x \in B_{si} \cap C_{si}$. In particular, it follows that $x \in B_s$, so to see that $x \in D_n$, it is enough to show that $x \in C_s$. Suppose, towards a contradiction, that there exists $y \in R_x$ such that $s \neq t$, where $t = \min_{\mathrm{lex}} S_n(y)$. As $x \in C_{si}$, it follows that $si \in S_{n+1}(x)$. As (1) ensures that $S_n(x) \subseteq S_n(y)$ and (2) ensures that $s \in S_n(x)$, it follows that $s \in S_n(y)$, thus $t <_{\mathrm{lex}} s$. As $t0 <_{\mathrm{lex}} si$ and $si = \min_{\mathrm{lex}} S_{n+1}(y)$, it follows that $t0 \notin S_{n+1}(y)$, so there exists $z \in R_y$ such that $|B_{t0} \cap R_z| < \aleph_0$. Similarly, since $t1 <_{\mathrm{lex}} si$ and $si = \min_{\mathrm{lex}} S_{n+1}(z)$, it follows that $t1 \notin S_{n+1}(z)$, so there exists $w \in R_z$ such that $|B_{t1} \cap R_w| < \aleph_0$. As the transitivity of R ensures that $R_w \subseteq R_z$, this implies that $|B_t \cap R_w| < \aleph_0$. As the transitivity of R implies also that $(y,w) \in R$, this contradicts our assumption that $t \in S_n(y)$. □

While each D_n is an R-complete section, we will show something stronger:

Sublemma 3.5. *$\forall x \in X \, \forall n \in \mathbb{N} \, (|D_n \cap R_x| = \aleph_0)$.*

Proof. Fix an enumeration $\langle s_i \rangle_{i<2^n}$ of $\{0,1\}^n$. For each $x \in X$, set $x_0 = x$, and given x_i, let x_{i+1} be any element of R_{x_i} such that $\min_{\mathrm{lex}} S_n(x_{i+1}) \neq \min_{\mathrm{lex}} S_n(x_i)$, if such an element exists. Otherwise, set $x_{i+1} = x_i$. Let $y = x_{2^n}$ and $s = \min_{\mathrm{lex}} S_n(y)$, and observe that $\forall z \in R_y \; (s = \min_{\mathrm{lex}} S_n(z))$, thus $y \in C_s$. As $s \in S_n(y)$, it follows that $|B_s \cap R_y| = \aleph_0$, and since $y \in C_s$, it follows that $B_s \cap R_y = B_s \cap C_s \cap R_y$, thus $|B_s \cap C_s \cap R_y| = \aleph_0$. As $B_s \cap C_s \subseteq D_n$ and the transitivity of R ensures that $R_y \subseteq R_x$, it follows that $|D_n \cap R_x| = \aleph_0$. □

Unfortunately, it need not be the case that the set $D = \bigcap_{n \in \mathbb{N}} D_n$ is empty. However, this is not so far from the truth:

Sublemma 3.6. *$\forall x, y \in D \; (x \neq y \Rightarrow (x,y) \notin R)$.*

Proof. Suppose, towards a contradiction, that there are distinct points $x, y \in D$ such that $(x,y) \in R$. Fix $n \in \mathbb{N}$ and $s \in 2^n$ such that $x \in B_s$ and $y \notin B_s$. As $x \in D_n$, it follows that $\min_{\mathrm{lex}} S_n(x) = \min_{\mathrm{lex}} S_n(y) = s$, so $y \notin D_n$, thus $y \notin D$, the desired contradiction. □

Now define $A_n = D_n \setminus D$. Sublemma 3.4 implies that these sets are decreasing, and they clearly have empty intersection, so it only remains to check that each A_n is an R-complete section. Towards this end, fix $x \in X$, and observe that two applications of Sublemma 3.5 ensure that there are distinct points $y \in D_n \cap R_x$ and $z \in D_n \cap R_y$. Sublemma 3.6 then ensures that $y \notin A_n \Rightarrow y \in D \Rightarrow z \notin D \Rightarrow z \in A_n$, and the transitivity of R then implies that $A_n \cap R_x \neq \emptyset$. $\square$

It is natural to ask whether the assumption of acyclicity in Theorem 3.1 can be weakened to $\chi(G) \leq 2$. This is not the case:

Example 3.7. There is a Borel graph G on a standard probability space (X, μ) which is locally countable, μ-hyperfinite, and satisfies both $\chi(G) = 2$ and $i_\mu(G) = 0$. To see this, set $X = 2^{\mathbb{N}}$, μ the usual product measure, and $xEy \Leftrightarrow x$ and y differ on exactly one coordinate. Then $E^* = E_0$, the equivalence relation of eventual agreement, which is hyperfinite. Certainly, $\chi(G) = 2$ as G contains no cycles of odd length. It remains only to see that $i_\mu(G) = 0$.

Suppose that $Y \subseteq X$ is a set of positive measure. Then, by Lebesgue density, there is a finite binary string s such that $\frac{\mu(Y \cap \mathcal{N}_s)}{\mu(\mathcal{N}_s)} > 1/2$, where $\mathcal{N}_s = \{x \in X : s \sqsubseteq x\}$. This implies that the set $\{x \in X : s0x \in Y \text{ and } s1x \in Y\}$ has positive measure, so Y cannot be independent for G.

(**C**) We now turn our attention to locally finite μ-hyperfinite graphs. In this context, counterexamples such as 3.7 do not arise:

Theorem 3.8. *Let G be a locally finite, μ-hyperfinite, Borel graph on a standard probability space (X, μ). Then $\chi_\mu^{\mathrm{ap}}(G) \leq \chi^*(G)$, and thus $i_\mu(G) \geq 1/\chi^*(G)$. If moreover G is m.p., then $\chi_\mu^{\mathrm{ap}}(G) = \chi^*(G)$.*

Proof. For the first part, discarding a null set if necessary, we may assume that G is hyperfinite and $\chi^*(G) = \chi(G)$. Fix $\varepsilon > 0$ and finite Borel equivalence relations $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ witnessing the hyperfiniteness of G. For each n, define the set

$$X_n = \{x \in X : \forall y \in X (xEy \Rightarrow xF_ny)\}.$$

Then, since G is locally finite, $X = \bigcup_n X_n$. Choose n such that $\mu(X_n) > 1 - \varepsilon$, and define $G' = G|X_n = (X_n, E')$. Since $E' \subseteq F_n$, we have $(E')^* \subseteq F_n$, and thus the connected components of G' are finite. Consequently $\chi_B(G') \leq \chi(G') \leq \chi(G)$, and therefore $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$.

In the m.p. case, we always have $\chi^*(G) \leq \chi_\mu^{\mathrm{ap}}(G)$, thus we have equality. $\square$

4 Graphs associated with group actions

(**A**) Consider a countable group Γ, which we will assume to be infinite, unless otherwise indicated. Let $A : \Gamma \times X \to X$ be a free Borel action of Γ on a standard Borel space. (Free means that $A(\gamma, x) \neq x$, $\forall \gamma \neq 1$, $\forall x \in X$.) We put $\gamma^A(x) = A(\gamma, x)$ and often write $\gamma \cdot x$ for $A(\gamma, x)$ if there is no danger of confusion.

If $S \subseteq \Gamma$ is a set of generators for Γ, where we always assume, unless otherwise indicated, that $1 \notin S$, we define the associated graph

$$G(S, A) = (X, E(S, A)),$$

by

$$(x, y) \in E(S, A) \Leftrightarrow \exists s \in S(y = s^{\pm 1} \cdot x).$$

Clearly this is a locally countable, Borel graph on X whose connected components are the Γ-orbits of the action A. Denote by $\chi(S, A)$, resp., $\chi_B(S, A)$ the associated with $G(S, A)$ chromatic, resp., Borel chromatic numbers. Let also $\mathrm{Cay}(\Gamma, S)$ be the (left) *Cayley graph* of Γ with respect to S, where γ, δ are connected by an edge iff there is $s \in S$ with $\delta = s^{\pm 1}\gamma$. If $x \in X$ then the map $\gamma \in \Gamma \mapsto \gamma \cdot x$ gives an isomorphism between $\mathrm{Cay}(\Gamma, S)$ and the connected component of x in $G(S, A)$. Thus

$$\chi(S, A) = \chi(\mathrm{Cay}(\Gamma, S)).$$

Let now (X, μ) be a standard probability space and let a be a free, measure-preserving action of Γ on (X, μ). This is an equivalence class of Borel actions of Γ on X that preserve the measure μ, where two actions $A(\gamma, x)$, $B(\gamma, x)$ are identified if $A(\gamma, x) = B(\gamma, x)$, a.e., $\forall \gamma$, and free means that $\forall \gamma \neq 1 (a(\gamma, x) \neq x$, a.e.). We again denote, for each set S of generators of Γ, by $G(S, a) = (X, E(S, a))$ the associated graph on X and by $\chi^*(S, a)$, $\chi_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$, $i_\mu(S, a)$ the corresponding numbers. It should be noted that $G(S, a)$ is only defined almost everywhere in the sense that if A, B are representatives for a, then there is a conull set Y which is invariant under both A and B on which A and B agree, thus Y is a set of connected components of both $G(S, A)$, $G(S, B)$ and $G(S, A)|Y = G(S, B)|Y$. It follows that the numbers χ^*, χ_μ, χ_μ^{ap}, i_μ are well-defined, i.e., depend only upon a (and S). Moreover, again, $\chi^*(S, a) = \chi(\mathrm{Cay}(\Gamma, S)) \leq \chi_\mu^{\mathrm{ap}}(S, a)$. Finally, we usually write E_a for $E^*(S, a)$, the equivalence relation induced by the action a. Clearly, $G(S, a)$ is measure preserving and it is ergodic iff a is ergodic.

We first note the following obvious inequality

$$i_\mu(S, a) \leq 1/2.$$

This is because if $A \subseteq X$ is independent and $s \in S$, then $A \cap s^a(A) = \emptyset$ and $\mu(A) = \mu(s^a(A))$.

Moreover, if for every $\Gamma_0 \leq \Gamma$ of index at most 2, the action $a|\Gamma_0 \in A(\Gamma_0, X, \mu)$ is ergodic (e.g., if a is weakly mixing), then there can be no Borel independent set A of measure exactly $1/2$ (so if $i_\mu(S, a) = 1/2$, the supremum is not attained). Otherwise $s \cdot A = X \setminus A$ for every $s \in S^{\pm 1}$ and so A is Γ_0-invariant, where $\Gamma_0 = \{t_1 t_2 \cdots t_{2n} : n \geq 0, t_i \in S^{\pm 1}\}$ and $S^{\pm 1} = \{s^{\pm 1} : s \in S\}$. Since $[\Gamma : \Gamma_0] \leq 2$, this gives a contradiction to our assumption. In particular, for such a, $\chi_\mu(S, a) \geq 3$.

Finally we note that by 2.2, (b) if S is an infinite set of generators and $a \in \mathrm{FR}(\Gamma, X, \mu)$ is mixing, then $i_\mu(S, a) = 0$.

We denote by $\mathrm{FR}(\Gamma, X, \mu)$ the space of free, measure-preserving actions of Γ on (X, μ) and we equip it with the weak topology in which $\mathrm{FR}(\Gamma, X, \mu)$ is a Polish space (see [24], 10).

Theorem 4.1. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then the map*

$$a \mapsto i_\mu(S, a)$$

is lower semicontinuous in $\mathrm{FR}(\Gamma, X, \mu)$.

Proof. Note that for $r \in \mathbb{R}$,

$$\begin{aligned} r < i_\mu(S, a) \quad \Leftrightarrow \quad & \exists \text{ Borel } A\ \exists \varepsilon > 0 \\ & \left(\mu(A) > r + \varepsilon \text{ and } \forall t \in S^{\pm 1} \left(\mu(A \cap t^a(A)) < \frac{\varepsilon}{n} \right) \right), \end{aligned}$$

where $|S^{\pm 1}| = n$. The direction from left to right is clear, because we can take A to be a Borel independent set of measure $r + \varepsilon$ for some $\varepsilon > 0$. Conversely, let A, ε satisfy the right-hand side. Then

$$B = A \setminus \bigcup_{t \in S^{\pm 1}} t^a(A)$$

is independent and $\mu(B) \geq \mu(A) - n \cdot \frac{\varepsilon}{n} > r$, so $i_\mu(S, a) > r$.

Since the map $a \mapsto \mu(A \cap \gamma^a(A))$ from $\mathrm{FR}(\Gamma, X, \mu)$ to $\mathbb{R}$ is continuous in the weak topology, for each $\gamma \in \Gamma$, $\{a \in \mathrm{FR}(\Gamma, X, \mu) : r < i_\mu(S, a)\}$ is open and the proof is complete. ☐

Recall that $a \in \mathrm{FR}(\Gamma, X, \mu)$ is *weakly contained* in $b \in \mathrm{FR}(\Gamma, X, \mu)$, in symbols $a \prec b$, if a is in the closure of the conjugacy class of b (see [24], 10, 10.1). So we have

Corollary 4.2. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then*

$$a \prec b \Rightarrow i_\mu(S, a) \leq i_\mu(S, b).$$

In particular, $i_\mu(S, a)$ is an invariant of weak equivalence, defined by $a \sim b \Leftrightarrow a \prec b$ and $b \prec a$.

Corollary 4.2 (and thus 4.1) may fail if S is infinite. Take for example the free group Γ with a (free) infinite generating set S. Then every action of Γ is weakly contained in a mixing action (this is a very special case of the result in [16]), so by 2.2, (b) if 4.2 was true in this case, then we would have $i_\mu(S, a) = 0$, for any free action a. But this contradicts, for example, 4.6 below. A similar remark applies to 4.3 and 4.13.

Concerning χ_μ^{ap} we have the following result.

Theorem 4.3. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then for any $a, b \in \mathrm{FR}(\Gamma, X, \mu)$,*

$$a \prec b \Rightarrow \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, b).$$

Proof. Assume $a \prec b$ and let $k = \chi_\mu^{\mathrm{ap}}(S, a)$, $n = |S^{\pm 1}|$. Fix $\varepsilon > 0$. Let then $A_1, \dots, A_k$ be Borel, pairwise disjoint, independent subsets of X with $\mu(A_1 \cup \dots \cup A_k) > 1 - \frac{\varepsilon}{k+1}$. Since $a \prec b$, there are Borel, pairwise disjoint subsets $B_1, \dots, B_k$ of X with $\mu(B_1 \cup \dots \cup B_k) > 1 - \frac{\varepsilon}{k+1}$ and

$$|\mu(s^a(A_i) \cap A_i) - \mu(s^b(B_i) \cap B_i)| < \frac{\varepsilon}{n(k+1)}$$

for all $s \in S^{\pm 1}$, thus $\mu(s^b(B_i) \cap B_i) < \frac{\varepsilon}{n(k+1)}$ for all $s \in S^{\pm 1}$. If $\overline{B}_i = B_i \setminus \bigcup_{s \in S^{\pm 1}} s^b(B_i)$, then $\overline{B}_i$, $1 \leq i \leq k$, are Borel, pairwise disjoint, independent (for the action b) sets, and $\mu(\overline{B}_i) \geq \mu(B_i) - \frac{\varepsilon}{k+1}$, so $\mu(\overline{B}_1 \cup \dots \cup \overline{B}_k) > 1 - \varepsilon$, therefore $k \geq \chi_\mu^{\mathrm{ap}}(S, b)$. □

It follows that $a \mapsto \chi_\mu^{\mathrm{ap}}(S, a)$ is also an invariant of weak equivalence. Next recall the following simple fact.

Proposition 4.4. *Let Γ be a countable group and $S \subseteq \Gamma$ a set of generators. Then the following are equivalent:*

(i) $\chi(\text{Cay}(\Gamma, S)) = 2$ *(i.e.,* $\text{Cay}(\Gamma, S)$ *is bipartite),*

(ii) *There is a homomorphism* $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ *that sends* S *to 1,*

(iii) $\{s_1 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$ *has index 2 in* Γ*,*

(iv) *For any* $s_1, \dots, s_{2n+1} \in S^{\pm 1}$*,* $n \geq 0$*, we have* $s_1 \cdots s_{2n+1} \neq 1$*.*

Moreover, a group Γ *admits a set of generators* $S \subseteq \Gamma$ *with* $\text{Cay}(\Gamma, S)$ *bipartite iff* $\mathbb{Z}/2\mathbb{Z}$ *is a factor of* Γ*.*

We now have:

Proposition 4.5. *Let* Γ *be a countable group and* $S \subseteq \Gamma$ *a set of generators. Let* $g = g_{\text{odd}}(\text{Cay}(\Gamma, S))$ *be the odd girth of the Cayley graph* $\text{Cay}(\Gamma, S)$*. Then for any* $a \in \text{FR}(\Gamma, X, \mu)$*, we have*

$$i_\mu(S, a) \leq 1/2 - 1/(2g).$$

Also if $\Gamma_0 = \{s_1 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$ *and* $a|\Gamma_0 \in \text{FR}(\Gamma_0, X, \mu)$ *is strongly ergodic, then* $i_\mu(S, a) < 1/2$*.*

Proof. We can assume that $g < \infty$, i.e., that the Cayley graph is not bipartite. Let $A \subseteq X$ be a Borel independent set and let $\mu(A) = 1/2 - \varepsilon$. Then for any $s, t \in S^{\pm 1}$, it is easy to see that $\mu(A \triangle st \cdot A) \leq 4\varepsilon$. So, by induction, if $\gamma = s_1 \cdots s_{2n}$, where $s_i \in S^{\pm 1}$, then $\mu(\gamma \cdot A \triangle A) \leq 4n\varepsilon$. If $g = 2n+1$, then for some $s, s_1, \dots, s_{2n} \in S^{\pm 1}$, we have that $s = s_1 \cdots s_{2n}$, so $\mu(s \cdot A \triangle A) \leq 4n\varepsilon$, thus $\varepsilon \geq 1/(4n+2) = 1/(2g)$, therefore $\mu(A) \leq 1/2 - 1/(2g)$.

In the case $a|\Gamma_0$ is strongly ergodic but $i_\mu(S, a) = 1/2$, there are Borel independent sets A_n with $\mu(A_n) \to 1/2$. Then for any finite $F \subseteq \Gamma_0$, $\varepsilon > 0$, and all large enough n, we have $\mu(\gamma \cdot A_n \triangle A_n) < \varepsilon$, $\forall \gamma \in F$, i.e., $a|\Gamma_0$ has non-trivial almost invariant sets, contradicting strong ergodicity. □

Thus in the context of 4.5, if $\text{Cay}(\Gamma, S)$ is not bipartite (so that $g < \infty$) or if $a|\Gamma_0$ is strongly ergodic, we have that $i_\mu(S, a) < 1/2$ and $\chi_\mu^{\text{ap}}(S, a) \geq 3$. Also recall that if $a|\Gamma_0$ is ergodic, e.g., if a is weak mixing, then $\chi_\mu(S, a) \geq 3$ (otherwise there would be an independent set of measure 1/2).

Applying 4.5 to $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \langle s, t | s^2 = 1, t^3 = 1 \rangle$, with $S = \{s, t\}$, we have $g = 3$, and thus $i_\mu(S, a) \leq 1/3$, for any $a \in \text{FR}(\Gamma, X, \mu)$. But by 2.19, since $d = 3$, $i_\mu(S, a) \geq 1/3$. So $i_\mu(S, a) = 1/3$ for any $a \in \text{FR}(\Gamma, X, \mu)$. This also shows that the upper bound in 4.5 cannot, in general, be improved. We will see at the end of Section 6, using also a result of Lyons-Nazarov [35], that for $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \langle s, t | s^3 = 1, t^3 = 1 \rangle$ and $S = \{s, t\}$, we also have $i_\mu(S, a) = 1/3$ for any $a \in \mathrm{FR}(\Gamma, X, \mu)$ and the sup is attained. Also $\chi_\mu^{\mathrm{ap}}(S, a) = 3$.

Theorem 4.6. *Let Γ be a countable group and $S \subseteq \Gamma$ a set of generators. Then the following are equivalent:*

(i) $\mathrm{Cay}(\Gamma, S)$ *is bipartite,*

(ii) *There is an action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S, a) = 1/2$,*

(iii) *There is an ergodic action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S, a) = 1/2$,*

(iv) *There is an action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu^{\mathrm{ap}}(S, a) = 2$,*

(v) *There is an ergodic action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu^{\mathrm{ap}}(S, a) = 2$,*

(vi) *There is an action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) = 2$,*

(vii) *There is an ergodic action $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) = 2$.*

Proof. (ii)$\Rightarrow$(i) follows from 4.5. Clearly (vii) $\Rightarrow$ (v) $\Rightarrow$ (iii), (vii) $\Rightarrow$ (vi), (v) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii).

Conversely, assume (i) in order to prove (vii). Let $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism that sends S to 1. Let $a_0 \in \mathrm{FR}(\Gamma, X, \mu)$ be weakly mixing, and let a_1 be the action of Γ on $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ given by $\gamma \cdot i = \varphi(\gamma) + i$. If $\nu(\{0\}) = \nu(\{1\}) = 1/2$, then the action of Γ on $\mathbb{Z}/2\mathbb{Z}$ is measure preserving and ergodic.

Let now $a \in \mathrm{FR}(\Gamma, X \times \{0, 1\}, \mu \times \nu)$ be the product action $a_0 \times a_1$, i.e.,

$$\gamma \cdot (x, i) = (\gamma \cdot x, \varphi(\gamma) + i).$$

Clearly, $X \times \{0\}$, $X \times \{1\}$ gives a measurable 2-coloring of the graph $G(S, a)$, so $\chi_\mu(S, a) = 2$. Finally, since a_0 is free and weakly mixing, the action a is free and ergodic. $\square$

(**B**) Consider now the case of (Γ, S), where S is a finite set of generators of Γ and Γ is amenable. Then if $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ are ergodic, by [24],13.2, $a \sim b$ and so $i_\mu(S, a) = i_\mu(S, b)$. Thus $i_\mu(S, a)$ is constant for all free, ergodic

a. Using the ergodic decomposition, this is true for all $a \in \mathrm{FR}(\Gamma, X, \mu)$. We denote by

$$i_\mu(\Gamma, S)$$

this constant value.

Similarly, $\chi_\mu^{\mathrm{ap}}(S, a)$ is constant for all $a \in \mathrm{FR}(\Gamma, X, \mu)$ and we denote this constant value by

$$\chi_\mu^{\mathrm{ap}}(\Gamma, S).$$

We now have:

Theorem 4.7. *Let Γ be a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators. Then*

$$\chi_\mu^{\mathrm{ap}}(\Gamma, S) = \chi(\mathrm{Cay}(\Gamma, S)).$$

Proof. This follows from Theorem 3.8. □

Theorem 4.8. *Let Γ be a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators. Then:*

(i) *If $\mathrm{Cay}(\Gamma, S)$ is bipartite, then $i_\mu(\Gamma, S) = 1/2$,*

(ii) *If $\mathrm{Cay}(\Gamma, S)$ is not bipartite, then $i_\mu(\Gamma, S) \leq 1/2 - 1/(2g) < 1/2$, where g is the odd girth of $\mathrm{Cay}(\Gamma, S)$.*

Proof. (i) follows from 3.8, and (ii) from 4.5. □

We can now identify $i_\mu(\Gamma, S)$ in terms of $\mathrm{Cay}(\Gamma, S)$. Recall that a *Følner sequence* in Γ is a sequence (F_n) of finite, non-empty subsets of Γ such that $\forall \gamma \in \Gamma$, $\frac{|\gamma F_n \triangle F_n|}{|F_n|} \to 0$. We will use the following result that is a consequence of the quasi-tiling machinery in Ornstein-Weiss [43] and is explicitly stated in Gromov [14], 1.3 and Lindenstrauss-Weiss [31], Appendix (see also Abért, Jaikin-Zapirain and Nikolov [2], Lemma 18).

Theorem 4.9 (Ornstein-Weiss [43], Gromov [14], Lindenstrauss-Weiss [31])**.** *Let Γ be an amenable group and (F_n) a Følner sequence in Γ. Let h be a positive real-valued function defined on all finite subsets of Γ such that h is subadditive (i.e., $h(A \cup B) \leq h(A) + h(B)$) and right-invariant ($h(A\gamma) = h(A), \forall \gamma \in \Gamma$). Then $\lim_{n\to\infty} \frac{h(F_n)}{|F_n|}$ exists (and is of course independent of (F_n)).*

For each finite $F \subseteq \Gamma$ denote by $i(F,S)$ the independence ratio of the induced subgraph $\mathrm{Cay}(\Gamma,S)|F$. Then it is easy to check that $h(F) = i(F,S)|F|$ (i.e., $h(F)$ is the maximal cardinality of an independent subset of $\mathrm{Cay}(\Gamma,S)|F$) is subadditive and right-invariant, thus for each Følner sequence (F_n) the limit

$$\lim_{n\to\infty} i(F_n,S) = i(\Gamma,S)$$

exists and is independent of (F_n). We call it the *independence number* of $\mathrm{Cay}(\Gamma,S)$. We now have:

Theorem 4.10. *Let Γ be a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators. Then*

$$i_\mu(\Gamma,S) = i(\Gamma,S).$$

Proof. We will use the following two results, the first of which is a consequence of the quasi-tiling machinery of Ornstein-Weiss [43] (see specifically II.§2, Theorem 5 and its subsequent remark, and also the proof of I.§2, Theorem 6) and the second is a very weak consequence of the mean ergodic theorem for Følner sequences (see, e.g., Nevo [41], 6.7).

Lemma 4.11. (i) *Let Γ be a countable, amenable group and $1 \in F_0 \subseteq F_1 \subseteq \cdots$ an increasing Følner sequence. Let $a \in \mathrm{FR}(\Gamma,X,\mu)$ and $\varepsilon > 0$. Then we can find $n_1 < \cdots < n_k$, so that letting $T_i = F_{n_i}, 1 \leq i \leq k$, we have the following:*

For each $1 \leq i \leq k$, there is $l_i \geq 1$, sets $T_{ij} \subseteq T_i$, and Borel sets $B_{ij} \subseteq X, 1 \leq j \leq l_i$, such that

(a) *The sets $T_{ij}B_{ij}, 1 \leq j \leq l_i$, are pairwise disjoint,*

(b) *The sets $tB_{ij}, t \in T_{ij}$, are pairwise disjoint,*

(c) *The sets $\bigcup_{j\leq l_i} T_{ij}B_{ij}, 1 \leq i \leq k$, are pairwise disjoint,*

(d) $\mu\left(\bigcup_{i\leq k}\bigcup_{j\leq l_i} T_{ij}B_{ij}\right) > 1-\varepsilon,$

(e) $\frac{|T_{ij}|}{|T_i|} > 1-\varepsilon, 1 \leq i \leq k, 1 \leq j \leq l_i.$

(ii) *Let Γ be a countable, amenable group and $a \in \mathrm{FR}(\Gamma,X,\mu)$ an ergodic action. Let (F_n) be a Følner sequence for Γ, let $\varepsilon > 0$ and let $A \subseteq X$ a Borel set. Then for some $n \in \mathbb{N}$ and $x \in X$ we have*

$$\left|\frac{|\{f \in F_n : f\cdot x \in A\}|}{|F_n|} - \mu(A)\right| < \varepsilon.$$

It is clearly enough to show that for any ergodic $a \in \mathrm{FR}(\Gamma, X, \mu)$, $i_\mu(S, a) = i(\Gamma, S)$. Fix a Følner sequence $1 \in F_0 \subseteq F_1 \subseteq \cdots$, in order to show that $\lim_{n\to\infty} i(F_n, S) = i_\mu(S, a)$. We proceed in two steps.

(A) $\lim_{n\to\infty} i(F_n, S) \leq i_\mu(S, a)$

Put $\alpha = \lim_{n\to\infty} i(F_n, S)$. We may of course assume that $\alpha > 0$. Fix $\alpha > \varepsilon > 0$. By (i) of the lemma, applied to an appropriate subsequence of (F_n) and $\varepsilon' << \varepsilon$, we can find $n_1 < \cdots < n_k$ such that letting $T_i = F_{n_i}, 1 \leq i \leq k$ and letting T_{ij}, B_{ij} be as in the lemma we have

(i) There are independent sets $A_{ij} \subseteq T_{ij}$ with $\left|\frac{|A_{ij}|}{|T_{ij}|} - \alpha\right| < \varepsilon$ and $A_{ij} \subseteq \{t \in T_{ij} : \forall s \in S^{\pm 1}(st \in T_{ij})\}$,

(ii) The family of sets $\{T_{ij}b\}$, where $b \in B_{ij}, 1 \leq i \leq k, 1 \leq j \leq l_i$, is pairwise disjoint and $\mu\left(\bigcup_{i\leq k}\bigcup_{j\leq l_i} T_{ij}B_{ij}\right) > 1 - \varepsilon$.

Let then $A = \bigcup_{i\leq k}\bigcup_{j\leq l_i} A_{ij}B_{ij}$. Then A is independent and

$$\begin{aligned} \mu(A) &= \sum_{i\leq k}\sum_{j\leq l_i} |A_{ij}|\mu(B_{ij}) \\ &\geq \sum_{i\leq k}\sum_{j\leq l_i}(\alpha - \varepsilon)|T_{ij}|\mu(B_{ij}) \\ &= (\alpha - \varepsilon)\sum_{i\leq k}\sum_{j\leq l_i} |T_{ij}|\mu(B_{ij}) \\ &> (\alpha - \varepsilon)(1 - \varepsilon), \end{aligned}$$

so $i_\mu(S, a) > (\alpha - \varepsilon)(1 - \varepsilon)$ and thus, letting $\varepsilon \to 0$, $i_\mu(S, a) \geq \alpha$.

(B) $\lim_{n\to\infty} i(F_n, S) \geq i_\mu(S, a)$.

Let $A \subseteq X$ be a Borel independent set. Fix $\varepsilon > 0$ and, by (ii) of the lemma applied to tail ends of (F_n), we can find $x_1, x_2, \ldots \in X$ and $n_1 < n_2 < \cdots$ such that

$$\left|\frac{|\{f \in F_{n_i} : f \cdot x_i \in A\}|}{|F_{n_i}|} - \mu(A)\right| < \varepsilon.$$

Let $A_i = \{f \in F_{n_i} : f \cdot x_i \in A\}$. Then A_i is independent in $\mathrm{Cay}(\Gamma, S)|F_{n_i}$, so $\frac{|A_i|}{|F_{n_i}|} \leq i(F_{n_i}, S)$, thus $\mu(A) < \frac{|A_i|}{|F_{n_i}|} + \varepsilon \leq i(F_{n_i}, S) + \varepsilon$. Letting $i \to \infty$, $\varepsilon \to 0$, we have $\mu(A) \leq \lim_{i\to\infty} i(F_{n_i}, S) = \lim_{n\to\infty} i(F_n, S)$. □

Remark 4.12. By a similar argument, one can see that if Γ is a countable, amenable group and m is a finitely additive, shift-invariant probability measure on Γ, then

$$\sup\{m(A) : A \subseteq \Gamma \text{ is independent in } \mathrm{Cay}(\Gamma, S)\} = i(S, a).$$

In particular, this supremum is independent of the choice of m.

(**C**) When Γ is not amenable, $i_\mu(S, a)$ and $\chi_\mu^{\mathrm{ap}}(S, a)$ might not be constant. For example, we have:

Proposition 4.13. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators with $\mathrm{Cay}(\Gamma, S)$ bipartite. Then the following are equivalent:*

(i) *Γ is amenable,*

(ii) *$i_\mu(S, a)$ is constant, for all $a \in \mathrm{FR}(\Gamma, X, \mu)$,*

(iii) *$\chi_\mu^{\mathrm{ap}}(S, a)$ is constant, for all $a \in \mathrm{FR}(\Gamma, X, \mu)$.*

Proof. We have seen that (i) ⇒ (ii), (iii). Assume now that Γ is not amenable. Then for the shift action s_Γ of Γ on 2^Γ, with the usual product measure, and Γ_0 as in Proposition 4.5, $a|\Gamma_0$ is strongly ergodic, so $i_\mu(S, s_\Gamma) < 1/2$ and thus also $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq 3$. On the other hand, by 4.6 there is $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S, a) = 1/2$ and $\chi_\mu^{\mathrm{ap}}(S, a) = 2$, giving the failure of (ii) and (iii). □

On the other hand, Abért and Weiss [3] showed that among all $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is a minimum one in the sense of weak containment, namely the shift action s_Γ of Γ on 2^Γ (with the usual product measure), and earlier Hjorth (unpublished) and (independently) Glasner-Thouvenot-Weiss [13] showed that there is a maximum one, denoted by $a_{\Gamma,\infty}$ (see also [24], 10.7). Similarly there is a free, ergodic action which is maximum in the sense of weak containment among all the free, ergodic actions, denoted by $a_{\Gamma,\infty}^{\mathrm{erg}}$ (see [24], 13.1). Then for any free, ergodic action a,

$$s_\Gamma \prec a \prec a_{\Gamma,\infty}^{\mathrm{erg}}.$$

Therefore for any finite generating set $S \subseteq \Gamma$, we have for any ergodic $a \in \mathrm{FR}(\Gamma, X, \mu)$,

$$i_\mu(S, s_\Gamma) \leq i_\mu(S, a) \leq i_\mu(S, a_{\Gamma,\infty}^{\mathrm{erg}}),$$

and

$$\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, a_{\Gamma,\infty}^{\mathrm{erg}}).$$

Thus, by the ergodic decomposition, for *any* free action a,

$$i_\mu(S, s_\Gamma) \leq i_\mu(S, a) \leq i_\mu(S, a_{\Gamma,\infty}^{\mathrm{erg}}),$$

and, using also the proof of 4.3,

$$\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, a_{\Gamma,\infty}^{\mathrm{erg}}).$$

Note also that if for $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta = 1$, we consider the convex combination $\alpha a + \beta b$, for any free actions a, b (see [24], 10 (**F**)), then trivially $i_\mu(S, \alpha a + \beta b) = \alpha i_\mu(S, a) + \beta i_\mu(S, b)$, therefore $\{i_\mu(S, a) : a \in \mathrm{FR}(\Gamma, X, \mu)\} = [i_\mu(S, s_\Gamma), i_\mu(S, a_{\Gamma,\infty}^{\mathrm{erg}})]$. For example, if $\mathrm{Cay}(\Gamma, S)$ is bipartite and Γ is not amenable, then this last interval is not trivial, so $i_\mu(S, a)$ takes continuum many values on $\mathrm{FR}(\Gamma, X, \mu)$ and thus, in particular, there are continuum many weak equivalence classes of free actions. Note also that for all these actions $c = \alpha a + \beta b$, $\alpha, \beta \neq 0$, the corresponding Koopman representations κ^c (see §4, (**D**) below) are all isomorphic (to $\kappa^a \oplus \kappa^b$). It is not clear however what is the range of $i_\mu(S, a)$ on the space of *ergodic*, free actions.

We next show that for (Γ, S) with $\mathrm{Cay}(\Gamma, S)$ bipartite, one can characterize whether Γ is amenable, has property (T) or the HAP in terms of the independence and approximate chromatic numbers of its actions. We start with the following characterization of amenability.

Theorem 4.14. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *is amenable,*

(ii) $i_\mu(S, a) = 1/2$*, for any* $a \in \mathrm{FR}(\Gamma, X, \mu)$*,*

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) = 2$*, for any* $a \in \mathrm{FR}(\Gamma, X, \mu)$*,*

(iv) $i_\mu(S, s_\Gamma) = 1/2$*,*

(v) $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) = 2$*.*

In particular, if Γ is a finitely generated group having $\mathbb{Z}/2\mathbb{Z}$ as a factor, then the following are equivalent:

(a) Γ *is amenable,*

(b) *There is a finite generating set* $S \subseteq \Gamma$ *such that* $i_\mu(S, s_\Gamma) = 1/2$,

(c) *As in* (b) *with* $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) = 2$.

Proof. This follows from 4.13 and its proof. □

We next consider property (T) and the HAP.

Theorem 4.15. *Let* Γ *be an infinite, countable group and* $S \subseteq \Gamma$ *a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *has property* (T),

(ii) $i_\mu(S, a) < 1/2$, *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$, *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$.

Also the following are equivalent:

(i*) Γ *does not have the* HAP,

(ii*) $i_\mu(S, a) < 1/2$, *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iii*) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$, *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$.

Proof. Suppose first that Γ has property (T). If $\Gamma_0 = \{s_1 s_2 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$, then Γ_0 has index 2 in Γ and thus Γ_0 itself has property (T). Moreover, if $a \in \mathrm{FR}(\Gamma, X, \mu)$ is weakly mixing, then $a|\Gamma_0 \in \mathrm{FR}(\Gamma_0, X, \mu)$ is ergodic, so strongly ergodic (see, e.g., [24] 11.2), thus $i_\mu(S, a) < 1/2$ by Proposition 4.5. So (i) ⇒ (ii) ⇒ (iii).

Assume now that Γ does not have property (T). By 4.6 there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S, b) = 1/2$ and $\chi_\mu^{\mathrm{ap}}(S, b) = 2$. By a result of Kerr-Pichot [27] (see also [24], 12.9), there is a weakly mixing $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \prec a$, so $i_\mu(S, b) \leq i_\mu(S, a)$, thus $i_\mu(S, a) = 1/2$, and $\chi_\mu^{\mathrm{ap}}(S, b) \geq \chi_\mu^{\mathrm{ap}}(S, a)$, therefore $\chi_\mu^{\mathrm{ap}}(S, a) = 2$.

If now Γ does not have the HAP and $a \in \mathrm{FR}(\Gamma, X, \mu)$ is mixing, then Γ_0 as above does not have the HAP, and $a|\Gamma_0 \in \mathrm{FR}(\Gamma_0, X, \mu)$ is mixing, so (see, e.g., [24] 11.1) it is strongly ergodic, thus $i_\mu(S, a) < 1/2$ as before. So (i*) ⇒ (ii*) ⇒ (iii*).

Conversely, if Γ has the HAP, then we can repeat the argument above (for the case that Γ does not have property (T)) using the result of Hjorth [16] (see also [24], 12.11) to replace in this argument weakly mixing by mixing. $\square$

(**D**) For any unitary representation $\pi : \Gamma \to U(H)$ of a countable group Γ on a Hilbert space H, and a finite set of generators $S \subseteq \Gamma$, one defines the *averaging operator* $T_{S,\pi}$ by

$$T_{S,\pi}(f) = \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} \pi(s)(f).$$

Clearly $T_{S,\pi}$ is a self-adjoint operator and $\|T_{s,\pi}\| \leq 1$. It is easy to check that if π, ρ are unitary representations and π is weakly contained in ρ (see, e.g, Bekka-de la Harpe-Valette [5], Appendix F), which is denoted by $\pi \prec \rho$, then $\|T_{s,\pi}\| \leq \|T_{s,\rho}\|$, i.e.,

$$\pi \prec \rho \Rightarrow \|T_{s,\pi}\| \leq \|T_{s,\rho}\|.$$

When Γ is amenable, Kesten [28] showed that $\|T_{S,\lambda_\Gamma}\| = 1$, where λ_Γ is the (left) regular representation of Γ.

For each $a \in \mathrm{FR}(\Gamma, X, \mu)$, consider the corresponding Koopman unitary representation κ^a on $L^2(X,\mu)$ and its restriction κ^a_0 on $L^2_0(X,\mu) = \{f \in L^2(X,\mu) : \int f \, d\mu = 0\} = \mathbb{C}^\perp$ (where $\mathbb{C}$ is identified with the subspace of constant functions in $L^2(X,\mu)$). Then for a finite generating set $S \subseteq \Gamma$, let

$$T_{S,a} = T_{S,\kappa^a_0}.$$

There is a well-known connection between norms of averaging operators and independence ratios in the case of finite graphs, due to Hoffman [18] (see, e.g., Davidoff-Sarnak-Valette [9], 1.5.3), and a version of this carries over to our context.

Proposition 4.16. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Let $a \in \mathrm{FR}(\Gamma, X, \mu)$ and let $T_{S,a}$ the corresponding averaging operator. If $\nu = \|T_{S,a}\|$, then*

$$i_\mu(S,a) \leq \frac{\nu}{1+\nu},$$

and thus

$$\chi^{\mathrm{ap}}_\mu(S,a) \geq \frac{1+\nu}{\nu}.$$

Proof. Let $A \subseteq X$ be independent and let $T = T_{S,a}$, $f = \chi_A - \mu(A)$. Then $f \in L^2_0(X, \mu)$ and

$$\|f\|^2 = \mu(A)(1 - \mu(A)).$$

Also,

$$\begin{aligned}
\langle T(f), f \rangle &= \int T(f)(x) f(x) \; d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} \int f(s \cdot x) f(x) \; d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} \int (\chi_{s^{-1} \cdot A}(x) - \mu(A))(\chi_A(x) - \mu(A)) \; d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} \int (-\mu(A)(\chi_A(x) + \chi_{s^{-1} \cdot A}(x)) + \mu(A)^2) \; d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} (-2\mu(A)^2 + \mu(A)^2) \\
&= -\mu(A)^2.
\end{aligned}$$

Since $|\langle T(f), f \rangle| \leq \|T\| \cdot \|f\|^2$, letting $\alpha = \mu(A)$, we have

$$\alpha^2 \leq \nu \cdot \alpha(1 - \alpha),$$

so

$$\alpha \leq \frac{\nu}{1 + \nu}.$$

□

Since for $a, b \in \mathrm{FR}(\Gamma, X, \mu)$,

$$a \prec b \Rightarrow \kappa_0^a \prec \kappa_0^b$$

(see [24], 10.5), it follows that

$$a \prec b \Rightarrow \|T_{s,a}\| \leq \|T_{s,b}\|$$

(in fact it is not hard to see that $a \mapsto \|T_{s,a}\|$ is lower semicontinuous), thus, since $s_\Gamma \prec a$, $\forall a \in \mathrm{FR}(\Gamma, X, \mu)$, $\|T_{S,s_\Gamma}\|$ is minimum among all such $\|T_{s,a}\|$. Now it is well known (see, e.g., [5], E.4.5) that $\kappa_0^{s_\Gamma} \sim \lambda_\Gamma$, thus $\|T_{S,s_\Gamma}\| = \|T_{S,\lambda_\Gamma}\|$.

Suppose now that $S = \{\gamma_1, \dots, \gamma_m\}$. Kesten [28] has shown that

$$\|T_{S,\lambda_\Gamma}\| \geq \frac{\sqrt{2m-1}}{m}$$

and if S is a free set of generators, so that $\Gamma = \mathbb{F}_m$, then

$$\|T_{S,\lambda_\Gamma}\| = \frac{\sqrt{2m-1}}{m}.$$

Also, if $S = \{\gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_n\}$, where $\gamma_1, \dots, \gamma_m$ are free and $\delta_1, \dots, \delta_n$ are free satisfying $\delta_i^2 = 1$, $i = 1, \dots, n$, so that $\mathrm{Cay}(\Gamma, S)$ is still acyclic and $\Gamma = \mathbb{F}_m * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (n times), then again

$$\|T_{S,\lambda_\Gamma}\| = \frac{2\sqrt{(2m+n)-1}}{2m+n}.$$

We then have:

Theorem 4.17. *Let $\Gamma = \mathbb{F}_m$ be the free group with a free set S of m generators, and s_Γ its shift action on 2^Γ, with the product measure μ. Then*

$$\frac{1}{2m} \leq i_\mu(S, s_\Gamma) \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}}$$

and

$$2m \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m + \sqrt{2m-1}}{\sqrt{2m-1}}.$$

Moreover,

$$2m + 1 \geq \chi_B(S, s_\Gamma)$$

(where we view in the last inequality s_Γ as the shift action restricted to its free part).

Proof. The first part follows from Theorem 2.19, Propositions 2.13, 4.16, and the preceding paragraphs. The last part follows from [22], 4.6. □

This, in particular, gives examples of m.p., ergodic, Borel graphs of bounded degree which are acyclic but the approximate chromatic numbers and thus the measurable and Borel chromatic numbers are finite but tend towards ∞.

An analogous result to Theorem 4.17 holds when $\Gamma = \mathbb{F}_m * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (n times).

It is mentioned in Lyons-Nazarov [35] that from results of Bollobás and Frieze-Luczak concerning random regular graphs, it follows that, *for large enough* m, one has for $\Gamma = \mathbb{F}_m$, and S a free set of generators, $i_\mu(S, s_\Gamma) \leq \frac{\log 2m}{m}$ and so $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m}{\log 2m}$. For references, see Section 5 of [35].

We do not know what are the exact values of $i_\mu(S, s_\Gamma)$, $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma)$ for $\Gamma = \mathbb{F}_m$ and S a free set of generators (similarly for $\chi_B(S, s_\Gamma)$, $\chi_\mu(S, s_\Gamma)$).

Concerning Borel chromatic numbers of shifts, denote below by s_Γ the restriction of the shift action of Γ on 2^Γ to its free part, and recall that for a generating set $S \subseteq \Gamma$, $\chi_B(S, s_\Gamma)$ denotes the Borel chromatic number associated with s_Γ. If $\Gamma = \mathbb{Z}^m$, with S the usual set of m generators, then Gao-Jackson [12] showed that $\chi_B(S, s_\Gamma) \in \{3, 4\}$, while of course $\chi(G(S, s_\Gamma)) = \chi(\mathrm{Cay}(\Gamma, S)) = 2$. For a generalization of the $m = 2$ case, see 5.15 below. Gao-Jackson-Miller (unpublished) and recently Adam Timar (private communication) have shown that in this setting $\chi_\mu(S, s_\Gamma) = 3$.

Is it true that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if Γ is amenable and S is any finite generating set, then $\chi_B(S, s_\Gamma) \leq f(\chi(\mathrm{Cay}(\Gamma, S)))$? (Note that, by 4.7, this is true for χ_μ^{ap} with $f = \mathrm{id}$.) We do not know a counterexample even for $f(n) = n + 1$.

On the other hand, if Γ is finitely generated with $\mathbb{F}_2 \leq \Gamma$ and $\mathbb{Z}/2\mathbb{Z}$ is a factor of Γ, then for each $\varepsilon > 0$, there is a finite generating set $S \subseteq \Gamma$ with $\chi(\mathrm{Cay}(\Gamma, S)) = 2$, but $i_\mu(S, s_\Gamma) < \varepsilon$ and so $\chi_B(S, s_\Gamma) \geq \chi_\mu(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) > 1/\varepsilon$. Indeed, choose m large enough so that $\frac{\sqrt{2m-1}}{m+\sqrt{2m-1}} < \varepsilon$ and let $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be a surjective homomorphism. Then $\Gamma_0 = \ker(\varphi)$ contains a free subgroup $\Delta = \langle a_1, \ldots, a_m \rangle$ with m free generators. Let $S_0 \supset \{a_1, \cdots, a_m\}$ be a finite set of generators for Γ_0 and let $a \notin \Gamma_0$. Put $S = \{a\} \cup aS_0$. Clearly S generates Γ and there are no odd cycles in $\mathrm{Cay}(\Gamma, S)$, so $\chi(\mathrm{Cay}(\Gamma, S)) = 2$. However, if A is an independent set in the graph associated with s_Γ, then it is independent for the graph associated with the action $s_\Gamma|\Delta$ and the set of generators $S_\Delta = \{a_1, \ldots, a_m\}$. One can again see that $\kappa_0^{s_\Gamma|\Delta} \sim \lambda_\Delta$, so

$$i_\mu(S, s_\Gamma) \leq i_\mu(S_\Delta, s_\Gamma|\Delta) \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}} < \varepsilon.$$

We should finally mention that although we have examples where

$$\chi_\mu^{\mathrm{ap}}(S, a) < \chi_\mu(S, a)$$

(see 2.8 or take any weakly mixing action a, for which therefore $\chi_\mu(S,a) \geq 3$, with $\chi_\mu^{\mathrm{ap}}(S,a) = 2$), we do not know any examples for which $\chi_\mu^{\mathrm{ap}}(S,a)+1 < \chi_\mu(S,a)$. R. Lyons (private communication) also asked if there are examples of strongly ergodic (also known as E_0-ergodic) actions a with $\chi_\mu^{\mathrm{ap}}(S,a) < \chi_\mu(S,a)$.

Remark 4.18. Let $S_m = \{\gamma_1, \dots, \gamma_m\}$ be a free set of generators for $\mathbb{F}_m, m \geq 1$. Denote by $i_m = i_\mu(S_m, s_{\mathbb{F}_m})$ the independence number of the shift action of $\mathbb{F}_m$, i.e., the minimum independence number of a free, measure-preserving action of $\mathbb{F}_m$. Here and below we abuse notation by using the same subscript μ for the associated measure of any action discussed below. From the preceding theorem we have that $i_m \to 0$ as $m \to \infty$. Let us also note the following:

(i) $\forall m (i_{m+1} \leq i_m)$.

To see this, consider the shift action $s_{\mathbb{F}_{m+1}}$ and its restriction $a = s_{\mathbb{F}_{m+1}} | \mathbb{F}_m$, where we view $\mathbb{F}_m$ as the subgroup generated by $\gamma_1, \dots, \gamma_m$. Clearly if $A \subseteq 2^{\mathbb{F}_{m+1}}$ is independent for $s_{\mathbb{F}_{m+1}}$, it is also independent for a, thus

$$i_{m+1} \leq i_\mu(S_m, a).$$

Moreover $a \cong (s_{\mathbb{F}_m})^{\mathbb{N}}$, where $(s_{\mathbb{F}_m})^{\mathbb{N}}$ is the product of countably many copies of $s_{\mathbb{F}_m}$, i.e., it is the action of $\mathbb{F}_m$ on $(2^{\mathbb{F}_m})^{\mathbb{N}}$, given by

$$\gamma \cdot (p_n) = (\gamma \cdot p_n), \forall \gamma \in \mathbb{F}_m.$$

Now $(s_{\mathbb{F}_m})^{\mathbb{N}} \cong s^*_{\mathbb{F}_m}$, where $s^*_{\mathbb{F}_m}$ is the shift action of $\mathbb{F}_m$ on $(2^{\mathbb{N}})^{\mathbb{F}_m}$. The isomorphism is given by the map

$$(p_0, p_1, \dots) \in (2^{\mathbb{F}_m})^{\mathbb{N}} \mapsto p \in (2^{\mathbb{N}})^{\mathbb{F}_m},$$

where $p(\gamma) = (p_0(\gamma), p_1(\gamma), \dots), \forall \gamma \in \mathbb{F}_m$. (Here all these product spaces have the product measures arising from the $(1/2, 1/2)$-measure on $2 = \{0,1\}$.) Now Bowen [6] has shown that $s_{\mathbb{F}_m} \sim s^*_{\mathbb{F}_m}$, thus

$$\begin{aligned} i_{m+1} &\leq i_\mu(S_m, a) \\ &= i_\mu(S_m, s^*_{\mathbb{F}_m}) \\ &= i_\mu(S_m, s_{\mathbb{F}_m}) \\ &= i_m. \end{aligned}$$

It follows that for infinitely many m, $i_{m+1} < i_m$. For such $m \geq 2$, one can see that there are at least three distinct values of $i_\mu(S_{m+1}, a)$, as a varies over *ergodic* actions in $\mathrm{FR}(\mathbb{F}_{m+1}, X, \mu)$. This is a small initial step towards trying to understand the possible values of the independence number of free, ergodic actions of a free group (see the penultimate paragraph preceding 4.14). Recalling that the maximum value of $i_\mu(S_{m+1}, a)$, for ergodic $a \in \mathrm{FR}(\mathbb{F}_{m+1}, X, \mu)$, is equal to 1/2, this will follow from the following fact:

(ii) Let $m \geq 2$. Then there is a free, ergodic action b of $\mathbb{F}_{m+1}$ such that

$$i_m \leq i_\mu(S_{m+1}, b) < 1/2.$$

To see this, let $\varphi : \mathbb{F}_{m+1} \to \mathbb{F}_m$ be the homomorphism defined by $\varphi(\gamma_i) = \gamma_i$, if $i \leq m$, $\varphi(\gamma_{m+1}) = \gamma_m$. Let $d = s_{\mathbb{F}_m}$ and let c be the lift of d to $\mathbb{F}_{m+1}$ via φ:

$$\gamma^c(x) = \varphi(\gamma)^d(x).$$

Clearly $c|\mathbb{F}_m = d$ and $i_\mu(S_{m+1}, c) = i_\mu(S_m, d)$. Put

$$b = c \times s_{\mathbb{F}_{m+1}}.$$

Then $b \in \mathrm{FR}(\mathbb{F}_{m+1}, X, \mu)$ and b is ergodic. We will show that b is strongly ergodic thus, by 4.5, $i_\mu(S_{m+1}, b) < 1/2$. But also

$$i_\mu(S_{m+1}, b) \geq i_\mu(S_{m+1}, c) = i_\mu(S_m, d) = i_m.$$

If b is not strongly ergodic, towards a contradiction, there exist almost invariant sets for $b|\Gamma_0$, where $\Gamma_0 \leq \mathbb{F}_{m+1}$ is the group of words of even length in $\{\gamma_1, \dots, \gamma_{m+1}\}$, and thus there exist almost invariant sets for $b|\Gamma_0'$, where $\Gamma_0' \leq \mathbb{F}_m$ is the analogous group of words of even length in $\{\gamma_1, \dots, \gamma_m\}$. But

$$\begin{aligned} b|\mathbb{F}_m &= (c \times s_{\mathbb{F}_{m+1}})|\mathbb{F}_m \\ &= (c|\mathbb{F}_m) \times (s_{\mathbb{F}_{m+1}}|\mathbb{F}_m) \\ &\cong d \times (s_{\mathbb{F}_m})^{\mathbb{N}} \\ &\sim s_{\mathbb{F}_m} \times s_{\mathbb{F}_m} \\ &\sim s_{\mathbb{F}_m}, \end{aligned}$$

so $b|\Gamma_0' \sim s_{\mathbb{F}_m}|\Gamma_0'$, which is strongly ergodic (see, e.g., [17] A4.1), a contradiction.

(**E**) We will next see some connections with finite graphs.

Let Γ be a countable group and fix a sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq \Gamma$ of finite, non-empty subsets of Γ with $\bigcup_n F_n = \Gamma$. Consider the space 2^Γ with the product topology. If $p \in 2^{F_n}$, let

$$\mathcal{N}_p = \{f \in 2^\Gamma : f|F_n = p\}.$$

Then $\{\mathcal{N}_p\}_{n\geq 1, p\in 2^{F_n}}$ is a clopen basis for the topology of 2^Γ. Let now $S \subseteq \Gamma$ be a set of generators for Γ. Consider the finite graph with loops $G_{S,n} = (2^{F_n}, E_{S,n})$, where

$$pE_{S,n}q \Leftrightarrow \exists s \in S(s^{\pm 1} \cdot \mathcal{N}_p \cap \mathcal{N}_q \neq \emptyset).$$

Here $\gamma \cdot f$ ($\gamma \in \Gamma$, $f \in 2^\Gamma$) refers to the shift action of Γ on 2^Γ. Thus, there is a loop from p to p iff $\exists s \in S(s \cdot \mathcal{N}_p \cap \mathcal{N}_p \neq \emptyset)$. We may view each $G_{S,n}$ as a finite approximation of the graph associated with the shift action of Γ. Below recall that for a finite graph with loops $G = (X, E)$ its *independence ratio* $i(G)$ is defined as the ratio of the largest size of an independent set divided by $|X|$. When we have a graph with loops we define an independent set to be one for which there are no edges between two (not necessarily distinct) elements of A (thus, A cannot contain any vertex incident with a loop).

Theorem 4.19. *For the graphs $G_{S,n}$ as above, $i(G_{S,n}) \leq i(G_{S,n+1})$, $\forall n \geq 1$, and*

$$\lim_{n\to\infty} i(G_{S,n}) = i_\mu(S, s_\Gamma),$$

where μ is the usual product measure on 2^Γ.

Proof. Let $A \subseteq 2^{F_n}$ be an independent set for $G_{S,n}$, i.e., for $p, q \in A$ (not necessarily distinct), $s \cdot \mathcal{N}_p \cap \mathcal{N}_q = \emptyset$, $\forall s \in S^{\pm 1}$. This is the same thing as saying that

$$s \cdot \left(\bigcup_{p\in A} \mathcal{N}_p\right) \cap \bigcup_{p\in A} \mathcal{N}_p = \emptyset, \forall s \in S^{\pm 1}.$$

Let $A' \subseteq 2^{F_{n+1}}$ be defined by

$$q \in A' \Leftrightarrow q|F_n \in A.$$

Then $\frac{|A|}{|2^{F_n}|} = \frac{|A'|}{|2^{F_{n+1}}|}$ and for $p \in A$, $\mathcal{N}_p = \bigcup_{q\in A', q|F_n = p} \mathcal{N}_q$, so $\bigcup_{p\in A} \mathcal{N}_p = \bigcup_{p\in A} \bigcup_{q\in A', q|F_n=p} \mathcal{N}_q = \bigcup_{q\in A'} \mathcal{N}_q$, so $s \cdot \left(\bigcup_{q\in A'} \mathcal{N}_q\right) \cap \bigcup_{q\in A'} \mathcal{N}_q = \emptyset, \forall s \in S^{\pm 1}$, i.e., A' is independent for $G_{S,n+1}$. Thus $i(G_{S,n}) \leq i(G_{S,n+1})$.

Also if A is independent for $G_{S,n}$ and $\hat{A} = \bigcup_{p \in A} \mathcal{N}_p$, then $\hat{A}$ is independent for $G(S, s_\Gamma)$ and $\frac{|A|}{|2^{F_n}|} = \mu(\hat{A})$, thus $i(G_{S,n}) \leq i_\mu(S, s_\Gamma)$.

Assume now that $\alpha < i_\mu(S, s_\Gamma)$ and let $B \subseteq 2^\Gamma$ be an independent Borel set for $G(S, s_\Gamma)$ with $\mu(B) > \alpha$. Let $\varepsilon > 0$ and let $K \subseteq 2^\Gamma$ be compact with $K \subseteq B$, $\mu(B \setminus K) < \varepsilon$. Then K is also independent, so $s \cdot K \cap K = \emptyset$, $\forall s \in S^{\pm 1}$. Since the shift action is continuous, there is an open set $U \supseteq K$ with $\mu(U \setminus K) < \varepsilon$ such that $s \cdot U \cap U = \emptyset$, $\forall s \in S^{\pm 1}$, i.e., U is also independent. By compactness, let now n be large enough and $A \subseteq 2^{F_n}$ be such that $K \subseteq \hat{A} \subseteq U$ (where, as before, $\hat{A} = \bigcup_{p \in A} \mathcal{N}_p$). Thus A is independent in $G_{S,n}$ and so $\alpha - \varepsilon \leq \mu(\hat{A}) = \frac{|A|}{|2^{F_n}|} \leq i(G_{S,n})$. Letting $\varepsilon \to 0$ we have that

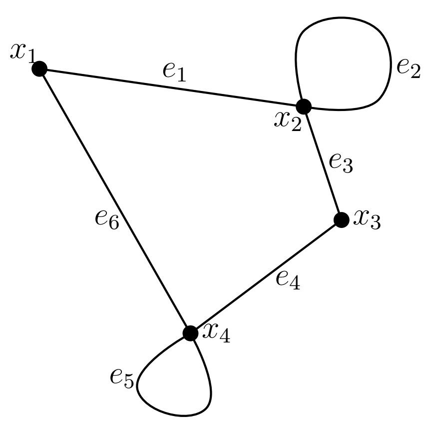
$$\alpha \leq \lim_{n \to \infty} i(G_{S,n}),$$

so

$$\lim_{n \to \infty} i(G_{S,n}) = i_\mu(S, s_\Gamma).$$

□

For the next result, if $G = (X, E)$ is a graph with loops, by a *cycle* we understand a sequence of distinct elements $x_1, \dots, x_n$ of X and distinct edges $e_1, \dots, e_m$ such that each e_i is either an edge connecting some x_j, x_{j+1}, $1 \leq k < n$, or x_n, x_0, or else a loop incident with some x_j, and moreover there is an edge e_i from each x_j to x_{j+1}, $1 \leq j < n$, and from x_n to x_0. The *length* of this cycle is the number m of edges. For example



is a cycle of length 6.

Theorem 4.20. *If* $\mathrm{Cay}(\Gamma, S)$ *is bipartite, then if*

$$X_{S,k,n} = \{p \in 2^{F_n} : p \text{ belongs to an odd cycle of length} \leq k \text{ in } G_{S,n}\},$$

we have $\frac{|X_{S,k,n}|}{|2^{F_n}|} \to 0$.

Proof. If $\hat{X}_{S,k,n} = \bigcup\{\mathcal{N}_p : p \in X_{S,k,n}\}$, we will show that

$$\mu(\hat{X}_{S,k,n}) = \frac{|X_{S,k,n}|}{|2^{F_n}|} \to 0.$$

Let $X \subseteq 2^\Gamma$ be the free part of the shift action of Γ on 2^Γ, so that $\mu(X) = 1$. It is enough to show that

$$\bigcap_{n \geq 1} \bigcup_{m \geq n} \hat{X}_{S,k,m} \subseteq 2^\Gamma \setminus X.$$

(Then $\lim_{n\to\infty} \mu\left(\bigcup_{m\geq n} \hat{X}_{S,k,m}\right) = 0$, so $\lim_{n\to\infty} \mu(\hat{X}_{S,k,n}) = 0$.)

Fix $x \in \bigcap_n \bigcup_{m \geq n} \hat{X}_{S,k,m}$. Then $x \in \hat{X}_{S,k,n_i}$, where $1 < n_1 < n_2 < \cdots$, so $x \in \mathcal{N}_{p_i}$, where $p_i \in X_{S,k,n_i}$. Then p_i belongs to some odd cycle of length $\leq k$, so, by going to a subsequence of (n_i), we may assume that every p_i belongs to a $(2l+1)$ cycle for some l with $2l+1 \leq k$. Then there are $p_i^0 = p_i, p_i^1, \ldots, p_i^{2l}$ in X_{S,k,n_i} and $s_i^0, s_i^1, \ldots, s_i^{2l}$ in $S^{\pm 1}$ with

$$\begin{aligned}
s_i^0 \cdot \mathcal{N}_{p_i^0} \cap \mathcal{N}_{p_i^1} &\neq \emptyset, \\
s_i^1 \cdot \mathcal{N}_{p_i^1} \cap \mathcal{N}_{p_i^2} &\neq \emptyset, \\
&\vdots \\
s_i^{2l-1} \cdot \mathcal{N}_{p_i^{2l-1}} \cap \mathcal{N}_{p_i^{2l}} &\neq \emptyset, \\
s_i^{2l} \cdot \mathcal{N}_{p_i^{2l}} \cap \mathcal{N}_{p_i^0} &\neq \emptyset.
\end{aligned}$$

By again passing to a subsequence of (n_i), we may assume that $s_i^0 = s^0, s_i^1 = s^1, \ldots, s_i^{2l} = s^{2l}$ do not depend upon i. Thus

$$\begin{aligned}
s^0 \cdot \mathcal{N}_{p_i^0} \cap \mathcal{N}_{p_i^1} &\neq \emptyset, \\
&\vdots \\
s^{2l} \cdot \mathcal{N}_{p_i^{2l}} \cap \mathcal{N}_{p_i^0} &\neq \emptyset.
\end{aligned}$$

By once again going to a subsequence of (n_i), we may assume that there are $x^0 = x, x^1, \dots, x^{2l} \in 2^\Gamma$ such that $x^k|i = p_i^k|i$ for all $k \le 2l$ and all i. Thus for each i,

$$\begin{aligned} s^0 \cdot \mathcal{N}_{x^0|i} \cap \mathcal{N}_{x^1|i} &\neq \emptyset, \\ &\vdots \\ s^{2l} \cdot \mathcal{N}_{x^{2l}|i} \cap \mathcal{N}_{x^0|i} &\neq \emptyset, \end{aligned}$$

therefore by the continuity of the shift action again,

$$s^0 \cdot x^0 = x^1, s^1 \cdot x^1 = x^2, \dots, s^{2l} \cdot x^{2l} = x^0,$$

i.e., $s^{2l}s^{2l-1}\cdots s^1 s^0 \cdot x = x$. Since $s^{2l}s^{2l-1}\cdots s^1 s^0 \neq 1$, we have $x \in 2^\Gamma \setminus X$. □

Remark 4.21. One can actually calculate quantitative upper bound estimates for $\frac{|X_{S,k,n}|}{|2^{F_n}|}$ in the preceding theorem.

Let

$$G_{S,k,n} = G_{S,n}|(2^{F_n} \setminus X_{S,k,n})$$

be the induced graph on $2^{F_n} \setminus X_{S,k,n}$. Then for n large enough (depending upon S, k), $2^{F_n} \setminus X_{S,k,n} \neq \emptyset$ and $G_{S,k,n}$ is an ordinary graph, i.e., has no loops. Moreover, the odd girth of $G_{S,k,n}$ is bigger than k, i.e.,

$$g_{\mathrm{odd}}(G_{S,k,n}) > k.$$

Furthermore, if $\delta_{S,k,n} = \frac{|X_{S,k,n}|}{|2^{F_n}|}$,

$$\begin{aligned} i(G_{S,k,n}) &\le \frac{1}{1-\delta_{S,k,n}} i(G_{S,n}) \\ &\le \frac{1}{1-\delta_{S,k,n}} i_\mu(S, s_\Gamma). \end{aligned}$$

Let now $\Gamma = \mathbb{F}_m$ with free generating set $S_m = \{a_1, \dots, a_m\}$, and let

$$G_{m,k,n} = G_{S_m,k,n}, \delta_{m,k,n} = \delta_{S_m,k,n}.$$

Then

$$i(G_{m,k,n}) \le \frac{1}{1-\delta_{m,k,n}} \cdot \frac{\sqrt{2m-1}}{m+\sqrt{2m-1}},$$

and $\delta_{m,k,n} \to 0$ as $n \to \infty$. Thus we have a new family of explicitly given (finite) graphs with large odd girth and small independence ratio, thus large chromatic number. For example,

Theorem 4.22. *Given m, k, for all large enough n (depending upon m, k),*

$$\begin{aligned} g_{\text{odd}}(G_{m,k,n}) &> k, \\ i(G_{m,k,n}) &\leq \frac{2\sqrt{2m-1}}{m+\sqrt{2m-1}}, \end{aligned}$$

and thus

$$\chi(G_{m,k,n}) \geq \frac{m+\sqrt{2m-1}}{2\sqrt{2m-1}}.$$

(**F**) There are many other actions of $\mathbb{F}_m$ that exhibit phenomena similar to those discussed in §4, (**D**), (**E**) before.

(a) An action $a \in A(\Gamma, X, \mu)$ is called *tempered* if $\kappa_0^a \prec \lambda_\Gamma$ (see Kechris [23]). It is clear that for such a free action a, we have that $\|T_{S,\lambda_\Gamma}\| = \|T_{S,a}\|$, for any finite set of generators $S \subseteq \Gamma$ and thus for $\Gamma = \mathbb{F}_m$ we have estimates for $i_\mu(S, a), \chi_\mu^{\text{ap}}(S, a)$ as in Theorem 4.17. Several examples of tempered actions of $\mathbb{F}_m$ are discussed in Kechris [23].

(b) It is shown in Lubotzky-Phillips-Sarnak, [32] that there are free actions a of $\mathbb{F}_m$, where $m = \frac{p+1}{2}$ with p prime, by rotations on the sphere S^2, for which the norm $\|T_{S,a}\|$ is given by the Kesten formula, i.e., is equal to $\frac{2\sqrt{p}}{p+1}$.

(c) Finally consider a countable, residually finite group Γ and a sequence $\Gamma_0 = \Gamma \geq \Gamma_1 \geq \cdots$ of decreasing normal subgroups which have finite index and $\bigcap_n \Gamma_n = \{1\}$. Then the action of Γ on the coset tree $T(\Gamma, (\Gamma_n))$ gives rise to an action of Γ on the boundary $\partial T(\Gamma, (\Gamma_n))$ of this tree. We can view $\partial T(\Gamma, (\Gamma_n))$ as a compact, metrizable, 0-dimensional group in which Γ is naturally embedded as a dense subgroup (for details, see Kechris [22], Section 2) and this action is simply the translation action of Γ on $\partial T(\Gamma, (\Gamma_n))$, so it is free and ergodic. Denote this action by $a_{\Gamma,(\Gamma_n)}$. Let S be a finite set of generators for Γ and let $T_{S,a_{\Gamma,(\Gamma_n)}} = T_{S,(\Gamma_n)}$ be the corresponding averaging operator. Let also $H_{S,n}$ be the Cayley graph of Γ/Γ_n, with respect to the generators which are the images of those in S under the canonical map of Γ onto Γ/Γ_n. When the graphs $H_{S,n}$ are not bipartite, it is known (see Lubotzky-Zuk [34], 2.6, where however the assumption that $H_{S,n}$ are not bipartite is inadvertently left out) that the chain (Γ_n) has property (τ) iff $\left\|T_{S,(\Gamma_n)}\right\| < 1$. Thus in this case the independence number of $a_{\Gamma,(\Gamma_n)}$ is less than $1/2$. When some $H_{S,n}$ is bipartite, then the independence number of $a_{\Gamma,(\Gamma_n)}$ is 1/2. It is not clear, e.g., in the case $\Gamma = \mathbb{F}_m$, what are the independence numbers of $a_{\Gamma,(\Gamma_n)}$, when (Γ_n) does not have property (τ). Could they all be equal to 1/2?

For certain free groups $\mathbb{F}_m$, one can actually construct (Γ_n) as above for which the norm of the corresponding averaging operator is given by the Kesten formula (see Margulis [36], Lubotzky-Phillips-Sarnak [33], Morgenstern [40]). Note that for the actions $a_{\Gamma,(\Gamma_n)}$ the graphs $H_{S,n}$ are analogs of the finite graphs $G_{S,n}$ discussed in §4 (**E**) above. In the case of the constructions of the three papers mentioned above, these are Ramanujan graphs.

(**G**) Suppose S is a finite set of generators for a group Γ, m is a probability measure on Γ supported by $S^{\pm 1}$ with $m(\gamma) = m(\gamma^{-1})$, and $\pi : \Gamma \to U(H)$ is a unitary representation. Then we can define again an averaging operator by

$$T_{S,m,\pi}(f) = \sum_{s \in S^{\pm 1}} m(s)\pi(s)(f)$$

and for $a \in \mathrm{FR}(\Gamma, X, \mu)$ let

$$T_{S,m,a} = T_{S,m,\kappa_0^a}.$$

If $\nu = \|T_{S,m,a}\|$, then the argument in 4.16 goes through and shows that $i_\mu(S,a) \leq \frac{\nu}{1+\nu}$.

Now it is easy to check that $(T_{S,m,a})^n = T_{S^n,m^{*n},a}$, where m^{*n} is the n-fold convolution of m, defined by

$$m^{*n}(\gamma) = \sum \{m(\gamma_1) \cdots m(\gamma_n) : \gamma_1 \cdots \gamma_n = \gamma\}.$$

It follows that if $\|T_{S,m,a}\| < 1$, then

$$\|T_{S^n,m^{*n},a}\| \leq \|T_{S,m,a}\|^n \to 0$$

as $n \to \infty$. It then follows that (exponentially) $i_\mu(S^n, a) \to 0$ as $n \to \infty$.

Take for example $\Gamma = \mathbb{F}_2$ with $S = \{a, b\}$ a free set of generators. Consider the graphs associated with (the free part of) the shift action s_Γ on 2^Γ with respect to the set of generators S^{2n+1} $(n \geq 1)$. Then (using as m the normalized counting measure on $S^{\pm 1}$, for which $T_{S,m,s_\Gamma} = T_{S,s_\Gamma}$) we see that $i_\mu(S^{2n+1}, s_\Gamma) \to 0$ as $n \to \infty$. Moreover, there are no odd cycles in these graphs. We can then repeat the arguments in 4.19 and 4.20 to find another infinite family of finite graphs $G'_{n,p,k}$ $(n, p, k \geq 1)$ such that for each n, k and p sufficiently large (depending upon n and k), we have $g_{\mathrm{odd}}(G'_{n,p,k}) > k$ and $i(G'_{n,p,k}) < \delta^n$, for some fixed constant $\delta < 1$ (here $\delta = \|T_{S,s_\Gamma}\|^2$).

5 On Brooks' Theorem

Recall that for a graph $G = (X, E)$, we let $\Delta(G)$ denote $\sup\{d_G(x) : x \in X\}$, where $d_G(x) = |\{y \in X : xEy\}|$. A point $x \in X$ is *monovalent* in G if $d_G(x) = 1$. In [26] it is shown that if $\Delta(G)$ is finite then $\chi_B(G) \leq \Delta(G) + 1$. For finite graphs G, Brooks' Theorem states that actually $\chi(G) \leq \Delta(G)$, unless G is an odd cycle or a complete graph. In this section we study Borel analogs of this bound.

Recall that E^* is the equivalence relation generated by E, whose classes are called the *connected components* of G, and that G is *connected* if E^* has only one class. We abbreviate $[x]_{E^*}$ by $[x]_G$. For a cardinal κ, we say G is *κ-connected* if $G|(X \setminus A)$ is connected for all $A \subseteq X$ with $|A| < \kappa$.

Also recall that we may view a graph G as inducing a metric ρ_G (informally called the *G-distance*) on each connected component of G by setting $\rho_G(x_0, x_1)$ equal to one less than the length of the shortest path from x_0 to x_1, where a *path* is a sequence of vertices, each G-related to the next.

A graph G on X is *vertex transitive* if its automorphism group acts transitively on X. We say G is *weakly 3-connected* if there exist $x_0, x_1 \in X$ such that $\rho_G(x_0, x_1) = 2$ and $G|(X \setminus \{x_0, x_1\})$ is connected. In the case that G is vertex transitive and not a complete graph, this is stronger than 2-connectivity and weaker than 3-connectivity.

Theorem 5.1. *Suppose that $G = (X, E)$ is a vertex-transitive Borel graph on a standard Borel space X whose connected components are each weakly 3-connected. Suppose further that $\Delta(G)$ is finite. Then $\chi_B(G) \leq \Delta(G)$.*

Proof. The argument is an amalgamation of the classical proof of Brooks' Theorem and the techniques involved in its analogue for approximate chromatic number (see Theorem 2.19). In addition to Lemma 2.18, we also require a technical lemma allowing us to find a nice subtree of G.

Lemma 5.2. *There is a Borel set $R \subseteq X$ and an acyclic Borel graph $T \subseteq G$, with vertex set X, such that*

1. *no two distinct points of R are within G-distance 3,*
2. *each connected component of T is finite,*
3. *each connected component of T contains exactly one point of R,*

4. *each point in R has two nonadjacent neighbors which are monovalent in T.*

Granting this, we may prove the theorem. Fix R and T as in Lemma 5.2. We think of R as a set of roots for the treed components of T. Let X_0 be the (Borel) set of neighbors of points in R granted by item 4 of the lemma. Let then $X_1 \subseteq X \setminus (X_0 \cup R)$ be those points monovalent in $T|(X \setminus X_0)$ and generally let $X_i \subseteq X \setminus (X_0 \cup \cdots \cup X_{i-1} \cup R)$ be those points monovalent in $T|(X \setminus (X_0 \cup \cdots \cup X_{i-1}))$. Item 2 of the lemma ensures that $X = R \sqcup \bigsqcup_{i \in \mathbb{N}} X_i$.

As X_0 is a G-independent set (by item 1), we may initially color every point in X_0 with color 0. Since every element of X_1 is adjacent to something closer (with respect to ρ_T) to R, each point in X_1 has degree less than $\Delta(G)$ in the restriction $G|(X_0 \cup X_1)$. Lemma 2.18 then allows us to extend our coloring to a Borel $\Delta(G)$-coloring of $X_0 \cup X_1$. Proceeding in this fashion, we extend our coloring in turn to each X_i until we have a Borel $\Delta(G)$-coloring of $X \setminus R$.

To complete the coloring, we simply need to choose colors for points in R. But each such point sees at most $\Delta(G)$ neighbors, and at least two of the neighbors receive color 0, so we may assign it the least color unused by its neighbors. □

Proof of Lemma 5.2. For convenience, fix some $x \in X$. Since G is weakly 3-connected, we may find nonadjacent neighbors y_0, y_1 of x such that $G|([x]_G \setminus \{y_0, y_1\})$ is connected. Fix $r \geq 2$ sufficiently large so that $G|(B_r(x) \setminus \{y_0, y_1\})$ is connected, where $B_r(x)$ denotes the ρ_G-ball of radius r about x. We may then fix a spanning tree H of $G|(B_r(x) \setminus \{y_0, y_1\})$ and subsequently extend it to a spanning tree H' of $G|B_r(x)$ by connecting y_0 and y_1 to x (leaving them monovalent).

Now, let R be a Borel maximal G^{2r}-independent subset of X, where G^{2r} is the graph relating two distinct points x_0, x_1 if $\rho_G(x_0, x_1) \leq 2r$. That is, no two distinct points of R are within G-distance $2r$, but every element of X is within G-distance $2r$ of something in R. Since for every $x_0, x_1 \in R$, $B_r(x_0) \cap B_r(x_1) = \emptyset$, we may "copy" in a Borel way H' onto each element of R to obtain an acyclic graph $T' \subseteq G$ connecting every element of $B_r(R) = \{x : \rho_G(x, R) \leq r\}$ to its nearest element of R.

We may extend this to an acyclic graph connecting every element of $B_{r+1}(R) = \{x : \rho_G(x, R) \leq r+1\}$ to exactly one element of R by connecting each element of $B_{r+1}(R) \setminus B_r(R)$ to one of its neighbors in $B_r(R)$. Continuing, we may extend step by step until we have an acyclic graph T connecting every element of $B_{2r}(R)$ to exactly one element of R. Since $B_{2r}(R) = X$, we are done once we know each connected component of T is finite. But since $[x]_T \subseteq B_{2r}(x)$ for all $x \in R$, each connected component of T must have cardinality at most $\Delta(G)^{2r}$. □

We spend the remainder of the section discussing graphs for which the hypotheses of Theorem 5.1 are met. Towards this end, we must recall some notions arising naturally in the study of connectivity of infinite graphs [21]. Given a graph $G = (X, E)$ and a subset $F \subseteq X$, we let ∂F denote the *(external) boundary* of F, defined as $\{x \in X \setminus F : \exists y \in F\ (xEy)\}$. In the notation of the proof of Theorem 5.1 we then have $\partial F = B_1(F) \setminus F$. We denote by F^e the *exterior* of F, defined as $X \setminus (F \cup \partial F)$. Equivalently, $F^e = X \setminus B_1(F)$.

Our first goal is a self-contained proof of the following:

Proposition 5.3 ([21]). *Suppose that $G = (X, E)$ is an infinite, connected, vertex-transitive graph with finite $\Delta(G) \geq 3$ and that F is a nonempty, finite subset of X. Then $|\partial F| \geq 3$.*

Proof. We let κ_f denote the smallest possible cardinality of a boundary of a finite nonempty set of vertices of G. That is,

$$\kappa_f = \min\{|\partial F| : F \subseteq X \text{ finite and nonempty}\}.$$

A *fragment* is a finite set F with $|\partial F| = \kappa_f$. Since $F \subseteq (F^e)^e \subseteq F \cup \partial F$ and also $\partial((F^e)^e) \subseteq \partial F$, we have that $(F^e)^e = F$ whenever F is a fragment.

Suppose now that F_1 and F_2 are two fragments. We have

$$\begin{aligned}
&|\partial(F_1 \cap F_2)| + |\partial(F_1 \cup F_2)| \\
&\qquad = |\partial(F_1 \cap F_2) \cap \partial(F_1 \cup F_2)| + |\partial(F_1 \cap F_2) \cup \partial(F_1 \cup F_2)| \\
&\qquad \leq |\partial F_1 \cap \partial F_2| + |\partial F_1 \cup \partial F_2| \\
&\qquad = |\partial F_1| + |\partial F_2| \\
&\qquad = 2\kappa_f.
\end{aligned}$$

In particular, if $F_1 \cap F_2$ is nonempty, it must be a fragment. We may therefore unambiguously define an *atom* as a minimal under inclusion fragment, noting that distinct atoms are disjoint. By transitivity, it follows that the atoms of G partition X.

It is therefore enough to show that $|\partial F| \geq 3$ for some atom F. Suppose that F_1 and F_2 are distinct atoms with adjacent vertices $x_1 E x_2$ such that $x_1 \in F_1$ and $x_2 \in F_2$. By reasoning as above, we see

$$\begin{aligned}
&|\partial(F_1 \cap F_2^e)| + |\partial(F_1^e \cap F_2)| \\
&\qquad = |\partial(F_1 \cap F_2^e) \cap \partial(F_1^e \cap F_2)| + |\partial(F_1 \cap F_2^e) \cup \partial(F_1^e \cap F_2)| \\
&\qquad \leq |\partial F_1 \cap \partial F_2| + |\partial F_1 \cup \partial F_2| \\
&\qquad = |\partial F_1| + |\partial F_2| \\
&\qquad = 2\kappa_f.
\end{aligned}$$

That is, if both $F_1 \cap F_2^e$ and $F_1^e \cap F_2$ were nonempty, then they would both be fragments. In particular, $F_1 \cap F_2^e = F_1$, i.e., $F_1 \subseteq F_2^e$, contradicting the fact that $F_1 \cap \partial F_2 \neq \emptyset$. Without loss of generality, we may assume $F_1^e \cap F_2$ is empty, and thus $F_2 \subseteq \partial F_1$. Certainly ∂F_1 cannot equal F_2 or (by transitivity) every atom would have a single atom as its boundary, forcing the graph to be finite (the union of at most two atoms). Thus $|\partial F_1| \geq |F_2| + 1$, which gives the desired bound as long as $|F_2| > 1$. But, of course, if $|F_2| = 1$, then $|\partial F_2| = \Delta(G) \geq 3$ as required. □

Remark 5.4. A more detailed investigation into the nature of fragments gives much more information about the connectivity of infinite graphs; see [21] for more details.

We will also need to borrow from the study of *ends* of a graph (see, e.g., [37], 11.4). Recall that we say a connected, locally finite graph $G(X, E)$ *has at most n ends* ($n \geq 0$) if for all finite $F \subseteq X$ the induced subgraph $G|(X \setminus F)$ has at most n infinite connected components. Then G *has n ends* if n is least such that G has at most n ends. If no such n exists, we say G has infinitely many ends. We may view the number of ends of G as the limit of the number of infinite components of $G|(X \setminus F_i)$, where $F_0 \subseteq F_1 \subseteq \cdots$ is an exhaustive sequence of finite subsets of X.

Recall that an connected, infinite, vertex-transitive graph has either one, two, or infinitely many ends (see [15], F64, p. 497). In this situation, knowing the number of ends of a graph can give information about its connectivity.

Proposition 5.5. *Suppose that $G = (X, E)$ is an infinite, connected, vertex-transitive graph with finite $\Delta(G)$ and assume that G has one end. Then G is 3-connected.*

Proof. Note that since G has one end, $\Delta(G) \geq 3$. Fix $F \subseteq X$ with $|F| \leq 2$. By Proposition 5.3, $G|(X \setminus F)$ has no finite connected components. Since G has one end, it follows that $G|(X \setminus F)$ is connected. □

On the other hand, knowledge of a graph's connectivity can translate into knowledge of its ends.

Proposition 5.6. *Suppose that $G = (X, E)$ is an infinite, connected, vertex-transitive graph with $\Delta(G) \geq 3$ which is not 2-connected. Then G has infinitely many ends.*

Proof. By transitivity, we have that $G|(X \setminus \{x\})$ is disconnected for every $x \in X$. Fix x_0, x_1 with $\rho_G(x_0, x_1) \geq 2$. Deleting x_0 results in at least two connected components, and further deleting x_1 splits its component into at least two subcomponents. Since Proposition 5.3 ensures that none of the components of $G|(X \setminus \{x_0, x_1\})$ is finite, $G|(X \setminus \{x_0, x_1\})$ has at least three infinite components. Thus, G has infinitely many ends. □

Recall that if Γ is a group with finite generating set S, the number of ends of $\mathrm{Cay}(\Gamma, S)$ is independent of the choice of S (see, e.g., [37], 11.4). We may thus say Γ has n ends exactly when $\mathrm{Cay}(\Gamma, S)$ has n ends (and similarly with infinitely many ends).

Proposition 5.7. *Suppose that Γ is a group with finite generating set S. Suppose further that Γ has two ends and is isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then $\mathrm{Cay}(\Gamma, S)$ is weakly 3-connected.*

Proof. It is well known that Γ has a finite index subgroup isomorphic to $\mathbb{Z}$ (see, e.g., [37], 11.4), a fact that we will use repeatedly below. In particular, this implies that every subgroup of Γ isomorphic to $\mathbb{Z}$ is of finite index. Moreover, there is an element $z \in \Gamma$ of infinite order such that either $z \in S$ or z is the product of two elements of S (see [45], p. 25 or [19]). Fix such a z such that $z^2 \notin S$.

Lemma 5.8. *There is some fixed N_z such that any vertex of $\mathrm{Cay}(\Gamma, S)$ is connected to an element of $\langle z \rangle$ by a path of length at most N_z.*

Proof. There must be a finite index subgroup $\Gamma_0 \leq \langle z \rangle$ such that Γ_0 is normal in Γ. Let $N_z = [\Gamma : \Gamma_0]$.

Now fix $x \in \Gamma$. Working in the quotient group Γ/Γ_0, there is a way of writing $x\Gamma_0$ as a word of the form $(s_1\Gamma_0)(s_2\Gamma_0) \cdots (s_k\Gamma_0)$ with each $s_i \in S$

and $k \leq N_z$. Now working in the Cayley graph $\mathrm{Cay}(\Gamma, S)$, we see that the path $(x, s_1^{-1}x, s_2^{-1}s_1^{-1}x, \ldots, s_k^{-1} \cdots s_1^{-1}x)$ connects x to some element of Γ_0, which is necessarily in $\langle z \rangle$. ☐

Suppose first that $z \in S$. The right cosets $\{\langle z \rangle\, a : a \in \Gamma\}$ partition the vertices of $\mathrm{Cay}(\Gamma, S)$ into finitely many sets, the graph's restriction to each resembling the Cayley graph of the integers.

Claim 5.9. *If $F \subseteq \Gamma$ is finite such that $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$ has two infinite connected components, then F meets every right coset of $\langle z \rangle$.*

Proof. By homogeneity, it is enough to prove that F meets $\langle z \rangle$. Assume not, towards a contradiction. If x has distance $> N_z$ from F, then 5.8 gives a path from x to $\langle z \rangle$ disjoint from F, so x is in the same component of $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$ as $\langle z \rangle$, thus $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$ has a unique infinite component, a contradiction. ☐

Recall that by Proposition 5.3, the deletion of two vertices of $\mathrm{Cay}(\Gamma, S)$ cannot result in a finite connected component. If we set $F_0 = \{z, z^{-1}\}$, then F_0 meets only one right coset of $\langle z \rangle$. Since Γ is not isomorphic to $\mathbb{Z}$, we may conclude from the claim that $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F_0)$ is connected, and thus $\mathrm{Cay}(\Gamma, S)$ is weakly 3-connected.

It remains to handle the case that no element of S has infinite order. Recall then that z is the product of two distinct elements of S, say $z = st$. We have a slightly weaker analog of the previous claim.

Claim 5.10. *If $F \subseteq \Gamma$ is finite such that $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$ has two infinite connected components, then $F \cup sF$ meets every right coset of $\langle st \rangle$.*

Proof. As before, each infinite connected component of $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$ meets each right coset of $\langle st \rangle$. Then for each right coset $\langle st \rangle\, a$, we have $b \in \langle st \rangle\, a$ and a path of the form $(b, tb, stb, tstb, \ldots, (st)^k b)$ in $\mathrm{Cay}(\Gamma, S)$ such that b and $(st)^k b$ are in distinct components of $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F)$. This means that some vertex x in the path must be an element of F.

If x is of the form $(st)^i b$, then $x \in \langle st \rangle\, a$. On the other hand, if x is of the form $t(st)^i b$, then $sx \in \langle st \rangle\, a$. Thus, $F \cup sF$ meets $\langle st \rangle\, a$. ☐

We now set $F_0 = \{t^{-1}, s\}$, and will show that $\mathrm{Cay}(\Gamma, S)|(\Gamma \setminus F_0)$ is connected. Suppose towards a contradiction that it is not connected; then by Proposition 5.3 it must have two infinite components. We see that F_0 meets a single right coset of $\langle st \rangle$, namely $\langle st \rangle\, s$. On the other hand, sF meets at most two other right cosets, $\langle st\rangle\, s^2$ and $\langle st\rangle\, st^{-1} = \langle st\rangle\, t^{-2}$. By the last claim, the union of these cosets must be the entire group (in particular $\Gamma = \langle s, t\rangle$), and so the identity must fall into one. The three equations

$$
\begin{aligned}
(st)^n &= s^{-1} \\
(st)^n &= s^{-2} \\
(st)^n &= t^2
\end{aligned}
$$

have solutions only when $n = 0$, otherwise the left-hand side has infinite order while the right-hand side has finite order. We conclude that at least one of s and t has order 2. Replacing st by $t^{-1}s^{-1}$ if necessary, we may assume without loss of generality that $t^2 = 1$.

Then $\Gamma = \langle st\rangle \cup \langle st\rangle\, s \cup \langle st\rangle\, s^2$, ensuring that s has order at most 3. If s has order 2, then $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, which is precluded by our hypothesis. Thus we may assume s has order 3.

Arguing as above, the right cosets $\langle st\rangle\, t$, $\langle st\rangle\, st$ and $\langle st\rangle\, s^2t$ are disjoint and cover Γ. It is clear that $\langle st\rangle = \langle st\rangle\, st$ and $\langle st\rangle\, t = \langle st\rangle\, s$, so $\langle st\rangle\, s^2 = \langle st\rangle\, s^2t$. The equation

$$(st)^n s^2 = s^2 t.$$

has a solution only when $n = 0$, since s^2ts^{-2} has finite order. Thus $s^2 = s^2t$ and consequently t is the identity, a contradiction. □

We finally apply these results to build a class of graphs satisfying Brooks' Theorem in the Borel context.

Theorem 5.11. *Suppose that G is a vertex-transitive Borel graph on a standard Borel space X with $\Delta(G)$ finite and whose connected components each have one end. Then $\chi_B(G) \leq \Delta(G)$.*

Proof. By Proposition 5.5, each connected component of G is 3-connected, and thus weakly 3-connected. Therefore, the hypotheses of Theorem 5.1 are met. □

Theorem 5.12. *Suppose that Γ is a countable infinite group isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Suppose further that Γ has finitely many ends. Let S be a finite set of generators for Γ and put $d = |S^{\pm 1}|$. Then for any free Borel action A of Γ on a standard Borel space X, we have $\chi_B(S, A) \leq d$.*

Proof. If Γ has one end, this is a consequence of Theorem 5.11. If Γ has two ends, Proposition 5.7 ensures that $G(S, A)$ meets the hypotheses of Theorem 5.1. ☐

Remark 5.13. The assumption that Γ is neither $\mathbb{Z}$ nor $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is necessary. If Γ is either of those groups equipped with its natural generating set S (with $|S^{\pm 1}| = 2$), then the free part of the shift action s_Γ of Γ on 2^Γ has Borel chromatic number 3.

Example 5.14. Finitely generated groups that have only finitely many ends, and thus Theorem 5.12 applies, include: Property (T) groups (see, e.g., [42]); groups of cost 1 (see Gaboriau [11]), and thus, in particular, amenable groups, direct products of two infinite groups, etc.; and groups not containing $\mathbb{F}_2$ (by Stallings' Theorem, see [44]). For a finitely generated torsion-free group Γ, Stallings' Theorem implies that Γ has infinitely many ends iff Γ is a non-trivial free product. So in this case 5.12 holds for any group that is not a non-trivial free product.

Example 5.15. Suppose now that Γ has a generating set S of cardinality 2 and finitely many ends but is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then for any free Borel action A of Γ which admits an invariant Borel probability measure with respect to which it is weakly mixing, we have $\chi_B(S, A) \in \{3, 4\}$. In particular, this holds for the free part of the shift action of Γ on 2^Γ. This generalizes a theorem of Gao-Jackson [12] and Miller, who proved this for $\Gamma = \mathbb{Z}^2$ (see §4, **(D)**).

Example 5.16. In Aldous-Lyons [4], 10.5, it is pointed out that for any sofic group Γ and finite generating set S, there is a free, measure preserving action a of Γ with $\chi_\mu(S, a) \leq d = |S^{\pm 1}|$.

It is still unknown whether the Brooks bound holds for groups with infinitely many ends, even in the torsion-free context. In fact, the following question remains unanswered.

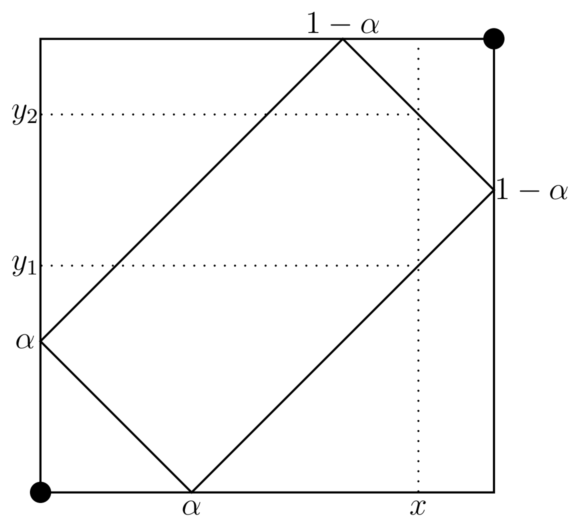
Question 5.17. *Does every graph corresponding to a Borel action of $\mathbb{F}_n$ $(n \geq 2)$, with a free set of generators, admit a Borel coloring with $2n$ colors?*

6 A matching problem

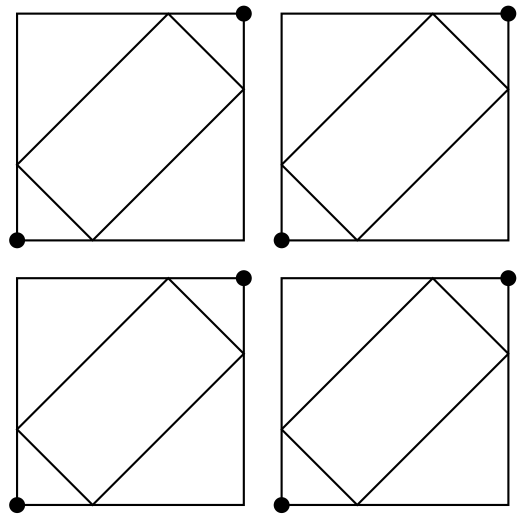
Consider a Borel bipartite graph $G = (X, E)$, i.e., $X = X_1 \sqcup X_2$ is a Borel partition and if $(x, y) \in E$ then one of x, y is in X_1 and the other is in X_2. If

$d(x) = k < \aleph_0$ for every $x \in X$, then by a theorem of König (a special case of Hall's Theorem), G admits a *matching*, i.e., a bijection $\varphi : X_1 \to X_2$ such that $(x, \varphi(x)) \in E$, $\forall x \in X$. The question was raised (see, e.g., Miller [38]) whether there is a Borel version of that theorem, more precisely, whether there is a Borel matching.

Laczkovich [30] provided the following counterexample for $k = 2$. Fix an irrational $0 < \alpha < 1$ and consider the set R consisting of the following rectangle inscribed in the unit square, together with the indicated two corner points.

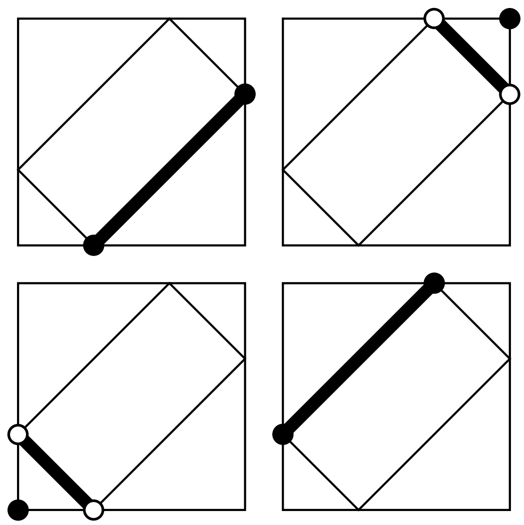


We take X_1, X_2 to be two disjoint copies of $[0, 1]$ and for $x \in X_1$ its two neighbors $y_1, y_2 \in X_2$ are such that $(x, y_i) \in R$. The two neighbors of any $y \in X_2$ are defined analogously. Clearly this is a Borel graph in which every vertex has degree 2, but Laczkovich showed that it does not have a Borel matching. In the paper Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [29], the authors argue that the following graph



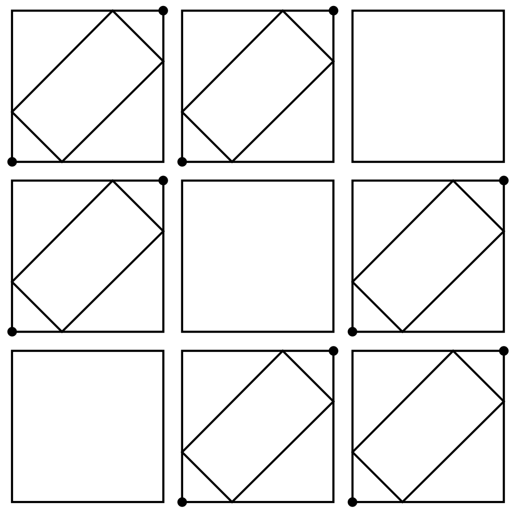
which consists of 4 "copies" of the preceding graph (actually, the authors discard finitely many connected components rather than adding dots at the corners, but it is clear that one graph has admits Borel matching if and only if the other does), and in which every vertex has degree 4 provides a counterexample for $k = 4$ (and similarly for all even k). They also raised the question of whether there is a counterexample for $k = 3$.

Lyons (private communication) showed that the above example actually does not work, as it has a Borel matching. A simpler argument is as follows:



The boldface segments and dots provide the matching, where as usual an endpoint of a segment is colored black if it is included, and is colored white if it is not included.

However, it turns out that there is a way to modify this construction to find counterexamples for every even k. For example, for $k = 4$, the idea is to construct a "Sudoku" version which is illustrated in the following picture:



Let us give a detailed argument. Fix a Borel bipartite graph $(X_1 \sqcup X_2, E)$ with degree $k = 2$ possessing no Borel matching. Define from this a new graph $(\overline{X}_1 \sqcup \overline{X}_2, \overline{E})$ as follows: $\overline{X}_1 = X_1 \times \{1,2,3\}$, $\overline{X}_2 = X_2 \times \{1,2,3\}$, and $(x,i)\overline{E}(y,j) \Leftrightarrow (i \neq j \text{ and } xEy)$. This has degree $k = 4$ and it is enough to show that if there is a Borel injection $\overline{f} : \overline{X}_1 \to \overline{X}_2$ such that $(x,i)\overline{E}\ \overline{f}(x,i)$, then there is a Borel injection $f : X_1 \to X_2$ with $xEf(x)$ (and similarly if we switch the roles of $\overline{X}_1, \overline{X}_2$). Granting this, if there is a Borel matching for $(\overline{X}_1 \sqcup \overline{X}_2, \overline{E})$, there are two Borel injections, from X_1 to X_2 and vice versa, whose graphs are contained in E, so, by a Schröder-Bernstein argument, there is a Borel matching for $(X_1 \sqcup X_2, E)$, a contradiction.

So fix $\overline{f}$ as above, which we will use to define f. Given $x \in X$, consider $\overline{f}(x,1) = (u,a)$, $\overline{f}(x,2) = (v,b)$, and $\overline{f}(x,3) = (w,c)$. Then xEu, xEv, and xEw. Since $(X_1 \sqcup X_2, E)$ has degree 2, at least two of u,v,w are equal. So there is a unique $y \in X_2$ such that for at least two distinct $i,j \leq 3$, we have $\overline{f}(x,i) = (y,k)$, $\overline{f}(x,j) = (y,l)$ (for some necessarily distinct k,l). Put $f(x) = y$; we claim that this works. To see this, take $x \neq x'$. If $f(x) = f(x') = y$, then let $i \neq j$ be such that $\overline{f}(x,i) = (y,k)$, $\overline{f}(x,j) = (y,l)$ and let $i' \neq j'$ be such that $\overline{f}(x',i') = (y,k')$, $\overline{f}(x',j') = (y,l')$. As before, $k \neq l$ and $k' \neq l'$. It follows that one of k,l is equal to one of k',l', contradicting the injectivity of $\overline{f}$.

The same proof works for degree $k = 6$ by dropping from the definition of

$\overline{E}$ the condition $i \neq j$ (i.e., in the preceding picture inscribing the rectangle into all nine of the small squares). In general, for degrees $k = 4n$ and $k = 4n + 2$ ($n \geq 1$) one uses the same argument with the $(2n+1) \times (2n+1)$ square.

As far as we know, the case $k = 3$ is open. We sketch below an alternative approach to the $k = 2$ case which adapts naturally to the $k = 3$ case, relating the question of whether a bipartite graph has no Borel matching to the calculation of the independence number associated with the shift action of an appropriate group. This was actually for us a motivation for looking at the independence number of such graphs.

Let $m \geq 2$ and $A = \{1, a, a^2, \ldots, a^{m-1}\}$ and $B = \{1, b, b^2, \ldots, b^{m-1}\}$ be two copies of the cyclic group of order m. Let $\Gamma_m = A * B$ and consider the shift action of Γ_m on 2^{Γ_m}, and let $Y \subseteq 2^{\Gamma_m}$ be its free part. Let $X_1 = Y/A$, the set of A-orbits under the shift action, and $X_2 = Y/B$. Then X_1 and X_2 are standard Borel spaces and let $X = X_1 \sqcup X_2$. Define the bipartite graph $G_m = (X, E)$ by

$$pEq \Leftrightarrow p \cap q \neq \emptyset.$$

If $p \in X_1$, $q \in X_2$ and $p \cap q \neq \emptyset$, then for some $y \in p \cap q$, $p = A \cdot y$ and $q = B \cdot y$. Since the action of Γ on Y is free, clearly $p \cap q = \{y\}$. Thus there is a canonical bijective correspondence between Y and E, namely

$$y \mapsto \{A \cdot y, B \cdot y\}$$

(we view E here as a set of unordered pairs). Clearly each vertex in G_m has degree exactly m.

Suppose now that $f : X_1 \to X_2$ is a Borel matching for G. By the above identification, f can be viewed as a Borel subset $M \subseteq Y$ and the condition of being a matching corresponds exactly to the assertion that M meets every A-orbit in exactly one point, and likewise meets every B-orbit. That is, M is a common transversal for the A- and B-orbits.

The set $S = (A \cup B) \setminus \{1\} \subseteq \Gamma_m$ is a set of generators for Γ_m and the above condition for M implies that M is a Borel independent set for the graph $G(S, s_{\Gamma_m})$. Moreover it is clear that for the product measure μ on 2^{Γ_m}, $\mu(M) = 1/m$. Thus, in particular, if there is a Borel matching in $G_m = (X, E)$, then $i_\mu(S, s_{\Gamma_m}) \geq 1/m$.

On the other hand, if $i_\mu(S, s_{\Gamma_m}) \geq 1/m$ and the supremum is attained, say by a Borel independent set C, then C must meet almost every A-orbit and almost ever B-orbit in exactly one point. It follows that the existence of an almost everywhere Borel matching in G_m is equivalent to the statement that $i_\mu(S, s_{\Gamma_m}) = 1/m$ and the supremum is attained.

If $m = 2$ this is impossible: since the action s_{Γ_2} is weakly mixing, there can be no independent set of measure $1/2$. Thus there is no Borel matching in the graph G_2, providing an alternate proof of Laczkovich's theorem. In fact, there is no almost everywhere Borel matching in G_2.

We do not know whether there is a Borel matching for G_3. In an earlier version of this paper, we have asked whether there is even an almost everywhere Borel matching in G_3. However, Lyons and Nazarov [35] have now shown that this is indeed the case, or equivalently that $i_\mu(S, s_{\Gamma_3}) = 1/3$ and the supremum is attained. From this it follows that in fact $i_\mu(S, a) = 1/3$ for any $a \in \mathrm{FR}(\Gamma_3, X, \mu)$. That $i_\mu(S, a) \leq 1/3$ is clear since any independent set contains at most one element in each A-orbit. Since $s_{\Gamma_3} \prec a$ we also have the reverse inequality. It also follows that $\chi_\mu^{\mathrm{ap}}(S, a) = 3$. It is enough to prove it for $a = s_\Gamma$. To see this let T be a set meeting each triangle in exactly one point, a.e.. This is one color. Removing T we get an acyclic degree 2 graph, so 2 colors are enough up to any ε.

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