ED NELSON'S WORK IN QUANTUM THEORY

BARRY SIMON*

Abstract. We review Edward Nelson's contributions to nonrelativistic quantum theory and to quantum field theory.

1. Introduction

It is a pleasure to contribute to this celebration of Ed Nelson's scientific work, not only because of the importance of that work but because it allows me an opportunity to express my gratitude and acknowledge my enormous debt to Ed. He and Arthur Wightman were the key formulative influences on my education, not only as a graduate student but during my early postdoctoral years. Thanks, Ed!

I was initially asked to talk about Ed's work in quantum field theory (QFT), but I'm going to exceed my assignment by also discussing Ed's impact on conventional nonrelativistic quantum mechanics (NRQM). There will be other talks on his work on unconventional quantum theory.

After discussion of NRQM and the Nelson model, I'll turn to the truly great contributions: the first control of a renormalization, albeit the Wick ordering that is now regarded as easy, and the seminal work on Euclidean QFT.

Many important ideas I'll discuss below involve crucial remarks of Ed that — in his typically generous fashion — he allowed others to publish.

2. NRQM

Ed has very little published specifically on conventional NRQM but he had substantial impact through his lectures, students, and ideas from his papers that motivated work on NRQM. In particular, two of

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* Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592.

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my own books [68, 73] have subject matter motivated by what I learned from Ed.

(a) **Quadratic Forms.** Reed-Simon [64] call the perturbation theorem for closed quadratic forms the KLMN theorem for Kato, Lions, Lax-Milgram, and Nelson. Ed was not the first to prove the KLMN theorem nor was he the first to use the scale of spaces that lies behind rigged Hilbert space theory, but so far as I know, he is the first to use scales in the context of studying selfadjointness of operators associated to quadratic forms, not only in the KLMN theorem but in the selfadjointness theorem [57] I'll discuss below.

(b) **Path Integrals.** Ed's best known published paper on path integrals deals with Feynman path integrals [53], that is, for e^{-itH} where he uses the Trotter product formula to write $(e^{-itH}\varphi)(x)$ as a limit (in L^2 sense in x) of Riemann integral approximations to a formal path integral. One cannot take the limit inside the integral and get a well-defined measure in the conventional sense, so these ideas have had limited use as an analytic tool. Still, they have conceptual uses and have been the starting point for other work on Feynman path integrals [4, 18, 24, 79].

From my point of view, the most significant contribution of [53] is the idea of using the Trotter product formula to prove Feynman-Kac-type formulae, an idea which is now standard.

Even more, Ed was a strong proponent of using path integrals in NRQM, an attitude which permeated Princeton in the 1970's, for example, Aizeman-Simon [1], Carmona [12], and Lieb [45].

(c) **Selfadjointness Theorems.** Proving (essential) selfadjointness of unbounded operators on a suitable domain is a basic part of mathematical quantum theory. Besides the KLMN theorem already mentioned, Ed is responsible for two general theorems and played a role in a third.

If A is a Hermitian operator, an analytic vector for A is a $\varphi \in \cap_n D(A^n)$ so that for some $t > 0$,

$$\sum \frac{t^n \|A^n\varphi\|}{n!} < \infty.$$

In [52], Ed proved that if $D(A)$ contains a dense set of analytic vectors, then A is essentially selfadjoint. This is basic to representation theory. For extensions of [52], see Nussbaum [62] and Masson-McClary [51].

In [57], Ed proved a result that essentially says that if $N \geq 1$ is a second operator which is selfadjoint and

$$\pm A \leq c_1(N+1)$$

$$\pm i[N, A] \leq c_2(N+1),$$

then A is essentially selfadjoint (I suppress the technical issue of what $[N, A]$ means; see [57] or [64, Section X.5]). Ed applied this to selfadjointness of time-smeared quantum fields, a result that Glimm-Jaffe [28] also proved using commutator estimates on the operator, commutator, and double commutator.

While Ed didn't apply his commutator theorem to NRQM, Faris-Lavine did [20].

Finally, I should mention the dog that didn't bark [19]: selfadjointness and hypercontractive semigroups (the later are discussed in Section 4 below). Segal [67], following up on Ed's work in [56], proved that if H_0 generated a hypercontractive semigroup on $L^2(M, d\mu)$ and if $V \in L^2$, $e^{-tV} \in L^2$ (for all $t > 0$) for some function, V, then $H_0 + V$ is essentially selfadjoint on $e^{-H_0}[L^\infty]$ (see also [41, 70]). Rosen [65] in a concrete setting had a similar idea to Segal [67].

(d) **Diamagnetic Inequalities.** These inequalities state that if $H(a, V)$ is a quantum Hamiltonian (any number of dimensions or particles, any masses, and any magnetic vector potential, a, and scalar potential V with enough regularity to define H), then

$$|(\exp(-tH(a, V))\varphi(x))| \leq (\exp(-tH(0, V))|\varphi|)(x). \tag{1}$$

I named them diamagnetic inequalities since they imply a finite temperature analog of the fact that $\inf \operatorname{spec}(H(a, V)) \geq \inf \operatorname{spec}(H(0, V))$, an expression of the fact that in the absence of spin (i.e., of magnetic moments) and/or fermi statistics, energies increase in a magnetic field. One author tried to name them Nelson-Simon inequalities but the name didn't stick, so I guess I should apologize to Ed for coming up with a name that had such a nice ring to it.

What was Ed's role in this? The story begins with two of the selfadjointness results of the last section. Before 1972, the conventional wisdom was that selfadjointness results for $-\Delta + V$ on $L^2(\mathbb{R}^\nu)$ required V to be at least locally L^p with $p > \nu/2$ (and $p \geq 2$). Since $-\Delta - c/r^2$ for c large and $\nu \geq 5$ is not essentially selfadjoint on $C_0^\infty(\mathbb{R}^\nu)$, this condition would seem to be close to optimal since $\int_{|r|\leq 1} (r^{-2})^p \, d^\nu r < \infty$ if $p < \nu/2$. What I discovered is that the "correct" conditions are asymmetric — positive singularities need only be L^2. I proved that if $V \in L^2(\mathbb{R}^\nu, e^{-x^2} \, d^\nu x)$ and $V \geq 0$, then $-\Delta + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^\nu)$ for any ν.

The proof [70] went as follows. By Ed's result on hypercontractivity of the fixed Hamiltonians [56], $H_0 = -\Delta + x^2$ generates a hypercontractive semigroup after translating to $L^2(\mathbb{R}^\nu, \Omega_0^2 \, d^\nu x)$ with Ω_0

the ground state of H_0. By Segal's theorem and a simple approximation argument, $N = H_0 + V$ is selfadjoint on $C_0^\infty(\mathbb{R}^\nu)$. Now use $[N, -\Delta + V] = [x^2, -\Delta + V]$ to verify the hypotheses of Nelson's commutator theorem [57] to conclude that $-\Delta + V$ is essentially selfadjoint. Actually, in [70], I used a different argument from the Nelson commutator theorem, but I could have used it!

I conjectured that the weak growth restriction implicit in $\int V(x)^2 e^{-x^2}\, dx < \infty$ was unnecessary and that $V \geq 0$ and $V \in L^2_{\text{loc}}(\mathbb{R}^\nu)$ implied $-\Delta + V$ was essentially selfadjoint on $C_0^\infty(\mathbb{R}^\nu)$. Kato took up this conjecture and found the celebrated Kato's inequality approach to selfadjointness [43]. This is not the right place to describe this in detail (see [43] or [64, 75]), but what is important is that between the original draft he sent me and the final paper, he added magnetic fields and that he used as an intermediate inequality

$$|(\nabla - ia)\varphi| \geq \nabla|\varphi| \tag{2}$$

pointwise in x. Formally, (2) is obvious; for if $\varphi = |\varphi|e^{i\eta}$, then $\mathrm{Re}(e^{-i\eta}(\nabla - ia)\varphi) = \mathrm{Re}((\nabla - ia + i(\nabla\eta))|\varphi|) = \nabla|\varphi|$.

What I realized two years later was that by integrating (2) in x, one has

$$(|\varphi|, H(0,V)|\varphi|) \leq (\varphi, H(a,V)\varphi), \tag{3}$$

which implies the diamagnetism of the ground state. The analog of (3) for finite temperature is

$$\mathrm{Tr}(e^{-\beta H(a,V)}) \leq \mathrm{Tr}(e^{-\beta H(0,V)})$$

and this led me to conjecture the diamagnetic inequality (1).

At the time, every Thursday the mathematical physicists at Princeton got together for a "brown bag lunch." During 1973–78, the postdocs/assistant professors included Michael Aizenman, Sergio Albeverio, Yosi Avron, Jürg Fröhlich, Ira Herbst, Lon Rosen, and Israel Sigal. Lieb, Wightman, and I almost always attended, and often Dyson and Nelson did. After lunch, various people talked about work in progress. I discussed (3) and my conjecture (1), explaining that I was working on proving it. After I finished, Ed announced: "Your conjecture is true; it follows from the correct variant of the Feynman-Kac formula with a magnetic field." So the first proof of (1) was Ed's. Characteristically, he refused my offer to coauthor the paper where this first appeared with another semigroup-based proof [72].

I should mention that the simplest proof of (1) and my favorite [75] is very Nelsonian in spirit: One uses the Trotter product formula to get the semigroup $(e^{-tH(a,V)})$ as a limit of products of one-dimensional operators $e^{+t(\partial_j - ia_j)^2/n}$ and uses the fact that one-dimensional magnetic fields can be gauged away. This is Nelsonian for two reasons. The use of Trotter product formula in such a context is due to Ed, but also the proof is a poor man's version of Ed's original proof: The gauge transformations are just a discrete approximation to an Itô stochastic integral.

(e) **Point Interactions.** The subject of point interactions has been heavily studied (see, e.g., [3]). So far as I know, Ed was the first to study point interactions as limits of potentials with supports shrinking to a point. He presented this in his courses; an extension of the ideas then appeared in the theses of his students, Alberto Alonso and Charles Friedman [22]. The basic points are:

(i) If $\nu \geq 4$ and V_n is any sequence of potentials, say, each bounded (but not uniformly bounded in n), supported in $\{x \mid |x| < n^{-1}\}$, then $-\Delta + V_n \to -\Delta$ in strong resolvent sense.
(ii) If $\nu \geq 2$ and $V_n \geq 0$, (i) remains true.
(iii) If $\nu = 1, 2, 3$, there are special negative V_n's that have strong limits different from $-\Delta$, many with a single negative eigenvalue. These are the point integrations.

(i) is an immediate consequence of the fact that $\{f \in C_0^\infty(\mathbb{R}^\nu) \mid f \equiv 0$ in a neighborhood of $0\}$ is an operator core of $-\Delta$ if $\nu \geq 4$. While (ii) can be obtained by similar consideration of form cores (and a suitable, somewhat subtle, limit theorem for quadratic forms), in typical fashion, Ed explained it not from this point of view, but by noting that in dimension 2 or more and $x, y \neq 0$, almost every Brownian path from x to y in fixed finite time t avoids 0. Thus, in a Feynman-Kac formula, if V_n has shrinking support, the integrand goes to one (i.e., $\exp(-\int_0^t V_n(\omega(\nu)\,ds) \to 1)$; $V_n \geq 0$ is needed to use the dominated convergence theorem in path space.

3. The Nelson Model

A search in MathSciNet on "Nelson model" turns up nineteen papers, many of them recent [2, 5, 6, 7, 8, 9, 11, 13, 14, 23, 25, 37, 38, 39, 40, 47, 48, 49, 78], so I'd be remiss to not mention the model, although I'll restrict myself to describing the model itself and noting that Ed introduced it in [54] and studied it further in [55].

The nucleon space $\mathcal{H}^{(N)}$ is $L^2(\mathbb{R}^{3n})$ where n is fixed (most later papers take $n = 1$) with elements in $\mathcal{H}^{(N)}$ written $\psi(x_1, \dots, x_n)$ and free nucleon Hamiltonian

$$H^{(N)} = -\sum_{j=1}^{n} \Delta_j.$$

The meson space is the Fock space, $\mathcal{H}^{(M)}$, on $\mathbb{R}^3$ with creation operators $a^\dagger(k)$ ($k \in \mathbb{R}^3$). The meson has mass μ (Ed took $\mu > 0$; many applications take $\mu = 0$) and free Hamiltonian

$$H^{(M)} = \int \omega(k) a^\dagger(k) a(k)\, d^3k,$$

where

$$\omega(k) = (k^2 + \mu^2)^{1/2}.$$

One defines the cutoff field for fixed $x \in \mathbb{R}^3$ by

$$\varphi_\chi(x) = 2^{-1/2}(2\pi)^{-3/2} \int \omega(k)^{-1/2}(a(k)e^{ik\cdot x} + a^\dagger(k)e^{-ik\cdot x})\chi(k)\, dk.$$

Ed took χ to be a sharp cutoff (characteristic function of a large ball); some later authors take other smoother χ's. One defines

$$\mathcal{H} = \mathcal{H}^{(N)} \otimes \mathcal{H}^{(M)}$$

and on $\mathcal{H}$,

$$H_I = g \sum_{j=1}^n \varphi_\kappa(x_j),$$

where g is a coupling constant and now x is the nucleon coordinate. The Nelson model is the Hamiltonian

$$H^{(N)} + H^{(M)} + H_I.$$

This has been a popular model because it is essentially the simplest example of an interacting field theory with an infinite number of particles.

4. Hypercontractivity

The next two sections concern outgrowths of Nelson's seminal paper [56]. This paper of only five pages (and because of the format of the conference proceedings, they are short pages; in J. Math. Phys., it would have been less than two pages!) is remarkable for its density of good ideas. The following abstracts a notion Ed discussed in [56]:

Definition. Let $H_0 \geq 0$ be a positive selfadjoint operator on the Hilbert space $L^2(M, d\mu)$ with $d\mu$ a probability measure. We say e^{-tH_0} is a *hypercontractive semigroup* if and only if

(1) $\|e^{-tH_0}\varphi\|_p \leq \|\varphi\|_p$, $1 \leq p \leq \infty$, $t > 0$

(2) For some T_0 and some $C < \infty$,

$$\|e^{-TH_0}\varphi\|_4 \leq C\|\varphi\|_2. \qquad (4)$$

Here the bounds are intended as a priori on $\varphi \in L^2 \cap L^p$. Ed's key discovery in [56] is that if V is a function with $e^{-V} \in \cap_{p<\infty} L^p$, then $H_0 + V$ is bounded below (to define $H_0 + V$ in reasonable cases, one usually assumes also $V \in L^2$ but for any V, and each $k < \infty$, $V_k = \max(V, -k)$ allows $H_0 + V_k$ to be defined as a form sum, and one has $\inf_k \min(H_0 + V_k) > -\infty$).

The simplest proof of this boundedness result follows from the formula

$$\|e^{A+B}\| \leq \|e^A e^B\| \tag{5}$$

for selfadjoint operators A and B. This formula is associated with the work of Golden, Thompson, and Segal (see the discussion of Section 8a in Simon [74]). It is proven by a suitable use of the Trotter product formula and the fact that $\|CD\| \leq \|C\| \, \|D\|$. Typically, in Ed's application, he appeals to a Feynman-Kac formula which has the Trotter formula built in and a use of Hölder's inequality which can replace $\|CD\| \leq \|C\| \, \|D\|$ because in the path integral formulation, the operators become functions.

I'd like to sketch a proof of (5) since it is not appreciated that it follows from Löwner's theorem on monotonicity of the square root ([50]; see also [36, 42]). We start with

$$C^{1/2}\varphi = \frac{1}{\pi} \int_0^\infty w^{-1/2} (C+w)^{-1} C\varphi \, dw \tag{6}$$

By the functional calculus, it suffices to prove (6) when C is a number and by scaling when $C = 1$, in which case, by a change of variables, it reduces to an arctan integral. Since $C(C+w)^{-1} = 1 - w(C+w)^{-1}$, we have

$$\begin{aligned} 0 \leq C \leq D &\Rightarrow (C+w)^{-1} \geq (D+w)^{-1} \\ &\Rightarrow C^{1/2} \leq D^{1/2} \end{aligned} \tag{7}$$

which is Löwner's result.

Let A, B be finite Hermitian matrices. Since

$$0 \leq C \leq D \Leftrightarrow \|C^{1/2} D^{-1/2}\| \leq 1$$

(7) can be rewritten

$$\|C^{1/2} D^{-1/2}\| \leq 1 \Rightarrow \|C^{1/4} D^{-1/4}\|^2 \leq 1$$

which, letting $C^{1/2} = e^A$, $D^{1/2} = e^{-B}$, implies

$$\|e^{A/2} e^{B/2}\|^2 \leq \|e^A e^B\| \tag{8}$$

Iterating (8) implies

$$\|(e^{A/2^n} e^{B/2^n})^{2^n}\| \leq \|e^{A/2^n} e^{B/2^n}\|^{2^n} \leq \|e^A e^B\|$$

Taking $n \to \infty$ and using the Trotter product formula implies (5) for bounded matrices, and then (5) follows by a limiting argument.

Once one has (5), one gets lower boundedness by noting

$$\begin{aligned}\|e^{-TV}e^{-TH_0}\varphi\|_2 &\le \|e^{-TV}\|_4\|e^{-TH_0}\varphi\|_4\\ &\le C\|e^{-TV}\|_4\|\varphi\|_2\end{aligned}$$

so hypercontractivity and $e^{-4TV} \in L^1$ implies $\|e^{-T(H_0+V)}\| < \infty$.

The term "hypercontractive" appeared in my paper with Høegh-Krohn [41], which systematized and extended the ideas of Nelson [56], Glimm [27], Rosen [65], and Segal [67]. The name stuck, and I recall Ed commenting to me one day, with a twinkle in his eye that many know, that after all "hypercontractive" was not really an accurate term since the theory only requires (4) with $C < \infty$, not $C \le 1$! That is, e^{-TH_0} is only bounded from L^2 to L^4, not contractive. We should have used "hyperbounded," not "hypercontractive."

Ed was correct (of course!), but I pointed out (correctly, I think!) that hypercontractive had a certain ring to it that hyperbounded just didn't have. There was, of course, a double irony in Ed's complaint.

The first involves an issue that wasn't explicitly addressed in [56]. What Ed proved, using L^p properties of the Mehler kernel, is that for the one-dimensional intrinsic oscillator, $H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx}$ on $L^2(\mathbb{R}, \pi^{-1/2}e^{-x^2}\,dx)$, e^{-tH_0} is bounded from L^2 to L^4 if t is large enough with a bound on the norm between those spaces of the form $1 + O(e^{-t})$ as $t \to \infty$. Ed then applied this to a free quantum field in a box with periodic boundary conditions. Because the eigenvalues of relevant modes go $\omega_\ell \sim \ell$, one has $\prod_{\ell=0}^{\infty}(1 + e^{-\omega_\ell t})$ convergent, so this application is legitimate — [56] does not discuss anything explicit about the passage to infinitely many degrees of freedom, but this step was made explicit in [21]. (I thank Lenny Gross for making this point to me at the conference in Vancouver.) To handle cases like H_0 in infinite volume, it is important to know that for t large enough, e^{-tH_0} is actually a contraction from L^2 to L^4, so the discreteness of modes doesn't matter. This was accomplished by Glimm [27] who showed that if $H_0 1 = 0$, $H_0 \restriction \{1\}^\perp \ge m_0$ and (4) for some C, then by increasing T, (4) holds with $C = 1$.

The second irony concerns Ed's second great contribution to hypercontractivity: the proof in [60] of optimal estimates for second quantized semigroups — exactly the kind of special H_0 in e^{-tH_0} he considered in [56]. He proved such an operator from L^p to L^q was either not bounded or it was contractive!

His precise result is if $H \geq a \geq 0$, then $\Gamma(e^{-tH_0})$ is a contraction from L^q to L^p if $e^{-ta} \leq (q-1)^{1/2}/(p-1)^{1/2}$ and is not bounded otherwise. Here $\Gamma(\,\cdot\,)$ is second quantization of operators; see [71].

Ed's work in these two papers on hypercontractive estimates spawned an industry, especially after the discovery of log Sobolev inequalities by Federbush [21] and Gross [31]. Brian Davies, in his work on ultracontractivity [17] and on Gaussian estimates on heat kernels [15], found deep implications of extensions of these ideas. While I dislike this way of measuring significance, I note that eighty papers in MathSciNet mention "hypercontractive" in their titles or reviews and Google finds 269 hits. See [16, 32] for reviews of the literature on this subject.

5. Taming Wick Ordering

There was a second element in [56] besides hypercontractivity, namely, the control of e^{-tV}. I want to schematically explain the difficulty and the way Ed solved it. In adding the interaction to a free quantum field, one might start with a spatial cutoff and want to consider

$$V_{un} = \int_{-L}^{L} \varphi^4(x)\,dx,$$

where φ is a free field. If $g_1, \dots, g_8$ are Gaussian variables, then

$$\langle g_1 \dots g_8 \rangle = \sum_{\text{pairings}} \langle g_{i_1} g_{j_1} \rangle \dots \langle g_{i_4} g_{j_4} \rangle \tag{9}$$

over all 105 pairings of $1, \dots, 8$. Thus, in computing $\langle V_{un}^2 \rangle$, one gets $\langle \varphi^4(x)\varphi^4(y) \rangle$ and the pairings $\langle \varphi(x)\varphi(x) \rangle$ are infinite, since they are $\int \frac{dk}{\sqrt{k^2+\mu^2}} = \infty$.

The solution is the very simplest of renormalizations, Wick ordering. If g is a finite Gaussian variable, one defines

$$:g^4: = g^4 - 6\langle g^2 \rangle g^2 + 3\langle g^2 \rangle^2. \tag{10}$$

The constants are exactly chosen, so in using (9) to compute $\langle :g^4::h^4: \rangle$, all cross terms involving $\langle g^2 \rangle$ drop out and

$$\langle :g^4::h^4: \rangle = 24\langle gh \rangle^4 \tag{11}$$

which allows one to prove that $V = \int_{-L}^{L} :\varphi^4(x):\,dx$ makes sense and defines a function in L^2, indeed, in $\cap_{p<\infty} L^p$. The difficulty is that (10) says

$$:g^4: = (g^2 - 3\langle g^2 \rangle)^2 - 6\langle g^2 \rangle^2 \tag{12}$$

is no longer positive, so since $\langle \varphi^2(x) \rangle = \infty$, V is no longer bounded below. What Ed realized is that it was still true that e^{-V} is integrable (and that using hypercontractivity, $H_0 + V$ is bounded below).

To prove e^{-V} integrable, Ed made a momentum cutoff in φ to get a φ_κ with $\langle \varphi_\kappa^2 \rangle \sim \log(\kappa)$ realizing the rate of divergence. He then wrote $V = V_\kappa + V'_\kappa$ where V_κ is just $\int_{-L}^{L} : \varphi_\kappa^4(x) : \, dx$ and V'_κ is the remainder. By using (12), V_κ is bounded below

$$V_\kappa \geq -C(\log \kappa)^2 \tag{13}$$

Moreover, it is easy to bound

$$\langle (V'_\kappa)^{2j} \rangle = \frac{(8j)!}{2^{4j}(4j)!} \, \langle (V'_\kappa)^2 \rangle^j \tag{14}$$

using Gaussian variable calculations. Here $\langle (V'_\kappa)^2 \rangle$ goes to zero as a negative power of κ. Since (13) implies

$$\begin{aligned} \operatorname{Prob}(V \leq -c(\log \kappa)^2 - 1) &\leq \operatorname{Prob}(|V'_\kappa| \leq 1) \\ &\leq \langle (V'_\kappa)^{2j} \rangle \end{aligned}$$

for any κ, one can choose j using the explicit formula in (14) to optimize this bound and find

$$\operatorname{Prob}(V \leq -x - 1) \leq \exp(-c_1 \exp(c_2 x^{1/2})),$$

which implies $\langle e^{-Vp} \rangle < \infty$ for all p.

This general idea of decomposing the interactions, undoing the renormalization to control one piece, and using L^p estimates on the other piece became a standard tool in much later work [29] in constructive quantum field theory.

6. Euclidean Quantum Field Theory

I've saved the best for last. In 1971–72, Ed, with an important boost from Guerra [33] (also [34]), in part following up on work of Schwinger [66] and Symanzik [76, 77], caused a revolution in mathematical quantum field theory, at least the model-building side. Ed's Euclidean Field Theory and the lattice approximation of Guerra-Rosen-Simon [35] that it motivated totally changed the objects studied. The structure of QFT in 1973 looked very different from 1971, although it looks very similar in 1973 and 2003! The way high energy physicists looked at quantum fields also changed to Euclidean and lattice models during this period. I'm not enough of a historian of the subject to know to what extent mathematicians' work had an impact on them.

Ed developed his ideas in the first part of 1971, gave a few lectures at Princeton (I only heard the first and didn't understand where it was heading or what it was good for!), and a lecture in Berkeley [58]. In retrospect, it is remarkable that — despite Ed talking in Princeton at a lecture attended by all or almost all the local experts and in Berkeley to many of the other workers — there was almost no reaction. I don't remember the details of the Princeton talk, but I assume it was close to the published Berkeley lecture [58], and rereading it, I can perhaps reconstruct my own reasons for not catching on.

I think Ed partly had me in mind when he stated in [58]: "Probabilistic methods ... have been used in quantum field theory particularly by Glimm, Jaffe, Rosen and myself. The usual reaction of workers in the field is to recoil in horror and to attempt to find alternate methods."

It's true that until Euclidean Field Theory changed my tune, I tended to think of probabilists as a priesthood who translated perfectly simple functional analytic ideas into a strange language that merely confused the uninitiated. In [71], the dedication says: "To Ed Nelson who taught me how unnatural it is to view probability theory as unnatural."

Ed's lecture has one result that should have made everyone sit up and take notice — a really simple proof of the linear lower bounds on the cutoff vacuum energy. But by dressing the proof in layers of functors, Markov properties, multiplicative functionals, and only sketching it (my brief sketch below is much more detailed than his!), he perhaps obscured its great simplicity. Because I had then recently found my own simple (but it turns nor nearly as simple as Ed's!) proof [69], I put his work aside.

Francesco Guerra was visiting Princeton at the time and was rather quiet and unassuming. At Guerra's request, Arthur Wightman set up a meeting with Lon Rosen and me in January 1972. Guerra began by listing on the blackboard what he was going to prove. It was as if he were from another planet. His first result was the linear lower bound that Ed and others had proven. All his other results (starting with existence of the limits) were beyond anything that the then current technology could prove. I remember thinking to myself: "Yeah, sure, you're gonna prove all that." He proceeded to say he needed Nelson's ideas and, in particular, something he called Nelson's symmetry. I'd seen this on the blackboard during Ed's talk eight months before, but thought it a curiosity. Within fifteen minutes, Guerra had explained his proofs of the results that I'd found impossible to believe! It was like a thunderbolt.

Within a week, Guerra, Rosen, and I had used these ideas to recover [34] some bounds of Glimm-Jaffe [28] whose proofs were regarded as very hard. The next week, Glimm visited to give a talk on this bound, sketching the strategy of their proof. After his talk, Lon and I cornered Glimm and described the new proof. He seemed to have the same jaw-dropping reaction I'd had. Euclidean Field Theory had arrived, but at least six months late. Six months may not seem like a lot, but in the next six and twelve and eighteen months, there was a flood of new results that came from exploiting the Euclidean point of view (the breathless pace is described in some detail in the introduction of [71]).

This is not the place to give a minicourse on Euclidean QFT, but I'd like to make some general remarks to explain what Ed did.

Prior to Ed's work, the usual way path integrals came in was to cut off the field theory to get a finite-dimensional system, write down a path integral for that, estimate, and try to get results that survived removal of the cutoff. From my point of view, what Ed did first of all was to view the free field semigroup as a positivity-preserving operator on an infinite-dimensional space. Such a positivity condition allows one to build up an abstract path integral which, for the free field, could be written as an explicit Gaussian process. This process is formally an analytic continuation of the quantum field from real time to imaginary time, and the continuation of Minkowski invariance to the Gausssian process is Euclidean invariance, that is, the process covariance $C(x-y)$ is invariant under rotations. Indeed, C is the integral kernel (Green's function) for $(-\Delta + m^2)^{-1}$.

It turns out that the process Ed wrote down had been written down somewhat earlier by Pitt [63] who discussed its properties as a multi-parameter Markov process but didn't consider any connection to QFT.

Ed's proof of the linear lower bound depended on this Euclidean invariance. First, let us describe the Feynman-Kac-Nelson formula, the Feynman-Kac formula for this case.

Let H_0 be a free quantum field Hamiltonian, Ω_0 its ground state, and $V \equiv V(\varphi(x,0))$ as functions of the time zero fields. Then

$$\begin{aligned} &\langle F_1(\varphi(x,0))\Omega_0, e^{-t(H_0+V)} F_2(\varphi(x,0))\Omega_0\rangle \\ &\qquad = \int F_1(\varphi(x,t))F_2(\varphi(x,0)) e^{-\int_0^t V(\varphi(x,s))\,ds}\,d\mu_0 \end{aligned} \tag{15}$$

where μ_0 is the Gaussian measure for the free Euclidean field. The proof is by the Trotter product formula method of Nelson [53].

In particular, if $V_\ell \equiv \int_{-\ell/2}^{\ell/2} F(\varphi(x,0))\,dx$, where $F(\varphi)$ is a local function of the field (think $:\varphi^4(x,0):$) and $H_\ell(\lambda) = H_0 + \lambda V_\ell$, then Euclidean invariance and (15) imply Nelson's symmetry

$$Q_{t,\ell}(\lambda) \equiv \langle \Omega_0, e^{-tH_\ell(\lambda)}\Omega_0\rangle = \langle \Omega_0, e^{-\ell H_t(\lambda)}\Omega_0\rangle. \tag{16}$$

To get the linear lower bound on

$$E_\ell(\lambda) = \inf \operatorname{spec}(H_\ell(\lambda)),$$

we need to use hypercontractivity, Ed's earlier idea, and Hölder's inequality in the FKN formula (15):

$$\begin{aligned} |\langle F_1\Omega_0, e^{-tH_\ell(\lambda)}F_2\Omega_0\rangle| &\le \langle \Omega_0, e^{-tH_\ell(4\lambda)}\Omega_0\rangle^{1/4}\langle |F_1|^{4/3}\Omega_0, e^{-tH_0}|F_2|^{4/3}\Omega_0\rangle^{3/4} \\ &\le Q_{t,\ell}(4\lambda)^{1/4}\|F_1\|_2\|F_2\|_2, \end{aligned} \tag{17}$$

where t is chosen so e^{-tH_0} is a contraction from $L^{4/3}$ to L^4 (if e^{-sH_0} is a contraction from L^2 to L^4, by duality, it is a contraction from $L^{4/3}$ to L^2 and so e^{-2sH_0} is a contraction from $L^{4/3}$ to L^4).

(17) says that

$$\begin{aligned} e^{-tE_\ell(\lambda)} &\le Q_{t,\ell}(4\lambda)^{1/4} \\ &= Q_{\ell,t}(4\lambda)^{1/4} \qquad \text{by (16)} \\ &\le e^{-\ell E_t(4\lambda)/4}. \end{aligned}$$

Thus

$$E_\ell(\lambda) \ge \frac{\ell}{4t}\, E_t(4\lambda),$$

which is the linear lower bound.

This is actually the hardest application of Nelson's symmetry. What Guerra shocked us with is actually much simpler! For any selfadjoint operator, A, bounded from below, and any unit vector φ, we have

$$\langle \varphi, e^{-tA}\varphi\rangle = \int_\alpha^\infty e^{-tx}\, d\mu(x)$$

for some α ($= \inf \operatorname{spec}(A)$) and probability measure, $d\mu$. Hölder's inequality thus implies

$$t \to \log\langle \varphi, e^{-tA}\varphi\rangle$$

is convex. Since $E_\ell = -\lim_{t\to\infty} \frac{1}{t} Q_{t,\ell}$ (we henceforth set $\lambda \equiv 1$ and drop it!), we see, by Nelson's symmetry, that $\ell \to E_\ell$ is concave. Concave functions, g, with a linear lower bound always have that $g(\ell)/\ell$ has a finite limit, e_∞, and $g(\ell) - \ell e_\infty$ is monotone, and so it also has a limit! In this way, Guerra showed that the energy per unit volume and the surface energy actually converged!

There is a postscript to Ed's breakthrough, a final significant contribution. Guerra, Rosen, and I realized that (15) made Euclidean QFT look like classical statistical mechanics (at least for bosons). A key tool in the rigorous statistical mechanics of the time were correlation inequalities, starting with Griffiths [30]. It was natural to try to prove them in EQFT by approximating with a discrete system. Since the free EQFT was the Gaussian process with covariance $(-\Delta + m^2)^{-1}(x, y)$, it was natural to use a Gaussian lattice theory with covariance $(-\Delta_\delta + m^2)^{-1}_{ij}$ where Δ_δ is a discrete Laplacian. In the Gaussian measure, the inverse of covariance matrix appears, so the Gaussian measure here has Δ_δ in it, that is, nearest neighbor interactions. So was born the lattice approximation [35]. Using ideas of Ginibre [26] for general classes of spin systems, we could get correlation inequalities for EQFT!

In the applications though, there was one catch. One of the nicest applications of Griffiths' work was the existence of the infinite volume limit. A system of spins σ_j in volume $\wedge$ with no spins outside $\wedge$ has correlations monotone in $\wedge$, so a limit existed. Our problem was that the $\delta \downarrow 0$ limit we could control was to the free, infinite volume $(-\Delta+m^2)^{-1}$ Gaussian process. The $\wedge$ dependence was where the local interaction was turned on. There was no monotonicity in $\wedge$ in this limit.

We could follow Griffiths and take a sharp cutoff lattice theory, take the limit for that as $\wedge \to \infty$, and then hope to take $\delta \downarrow 0$, but the lattice cutoff destroyed rotation symmetry.

Ed made a crucial remark. The finite $\wedge$ theory with sharp cutoff had a limit as $\delta \downarrow 0$. It was just a theory where the $(-\Delta + m^2)^{-1}$ Gaussian process was replaced by $(-\Delta_\wedge + m^2)^{-1}$ with $\Delta_\wedge$ a Dirichlet Laplacian! One needs to take a $\wedge$-independent local interaction so Wick order is done relative to Δ not $\Delta_\wedge$ ([35] calls this "half-Dirichlet" boundary conditions). With this remark, it was easy to get a Euclidean invariant infinite volume limit, and so, using other ideas of Nelson [59], the first interesting field theory obeying all the Wightman axioms except perhaps uniqueness of the vacuum.

Many would have insisted on publishing this crucial remark in their own name, but Ed urged us to use it and we, of course, acknowledged his contribution. [61] does include this remark, but it is unclear from the discussion there that the remark is due to Ed and not to GRS!

Kuhn [44] regards the hallmark of a scientific revolution as a change of paradigm. The only way to think of the change in rigorous QFT produced by Ed's introduction of EQFT ideas is as such a change of paradigm.

7. Some Concluding Remarks

Lipman Bers in the introduction to Löwner's "Complete Works" [10] described Löwner in a way that so accurately describes Ed Nelson that I'll quote it here: "... was a man whom everyone liked, perhaps because he was a man at peace with himself. He conducted a life-long passionate love affair with mathematics, but was neither competitive, nor jealous, nor vain. His kindness and generosity in scientific matters, to students and colleagues alike, were proverbial. He seemed to be incapable of malice. His manners were mild and even diffident, but those hid a will of steel ... But first and foremost, he was a mathematician." Contemplate how many really first-class mathematicians for whom one can say they are "neither competitive, nor jealous, nor vain" and appreciate Ed for who he is!

References

[1] M. Aizenman and B. Simon, *Brownian motion and Harnack's inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–273.

[2] S. Albeverio, *Scattering theory in a model of quantum fields. I*, J. Math. Phys. **14** (1973), 1800–1816.

[3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Texts and Monographs in Physics, Springer-Verlag, New York, 1988.

[4] S. Albeverio and R. Høegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics, Vol. 523, Springer-Verlag, Berlin-New York, 1976.

[5] Z. Ammari, *Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: The Nelson model*, Math. Phys. Anal. Geom. **3** (2000), 217–285.

[6] A. Arai, *Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation*, Rev. Math. Phys. **13** (2001), 1075–1094.

[7] ———, *The massless Nelson model without infrared cutoff in a non-Fock representation* in "Analytical Study of Quantum Information and Related Fields" (Kyoto, 2001), pp. 82–93, Sūrikaisekikenkyūsho Kōkyūroku, 1266, (2002). [Japanese]

[8] G. Artbazar, A. Jensen, and K. Yajima, *The Nelson model with few photons*, in "Spectral and Scattering Theory and Related Topics" (Kyoto, 2001), pp. 124–143, Sūrikaisekikenkyūsho Kōkyūroku, 1255, (2002). [Japanese]

[9] ———, *The Nelson model with less than two photons*, Ann. Henri Poincaré **4** (2003), 239–273.

[10] L. Bers, *Introduction to "Charles Loewner: Collected Papers"*, Contemporary Mathematicians, Birkhäuser, Boston, Mass., 1988.

[11] V. Betz, F. Hiroshima, J. Lörinczi, R.A. Minlos, and H. Spohn, *Ground state properties of the Nelson Hamiltonian: A Gibbs measure-based approach*, Rev. Math. Phys. **14** (2002), 173–198.

[12] R. Carmona, *Schrödinger eigenstates*, Comm. Math. Phys. **62** (1978), 97–106.

[13] M. Davidson, *On the equivalence of quantum mechanics and a certain class of Markov processes*, J. Math. Phys. **19** (1978), 1975–1978.

[14] ———, *The generalized Fényes-Nelson model for free scalar field theory*, Lett. Math. Phys. **4** (1980), 101–106.

[15] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, 92, Cambridge Univ. Press, Cambridge, 1989.

[16] E.B. Davies, L. Gross, and B. Simon, *Hypercontractivity: A bibliographic review*, in "Proc. Høegh-Krohn Memorial Conference, Ideas and Methods in Quantum and Statistical Physics" (Oslo, 1988), pp. 370-389, Cambridge Univ. Press, Cambridge, 1992.

[17] E.B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395.

[18] C. DeWitt-Morette, *Feynman path integrals. I. Linear and affine techniques; II. The Feynman-Green function*, Comm. Math. Phys. **37** (1974), 63–81.

[19] A. Conan Doyle, *"The Adventure of Silver Blaze"* (Sherlock Holmes).

[20] W.G. Faris and R.B. Lavine, *Commutators and self-adjointness of Hamiltonian operators*, Comm. Math. Phys. **35** (1974), 39–48.

[21] P. Federbush, *Partially alternate derivation of a result of Nelson*, J. Math. Phys. **10** (1969), 50–52.

[22] C.N. Friedman, *Perturbations of the Schroedinger equation by potentials with small support*, J. Funct. Anal. **10** (1972), 346–360.

[23] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field*, Adv. Math. **164** (2001), 349–398.

[24] D. Fujiwara, *Remarks on convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), 559–600.

[25] C. Gérard, *On the scattering theory of massless Nelson models*, Rev. Math. Phys. **14** (2002), 1165–1280.

[26] J. Ginibre, *General formulation of Griffiths' inequalities*, Comm. Math. Phys. **16** (1970), 310–328.

[27] J. Glimm, *Boson fields with nonlinear self-interaction in two dimensions*, Comm. Math. Phys. **8** (1968), 12–25.

[28] J. Glimm and A. Jaffe, *The $\lambda\phi_2^4$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian*, J. Math. Phys. **13** (1972), 1568–1584.

[29] ———, *Quantum Physics. A Functional Integral Point of View*, Springer-Verlag, New York-Berlin, 1981.

[30] R.B. Griffiths, *Correlations in Ising ferromagnets. I*, J. Math. Phys. **8** (1967), 478–483.

[31] L. Gross, *Logarithmic Sobolev inequalities*, Am. J. Math. **97** (1975), 1061–1083.

[32] L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups*, in "Dirichlet Forms" (Varenna, 1992), pp. 54–88, Lecture Notes in Math., 1563, Springer, Berlin, 1993.

[33] F. Guerra, *Uniqueness of the vacuum energy density and van Hove phenomenon in the infinite-volume limit for two-dimensional self-coupled Bose fields*, Phys. Rev. Lett. **28** (1972), 1213–1215.

[34] F. Guerra, L. Rosen, and B. Simon, *Nelson's symmetry and the infinite volume behavior of the vacuum in $P(\phi)_2$*, Commun. Math. Phys. **27** (1972), 10–22.

[35] ———, *The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics*, Ann. of Math. (2) **101** (1975), 111–189; ibid. **101** (1975), 191–259.

[36] E. Heinz, *Beiträge zur Störunstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.

[37] F. Hiroshima, *Diamagnetic inequalities for systems of nonrelativistic particles with a quantized field*, Rev. Math. Phys. **8** (1996), 185–203.

[38] ———, *Asymptotic behaviors of the interaction Hamiltonians of quantum fields and particles*, in "Proc. Second World Congress of Nonlinear Analysts, Part 8" (Athens, 1996), Nonlinear Anal. **30** (1997), 4863–4874.

[39] ———, *Strong and weak coupling limits of interaction models of quantum fields and particles*, in " Quantum Probability Theory and Entropy Analysis" (Kyoto, 1997), pp. 207–224, Sūrikaisekikenkyūsho Kōkyūroku, 1013, (1997). [Japanese]

[40] R. Høegh-Krohn, *Asymptotic fields in some models of quantum field theory. II, III*, J. Math. Phys. **10** (1969), 639–643; ibid. **11** (1969), 185–188.

[41] R. Høegh-Krohn and B. Simon, *Hypercontractive semigroups and two-dimensional self-coupled Bose fields*, J. Funct. Anal. **9** (1972), 121–180.

[42] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.

[43] ———, *Schrödinger operators with singular potentials*, Israel J. Math. **13** (1972), 135–148 (1973).

[44] T.S. Kuhn, *The Structure of Scientific Revolutions*, Univ. of Chicago Press, Chicago, 1962.

[45] E. Lieb, *Bounds on the eigenvalues of the Laplace and Schroedinger operators*, Bull. Amer. Math. Soc. **82** (1976), 751–753.

[46] C. Loewner, *Collected Papers*, (L. Bers, ed.), Contemporary Mathematicians, Birkhäuser, Boston, Mass., 1988.

[47] J. Lörinczi and R.A. Minlos, *Gibbs measures for Brownian paths under the effect of an external and a small pair potential*, J. Statist. Phys. **105** (2001), 605–647.

[48] J. Lörinczi, R.A. Minlos, and H. Spohn, *Infrared regular representation of the three-dimensional massless Nelson model*, Lett. Math. Phys. **59** (2002), 189–198.

[49] ———, *The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field*, Ann. Henri Poincaré **3** (2002), 269–295.

[50] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177–216.

[51] D. Masson and W.K. McClary, *Classes of C^∞ vectors and essential self-adjointness*, J. Funct. Anal. **10** (1972), 19–32.

[52] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.

[53] ———, *Feynman integrals and the Schrödinger equation*, J. Math. Phys. **5** (1964), 332–343.

[54] ———, *Schrödinger particles interacting with a quantized scalar field*, in "Analysis in Function space," pp. 87–120, M.I.T. Press, Cambridge, Mass., 1964.

[55] ———, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), 1190–1197.

[56] ———, *A quartic interaction in two dimensions*, in "Mathematical Theory of Elementary Particles" (Dedham, Mass., 1965), pp. 69–73, M.I.T. Press, Cambridge, Mass., 1966.

[57] ———, *Time-ordered operator products of sharp-time quadratic forms*, J. Funct. Anal. **11** (1972), 211–219.

[58] ———, *Quantum fields and Markoff fields*, in "Partial Differential Equations" (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), pp. 413–420, American Mathematical Society, Providence, R.I., 1973.

[59] ———, *Construction of quantum fields from Markoff fields*, J. Funct. Anal. **12** (1973), 97–112.

[60] ———, *The free Markoff field*, J. Funct. Anal. **12** (1973), 211–227.

[61] ———, *Probability theory and Euclidean field theory*, in "Constructive Quantum Field Theory," Proc. 1973 "Ettore Majorana" International School of Mathematical Physics (Erice, 1973), pp. 94–124, (G. Velo and A. Wightman, eds.), Lecture Notes in Physics, 25, Springer-Verlag, Berlin-New York, 1973.

[62] A.E. Nussbaum, *Quasi-analytic vectors*, Ark. Mat. **6** (1965), 179–191.

[63] L.D. Pitt, *A Markov property for Gaussian processes with a multidimensional parameter*, Arch. Rational Mech. Anal. **43** (1971), 367–391.

[64] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[65] L. Rosen, *A $\lambda\phi^{2n}$ field theory without cutoffs*, Comm. Math. Phys. **16** (1970), 157–183.

[66] J. Schwinger, *On the Euclidean structure of relativistic field theory*, Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 956–965.

[67] I. Segal, *Construction of non-linear local quantum processes. I*, Ann. of Math. (2) **92** (1970), 462–481.

[68] B. Simon, *Quantum Mechanics for Hamiltonians Defined by Quadratic Forms*, Princeton Series in Physics, Princeton Univ. Press, Princeton, N.J., 1971.

[69] ———, *On the Glimm-Jaffe linear lower bound in $P(\phi)_2$ field theories*, J. Funct. Anal. **10** (1972), 251–258.

[70] ———, *Essential self-adjointness of Schrödinger operators with positive potentials*, Math. Ann. **201** (1973), 211–220.

[71] ———, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton Series in Physics, Princeton Univ. Press, Princeton, N.J., 1974.

[72] ———, *An abstract Kato's inequality for generators of positivity preserving semigroups*, Ind. Math. J. **26** (1977), 1067–1073.

[73] ———, *Functional Integration and Quantum Physics*, Pure and Applied Mathematics, 86, Academic Press, New York, 1979.

[74] ———, *Trace Ideals and Their Applications*, London Math. Society Lecture Note Series, 35, Cambridge Univ. Press, Cambridge, 1979.

[75] ———, *Maximal and minimal Schrödinger forms*, J. Oper. Theory **1** (1979), 37–47.

[76] K. Symanzik, *Euclidean quantum field theory. I. Equations for a scalar model*, J. Math. Phys. **7** (1966), 510–525.

[77] ———, *Euclidean quantum field theory*, in "Local Quantum Theory," (R. Jost, ed.), pp. 152–226, Academic Press, New York, 1969.

[78] S. Teufel, *Effective N-body dynamics for the massless Nelson model and adiabatic decoupling without spectral gap*, Ann. Henri Poincaré **3** (2002), 939–965.

[79] A. Truman, *The polygonal path formulation of the Feynman path integral*, in "Feynman Path Integrals" (Proc. Internat. Colloq., Marseille, 1978), pp. 73–102, Lecture Notes in Phys., 106, Springer, Berlin-New York, 1979.