INVERSE SPECTRAL ANALYSIS WITH PARTIAL INFORMATION ON THE POTENTIAL, III. UPDATING BOUNDARY CONDITIONS

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ABSTRACT. We discuss results where information on parts of the discrete spectra of one-dimensional Schrödinger operators $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ or of a finite Jacobi matrix together with partial information on q uniquely determines q a.e. on $[0,1]$. These extend classical results of Borg and Hochstadt-Lieberman as well as results in paper II of this series.

§1. Introduction

This paper is a postscript to two earlier papers [5, 6] in that it provides a new way of looking at the problems considered in those papers that allows the same methods to prove additional results.

To explain our results, we recall earlier theorems of Borg [1] (see also [8, 10–14]) and of Hochstadt-Lieberman [9] (see also [7, 15]). Throughout this paper assume $q \in L^1((0,1))$ to be real-valued and consider the operator $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions

$$u'(0) + h_0 u(0) = 0, \tag{1.1}$$

$$u'(1) + h_1 u(1) = 0, \tag{1.2}$$

where $h_j \in \mathbb{R} \cup \{\infty\}$, $j = 0, 1$ (with $h_0 = \infty$ shorthand for the boundary condition $u(0) = 0$). Fix $h_1 \in \mathbb{R}$ but think of $H(h_0)$ as a family of operators depending on h_0 as a parameter. Then Borg's and Hochstadt-Lieberman's results can be paraphrased as follows:

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Borg [1]. The spectra of $H(h_0)$ for two values of h_0 determine q.

Hochstadt-Lieberman [9]. The spectra of $H(h_0)$ for one value of h_0 and q on $[0, \frac{1}{2}]$ determine q.

In [6], two of us proved a result that can be paraphrased as

Theorem of [6]. *Half the spectra of one $H(h_0)$ and q on $[0, \frac{3}{4}]$ determine q.*

One of our goals in this note is to prove

New Result. The spectrum of one $H(h_0)$ and half the spectrum of another $H(h_0)$ and q on $[0, \frac{1}{4}]$ determine q.

We will also show that

New Result. Two-thirds of the spectra of three $H(h_0)$ determine q.

Our point is as much a new way of looking at the argument in [6] as these new results. Fundamental to our approach here and in [5, 6] is the Titchmarsh-Weyl m-function defined by

$$m_{h_1}(z) = \frac{u'_{h_1}(z,0)}{u_{h_1}(z,0)},$$

where $u_{h_1}(z,x)$ solves $-u''(z,x) + q(x)u(z,x) = zu(z,x)$ with the boundary condition (1.2). m_{h_1} is a meromorphic function on $\mathbb{C}$ (in fact, a Herglotz function) with all its zeros and poles on the real axis. Since $h_1 \in \mathbb{R}$ will be fixed throughout this paper, we will delete the subscript h_1 from now on and simply write $m(z)$ instead. Moreover, due to the assumption $h_1 \in \mathbb{R}$, we will index the eigenvalues of $H(h_0)$ by $\{\lambda_n\}_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A fundamental result of Marchenko [16] (see also [2, 3, 17]) says

Theorem 1.1. *$m(z)$ uniquely determines q a.e. on $[0, 1]$.*

Our fundamental strategy can be described as follows:

(a) Note that λ is an eigenvalue of $H(h_0)$ if and only if $m(\lambda) = -h_0$.

(b) Prove a general theorem that knowing m at points $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ determines m as long as $\{\lambda_n\}_{n\in\mathbb{N}_0}$ has sufficient density. Given (a), this will allow one to prove that if $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ have sufficient density, an infinite sequence of pairs $\{(\lambda_n, \alpha_n)\}_{n\in\mathbb{N}_0}$ and the knowledge that $H(h_0 = \alpha_n)$ has an eigenvalue at λ_n determines m (and so q a.e. on $[0, 1]$ by Theorem 1.1).

(c) Use scaling covariance to extend the $[0, 1]$ result to one for $[x, 1]$ for any $x \in (0, 1)$.

(d) Note that a knowledge of q a.e. on $[0, x]$ allows one to update boundary conditions. Explicitly, let $H(h_x)$ be the operator in $L^2((x, 1))$ with boundary condition (1.2) but (1.1) replaced by

$$u'(x) + h_x u(x) = 0. \tag{1.3}$$

Then λ_n is an eigenvalue of $H(h_0 = \alpha_n)$ if and only if it is an eigenvalue of $H(h_{x_0} = \beta_n)$, where β_n is obtained by solving $m_n'(x) = q(x) - \lambda_n - m_n^2$ on $[0, x_0]$ with the boundary condition $m_n(x = 0) = -\alpha_n$ and setting $\beta_n = -m_n(x = x_0)$.

We will present steps (b) and (c) in Sections 2 and 3 and then step (d) in Section 4.

We will not explicitly derive them, but the results in [6] that treat operators on $(0,1)$ and that allow one to trade C^{2k} conditions on q for k eigenvalues can be extended to the context we discuss here.

We also note that the ideas in this paper extend to Jacobi matrices.

Finally, while the present paper and [5, 6] concentrate on discrete spectra, we might point out that our m-function strategy also applies in certain cases involving absolutely continuous spectra, see [4].

§2. Zeros of the m-function

If $a \in \mathbb{R}$, let $a_+ = \max(a,0)$. Then

Theorem 2.1. *Let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ be a sequence of distinct positive real numbers satisfying*

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - \frac{1}{4}\pi^2 n^2)_+}{n^2} < \infty. \tag{2.1}$$

Let m_1, m_2 be the m-functions for two operators $H_j = -\frac{d^2}{dx^2} + q_j$ in $L^2((0,1))$ with boundary conditions

$$u'(1) + h_1^{(j)} u(1) = 0$$

and $h_1^{(j)} \in \mathbb{R}$, $j = 1,2$. Suppose that $m_1(\lambda_n) = m_2(\lambda_n)$ for all $n \in \mathbb{N}_0$. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$).

Remarks. 1. In our examples, $\lambda_n \sim \pi^2 n^2 + C$ as $n \to \infty$ (cf. (3.1)), so (2.1) is satisfied, for instance, by considering two distinct spectra of $H(h_0)$.

2. We allow the case $m_1(\lambda_n) = m_2(\lambda_n) = \infty$.

As a preliminary result we note the following

Lemma 2.2. *Suppose $\{\lambda_n\}_{n\in\mathbb{N}_0}$ is a sequence of positive real numbers satisfying* (2.1) *and*

$$\sum_{n=0}^{\infty} \lambda_n^{-1} < \infty. \tag{2.2}$$

Define $f(z) := \prod_{n=0}^{\infty}(1 - \frac{z}{\lambda_n})$, then

$$\varlimsup_{\substack{|y|\to\infty \\ y\in\mathbb{R}}} \frac{|y|^{1/2}\sinh(2|y|^{1/2})}{|f(iy)|} < \infty. \tag{2.3}$$

Proof. Let $y \in \mathbb{R}$. Then $\sinh(2|y|^{1/2})/|y|^{1/2} = |\sin(2i|y|^{1/2})/|y|^{1/2}|$ and

$$\frac{\sin(2\sqrt{z}\,)}{2\sqrt{z}} = \prod_{n=1}^{\infty}\left(1 - \frac{4z}{\pi^2 n^2}\right),$$

so (2.3) becomes

$$\overline{\lim_{|y|\to\infty}} \frac{|y|}{1+\frac{|y|}{\lambda_0}} \prod_{n=1}^{\infty} \left[\frac{(1+\frac{4|y|}{\pi^2 n^2})}{(1+\frac{|y|}{\lambda_n})} \right] < \infty \tag{2.4}$$

using $2^{-1/2}(1+|x|) \leq (1+x^2)^{1/2} \leq (1+|x|)$.

If $0 \leq a \leq b$, then $(\frac{1+a|y|}{1+b|y|}) \leq 1$, and if $a > b > 0$, then

$$\frac{(1+a|y|)}{1+b|y|} = 1 + \frac{(a-b)|y|}{1+b|y|} \leq 1 + \frac{a-b}{b} = \frac{a}{b},$$

$$\prod_{n=1}^{\infty} \frac{(1+\frac{4|y|}{\pi^2 n^2})}{(1+\frac{|y|}{\lambda_n})} \leq \prod_{n:\lambda_n > \frac{1}{4}\pi^2 n^2} \frac{4\lambda_n}{\pi^2 n^2} = \prod_{n=1}^{\infty} \left[1 + \frac{(\lambda_n - \frac{1}{4}\pi^2 n^2)_+}{\frac{1}{4}\pi^2 n^2} \right] < \infty$$

if (2.1) holds. □

Proof of Theorem 2.1. We follow the arguments in [5, 6] fairly closely. One can write $m_j(z) = \frac{Q_j(z)}{P_j(z)}$, $j = 1, 2$, where

(1) P_j, Q_j are entire functions satisfying

$$|P_j(z)| \leq C \exp(\sqrt{|z|}\,), \tag{2.5a}$$

$$|Q_j(z)| \leq C(1+\sqrt{|z|}\,) \exp(\sqrt{|z|}\,). \tag{2.5b}$$

(2)

$$m_j(z) = \pm i\sqrt{z} + o(1) \text{ as } z \to \pm i\infty. \tag{2.6}$$

(We use the square root branch with $\operatorname{Im}(\sqrt{z}) \geq 0$.)

Suppose $m_1 \neq m_2$. Then $P_2(z)Q_1(z) - P_1(z)Q_2(z) := H(z)$ is an entire function of order at most $\frac{1}{2}$ and not identically zero. Since $H(\lambda_n) = 0$, we conclude that $\sum_{n\in\mathbb{N}_0} \lambda_n^{-a} < \infty$ if $a > \frac{1}{2}$. In particular, (2.2) holds, and we can define $f(z) = \prod_{n=0}^{\infty}(1 - \frac{z}{\lambda_n})$. Next, define

$$G(z) := \frac{H(z)}{f(z)} = \frac{P_1(z)P_2(z)}{f(z)}\,(m_1(z) - m_2(z)). \tag{2.7}$$

Since $H(\lambda_n) = 0$, $G(z)$ is an entire function. By (2.3),

$$\overline{\lim_{|y|\to\infty}} \frac{|y|^{1/2} \exp(2|y|^{1/2})}{|f(iy)|} < \infty,$$

so by (2.5) and (2.6),

$$|G(iy)| \leq \frac{\exp(2|y|^{1/2})}{f(iy)}\,|m_1(iy) - m_2(iy)| = o(|y|^{-1/2})$$

goes to zero as $|y| \to \infty$. The Phragmén-Lindelöf argument of [6] then yields the contradiction $G(z) \equiv 0$, that is, $m_1 = m_2$. □

Remark. The above yields $o(|y|^{-1/2})$ even though $o(1)$ would have been sufficient. We have thrown away half a zero. That means one can prove the following result.

Theorem 2.2. *Let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ and $\{\mu_n\}_{n\in\mathbb{N}_0}$ be two sequences of real numbers satisfying*

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - \pi^2 n^2)_+}{n^2} < \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(\mu_n - \pi^2 n^2)_+}{n^2} < \infty, \tag{2.8}$$

with $\mu_m \neq \lambda_n$ for all $m, n \in \mathbb{N}_0$. Let m_1, m_2 be the m-functions for two operators $H_j = -\frac{d^2}{dx^2} + q_j$, $j = 1, 2$ in $L^2((0,1))$ with boundary conditions

$$u'(1) + h_1^{(j)} u(1) = 0$$

and $h_1^{(j)} \in \mathbb{R}$, $j = 1, 2$. Suppose that $m_1(z) = m_2(z)$ for all z in $\{\lambda_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}$ except perhaps for one. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$).

By scaling, one sees the following analog of Theorem 2.1 holds (there is also an analog of Theorem 2.2):

Theorem 2.3. *Let $a < b$ and $\{\lambda_n\}_{n\in\mathbb{N}_0}$ be a sequence of distinct positive real numbers satisfying*

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - \frac{\pi^2 n^2}{4(b-a)^2})_+}{n^2} < \infty. \tag{2.9}$$

Let m_1, m_2 be the m-functions for two operators $H_j = -\frac{d^2}{dx^2} + q_j$, $j = 1, 2$ in $L^2((a,b))$ with boundary conditions (1.3) at $x = a$ and

$$u'(b) + h_b^{(j)} u(b) = 0,$$

where $h_b^{(j)} \in \mathbb{R}$, $j = 1, 2$. Suppose that $m_1(\lambda_n) = m_2(\lambda_n)$ for all $n \in \mathbb{N}_0$. Then $m_1 = m_2$ (and hence $q_1 = q_2$ a.e. on $[a,b]$ and $h_b^{(1)} = h_b^{(2)}$).

§3. Whole Interval Results

Fix $h_1 \in \mathbb{R}$, let $H(h_0)$ be the operator on $L^2((0,1))$ with $u'(1)+h_1u(1) = 0$ and $u'(0)+h_0u(0) = 0$ boundary conditions, and denote by $\lambda_n(h_0)$ the corresponding eigenvalues of $H(h_0)$. Then, for $h_0 \in \mathbb{R}$, it is known (see, e.g., the references in [6]) that

$$\lambda_n = (n\pi)^2 + 2(h_1 - h_0) + \int_0^1 q(x)\,dx + o(1) \text{ as } n \to \infty \tag{3.1}$$

and for $h_0 = \infty$,

$$\lambda_n = [(n + \tfrac{1}{2})\pi]^2 + 2h_1 + \int_0^1 q(x)\,dx + o(1) \text{ as } n \to \infty. \tag{3.2}$$

To say that $H(h_0)$ has eigenvalue λ is equivalent to $m(\lambda) = -h_0$. Thus, Theorem 2.1 implies

Theorem 3.1. *Let $H_1(h_0), H_2(h_0)$ be associated with two potentials q_1, q_2 on $[0,1]$ and two potentially distinct boundary conditions $h_1^{(1)}, h_1^{(2)} \in \mathbb{R}$ at $x = 1$. Suppose that $\{(\lambda_n, h_0^{(n)})\}_{n\in\mathbb{N}_0}$ is a sequence of pairs with $\lambda_0 < \lambda_1 < \cdots \to \infty$ and $h_0^{(n)} \in \mathbb{R}\cup\{\infty\}$ so that both $H_1(h_0^{(n)})$ and $H_2(h_0^{(n)})$ have eigenvalues at λ_n. Suppose that (2.1) holds. Then $q_1 = q_2$ a.e. on $[0,1]$ and $h_1^{(1)} = h_1^{(2)}$.*

Given (3.1), (3.2) we immediately have Borg's theorem [1] as a corollary (this is essentially the usual proof), but more is true. For example, by using Theorem 2.2 one infers:

Corollary 3.2 [1]. *Fix $h_0^{(1)}, h_0^{(2)} \in \mathbb{R}$. Then all the eigenvalues of $H(h_0^{(1)})$ and all the eigenvalues of $H(h_0^{(2)})$, save one, uniquely determine q a.e. on $[0,1]$.*

Corollary 3.3. *Let $h_0^{(1)}, h_0^{(2)}, h_0^{(3)} \in \mathbb{R}$ and denote by $\sigma_j = \sigma(H(h_0^{(j)}))$ the spectra of $H(h_0^{(j)})$, $j = 1,2,3$. Assume $S_j \subseteq \sigma_j$, $j = 1,2,3$ and suppose that for all sufficiently large $\lambda_0 > 0$ we have*

$$\#\{\lambda \in \{S_1\cup S_2\cup S_3\} \text{ with } \lambda\le\lambda_0\} \ge \tfrac{2}{3}\#\{\lambda\in\{\sigma_1\cup\sigma_2\cup\sigma_3\} \text{ with } \lambda\le\lambda_0\} - 1.$$

Then q is uniquely determined a.e. on $[0,1]$.

In particular, two-thirds of three spectra determine q.

§4. Updating m

We are now able to understand why partial information on q — knowing it on $[0,a]$ — lets us get away with less information on eigenvalues, a phenomenon originally discovered by Hochstadt-Lieberman [9] in the special case where $a = \frac{1}{2}$. We note that $m(z,x)$ satisfies the Ricatti-type equation

$$m'(z,x) = q(x) - z - m^2(z,x). \tag{4.1}$$

If we know that λ is an eigenvalue of $H(h_0)$, then $m(\lambda,0) = -h_0$. If we know q on $[0,a]$, we can use (4.1) to compute $m(\lambda,a) := -h_a$ and so infer that λ is an eigenvalue of $H(h_a)$, the operator in $L^2((a,1))$. By Theorem 2.3, that means we only need a lower density of eigenvalues of the various $H(h_a)$. A typical result is the following

Theorem 4.1. *Let σ_N and σ_D be the eigenvalues of $H(h_0 = 0)$ and $H(h_0 = \infty)$, respectively. Let $S_N \subseteq \sigma_N$, $S_D \subseteq \sigma_D$. Fix $a \in (0,1)$. Suppose for $\lambda_0 > 0$ sufficiently large that*

$$\#\{\lambda\in\{S_N\cup S_D\} \text{ with } \lambda\le\lambda_0\} \ge (1-a)\#\{\lambda\in\{\sigma_N\cup\sigma_D\} \text{ with } \lambda\le\lambda_0\}.$$

Then S_N, S_D and q on $[0,a]$ uniquely determine q a.e. on $[0,1]$.

This follows immediately from the updating idea. For example, if $a = \frac{3}{4}$, we can recover Theorem 1.3 of [6] (it is essentially a reworking of the proof in [6]); but for

$a \in (0, \frac{1}{2})$, the result is new and implies, for example, that q on $[0, \frac{1}{4}]$, all the Neumann eigenvalues, and half the Dirichlet eigenvalues determine q a.e. on $[0, 1]$.

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