REGULARITY AND THE CESÀRO–NEVAI CLASS

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Abstract. We consider OPRL and OPUC with measures regular in the sense of Ullman–Stahl–Totik and prove consequences on the Jacobi parameters or Verblunsky coefficients. For example, regularity on $[-2,2]$ implies $\lim_{N\to\infty} N^{-1}[\sum_{n=1}^N (a_n-1)^2+b_n^2]=0$.

1. Introduction and Background

This paper concerns the general theory of orthogonal polynomials on the real line, OPRL (see [26, 1, 8, 23]), and the unit circle, OPUC (see [26, 9, 18, 19]). Ullman [27] introduced the notion of regular measure on $[-2,2]$ (he used $[-1,1]$; we use the normalization more common in the spectral theory literature): a measure, $d\mu$, on $\mathbb{R}$ with

$$\operatorname{supp}(d\mu)=[-2,2] \tag{1.1}$$

and ($\{a_n,b_n\}_{n=1}^\infty$ are the Jacobi parameters of $d\mu$)

$$\lim_{n\to\infty}(a_1\dots a_n)^{1/n}=1 \tag{1.2}$$

Here we will look at the larger class with (1.1) replaced by

$$\sigma_{\text{ess}}(d\mu)=[-2,2] \tag{1.3}$$

(i.e., $\operatorname{supp}(d\mu)$ is $[-2,2]$ plus a countable set whose only limit points are a subset of $\{\pm 2\}$).

Our goal is to explore what restrictions regularity places on the Jacobi parameters. At first sight, one might think (1.2) is the only restriction but, in fact, the combination of both (1.2) and (1.3) is quite strong. This should not be unexpected. After all, it is well known

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(going back at least to Nevai [15]; see also [19, Sect. 13.3]) that (1.1) plus $\liminf(a_1 \dots a_n) > 0$ implies

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \tag{1.4}$$

One can use variational principles to deduce some restrictions on the a's and b's. For example, picking φ_n to be the vector in $\ell^2(\{1, 2, \dots\})$

$$\varphi_{n,j} = \begin{cases} \frac{1}{\sqrt{n}} & j \leq n \\ 0 & j \geq n+1 \end{cases} \tag{1.5}$$

and using the Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.6}$$

one sees, for example, that (1.3) implies (see also Theorem 1.2 below)

$$b_n \equiv 0 \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} a_j \leq 1 \tag{1.7}$$

$$a_n \equiv 1 \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} b_j = 0 \tag{1.8}$$

In fact, we will prove much more:

Theorem 1.1. *If μ obeys (1.3) and (1.2), then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (|a_j - 1| + |b_j|) = 0 \tag{1.9}$$

Following the terminology for the OPUC analog of this in Golinskii–Khrushchev [10], we call (1.9) the Cesàro–Nevai condition and $\{a_j, b_j\}_{j=1}^{\infty}$ obeying (1.9) the Cesàro–Nevai class. It, of course, contains the Nevai class (named after [15]) where $|a_j - 1| + |b_j| \to 0$.

Noting that $\mathrm{supp}(d\mu)$ bounded implies

$$A = \sup_n (|a_n - 1| + |b_n|) < \infty \tag{1.10}$$

and that, by the Schwarz inequality,

$$\left(\frac{1}{n} \sum_{j=1}^{n} |a_j - 1| + |b_j| \right)^2 \leq \frac{2}{n} \sum_{j=1}^{n} (a_j - 1)^2 + (b_j)^2$$

$$\leq 2A\,\frac{1}{n}\sum_{j=1}^{n}(|a_j-1|+|b_j|) \tag{1.11}$$

we see

$$(1.9) \Leftrightarrow \frac{1}{n}\sum_{j=1}^{n}(a_j-1)^2+(b_j)^2 \to 0 \tag{1.12}$$

While Theorem 1.1 has a lot of information, it is not the whole story. For example, if $a_n \equiv 1$, then by the same variational principle, for any $j_k \to \infty$,

$$\frac{1}{n}\sum_{j_k}^{j_k+n} b_j \to 0$$

It would be interesting to see what else can be said.

A major theme we explore is what can be said if $[-2,2]$ is replaced by a more general set, $\mathfrak{e}$. In Section 5, we define Nevai and CN classes for finite gap sets $\mathfrak{e}$ and state a general conjecture which we prove in the special case where d has p components, each of harmonic measure $1/p$, that is, the periodic case with all gaps open.

In Section 3, we extend Theorem 1.1 to the matrix OPRL case on $[-2,2]$, and in Section 6, we use this and ideas of Damanik–Killip–Simon [6] to obtain the result in the last paragraph. Section 4 has a brief discussion of OPUC.

We should close by noting an earlier result of Máté–Nevai–Totik [14] related to—but neither stronger nor weaker than—Theorem 1.2:

Theorem 1.2 ([14])**.** *Suppose μ obeys* (1.1) *and $a_n \to 1$ as $n \to \infty$. Then $b_n \to 0$ as $n \to \infty$.*

Remarks. 1. μ need only obey (1.3) as seen by Remark 3 below.

2. This strengthens (1.8). There is no similar strengthening of (1.7).

3. One way of seeing this is as follows: By Last–Simon [13], any right limit of a J obeying (1.3) has $\sigma(J_r) \subset [-2,2]$ and has $a_n \equiv 1$. By a result of Killip–Simon [11] (see also [3, 4, 5]), any such J_r has $b_n \equiv 0$. By compactness, $b_n \to 0$ for the original J.

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2. OPRL on $[-2,2]$

Our goal here is to prove Theorem 1.1.

Lemma 2.1. *Suppose $a_n \in (0,\infty)$ is a sequence so that*

$$\text{(i)} \qquad \liminf_{N\to\infty}\, (a_1 \dots a_N)^{1/N} \geq 1 \tag{2.1}$$

$$\text{(ii)} \qquad \limsup_{N\to\infty}\, \frac{1}{N}\sum_{n=1}^N a_n^2 \leq 1 \tag{2.2}$$

Then, as $N\to\infty$,

$$\frac{1}{N}\sum_{n=1}^N a_n \to 1 \qquad\qquad \frac{1}{N}\sum_{n=1}^N a_n^2 \to 1 \tag{2.3}$$

$$\frac{1}{N}\sum_{n=1}^N (a_n-1)^2 \to 0 \tag{2.4}$$

Proof. By concavity of $\log x$ for all $x\in(0,\infty)$,

$$\log x \leq x-1$$

so (2.1) implies

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N a_n \geq 1 + \liminf_{N\to\infty}\, \log(a_1\dots a_N)^{1/N} \geq 1$$

Thus,

$$\limsup \frac{1}{N}\sum_{n=1}^N (a_n-1)^2 \leq 1-2+1 = 0$$

so (2.4) holds.

By the Schwarz inequality,

$$\frac{1}{N}\sum_{n=1}^N |a_n-1| \leq \biggl[\frac{1}{N}\sum_{n=1}^N (a_n-1)^2\biggr]^{1/2} \to 0$$

which implies the first limit in (2.3). (2.4) and that limit imply (2.3). $\square$

Proposition 2.2. *Let $\{a_n,b_n\}_{n=1}^\infty$ be the Jacobi parameters for a regular measure with $\sigma_{\text{ess}}(J)=[-2,2]$. Then*

$$\frac{1}{N}\biggl[2\sum_{n=1}^{N-1} a_n^2 + \sum_{n=1}^N b_n^2\biggr] \to 2 \tag{2.5}$$

as $N\to\infty$.

Proof. Let $\{x_j^{(N)}\}_{j=1}^N$ be the zeros of the OPRL $p_N(x)$ associated to the Jacobi parameters. Let $d\rho_{[-2,2]}$ be the equilibrium measures for $[-2,2]$ (see [12, 16, 22] for potential theory notions). Since regularity implies that the density of zeros converges to $d\rho_{[-2,2]}$ (see [25, 22]), we have

$$\frac{1}{N}\sum_{n=1}^{N}(x_j^{(N)})^2 \to \int x^2 d\rho_{[-2,2]}(x) \tag{2.6}$$

Since $\{x_j^{(N)}\}_{j=1}^N$ are the eigenvalues of the finite Jacobi matrix

$$J_{N;F} = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & b_{N-1} & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix} \tag{2.7}$$

we have that

$$\begin{aligned} \text{LHS of (2.6)} &= \frac{1}{N}\operatorname{Tr}(J_{N;F}^2) \\ &= \frac{1}{N}\left[\sum_{n=1}^{N} b_n^2 + 2\sum_{n=1}^{N-1} a_n^2\right] \end{aligned} \tag{2.8}$$

Thus (2.5) is equivalent to

$$\int x^2 d\rho_{[-2,2]}(x) = 2 \tag{2.9}$$

This can be seen either by using the explicit formula for $d\rho_{[-2,2]}$ (and $\int_0^\pi (2\cos\theta)^2 \frac{d\theta}{\pi} = 2$) or by considering the special case $a_n \equiv 1$, $b_n \equiv 0$ since the limit in (2.5) is the same for all regular J's. □

Proof of Theorem 1.1. By regularity,

$$\liminf_{N\to\infty} (a_1 \dots a_N)^{1/N} = 1 \tag{2.10}$$

and by Proposition 2.2 and

$$\limsup \frac{1}{N}\sum_{n=1}^{N-1} a_n^2 \le 1 \tag{2.11}$$

By Lemma 2.1, we have (2.4), and this and (2.5) imply

$$\frac{1}{N}\sum_{n=1}^{N} b_n^2 \to 0 \tag{2.12}$$

By(1.12), we get (1.9). □

3. MOPRL on $[-2,2]$

In this section, both for its own sake and because of the application in Section 6, we want to consider matrix-valued measures for $[-2,2]$. Our reference for the associated OPRL will be [7] which discusses regular measures. ℓ is fixed and finite, and we have a block Jacobi matrix of the form

$$J = \begin{pmatrix} B_1 & A_1 & 0 & \cdots \\ A_1^\dagger & B_2 & A_2 & \cdots \\ 0 & A_2^\dagger & B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1}$$

where A_j and B_j are $\ell \times \ell$ matrices and † is Hermitian conjugate. One requires each A_j is nonsingular.

Two sets of Jacobi parameters, $\{A_j, B_j\}_{j=1}^\infty$ and $\{\tilde A_j, \tilde B_j\}_{j=1}^\infty$, are called equivalent if there exist $\ell \times \ell$ unitaries, $u_1 \equiv \mathbf{1}, u_2, u_3, \dots$ so that

$$\tilde B_j = u_j^\dagger B_j u_j \qquad \tilde A_j = u_j^\dagger A_j u_{j+1} \tag{3.2}$$

It is known (see [7, Thm. 2.11]) that there is a one-one correspondence between nontrivial $\ell\times\ell$ matrix-valued measures, $d\mu$, (with nontriviality suitably defined) and equivalence classes of Jacobi parameters.

$\{A_j, B_j\}_{j=1}^\infty$ is called type 1 (resp. type 3) if each A_j is positive (resp. A_j is lower triangular and positive on diagonal). Moreover ([7, Thm. 2.8]), each equivalence class has exactly one representative of type 1 and one of type 3. An $\ell \times \ell$ matrix-valued measure is called regular ([7, Ch. 5]) for $[-2,2]$ if and only if

$$\sigma_{\text{ess}}(d\mu) = [-2,2] \tag{3.3}$$

and

$$\biggl[\prod_{n=1}^N \lvert\det(A_n)\rvert\biggr]^{1/N} \to 1 \tag{3.4}$$

Our basic result for such MOPRL is:

Theorem 3.1. *If $\{A_n, B_n\}_{n=1}^\infty$ are the Jacobi parameters for an $\ell \times \ell$ matrix-valued measure which is regular for $[-2,2]$ and are either of type 1 or type 3, then*

$$\frac{1}{N}\sum_{n=1}^N \lVert A_n - \mathbf{1}\rVert + \lVert B_n\rVert \to 1 \tag{3.5}$$

Remark. (3.5) does not hold for all equivalent $\{\tilde{A}_n, \tilde{B}_n\}_{n=1}^\infty$, but it is easy to see that

$$\frac{1}{N}\sum_{n=1}^{N} \|A_n^* A_n - \mathbf{1}\| + \|B_n\| \to 0 \tag{3.6}$$

is equivalence class independent and implied by (3.5) for the type 1 or type 3 representative.

Proof. We consider type 3 first. By Thm. 5.2 of [7], the density of zeros converges to the equilibrium measure, so analogously to (2.5),

$$\frac{1}{N\ell}\biggl[2\sum_{n=1}^{N-1} \operatorname{Tr}(A_n^* A_n) + \sum_{n=1}^{N} \operatorname{Tr}(B_n^* B_n)\biggr] \to 2 \tag{3.7}$$

In the type 3 case, (3.4) says

$$\biggl[\prod_{n=1}^{N}\prod_{j=1}^{\ell} (A_n)_{jj}\biggr]^{1/N\ell} \to 1 \tag{3.8}$$

so as in the proof of Theorem 1.1, we find

$$\frac{1}{N\ell}\sum_{n=1}^{N}\sum_{j=1}^{\ell} |(A_n)_{jj} - 1|^2 \to 0 \tag{3.9}$$

and then that

$$\frac{1}{N}\sum_{n=1}^{N} \operatorname{Tr}(B_n^* B_n) \to 0 \tag{3.10}$$

and

$$\frac{1}{N}\sum_{n=1}^{N} |\operatorname{Tr}(A_n^* A - \mathbf{1})| \to 0 \tag{3.11}$$

In the type 1 case, one uses the inequality

$$A \geq 0 \Rightarrow \det(A) \leq \prod_{j=1}^{\ell} A_{jj} \tag{3.12}$$

(see Simon [20, Cor. 8.10]) and the fact that Lemma 2.1 only requires an inequality in (2.1). □

4. OPUC

Here we will prove two results about OPUC. Recall $d\mu$ on $\partial\mathbb{D}$ with $\sigma_{\text{ess}}(d\mu) = \mathfrak{e}$ is called regular if and only if

$$\lim_{N\to\infty} \left(\prod_{j=0}^{N-1} \rho_j \right)^{1/N} = C(\mathfrak{e}) \tag{4.1}$$

the capacity of $\mathfrak{e}$ where $\rho_j = (1-|\alpha_j|^2)^{1/2}$ and $\{\alpha_j\}_{j=0}^\infty$ are the Verblunsky coefficients.

Theorem 4.1. *Let $d\mu$ be a measure of $\partial\mathbb{D}$ regular for $\mathfrak{e} = \partial\mathbb{D}$. Then, as $N \to \infty$,*

$$\frac{1}{N} \sum_{j=0}^{N-1} |\alpha_j| \to 0 \tag{4.2}$$

Remark. This is the original CN class of [10].

Proof. $C(\partial\mathbb{D}) = 1$, so by Lemma 2.1 and

$$\frac{1}{N} \sum_{j=0}^{N-1} \rho_j^2 \le 1 \tag{4.3}$$

we obtain

$$\frac{1}{N} \sum_{j=0}^{N-1} (1-\rho_j^2) \to 0 \tag{4.4}$$

which implies (4.2) by the Schwarz inequality. □

For $a \in (0,1)$, let Γ_a be the arc

$$\{z \in \partial\mathbb{D} \mid z = e^{i\theta}, \, \pi \ge |\theta| > 2\arcsin(a)\} \tag{4.5}$$

which has capacity a. Then

Theorem 4.2. *Let $d\mu$ be a measure on $\partial\mathbb{D}$, regular for $\mathfrak{e} = \Gamma_a$. Then as $N \to \infty$,*

(a)
$$\frac{1}{N} \sum_{j=0}^{N-1} (|\alpha_j| - a)^2 \to 0 \tag{4.6}$$

(b)
$$\frac{1}{N} \sum_{j=0}^{N-1} |\alpha_{j+1} - \alpha_j|^2 \to 0 \tag{4.7}$$

For any k,

$$\text{(c)} \qquad \frac{1}{N}\sum_{j=0}^{N-1} \min_\theta \biggl(\sum_{\ell=1}^{k} |\alpha_{j+\ell} - ae^{i\theta}|^2 \biggr) \to 0 \tag{4.8}$$

Remark. The isospectral torus for Γ_a is exactly $\{\{\alpha_j \equiv ae^{i\theta}\}\}_{\theta\in[0,2\pi)}$, that is, the constant sequence of Verblunsky coefficients, so (c) involves an approach to an isospectral torus.

Proof. By regularity and the connection between zeros of POPUC and eigenvalues of finite CMV matrices as defined in [21], one has that

$$\frac{1}{N}\sum_{n=0}^{N-1} -\bar\alpha_{n+1}\alpha_n \to c \tag{4.9}$$

where c is the first moment of the equilibrium measure, that is, $\int z\, d\rho_{\Gamma_a}(z)$. Specializing to the case $\alpha_n \equiv a$ to evaluate c, we see that

$$\frac{1}{N}\sum_{n=0}^{N-1} \bar\alpha_{n+1}\alpha_n \to a^2 \tag{4.10}$$

On the other hand, by regularity,

$$\frac{1}{N}\sum_{n=0}^{N-1} \log(1-|\alpha_n|^2) \to \log(1-|a|^2) \tag{4.11}$$

and by concavity of log,

$$\log(1-x) - \log(1-|a|^2) \le \frac{1}{1-|a|^2}\,(|a|^2 - x)$$

so

$$\liminf \frac{1}{N}\sum_{n=0}^{N-1} (|a|^2 - |\alpha_n|^2) \ge 0 \tag{4.12}$$

and thus,

$$\limsup \frac{1}{N}\sum_{n=0}^{N-1} |\alpha_n|^2 \le a^2 \tag{4.13}$$

By (4.10) and the Schwarz inequality,

$$\liminf \frac{1}{N}\sum_{n=0}^{N-1} |\alpha_n|^2 \ge a^2 \tag{4.14}$$

so

$$\frac{1}{N}\sum_{n=0}^{N-1} |\alpha_n|^2 \to a^2 \tag{4.15}$$

For $y \in (0,1]$ (by Taylor's theorem with remainder and $\max_{(0,1]} \frac{d^2}{dy^2} \log(y) = -1$),

$$\log(y) - \log(1-|a|^2) - \left[\frac{y-(1-|a|^2)}{1-|a|^2}\right] \leq -\frac{1}{2}\,(y-(1-|a|^2))^2$$

so (4.11) and (4.15) imply

$$\frac{1}{N}\sum_{n=0}^{N-1} ||\alpha|^2 - a^2| \to 0$$

which implies (4.7).

(4.15) and (4.10) imply (4.7). Finally, (4.6) and (4.7) imply (4.8). $\square$

5. The Nevai and CN Classes

In [19], I proposed using approach to an isospectral torus as a replacement for the Nevai class when $[-2,2]$ is replaced by the spectrum of a periodic Jacobi matrix. This idea was then implemented in Last–Simon [13] and Damanik–Killip–Simon [6]. The latter discussed extending this notion to a general finite gap set, and this idea was further developed in Remling [17].

$\mathfrak{e}$ will denote a finite gap set, that is,

$$\mathfrak{e} = [\alpha_1,\beta_1] \cup [\alpha_2,\beta_2] \cup \cdots \cup [\alpha_{\ell+1},\beta_{\ell+1}] \subset \mathbb{R} \tag{5.1}$$

where

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{\ell+1} \tag{5.2}$$

Given such a set, there is a natural torus, $\mathcal{T}_\mathfrak{e}$, of almost periodic Jacobi matrices, discussed, for example, in [24, 2]; it can be described [17] as the restriction to $\{1,\dots\}$ of the two-sided reflectionless Jacobi matrices, $J^\sharp$, with

$$\sigma(J^\sharp) = \mathfrak{e} \tag{5.3}$$

All $J \in \mathcal{T}_\mathfrak{e}$ have

$$\sigma_{\text{ess}}(J) = \mathfrak{e} \tag{5.4}$$

$\mathcal{T}_\mathfrak{e}$ is a torus in the uniform topology as well as the product topology.

Given a pair of bounded Jacobi parameters, $J = \{a_n,b_n\}_{n=1}^\infty$, $\tilde J = \{\tilde a_n,\tilde b_n\}_{n=1}^\infty$, define $d_m(J,\tilde J)$ by

$$d_m(J,\tilde J) = \sum_{k=0}^\infty e^{-|k|}(|a_{m+k} - \tilde a_{m+k}| + |b_{m+k} - \tilde b_{m+k}|) \tag{5.5}$$

If $\mathcal{T}_\mathfrak{e}$ is an isospectral torus, let

$$d_m(J,\mathcal{T}_\mathfrak{e}) = \inf_{\tilde J \in \mathcal{T}_\mathfrak{e}} d_m(J,\tilde J) \tag{5.6}$$

Definition. If $\mathfrak{e} \subset \mathbb{R}$ is a finite gap set, we define the Nevai class $N(\mathfrak{e})$ to be those J's with

$$\lim_{m\to\infty} d_m(J, \mathcal{T}_{\mathfrak{e}}) = 0 \tag{5.7}$$

This is equivalent (by compactness) to saying all the right limits of J lie in $\mathcal{T}_{\mathfrak{e}}$.

It is a theorem of Last–Simon [13] that

$$J \in N(\mathfrak{e}) \Rightarrow \sigma_{\text{ess}}(J) = \mathfrak{e} \tag{5.8}$$

and of Remling [17] that

$$\sigma_{\text{ess}}(J) = \sigma_{\text{ac}}(J) = \mathfrak{e} \Rightarrow J \in N(\mathfrak{e}) \tag{5.9}$$

It is not hard to see that

$$J \in N(\mathfrak{e}) \Rightarrow J \text{ is regular for } \mathfrak{e}$$

Analogously, we define the Cesàro–Nevai class, $CN(\mathfrak{e})$, as those J with

$$\frac{1}{N} \sum_{m=1}^{N} d_m(J, \mathcal{T}_{\mathfrak{e}}) \to 0 \tag{5.10}$$

A main conjecture we make in this note is:

Conjecture 5.1. If J is regular for $\mathfrak{e}$, that is, $\sigma_{\text{ess}}(J) = \mathfrak{e}$, and $(a_1 \dots a_N)^{1/N} \to C(\mathfrak{e})$, then $J \in CN(\mathfrak{e})$.

In the next section, we will prove this for a special class of $\mathfrak{e}$'s. Of course, we make a similar conjecture for finite gap OPUC. Indeed, Theorem 4.2 is the case of OPUC with one gap!

6. Generic Periodic Spectrum

Our goal is to prove:

Theorem 6.1. *Let $\mathfrak{e}$ be a finite gap set so that each $[\alpha_j, \beta_j]$ has harmonic measure $(\ell+1)^{-1}$ (equivalently, there is a J_0 with period $\ell+1$ so $\mathfrak{e} = \sigma_{\text{ess}}(J_0)$). Let J be a Jacobi matrix with regular spectral measure so that $\sigma_{\text{ess}}(J) = \mathfrak{e}$. Then $J \in CN(\mathfrak{e})$.*

We use p for $\ell+1$, the period of J_0.

Following [6], we exploit $\Delta_{J_0}(J)$ where Δ_{J_0} is the discriminant [6, 23] of J_0, a polynomial of degree p. If J is any Jacobi matrix, $\Delta_{J_0}(J)$ is a $p \times p$ block Jacobi matrix of type 3. We use $A_{J_0,k}(J)$ and $B_{J_0,k}(J)$ to denote the $p \times p$ matrix blocks in $\Delta(J)$.

[6] proved the following theorem (their Thm. 11.12); here $\|\cdot\|$ is the Hilbert–Schmidt norm.

Theorem 6.2 ([6])**.** *Fix J_0 periodic with $\sigma_{\text{ess}}(J_0) = \mathfrak{e}$ and J an arbitrary bounded Jacobi matrix. Then*

$$\sum_{k=1}^{\infty} \|A_{J_0,k}(J) - \mathbf{1}\|_2^2 + \|B_{J_0,k}(J)\|_2^2 < \infty \tag{6.1}$$

if and only if

$$\sum_{k=1}^{\infty} d_k(J, \mathcal{T}_{\mathfrak{e}})^2 < \infty \tag{6.2}$$

Because this comparison is local, the exact same proof shows

Theorem 6.3. *Let J_0 be periodic with $\sigma_{\text{ess}}(J_0) = \mathfrak{e}$ and J an arbitrary Jacobi matrix. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} [\|A_{J_0,k}(J) - \mathbf{1}\|_2^2 + \|B_{J_0,k}(J)\|^2] = 0 \tag{6.3}$$

if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} d_k(J, \mathcal{T}_{\mathfrak{e}})^2 \to 0 \tag{6.4}$$

With this and Theorem 3.1, we can prove Theorem 6.1.

Proof of Theorem 6.1. $\Delta(x)$ has the form

$$\Delta(x) = (a_{0,1}\, a_{0,2} \dots a_{0,p})^{-1} x^p + \text{lower order}$$

so the diagonal matrix elements of $\Delta(J)$ are

$$\frac{a_j a_{j+1} \dots a_{j+p}}{a_{0,j} \dots a_{0,j+p}} \equiv \alpha_{jj}$$

If J is regular, for $\mathfrak{e}$,

$$\left[\frac{a_1 \dots a_n}{C(\mathfrak{e})^n}\right]^{1/n} \to 1 \tag{6.5}$$

But $a_{0,j} \dots a_{0,j+p} = C(\mathfrak{e})^p$ for periodic Jacobi matrices, so (6.5) implies

$$(\alpha_{11}\alpha_{22} \dots \alpha_{nn})^{1/n} \to 1$$

which implies that $\Delta(J)$ is a regular block Jacobi matrix.

By Theorem 3.1, (6.3) holds and so, by Theorem 6.3, we have the $CN(\mathfrak{e})$ condition (6.4). □

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and Its Applications **13**, Gordon and Breach, New York-London-Paris, 1978. MR0481884

[2] J. S. Christiansen, B. Simon, and M. Zinchenko, *Finite gap Jacobi matrices, I. The isospectral torus*, in preparation.

[3] D. Damanik, D. Hundertmark, R. Killip, and B. Simon, *Variational estimates for discrete Schrödinger operators with potentials of indefinite sign*, Comm. Math. Phys. **238** (2003), 545–562. MR1993385

[4] D. Damanik, D. Hundertmark, and B. Simon, *Bound states and the Szegő condition for Jacobi matrices and Schrödinger operators*, J. Funct. Anal. **205** (2003), 357–379. MR2017691

[5] D. Damanik, R. Killip, and B. Simon, *Necessary and sufficient conditions in the spectral theory of Jacobi matrices and Schrödinger operators*, Int. Math. Res. Not. (2004), no. 22, 1087–1097. MR2041649

[6] D. Damanik, R. Killip, and B. Simon, *Perturbations of orthogonal polynomials with periodic recursion coefficients*, preprint.

[7] D. Damanik, A. Pushnitski, and B. Simon, *The analytic theory of matrix orthogonal polynomials*, preprint.

[8] G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford-New York, 1971.

[9] Ya. L. Geronimus, *Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval*, Consultants Bureau, New York, 1961. MR0133643

[10] L. Golinskii and S. Khrushchev, *Cesàro asymptotics for orthogonal polynomials on the unit circle and classes of measures*, J. Approx. Theory **115** (2002), 187–237. MR1901215

[11] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. **158** (2003), 253–321. MR1999923

[12] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin-New York, 1972. MR0350027

[13] Y. Last and B. Simon, *The essential spectrum of Schrödinger, Jacobi, and CMV operators*, J. Anal. Math. **98** (2006), 183–220. MR2254485

[14] A. Máté, P. Nevai, and V. Totik, *Extensions of Szegő's theory of orthogonal polynomials, II*, Constr. Approx. **3** (1987), 51–72. MR0892168

[15] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **18** (1979), no. 213, 185 pp. MR0519926

[16] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts **28**, Cambridge University Press, Cambridge, 1995. MR1334766

[17] C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, preprint.

[18] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Publications **54.1**, American Mathematical Society, Providence, R.I., 2005. MR2105088

[19] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Publications **54.2**, American Mathematical Society, Providence, R.I., 2005. MR2105089

[20] B. Simon, *Trace Ideals and Their Applications*, second edition, Mathematical Surveys and Monographs **120**, American Mathematical Society, Providence, R.I., 2005. MR2105089

[21] B. Simon, *CMV matrices: Five years after*, J. Comput. Appl. Math. **208** (2007), 120–154.

[22] B. Simon, *Equilibrium measures and capacities in spectral theory*, to appear in Inverse Problems and Imaging.

[23] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, in preparation; to be published by Princeton University Press.

[24] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), 387–435. MR1674798

[25] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications **43**, Cambridge University Press, Cambridge, 1992. MR1163828

[26] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23**, American Mathematical Society, Providence, R.I., 1939; third edition, 1967. MR0000077

[27] J. L. Ullman, *On the regular behaviour of orthogonal polynomials*, Proc. London Math. Soc. (3) **24** (1972), 119–148. MR0291718