WEAK CONVERGENCE OF CD KERNELS AND APPLICATIONS

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ABSTRACT. We prove a general result on equality of the weak limits of the zero counting measure, $d\nu_n$, of orthogonal polynomials (defined by a measure $d\mu$) and $\frac{1}{n}K_n(x,x)\,d\mu(x)$. By combining this with Máté–Nevai and Totik upper bounds on $n\lambda_n(x)$, we prove some general results on $\int_I \frac{1}{n}K_n(x,x)\,d\mu_{\rm s} \to 0$ for the singular part of $d\mu$ and $\int_I |\rho_E(x) - \frac{w(x)}{n}K_n(x,x)|\,dx \to 0$, where ρ_E is the density of the equilibrium measure and $w(x)$ the density of $d\mu$.

1. Introduction

We will discuss here orthogonal polynomials on the real line (OPRL) and unit circle (OPUC) (see [30, 10, 9, 21, 22, 27]). $d\mu$ will denote a measure on $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ (positive but not necessarily normalized), $\Phi_n(z, d\mu)$ and $\varphi_n(z, d\mu)$ its monic and normalized orthogonal polynomials, and $\{\alpha_n\}_{n=0}^\infty$ its Verblunsky coefficients determined by ($\Phi_n^*(z) \equiv z^n\, \overline{\Phi_n(1/\bar z)}$)

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar\alpha_n \Phi_n^*(z) \tag{1.1}$$

and

$$\|\Phi_n\|_{L^2(\partial\mathbb{D}, d\mu)} = \mu(\partial\mathbb{D})^{1/2} \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)^{1/2} \tag{1.2}$$

$d\mu$ will also denote a measure on $\mathbb{R}$ of compact support, $P_n(x, d\mu)$ and $p_n(x, d\mu)$ its monic and normalized orthogonal polynomials. $\{a_n, b_n\}_{n=1}^\infty$ are its Jacobi parameters defined by

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) \tag{1.3}$$

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and

$$\|P_n\|_{L^2(\mathbb{R},d\mu)} = \mu(\mathbb{R})^{1/2}(a_1 \dots a_n) \tag{1.4}$$

The CD kernel is defined by (some authors sum only to $n-1$)

$$K_n(z,w) = \sum_{j=0}^{n} \overline{\varphi_n(z)}\, \varphi_n(w) \tag{1.5}$$

$$K_n(x,y) = \sum_{j=0}^{n} p_j(x)p_j(y) \tag{1.6}$$

The Lebesgue decomposition

$$d\mu(e^{i\theta}) = w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\rm s}(e^{i\theta}) \tag{1.7}$$

$$d\mu(x) = w(x)\, dx + d\mu_{\rm s}(x) \tag{1.8}$$

with $d\mu_{\rm s}$ Lebesgue singular will enter.

To model the issues that concern us here, we recall two consequences of the Szegő condition for OPUC, namely,

$$\int \log(w(\theta))\, \frac{d\theta}{2\pi} > -\infty \tag{1.9}$$

Here are two central results:

Theorem 1.1 (Szegő, 1920 [29])**.** *If the Szegő condition holds, then*

$$\lim_{n\to\infty} \int |\varphi_n(e^{i\theta})|^2\, d\mu_{\rm s} = 0 \tag{1.10}$$

Remark. In distinction, if $w=0$, $\int |\varphi_n(e^{i\theta})|^2\, d\mu_{\rm s} \equiv 1$.

Theorem 1.2 (Máté–Nevai–Totik [16])**.** *If the Szegő condition holds, then for a.e. θ,*

$$w(\theta)\, \frac{1}{n+1}\, K_n(e^{i\theta}, e^{i\theta}) \to 1 \tag{1.11}$$

They state the result in an equivalent form involving the Christoffel function

$$\lambda_n(z_0) = \inf\biggl\{ \int |Q_n(e^{i\theta})|^2\, d\mu(\theta) \biggm| \deg Q_n \le n;\ Q_n(z_0) = 1 \biggr\} \tag{1.12}$$

The minimizer is

$$Q_n(e^{i\theta}) = \frac{K_n(z_0, e^{i\theta})}{K_n(z_0,z_0)} \tag{1.13}$$

$$\lambda_n(z_0) = K_n(z_0,z_0)^{-1} \tag{1.14}$$

and since (1.9) implies $w(\theta) > 0$ for a.e. θ, (1.11) is equivalent to

$$n\lambda_{n-1}(e^{i\theta}) \to w(\theta) \tag{1.15}$$

It is also known under a global $w(\theta) > 0$ condition that there are similar results that come from what is called Rakhmanov theory (see [22, Ch. 9] and references therein):

Theorem 1.3. *If $w(\theta) > 0$ for a.e. θ, then*
(i) (1.10) *holds.*
(ii)

$$\lim_{n\to\infty} \int \bigl| w(\theta)|\varphi_n(e^{i\theta})|^2 - 1 \bigr| \frac{d\theta}{2\pi} = 0 \tag{1.16}$$

Remark. (i) is due to Rakhmanov [18, 19]; (ii) is due to Máté, Nevai, and Totik [15].

Theorems 1.1 and 1.2 are known to hold under a local Szegő condition together with regularity in the sense of Stahl–Totik [28] (see also [24] and below).

One of our goals here is to prove the first results of these genres with neither a local Szegő condition nor a global a.c. condition. While we focus on the OPRL case, for comparison with the above theorems, here are our new results for OPUC:

Theorem 1.4. *If I is an interval on $\partial\mathbb{D}$ so that*
(a) *$w(\theta) > 1$ a.e. on I*
(b) *$d\mu$ is regular for $\partial\mathbb{D}$, that is,*

$$(\rho_1 \dots \rho_n)^{1/n} \to 1 \tag{1.17}$$

then

$$\text{(i)} \qquad \int_I \frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta})\, d\mu_{\mathrm{s}}(\theta) \to 0 \tag{1.18}$$

$$\text{(ii)} \qquad \int_I \left| 1 - w(\theta)\, \frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta}) \right| \frac{d\theta}{2\pi} \to 0 \tag{1.19}$$

Remarks. 1. We do not have pointwise convergence (1.11), but do have "one-half" of it, namely, for $e^{i\theta} \in I$,

$$\liminf\, w(\theta)\, \frac{1}{n+1}\, K_n(e^{i\theta}, e^{i\theta}) \to 1 \tag{1.20}$$

2. One associates existence of limits of $|\varphi_n|^2\, d\mu$ with Rakhmanov which we only expect when $d\mu$ has support on all of $\partial\mathbb{D}$ or a single interval of $\mathbb{R}$. At best, with multiple intervals, one expects almost periodicity of $|\varphi_n|^2\, d\mu$ rather than existence of the limits. For this reason, the Cesàro averages of Theorem 1.4 are quite natural.

There is nothing sacred about $\partial\mathbb{D}$—regularity is defined for any set, and for both OPRL and OPUC, all we need is regularity plus $w(\theta) > 0$ on an interval. We also have interesting new bounds on the density of zeros when regularity fails—these generalize a theorem of Totik–Ullman [33].

These new theorems do not involve tweaking the methods used to prove Theorems 1.1–1.3, but a genuinely new technique (plus one general method of Máté–Nevai used in the proof of Theorem 1.2). Our point here is as much to emphasize this new technique as to prove the results. The new technique is the following:

Theorem 1.5. *Let $d\mu$ be a measure on $\mathbb{R}$ with bounded support or a measure on $\partial\mathbb{D}$. For $\mathbb{R}$, let $d\nu_n$ be the normalized zero counting measure for the OPRL and for $\partial\mathbb{D}$ for the zeros of the paraorthogonal polynomials (POPUC). Let $n(j)$ be a subsequence $n(1) < n(2) < \dots$. Then*

$$d\nu_{n(j)+1} \xrightarrow{w} d\nu_\infty \Longleftrightarrow \frac{1}{n(j)+1}\, K_{n(j)}(x,x)\, d\mu(x) \xrightarrow{w} d\nu_\infty \tag{1.21}$$

Remarks. 1. We will discuss POPUC and related objects on $\partial\mathbb{D}$ in Section 2.

2. As we will discuss, $d\nu_{n(j)}$ and $d\nu_{n(j)+1}$ have the same limits.

In one sense, this result is more than twenty-five years old! It is a restatement of the invariance of the density of states under change of boundary conditions proven in this context first by Avron–Simon [2]. But this invariance is certainly not usually stated in these terms. We also note that for OPUC, I noted this result in my book (see [22, Thm. 8.2]) but did not appreciate its importance.

We note that for OPUC, limits of $\frac{1}{n+1}K_n(e^{i\theta}, e^{i\theta})\, d\mu(\theta)$ have been studied by Golinskii–Khrushchev [12] without explicitly noting the connection to CD kernels. Their interesting results are limited to the case of OPUC and mainly to situations where the support is all of $\partial\mathbb{D}$.

By itself, Theorem 1.4 is interesting (e.g., it could be used to streamline the proof of a slightly weaker version of Corollary 2 of Totik [31]), but it is really powerful when used with the following collection of results:

Theorem 1.6 (Máté–Nevai [14])**.** *For any measure on $\partial\mathbb{D}$,*

$$\limsup\, n\lambda_{n-1}(e^{i\theta}) \le w(\theta) \tag{1.22}$$

Máté–Nevai–Totik [16] noted that this applies to OPRL on $[-1,1]$ using the Szegő mapping, and Totik, in a brilliant paper [31], shows how to extend it to any measure of bounded support, E, on $\mathbb{R}$ so long as E contains an interval:

Theorem 1.7 (Totik [31])**.** *Let $I \subset E \subset \mathbb{R}$ where $I = (a, b)$ is an interval and E is compact. Suppose $d\mu$ is a measure with support contained in E so that*

$$d\mu(x) = w(x)\,dx + d\mu_{\mathrm{s}}(x) \tag{1.23}$$

Let $d\rho_E(x)$ be the potential theoretic equilibrium measure for E (so it is known $d\rho_E \restriction I = \rho_E(x)\,dx$ for some strictly positive, real analytic weight ρ_E). Then for Lebesgue a.e. $x \in I$,

$$\limsup\, n\lambda_{n-1}(x) \le \frac{w(x)}{\rho_E(x)} \tag{1.24}$$

Remarks. 1. Totik concentrated on the deeper and more subtle fact that if there is a local Szegő condition on I and μ is regular for E, then the limit exists and equals $w(x)/\rho_E(x)$ for a.e. x. But along the way he proved (1.24).

2. We will actually prove equality in (1.24) (for $\limsup$) when $E = \mathrm{supp}(d\mu)$ and μ is regular.

3. Later (see Section 8), we will prove the analog of Theorem 1.7 for closed subsets of $\partial\mathbb{D}$.

Some of our results assume regularity of μ so we briefly summarize the main results from that theory due largely to Stahl–Totik [28] in their book; see my recent paper [24] for an overview.

A measure μ on $\mathbb{R}$ is called *regular* if $E = \mathrm{supp}(d\mu)$ is compact and

$$\lim_{n\to\infty} (a_1 \dots a_n)^{1/n} = C(E) \tag{1.25}$$

where $C(E)$ is the (logarithmic) capacity of E. For OPUC, (1.25) is replaced by

$$\lim_{n\to\infty} (\rho_1 \dots \rho_n)^{1/n} = C(E) \tag{1.26}$$

Some insight is gained if one knows in both cases, for any μ (supported on E compact in $\mathbb{R}$ or $\partial\mathbb{D}$), the $\limsup$ is bounded above by $C(E)$. For this paper, regularity is important because of (this is from [28, Sect. 2.2] or [24, Thm. 2.5]).

Theorem 1.8. *If $d\mu$ is regular, then with $d\nu_n$, the density of zeros of the OPRL or of the POPUC, we have*

$$d\nu_n \xrightarrow{w} d\rho_E \tag{1.27}$$

the equilibrium measure for E. Conversely, if (1.27) holds, either μ is regular or μ is supported on a set of capacity zero.

We should mention one criterion for regularity that goes back to Erdös–Turán [8] for $[-1,1]$ and Widom [35] for general E:

Theorem 1.9. *If $E = \operatorname{supp}(d\mu)$ and*

$$d\mu(x) = f(x)\, d\rho_E(x) + d\mu_{\rm s}(x)$$

where $\mu_{\rm s}$ is ρ_E-singular and $f(x) > 0$ for ρ_E-a.e. x, then μ is regular.

There is a proof of Van Assche [34] of the general case presented in [24] that is not difficult. But in the case where $E^{\rm int}$ (interior in the sense of $\mathbb{R}$) differs from E by a set of capacity zero (e.g., $E = [-1,1]$), we will find a proof that uses only our Theorem 1.5/1.7 strategy. In particular, we will have a proof of the Erdös–Turán result that uses neither potential theory nor polynomial inequalities.

We can now describe the content of this paper. In Section 2, we prove the main weak convergence result, Theorem 1.5. In Section 3, we prove the analog of the Erdös–Turán result for $\partial\mathbb{D}$ and illustrate how these ideas are connected to regularity criterion of Stahl–Totik [28]. Section 4 proves Theorem 1.4 for μ regular on $\partial\mathbb{D}$. Sections 5 and 6 then parallel Sections 3 and 4 but for general compact sets $E \subset \mathbb{R}$. Section 7, motivated by work of Totik–Uhlmann [33], provides a comparison result about densities of zeros. Section 9 does the analog of Sections 5 and 6 for general $E \subset \partial\mathbb{D}$. To do this, we need Totik's result (1.24) in that situation. This does not seem to be in the literature, so Section 8 fills that need.

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2. Weak Convergence

Our main goal here is to prove a generalization of Theorem 1.5 (and so also that theorem). We let μ be a measure of compact support in $\mathbb{C}$ and

$$N(\mu) = \sup\{|z| \mid z \in \operatorname{supp}(d\mu)\} \tag{2.1}$$

We let M_z be multiplication by z on $L^2(\mathbb{C}, d\mu)$, so

$$\|M_z\| = N(\mu) \tag{2.2}$$

and let Q_n be the $(n+1)$-dimensional orthogonal projection onto polynomials of degree n or less. All estimates here depend on

Proposition 2.1. *Fix $\ell = 1, 2, \dots$. Then*

$$Q_n M_z^\ell Q_n - (Q_n M_z Q_n)^\ell \tag{2.3}$$

is an operator of rank at most ℓ and norm at most $2N(\mu)^\ell$.

Proof. Let X be the operator in (2.3). Clearly, $X = 0$ on $\operatorname{ran}(1-Q_n) = \ker(Q_n)$. Since M_z maps $\operatorname{ran}(Q_j)$ to $\operatorname{ran}(Q_{j+1})$, $X = 0$ on $\operatorname{ran}(Q_{n-\ell})$. This shows that X has rank at most ℓ. The norm estimate is immediate from (2.2). $\square$

Here is the link to K_n and to $d\nu_{n+1}$. Let $X_j(z, d\mu)$ be the monic OPs for μ and let $x_n = X_n/\|X_n\|$.

Proposition 2.2. (i) *We have for all $w \in \mathbb{C}$,*

$$\det\nolimits_{Q_n}(w - Q_n M_z Q_n) = X_{n+1}(w, d\mu) \tag{2.4}$$

In particular, if $d\nu_{n+1}$ is the zero counting measure for X_{n+1}, then

$$\frac{1}{n+1}\operatorname{Tr}((Q_n M_z Q_n)^\ell) = \int z^\ell \, d\nu_{n+1}(z) \tag{2.5}$$

(ii) *Let*

$$K_n(z, w) = \sum_{j=0}^{n} \overline{x_j(z)}\, x_j(w) \tag{2.6}$$

Then

$$\operatorname{Tr}(Q_n M_z^\ell Q_n) = \int z^\ell K_n(z, z)\, d\mu(z) \tag{2.7}$$

Remark. The proof of (i) is due to Davies and Simon [7].

Proof. (i) Let z_0 be a zero of X_{n+1} of order ℓ. Let $\varphi(z) = X_{n+1}(z)/(z - z_0)^\ell$. Since $Q_n[X_{n+1}] = 0$, we see, with $M_z^{(n)} = Q_n M_z Q_n$,

$$(M_z^{(n)} - z_0)^\ell \varphi = 0 \qquad (M_z^{(n)} - z_0)^{\ell-1}\varphi \neq 0 \tag{2.8}$$

showing that z_0 is a zero of $\det_{Q_n}(w - M_z^{(n)})$ of order at least ℓ. In this way, we see $X_{n+1}(w)$ and the det have the same zeros. Since both are monic, we obtain (2.4).

In particular, this shows

$$\operatorname{Tr}((M_z^{(n)})^\ell) = \sum_{\substack{\text{zeros } z_j \text{ of} \\ \text{multiplicity } m_j}} m_j z_j^\ell = (n+1)\int z^\ell \, d\nu_{n+1}(z)$$

proving (2.5)

(ii) In $L^2(\mathbb{C}, d\mu)$, $\{x_j\}_{j=0}^n$ span $\operatorname{ran}(Q_n)$ and are an orthonormal basis, so

$$\begin{aligned}\operatorname{Tr}(Q_n M_z^\ell Q_n) &= \sum_{j=0}^n \langle x_j, z^\ell x_j\rangle \\ &= \int \sum_{j=0}^n z^\ell |x_j(z)|^2\, d\mu(z)\end{aligned}$$

proving (2.7). $\square$

Proposition 2.3. *Let $d\eta_n$ be the probability measure*

$$d\eta_n(z) = \frac{1}{n+1}\, K_n(z,z)\, d\mu(z) \tag{2.9}$$

Then for $\ell = 0, 1, 2, \dots,$

$$\biggl| \int z^\ell\, d\eta_n(z) - \int z^\ell\, d\nu_{n+1}(z) \biggr| \leq \frac{2\ell N(\mu)^\ell}{n+1} \tag{2.10}$$

Suppose there is a compact set, $K \subset \mathbb{C}$, containing the supports of all $d\nu_n$ and the support of $d\mu$ so that $\{z^\ell\}_{\ell=0}^\infty \cup \{\bar z^\ell\}_{\ell=0}^\infty$ are $\|\cdot\|_\infty$-total in the continuous function on K. Then for any subsequence $n(j)$, $d\eta_{n(j)} \to d\nu_\infty$ if and only if $d\nu_{n(j)+1} \to d\nu_\infty$.

Proof. (2.10) is immediate from (2.5) and (2.7) if we note that, by Proposition 2.1,

$$|\operatorname{Tr}(Q_n M_z^\ell Q_n) - \operatorname{Tr}((Q_n M_z Q_n)^\ell)| \leq 2N(\mu)^\ell \ell \tag{2.11}$$

(2.10) in turn implies that we have the weak convergence result. $\square$

While we were careful to use $n(j)$ and $n(j)+1$, we note that since $|x_j(z)|^2\, d\mu$ is a probability measure, we have

$$\|\eta_n - \eta_{n+1}\| \leq \frac{1}{n+1} + n\biggl(\frac{1}{n} - \frac{1}{n+1}\biggr) \leq \frac{2}{n+1} \tag{2.12}$$

so we could just as well have discussed weak limits of $\eta_{n(j)+1}$ and $\nu_{n(j)+1}$. Here is Theorem 1.5 for OPRL:

Theorem 2.4. *For OPRL, $d\eta_{n(j)}$ converges weakly to $d\nu_\infty$ if and only if $d\nu_{n(j)+1}$ converges weakly to $d\nu_\infty$.*

Proof. $\{x_j\}_{j=0}^\infty$ are total in $C([\alpha,\beta])$ for any real interval $[\alpha,\beta]$, so Proposition 2.3 is applicable. $\square$

For OPUC, we need a few preliminaries: Define $\mathcal{P}\colon C(\partial\mathbb{D}) \to C(\mathbb{D})$ (with $\mathbb{D} = \{z \mid |z| \leq 1\}$) by

$$(\mathcal{P}f)(re^{i\theta}) = \int \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}\, f(e^{i\theta})\, \frac{d\varphi}{2\pi} \tag{2.13}$$

for $r < 1$ and $(\mathcal{P}f)(e^{i\theta}) = f(e^{i\theta})$ and $\mathcal{P}^*\colon \mathcal{M}_{+,1}(\mathbb{D}) \to \mathcal{M}_{+,1}(\partial\mathbb{D})$, the balayage, by duality. Then the following is well known and elementary:

Proposition 2.5. *$\mathcal{P}^*(d\nu)$ is the unique measure, η, on $\partial\mathbb{D}$ with*

$$\int e^{i\ell\theta}\, d\eta(\theta) = \int z^\ell\, d\nu(z) \tag{2.14}$$

for $\ell = 0, 1, 2, \dots$.

Widom [35] proved that if μ is supported on a strict subset of $\partial\mathbb{D}$, then the zero counting measure, $d\nu_n$, has weak limits supported on $\partial\mathbb{D}$,

$$\mathcal{P}^*(d\nu_n) - d\nu_n \xrightarrow{w} 0 \tag{2.15}$$

Finally, we note the following about paraorthogonal polynomials (POPUC) defined for $\beta \in \partial\mathbb{D}$ by

$$P_{n+1}(z,\beta) = z\Phi_n(z) - \bar\beta\Phi_n^*(z) \tag{2.16}$$

defined in [13] and studied further in [4, 5, 6, 11, 25, 36].

Proposition 2.6. *Let β_n be an arbitrary sequence $\partial\mathbb{D}$ and let $d\nu_{n+1}^{(\beta_n)}$ be the zero counting measure for $P_{n+1}(z,\beta_n)$ (known to live on $\partial\mathbb{D}$; see, e.g.* [25]*). Then for any $\ell \geq 0$,*

$$\left| \int z^\ell\, d\nu_{n+1} - \int z^\ell\, d\nu_{n+1}^{(\beta_n)} \right| \to 0 \tag{2.17}$$

Proof. Let $\mathcal{C}_{n+1,F}$ be the truncated CMV matrix of size $n+1$ whose eigenvalues are the zeros of $\Phi_{n+1}(z)$ (see [21, Ch. 4]) and let $\mathcal{C}_{n+1,F}^{(\beta_n)}$ be the unitary dialation whose eigenvalues are the zeros of $P_{n+1}(z,\beta_n)$ (see [6, 25, 23]). Then $\mathcal{C}_{n+1,F} - \mathcal{C}_{n+1,F}^{(\beta_n)}$ is rank one with norm bounded by 2, so $\mathcal{C}_{n+1,F}^\ell - [\mathcal{C}_{n+1,F}^{(\beta)}]^\ell$ is rank at most ℓ with norm 2. Thus

$$|\mathrm{Tr}(\mathcal{C}_{n+1,F}^\ell - (\mathcal{C}_{n+1,F}^{(\beta)})^\ell)| \leq 2\ell \tag{2.18}$$

and

$$|\text{LHS of (2.17)}| \leq \frac{2\ell}{n+1}$$

proving (2.17). $\square$

Theorem 2.7. *For OPUC, $d\eta_{n(j)}$ converges weakly to $d\nu_\infty$ if and only if each of the following converges weakly to $d\nu_\infty$:*

(i) $d\nu_{n+1}^{(\beta_n)}$ *for any* β_n.
(ii) $\mathcal{P}(d\nu_{n+1})$, *the balayage of* $d\nu_{n+1}$.
(iii) $d\nu_{n+1}$ *if* $\operatorname{supp}(d\mu) \neq \partial\mathbb{D}$.

Proof. Immediate from Proposition 2.3 and the above remark. $\square$

When I mentioned Theorem 2.4 to V. Totik, he found an alternate proof, using tools more familiar to OP workers, that provides some insight. With his permission, I include this proof. Let me discuss the result for convergence of sequences rather than subsequences and then give some remarks to handle subsequences.

The key to Totik's proof is Gaussian quadratures which says that if $\{x_j^{(n)}\}_{j=1}^n$ are the zeros of $p_n(x, d\mu)$, and $\lambda_{n-1}(x) = K_{n-1}(x,x)^{-1}$, then for any polynomial R_m of degree $m \leq 2n-1$,

$$\int R_m(x)\, d\mu(x) = \sum_{j=1}^n \lambda_{n-1}(x_j^{(n)}) R_m(x_j^{(n)}) \tag{2.19}$$

We will also need the Christoffel variation principle

$$n \leq q \Rightarrow \lambda_q(x) \leq \lambda_n(x) \tag{2.20}$$

which is immediate from (1.12).

Suppose we know $d\nu_n \to d\nu_\infty$. Fix a polynomial Q_m of degree m with $Q_m \geq 0$ on $\operatorname{cvh}(\operatorname{supp}(d\mu))$. Let

$$R_{m+2n}(x) = \frac{1}{n+1}\, Q_m(x) K_n(x,x)$$

which has degree $m+2n$. Thus, if $N = n+m$, we have by (2.19) that

$$\begin{aligned} &\int Q_m(x) \left[\frac{1}{n+1}\, K_n(x,x)\right] d\mu \\ &\qquad = \sum_{j=1}^N \lambda_{N-1}(x_j^{(N-1)}) Q_m(x_j^{(N-1)}) \left(\frac{1}{n+1}\right) \lambda_n(x_j^{(N-1)})^{-1} \end{aligned} \tag{2.21}$$

$$\leq \left(\frac{N+1}{n+1}\right) \frac{1}{N+1} \sum_{j=1}^N Q_m(x_j^{(N-1)}) \tag{2.22}$$

by $Q_m \geq 0$ and (2.21) (so $\lambda_{N-1}/\lambda_n \leq 1$). As $n \to \infty$, $N/n \to 1$. So, by hypothesis,

$$\text{RHS of (2.22)} \to \int Q_m(y)\, d\nu_\infty(y) \tag{2.23}$$

We conclude by (2.21) that

$$\limsup \int Q_m(x)\,d\eta_n(x) \leq \int Q_m(y)\,d\nu_\infty(y) \tag{2.24}$$

If $0 \leq Q_m \leq 1$ on $\operatorname{cvh}(\operatorname{supp}(d\mu))$, we can apply this also to $1 - Q_m$ and so conclude for such Q_m that

$$\lim \int Q_m\,d\eta_n(x) = \int Q_m\,d\nu_\infty$$

which implies $\text{w-lim}\, d\eta_n = d\nu_\infty$.

The same inequality (2.22) can be used to show that if $d\eta_n \xrightarrow{w} d\eta_\infty$, then

$$\liminf \int Q_m(y)\,d\nu_m(y) \geq \int Q_m(y)\,d\eta_\infty(y)$$

and thus, by the same $1 - Q$ trick, we get $d\nu_n \to d\eta_\infty$.

To handle subsequences, we only need to note that, by (2.12), if $d\eta_{n(j)} \to d\eta_\infty$, then $d\eta_{n(j)+\ell} \to d\eta_\infty$ for $\ell = 0, \pm 1, \pm 2, \dots$. Similarly, by zero interlacing, if $d\nu_{n(j)} \to d\nu_\infty$, then $d\nu_{n(j)+\ell} \to d\nu_\infty$ for fixed ℓ.

By using the operator theoretic proof of Gaussian quadrature (see, e.g., [21, Sect. 1.2]), one sees this proof is closely related to our proof above.

3. Regularity for $\partial\mathbb{D}$: The Erdös–Turán Theorem

Our goal in this section is to prove:

Theorem 3.1. *Let $d\mu$ on $\partial\mathbb{D}$ have the form* (1.7) *with $w(\theta) > 0$ for a.e. θ. Then μ is regular, that is,* (1.26) *holds with $E = \partial\mathbb{D}$ (so $C(E) = 1$).*

Remarks. 1. This is an analog of a theorem for $[-1, 1]$ proven by Erdös and Turän [8]. Our proof here seems to be new.

2. This is, of course, weaker than Rakhmanov's theorem (see [22, Ch. 9] and references therein), but as we will see, this extends easily to some other situations.

The proof will combine the Máté–Nevai theorem (Theorem 1.6) and Proposition 2.5. It is worth noting where the Máté–Nevai theorem comes from. By (1.12), if

$$Q_n(e^{i\theta}) = \frac{1}{n+1} \sum_{j=0}^{n} e^{ij(\theta - \varphi)} \tag{3.1}$$

then

$$\lambda_n(e^{i\varphi}) \leq \int |Q_n(e^{i\theta})|^2\,d\mu(\theta) \tag{3.2}$$

Recognizing $(n+1)|Q_n|^2$ as the Fejér kernel, (1.22) is a standard maximal function a.e. convergence result.

Proposition 3.2. *For any measure on $\partial\mathbb{D}$ for a.e. θ,*

$$\liminf_{n\to\infty} \frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta}) \geq w(\theta)^{-1} \tag{3.3}$$

On the set where $w(\theta) > 0$,

$$\liminf_{n\to\infty} \frac{1}{n+1} w(\theta) K_n(e^{i\theta}, e^{i\theta}) \geq 1 \tag{3.4}$$

Remark. If $w(\theta) = 0$, (3.3) is interpreted as saying that the limit is infinite. In that case, of course, (3.4) does not hold.

Proof. (3.3) is immediate from (1.14) and (1.22). If $w \neq 0$, $ww^{-1} = 1$, so (3.3) implies (3.4). □

Proof of Theorem 3.1. The hypothesis $w > 0$ for a.e. θ implies $d\mu$ is not supported on a set of capacity zero. Thus, by Theorem 1.8, regularity holds if we prove that the density of zeros of POPUC converges to $\frac{d\theta}{2\pi}$. By Proposition 2.5, this follows if we prove that $\frac{1}{n+1} K_{n+1}\, d\mu \to \frac{d\theta}{2\pi}$.

Suppose $n(j) \to \infty$ is a subsequence with $\frac{1}{n+1} K_{n+1}(e^{i\theta}, e^{i\theta}) w(\theta) \frac{d\theta}{2\pi} \to d\nu_1$ and $\frac{1}{n+1} K_{n+1}(e^{i\theta}, e^{i\theta}) d\mu_{\mathrm{s}} \to d\nu_2$. Then since $\frac{1}{n+1} K_n\, d\mu$ is normalized,

$$\int [d\nu_1 + d\nu_2] = 1 \tag{3.5}$$

On the other hand, by Fatou's lemma and (3.4) for any continuous $f \geq 0$ and the hypothesis $w(\theta) > 0$ a.e. θ,

$$\begin{aligned} \int f\, d\nu_1 &= \lim \int f \left[\frac{1}{n+1} w(\theta) K_n(e^{i\theta}, e^{i\theta}) \right] \frac{d\theta}{2\pi} \\ &\geq \int \liminf \left[f\, \frac{1}{n+1} w(\theta) K_n(e^{i\theta}, e^{i\theta}) \right] \frac{d\theta}{2\pi} \\ &\geq \int f(\theta)\, \frac{d\theta}{2\pi} \end{aligned} \tag{3.6}$$

Thus,

$$d\nu_1 \geq \frac{d\theta}{2\pi} \tag{3.7}$$

By (3.5), this can only happen if

$$d\nu_1 = \frac{d\theta}{2\pi} \qquad d\nu_2 = 0 \tag{3.8}$$

Compactness of the space of measures proves that $\frac{1}{n+1} K_{n+1}\, d\mu \to d\theta/2\pi$. □

Along the way, we also proved $d\nu_2 = 0$, that is,

Theorem 3.3. *Under the hypotheses of Theorem 3.1,*

$$\frac{1}{n+1}\int\sum_{j=0}^{n}|\varphi_j(e^{i\theta})|^2\,d\mu_{\mathrm{s}}(\theta)\to 0$$

It would be interesting to see if these methods provide an alternate proof of the theorem of Stahl–Totik [28] that, if for all $\eta>0$,

$$\lim_{n\to\infty}\bigl|\bigl\{\theta\mid\mu(\{\psi\mid|e^{i\theta}-e^{i\psi}|\le\tfrac{1}{n}\})\le e^{-n\eta}\bigr\}\bigr|=0 \tag{3.9}$$

then μ is regular. The point is that using powers of the Fejér kernel, one can get trial functions localized in an interval of size $O(\frac{1}{n})$, and off a bigger interval of size $O(\frac{1}{n})$, it is exponentially small. (3.9) should say that the dominant contribution comes from an $O(\frac{1}{n})$ interval. On the other hand, the translates of these trial functions are spread over $O(\frac{1}{n})$ intervals, so one gets lower bounds on $\frac{1}{n}K_n(e^{i\theta},e^{i\theta})$ of order $\mu(e^{i\theta}-\frac{c}{n},e^{i\theta}+\frac{c}{n})^{-1}$ which are then integrated against $d\mu$ canceling this inverse and hopefully leading to (3.7), and so regularity.

4. Localization on $\partial\mathbb{D}$

In this section, we will prove Theorem 1.4. Instead of using the Máté–Nevai bound to prove regularity, we combine it with regularity to get information.

Proof of Theorem 1.4. As in the proof of Theorem 3.1, we let $d\nu_1, d\nu_2$ be weak limits of $\frac{1}{n}K(e^{i\theta},e^{i\theta})w(\theta)\frac{d\theta}{2\pi}$ and $\frac{1}{n}K_n(e^{i\theta},e^{i\theta})d\mu_{\mathrm{s}}(\theta)$. On I, the same arguments as above imply

$$d\nu_1\restriction I\ge\frac{d\theta}{2\pi}\restriction I \tag{4.1}$$

By regularity, globally

$$d\nu_1+d\nu_2=\frac{d\theta}{2\pi} \tag{4.2}$$

Thus, on I,

$$\nu_1\restriction I=\frac{d\theta}{2\pi}\qquad \nu_2\restriction I=0 \tag{4.3}$$

The second implies (1.18). The first implies that

$$\int_a^b w(\theta)\,\frac{1}{n+1}\,K_n(e^{i\theta},e^{i\theta})\,\frac{d\theta}{2\pi}\to(b-a) \tag{4.4}$$

Since (3.4) and Fatou imply

$$\int_a^b \left[w(\theta)\, \frac{1}{n+1}\, K_n(e^{i\theta}, e^{i\theta}) - 1 \right]_- \frac{d\theta}{2\pi} \to 0 \tag{4.5}$$

we obtain (1.19). □

Notice that (4.4) and (3.4) imply a pointwise a.e. result (which we stated as (1.20))

$$\liminf_{n\to\infty} \frac{1}{n+1}\, w(\theta) K_n(e^{i\theta}, e^{i\theta}) = 1 \tag{4.6}$$

We do not know how to get a pointwise result on lim sup under only the condition $w(\theta) > 0$ (but without a local Szegő condition).

5. Regularity for $E \subset \mathbb{R}$: Widom's Theorem

In this section and the next, our goal is to extend the results of the last two sections to situations where $\partial\mathbb{D}$ is replaced by fairly general closed sets in $\mathbb{R}$. The keys will be Theorem 2.4 and Totik's Theorem 1.7. We begin with a few remarks on where Theorem 1.7 comes from (see also Section 8).

Since the hypothesis is that $\text{supp}(d\mu) \subset E$, not $= E$, it suffices to find $E_n \supset E$ so that $\rho_{E_n}(x) \to \rho_E(x)$ on I and for which (1.21) can be proven. By using $\tilde{E}_n = \{x \mid \text{dist}(x, E) \leq \frac{1}{n}\}$, one first gets approximation by a finite union of intervals and then, by a theorem proven by Bogatyrëv [3], Peherstorfer [17], and Totik [32], one finds $\tilde{E}_n \subset E_n$ where the E_n's are finite unions of intervals with rational harmonic measure. For rational harmonic measures, one can use as trial polynomials $K_m(x, x_0)/K_m(x_0, x_0)$ where K_m is the CD kernel of a measure in the periodic isospectral torus and Floquet theory. (This is the method from Simon [26]; Totik [31] instead uses polynomial mapping.)

Theorem 5.1. *Let $E \subset \mathbb{R}$ be a compact set with $\partial E \equiv E \setminus E^{\text{int}}$ (E^{int} means interior in $\mathbb{R}$) having capacity zero (e.g., a finite version of closed intervals). Let $d\mu$ be a measure with $\sigma_{\text{ess}}(d\mu) = E$ and*

$$d\mu = f(x)\, d\rho_E + d\mu_{\text{s}} \tag{5.1}$$

where $d\mu_{\text{s}}$ is $d\rho_E$-singular. Suppose $f(x) > 0$ for $d\rho_E$-a.e. x. Then $d\mu$ is regular.

Remarks. 1. In this case, $d\rho_E$ is equivalent to $\chi_E\, dx$.

2. For any compact E, this is a result of Widom [35]; see also Van Assche [34], Stahl–Totik [28], and Simon [24].

Proof. Essentially identical to Theorem 3.1. By Theorem 1.8 and the fact that $d\mu$ is clearly not supported on sets of capacity zero, it suffices to prove that $d\nu_n \to d\rho_E$. Pick $n(j) \to \infty$ so $\frac{1}{n(j)+1} K_{n(j)}(x,x) f(x)\, d\rho_E$ and $\frac{1}{n(j)+1} K_{n(j)}(x,x)\, d\mu_{\rm s}$ separately have limits $d\nu_1$ and $d\nu_2$. (1.21) says that (given that $f(x) > 0$ for a.e. x)

$$\liminf \frac{1}{n(j)+1} K_{n(j)}(x,x) f(x) \geq 1 \tag{5.2}$$

By Fatou's lemma on $E^{\rm int}$,

$$d\nu_1 \geq d\rho_E \tag{5.3}$$

Since also $\int (d\nu_1 + d\nu_2) = 1$ and $\int_{E^{\rm int}} d\rho_E = 1$ (since $C(E \setminus E^{\rm int}) = 0$), we conclude $d\nu_1 = d\rho_E$, $d\nu_2 = 0$. By compactness of probability measures, $\frac{1}{n(j)+1} K_{n(j)}(x,x)\, d\mu \xrightarrow{w} d\rho_E$, implying regularity. $\square$

6. Localization on $\mathbb{R}$

Here is an analog of Theorem 1.4 for any $E \subset \mathbb{R}$.

Theorem 6.1. *Let $I = [a,b] \subset E \subset \mathbb{R}$ with $a < b$ and E compact. Let $d\mu$ be a measure on $\mathbb{R}$ so that $\sigma_{\rm ess}(d\mu) = E$ and μ is regular for E. Suppose*

$$d\mu = w(x)\, dx + d\mu_{\rm s} \tag{6.1}$$

with $d\mu_{\rm s}$ Lebesgue singular. Suppose $w(x) > 0$ for a.e. $x \in I$. Then

(i) $\frac{1}{n+1} K_n(x,x)\, d\mu_{\rm s} \xrightarrow{w} 0$
(ii) $\int_I |\rho_E(x) - \frac{1}{n+1} w(x) K_n(x,x)|\, dx \to 0$

Proof. By $w(x) > 0$ a.e. on I and (1.17),

$$\liminf w(x) \frac{1}{n+1} K_n(x,x) \geq \rho_E(x)$$

for a.e. $x \in I$. From this, one can follow exactly the proofs in Section 4. $\square$

7. Comparisons of Density of Zeros

In [33], Totik–Ullman proved the following (we take $[-a,a]$ rather than $[a,b]$ only for notational simplicity):

Theorem 7.1 ([33])**.** *Let $d\mu$ be a measure supported on a subset of $[-1,1]$ where*

$$d\mu = w(x)\, dx + d\mu_{\rm s} \tag{7.1}$$

with

$$w(x) > 0 \quad \textit{for a.e. } x \in [-a,a]$$

for some $a \in (0,1)$. Let $d\nu_\infty$ be any limit point of the zero counting measures for $d\mu$. Then on $(-a,a)$, we have that

$$(2\pi)^{-1}(1-x^2)^{-1/2}\,dx \le d\nu_\infty(x) \le (2\pi)^{-1}(a^2-x^2)^{-1/2}\,dx \tag{7.2}$$

Our goal here is to prove the following, which we will show implies Theorem 7.1 as a corollary:

Theorem 7.2. *Let $d\mu_1, d\mu_2$ be two measures on $\mathbb{R}$ of compact support. Suppose that for some interval $I=(\alpha,\beta)$,*

$$\text{(i)} \qquad d\mu_1 \le d\mu_2 \tag{7.3}$$

$$\text{(ii)} \qquad d\mu_1 \restriction (\alpha,\beta) = d\mu_2 \restriction (\alpha,\beta) \tag{7.4}$$

Let $n(j)\to\infty$ and suppose $d\nu_{n(j)}^{(k)} \to d\nu_\infty^{(k)}$ for $k=1,2$, where $d\nu_n^{(k)}$ is the zero counting measure for $d\mu_k$. Then on (α,β),

$$d\nu_\infty^{(2)} \restriction (\alpha,\beta) \le d\nu_\infty^{(1)} \restriction (\alpha,\beta) \tag{7.5}$$

Proof. By (1.12),

$$\lambda_n(x,d\mu_1) \le \lambda_n(x,d\mu_2) \tag{7.6}$$

so, by (1.14),

$$\frac{1}{n+1}\,K_n^{(2)}(x,x) \le \frac{1}{n+1}\,K_n^{(1)}(x,x) \tag{7.7}$$

for all x. By (7.4) on (α,β),

$$\frac{1}{n+1}\,K_n^{(2)}(x,x)\,d\mu_2 \le \frac{1}{n+1}\,K_n^{(1)}(x,x)\,d\mu_1 \tag{7.8}$$

By Theorem 2.4, this implies (7.5). $\square$

Proof of Theorem 7.1. Let $d\mu_2 = d\mu$ and let $d\mu_1 = \chi_{(-a,a)}[d\mu]$ so $d\mu_1 \le d\mu_2$ with regularity on $[-a,a]$. By Theorem 5.1, $d\mu_1$ is regular for $[a,a]$, so $d\nu_n^{(2)} \to (2\pi)^{-1}(a^2-x^2)^{-1/2}\,dx$, the equilibrium measure for $[-a,a]$. Thus (7.5) implies the second inequality in (7.2).

On the other hand, let $d\mu_1 = d\mu$ and let $d\mu_2 = [\chi_{(-1,1)} - \chi_{(-a,a)}]\,dx + d\mu$. Then $d\mu_1 \le d\mu_2$ with equality on $(-a,a)$. Moreover, $d\mu_2$ is regular for $[-1,1]$ by Theorem 5.1 and the hypothesis $\sigma(d\mu)\subset[-1,1]$. Thus, $d\nu_n^{(1)} \to (2\pi)^{-1}(1-x^2)^{-1/2}\,dx$. Thus (7.5) implies the first inequality in (7.2). $\square$

Remarks. 1. Theorem 7.1 only requires $\sigma_{\text{ess}}(d\mu)\subset[-1,1]$.

2. The theorems in [33] are weaker than Theorem 7.1 in one respect and stronger in another. They are weaker in that, because of their dependence on potential theory, they require that one of the comparison measures be regular. On the other hand, they are stronger in that our

reliance on weak convergence limits us to open sets like (α, β), while they can handle more general sets.

3. [24] has an example of a measure, $d\mu$, on $[-1,1]$ where $w(x) > 0$ on $[-1,0]$ and the zero counting measures include among its limit points the equilibrium measures for $[-1,0]$ and for $[-1,1]$. This shows in the $[a,b]$-form of Theorem 7.1, both inequalities in (7.2) can be saturated!

8. Totik's Bound for OPUC

As preparation for applying our strategy to subsets of $\partial\mathbb{D}$, we need to prove an analog of Totik's bound (1.23) for closed sets on $\partial\mathbb{D}$. Given $a, b \in \partial\mathbb{D}$, we let $I = (a,b)$ be the "interval" of all points "between" a and b, that is, going counterclockwise from a to b, so $-1 \in (e^{i\theta}, e^{i(2\pi-\theta)})$ for $0 < \theta < \pi$ but $-1 \notin (e^{i(2\pi-\theta)}, e^{i\theta})$. Given $E \subset \partial\mathbb{D}$ closed, we let $d\rho_E$ be its equilibrium measure. If $I \subset E \subset \partial\mathbb{D}$ is a nonempty open interval, then

$$d\rho_E \restriction I = \rho_E(\theta)\, dm(\theta) \tag{8.1}$$

where $dm = d\theta/2\pi$. The main theorem of this section is

Theorem 8.1. *Let $I \subset E \subset \partial\mathbb{D}$ where $I = (a,b)$ is an interval and E is closed. Let $d\mu$ be a measure with support in E so that*

$$d\mu(\theta) = w(\theta)\, dm + d\mu_{\mathrm{s}}(\theta) \tag{8.2}$$

Then for dm-a.e. $\theta \in I$, we have

$$\limsup\, n\lambda_{n-1}(e^{i\theta}) \le \frac{w(\theta)}{\rho_E(\theta)} \tag{8.3}$$

Following Totik's strategy [31, 32] for OPRL, we do this in two steps:

Theorem 8.2. (8.3) *holds if* $\operatorname{supp}(d\mu) \subset E^{\mathrm{int}}$ *and E is a finite union of intervals whose relative harmonic measures are rational.*

Theorem 8.3. *For any closed E in $\partial\mathbb{D}$, we can find E_n with*

(i) *Each E_n is a finite union of intervals whose relative harmonic measures are rational.*

(ii)
$$E \subset E_n^{\mathrm{int}} \tag{8.4}$$

(iii)
$$C(E_n \setminus E) \to 0 \tag{8.5}$$

Remark. If $E = I_1 \cup \cdots \cup I_\ell$, the relative harmonic measures are $\rho_E(I_j)$ which sum to 1.

Proof of Theorem 8.1 given Theorems 8.2 and 8.3. By general principles, since I is an interval, $\rho_{E_n}(\theta)$ and $\rho_E(\theta)$ are real analytic with bounded derivatives. (8.5) implies $d\rho_{E_n} \xrightarrow{w} d\rho_E$ and then the bounded derivative implies $\rho_{E_n}(\theta) \to \rho_E(\theta)$ uniformly on compact subsets of I. By Theorem 8.2, LHS of (8.3) $\leq w(\theta)/\rho_{E_n}(\theta)$ for each n. Since $\rho_{E_n}(\theta) \to \rho_E(\theta)$, we obtain (8.3). $\square$

Our proof of Theorems 8.2 and 8.3 diverges from the Totik strategy in two ways. He obtains Theorem 8.2 by using polynomial maps. Instead, following Simon [26], we use Floquet solutions.

Second, Totik shows if E has ℓ gaps, one can find E_n with rational relative harmonic measures also with ℓ gaps obeying (8.4) and (8.5). This is a result with rather different proofs by Bogatyrëv [3], Peherstorfer [17], and Totik [32]. I believe any of these proofs will extend to OPUC, but we will settle for a weaker result—our E_n's will contain up to 2ℓ intervals, the first ℓ each containing one of the ℓ intervals of E and an additional ℓ or fewer exponentially small intervals. This will allow us to "get away" with following only the easier part of Peherstorfer's strategy.

Proof of Theorem 8.2. By Theorem 11.4.5 of [22], E is the essential spectrum of an isospectral torus of Verblunsky coefficients periodic up to a phase, that is, for suitable p (chosen so that $p\rho_E(I_j)$ is an integer for each j),

$$\alpha_{n+1} = \lambda\alpha_n$$

Let μ_E be the measure associated to a point on the isospectral torus.

By Floquet theory (see [22, Sect. 11.2]), for $z \in E^{\text{int}}$, φ_n is a sum of two functions each periodic up to a phase and, by [22, Sect. 11.12], on compact subsets, K, of E^{int},

$$\sup_{z\in K,n} |\varphi_n(z)| < \infty \tag{8.6}$$

It follows from the Christoffel–Darboux formula (see [21, Sect. 2.2]) that

$$\sup_{z,w\in K} |K_n(z,w)| \leq C|z-w|^{-1} \tag{8.7}$$

The almost periodicity of φ implies uniformly on K, $\frac{1}{n+1}K_n(z,z)$ has a finite nonzero limit, and then, by Theorem 2.7, the limit must be $\rho_E(\theta)/w_E(\theta)$, that is, uniformly on K,

$$\lim_{n\to\infty} \frac{1}{n} K_n(e^{i\theta},e^{i\theta}) = \frac{\rho_E(\theta)}{w_E(\theta)} \tag{8.8}$$

where $w_E(\theta)$ is the weight for $d\rho_E(\theta)$. Moreover, as proven in Simon [26], for any $A > 0$, uniformly on $e^{i\theta} \in K$ and $n|\varphi - \theta| < A$,

$$Q_n(e^{i\varphi}) \equiv \frac{K_n(e^{i\theta}, e^{i\varphi})}{K_n(e^{i\theta}, e^{i\theta})} = \frac{\sin(n\rho_E(\theta)(\theta - \varphi))}{n(\theta - \varphi)\rho_E(\theta)} (1 + O(1)) \tag{8.9}$$

Now use $Q_n(e^{i\varphi})$ as a trial function in (1.12). By (8.7) and (8.8), by taking A large, the contribution of $n|\varphi - \theta| > A$ can be made arbitrarily small. Maximal function arguments and (8.9) show that the contribution of the region $n|\varphi-\theta| < A$ to $n\lambda_{n-1}$ is close to $w(\theta)/\rho_E(\theta)$. This proves (8.3) for $d\mu$. $\square$

Given $E \subset \partial\mathbb{D}$ compact, define

$$\tilde{E}_n = \{e^{i\theta} \in \mathbb{D} \mid \operatorname{dist}(e^{i\theta}, E) \leq \tfrac{1}{n}\} \tag{8.10}$$

It is easy to see that $C(\tilde{E}_n \setminus E) \to 0$ and $\tilde{E}_n$ is a union of $\ell(n) < \infty$ closed intervals. It thus suffices to prove Theorem 8.3 when E is already a union of finitely many ℓ disjoint closed intervals, and it is that we are heading towards. (Parenthetically, we note that we could dispense with this and instead prove the analog of Theorem 8.2 for a finite union of intervals using Jost solutions for the isospectral torus associated to such finite gap sets, as in Simon [26].)

Define $\mathcal{P}_n$ to be the set of monic polynomials all of whose zeros lie in $\partial\mathbb{D}$. Since

$$e^{i\theta/2} + e^{i\varphi}e^{-i\theta/2} = e^{i\varphi/2}[e^{i(\theta-\varphi)/2} + e^{-i(\theta-\varphi)/2}]$$

if $P \in \mathcal{P}_n$, there is a phase factor $e^{i\eta}$ so

$$z^{-n/2}e^{i\eta}P(z) \text{ is real on } \partial\mathbb{D} \tag{8.11}$$

Define the restricted Chebyshev polynomials, $\tilde{T}_n$, associated to $E \subset \partial\mathbb{D}$ by requiring that $\tilde{T}_n$ miminize

$$\|P_n\|_E = \sup_{z\in E} |P_n(z)| \tag{8.12}$$

over all $P_n \in \mathcal{P}_n$. We will show that for n large, $2\tilde{T}_{2n}/\|\tilde{T}_{2n}\|_E$ are the rotated discriminants associated to sets E_n that approximate an E which is a finite union of intervals. An important input is

Lemma 8.4. *Let $I = (z_0, z_1)$ be an interval in $\partial\mathbb{D}$. For any small φ, let $I_\varphi = (z_0e^{i\varphi}, z_1e^{-i\varphi})$, so for $\varphi > 0$, I_φ is smaller than I. For $\varphi < 0$,*

$$|(z - z_0e^{i\varphi})(z - z_1e^{-i\varphi})|$$

decreases on $\partial\mathbb{D} \setminus I$ and increases on I as φ decreases in $(-\varepsilon, 0)$.

Proof. Take $z_1 = \bar{z}_0$ and then use some elementary calculus. $\square$

Theorem 8.5. *Let E be a finite union of disjoint closed intervals on $\partial\mathbb{D}$, $E = I_1 \cup \cdots \cup I_\ell$ and let $\partial\mathbb{D} \setminus E = G_1 \cup \cdots \cup G_\ell$ has ℓ gaps. Then*

(i) *Each $\tilde{T}_n$ has at most one zero in each G_ℓ.*

(ii) *If $z_j^{(n)}$ is the zero of $\tilde{T}_n$ in G_j, then on any compact $K \subset G_j$, we have*

$$\lim_{n\to\infty} \inf_{z\in K} \left(\left| \frac{\tilde{T}_n(z)}{(z - z_j^{(n)})\|F_n\|_E} \right| \right)^{1/n} > 1 \tag{8.13}$$

(iii) *At any local maximum, $\tilde{z}$, of $|\tilde{T}_n(z)|$ in some I_j, we have*

$$|\tilde{T}_n(\tilde{z})| = \|\tilde{T}_n\|_E \tag{8.14}$$

Proof. (i) If there are two zeros in some G_j, we can symmetrically move the zeros apart. Doing that increases $\tilde{T}_n$ on G_j which is disjoint from E, but it decreases $\|\tilde{T}_n\|_E$, contradicting the minimizing definition.

(ii) The zero counting measure for $\tilde{T}_n$ converges to the equilibrium measure on E. For standard T_n's and $E \subset \mathbb{R}$, this result is proven in [1, 20, 24]. A small change implies this result for $\tilde{T}_n$. This, in turn, says that uniformly on G_j,

$$\left| \frac{\tilde{T}_n(z)}{(z - z_j^{(n)})\|\tilde{T}_n\|_E} \right|^{1/n} \to \exp(-\Phi_{\rho_E}(z)) \tag{8.15}$$

which implies (8.13).

(iii) Since $z^{-n/2}\tilde{T}_n(z)$ is real up to a phase, the local maxima of $|\tilde{T}_n(z)|$ on $\partial\mathbb{D}$ alternate with the zeros of $\tilde{T}_n(z)$. If a local maximum is smaller than $\|\tilde{T}_n\|_E$, we move the nearest zeros, say z_0, z_1, apart. That decreases $\|\tilde{T}_n\|_{E\setminus(z_0,z_1)}$ and increases $\|\tilde{T}_n\|_{(z_0,z_1)}$. Since the latter is assumed smaller than $\|\tilde{T}_n\|_E$, it decreases $\|\tilde{T}_n\|_E$ overall, violating the minimizing definition. Thus, $|\tilde{T}_n(z)| \geq \|\tilde{T}_n\|_E$. But since $\tilde{z} \in E$, $|\tilde{T}_n(z)| \leq \|\tilde{T}_n\|_E$. $\square$

Now define

$$\Delta_n(z) = \frac{2e^{i\varphi_n}\tilde{T}_{2n}(z)}{\|\tilde{T}_{2n}\|_E} \tag{8.16}$$

where φ_n is chosen to make Δ_n real on $\partial\mathbb{D}$. By (8.14), maxima in E occur with $\Delta_n(z) = \pm 2$, and by (8.13), maxima in $\partial\mathbb{D} \setminus E$ occur at points where $|\Delta_n(z)| > 2$. Thus, up to a phase, $\Delta_n(z)$ looks like a discriminant. So, by Theorem 11.4.5 of [22], $E_n \equiv \Delta_n^{-1}([-2,2])$ is the essential spectrum of a CMV matrix whose Verblunsky coefficients obey $\alpha_{m+p} = \lambda\alpha_m$ for $|\lambda| = 1$.

Proof of Theorem 8.3. E_n has at most 2ℓ components, ℓ containing $I_1, \dots, I_\ell$ (call them $I_1^{(n)}, \dots, I_\ell^{(n)}$), and ℓ possible components, $J_1^{(n)}, \dots, J_\ell^{(n)}$, one in each gap. Since capacities are bounded by $\frac{1}{4}$ times Lebesgue measure, it suffices to show that

$$\sum_{j=1}^{\ell} |I_j^{(n)} \setminus I_j| + |J_j^{(n)}| \to 0$$

to prove (8.5) and complete the proof. Since, on $\partial\mathbb{D}$,

$$-\Phi_{\rho_E}(x) \geq c \operatorname{dist}(x, E)^{1/2} \tag{8.17}$$

by (8.15), we have $|I_j^{(n)} \setminus I_j| \to 0$ and $|J_j^{(n)}| \to 0$. $\square$

9. Theorems for Subsets of $\partial\mathbb{D}$

Given Theorem 8.1 and our strategies in Sections 3–6, we immediately have

Theorem 9.1. *Let $E \subset \partial\mathbb{D}$ with $\partial E = E \setminus E^{\text{int}}$ (E^{int} means interior in $\partial\mathbb{D}$) having capacity zero. Let $d\mu$ be a measure on $\partial\mathbb{D}$ with $\sigma_{\text{ess}}(d\mu) = E$ and*

$$d\mu = f(x)\, d\rho_E + d\mu_{\text{s}} \tag{9.1}$$

where $d\mu_s$ is $d\rho_E$-singular. Suppose $f(x) > 0$ for $d\rho_E$-a.e. x. Then $d\mu$ is regular.

Theorem 9.2. *Let $I \subset E \subset \partial\mathbb{D}$ with I a nonempty closed interval and E closed. Let $d\mu$ be a measure on $\partial\mathbb{D}$ so $\sigma_{\text{ess}}(d\mu) = E$ and μ is regular for E. Suppose*

$$d\mu = w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\text{s}} \tag{9.2}$$

and $w(\theta) > 0$ for a.e. $e^{i\theta} \in I$. Then

$$\text{(i)} \qquad \frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta})\, d\mu_{\text{s}}(\theta) \xrightarrow{w} 0$$

$$\text{(ii)} \qquad \int_I \left| \rho_E(\theta) - \frac{1}{n+1} w(\theta) K_n(e^{i\theta}, e^{i\theta}) \right| \frac{d\theta}{2\pi} \to 0$$

where $d\rho(\theta) = \rho_E(\theta) \frac{d\theta}{2\pi}$ on I.

References

[1] V. V. Andrievskii and H.-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2002.

[2] J. Avron and B. Simon, *Almost periodic Schrödinger operators. II. The integrated density of states*, Duke Math. J. **50** (1983), 369–391.

[3] A. B. Bogatyrëv, *On the efficient computation of Chebyshev polynomials for several intervals*, Sb. Math. **190** (1999), 1571–1605; Russian original in Mat. Sb. **190** (1999), no. 11, 15–50.

[4] M. J. Cantero, L. Moral, and L. Velázquez, *Measures and para-orthogonal polynomials on the unit circle*, East J. Approx. **8** (2002), 447–464.

[5] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56.

[6] M. J. Cantero, L. Moral, and L. Velázquez, *Measures on the unit circle and unitary truncations of unitary operators*, J. Approx. Theory **139** (2006), 430–468.

[7] E. B. Davies and B. Simon, unpublished.

[8] P. Erdös and P. Turán, *On interpolation. III. Interpolatory theory of polynomials*, Ann. of Math. (2) **41** (1940), 510–553.

[9] G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford-New York, 1971.

[10] Ya. L. Geronimus, *Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval*, Consultants Bureau, New York, 1961.

[11] L. Golinskii, *Quadrature formula and zeros of para-orthogonal polynomials on the unit circle*, Acta Math. Hungar. **96** (2002), 169–186.

[12] L. Golinskii and S. Khrushchev, *Cesàro asymptotics for orthogonal polynomials on the unit circle and classes of measures*, J. Approx. Theory **115** (2002), 187–237.

[13] W. B. Jones, O. Njåstad, and W. J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21** (1989), 113–152.

[14] A. Máté and P. Nevai, *Bernstein's inequality in L^p for $0 < p < 1$ and $(C, 1)$ bounds for orthogonal polynomials*, Ann. of Math. (2) **111** (1980), 145–154.

[15] A. Máté, P. Nevai, and V. Totik, *Strong and weak convergence of orthogonal polynomials*, Am. J. Math. **109** (1987), 239–281.

[16] A. Máté, P. Nevai, and V. Totik, *Szegő's extremum problem on the unit circle*, Ann. of Math. **134** (1991), 433–453.

[17] F. Peherstorfer, *Deformation of minimal polynomials and approximation of several intervals by an inverse polynomial mapping*, J. Approx. Theory **111** (2001), 180–195.

[18] E. A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Math. USSR Sb. **32** (1977), 199–213.

[19] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials, II*, Math. USSR Sb. **46** (1983), 105–117.

[20] E. B. Saff and V. Totik, *Logarithmic Potentials With External Eields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **316**, Springer-Verlag, Berlin, 1997.

[21] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, **54.1**, American Mathematical Society, Providence, RI, 2005.
[22] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.
[23] B. Simon, *CMV matrices: Five years after*, to appear in the Proceedings of the W. D. Evans 65th Birthday Conference.
[24] B. Simon, *Equilibrium measures and capacities in spectral theory*, to appear in Inverse Problems and Imaging.
[25] B. Simon, Rank one perturbations and the zeros of paraorthogonal polynomials on the unit circle, to appear in J. Math. Anal. Appl.
[26] B. Simon, *Two extensions of Lubinsky's universality theorem*, preprint
[27] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, in preparation; to be published by Princeton University Press.
[28] H. Stahl and V. Totik, *General Orthogonal Polynomials*, in "Encyclopedia of Mathematics and its Applications," **43**, Cambridge University Press, Cambridge, 1992.
[29] G. Szegő, *Beiträge zur Theorie der Toeplitzschen Formen, I, II*, Math. Z. **6** (1920), 167–202; **9** (1921), 167–190.
[30] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., **23**, American Mathematical Society, Providence, RI, 1939; 3rd edition, 1967.
[31] V. Totik, *Asymptotics for Christoffel functions for general measures on the real line*, J. Anal. Math. **81** (2000), 283–303.
[32] V. Totik, *Polynomial inverse images and polynomial inequalities*, Acta Math. **187** (2001), 139–160.
[33] V. Totik and J. L. Ullman, *Local asymptotic distribution of zeros of orthogonal polynomials*, Trans. Amer. Math. Soc. **341** (1994), 881–894.
[34] W. Van Assche, *Invariant zero behaviour for orthogonal polynomials on compact sets of the real line*, Bull. Soc. Math. Belg. Ser. B **38** (1986), 1–13.
[35] H. Widom, *Polynomials associated with measures in the complex plane*, J. Math. Mech. **16** (1967), 997–1013.
[36] M.-W. L. Wong, *First and second kind paraorthogonal polynomials and their zeros*, to appear in J. Approx. Theory.