EIGENVALUE BOUNDS FOR PERTURBATIONS OF SCHRÖDINGER OPERATORS AND JACOBI MATRICES WITH REGULAR GROUND STATES

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Abstract. We prove general comparison theorems for eigenvalues of perturbed Schrödinger operators that allow proof of Lieb–Thirring bounds for suitable non-free Schrödinger operators and Jacobi matrices.

1. Introduction

Consider a Schrödinger operator,

$$H_0 = -\Delta + V_0 \tag{1.1}$$

on $L^2(\mathbb{R}^\nu)$ which we suppose obeys

$$\inf \operatorname{spec}(-\Delta + V_0) = 0 \tag{1.2}$$

(By subtracting a constant, we can always arrange this, and by assuming this, the notation simplifies.) We are interested in controlling the negative eigenvalues of

$$H = H_0 + V \tag{1.3}$$

We let

$$E_1(V_0; V) \le E_2(V_0; V) \le \cdots \le E_n(V_0; V) \le \cdots \tag{1.4}$$

be either the negative eigenvalues or 0, that is,

$$E_j(V_0; V) = \min(0; \inf\{\lambda, \dim P_{(-\infty,\lambda]}(H) \ge j\}) \tag{1.5}$$

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We say that H_0 has a *regular ground state* if and only if there exists a function, u_0, on $\mathbb{R}^\nu$ obeying

$$(-\Delta + V_0)u_0 = 0 \tag{1.6}$$

$$0 < c_1 \leq u(x) \leq c_2 < \infty \tag{1.7}$$

for some c_1, c_2. We take $c_1 = \inf u$, $c_2 = \sup u$, and let

$$\beta(V_0) = \left(\frac{c_2}{c_1}\right)^2 \tag{1.8}$$

Our main result is that any bound on the number or sums of eigenvalues for the operator $-\Delta + V$ can be carried (with a change in the coupling constant) to the operator $H_0 + V$. This is based on the following observation:

Theorem 1.1. *For any $V \leq 0$ and V_0 obeying* (1.2) *with regular ground state, we have for all j,*

$$|E_j(0;\beta^{-1}V)| \leq |E_j(V_0;V)| \leq |E_j(0;\beta V)| \tag{1.9}$$

This result is remarkable for its generality and also for the simplicity of its proof. We will see in Section 2 that it can be used to compare not only V_0 and 0 but two arbitrary V_0's with relatively bounded ground states.

Of course, (1.9) immediately implies bounds on moments of bound states:

$$S_\gamma(V_0;V) = \sum_{j=1}^\infty |E_j(V_0;V)|^\gamma \tag{1.10}$$

where we look at this only for $\gamma \geq 0$ and interpret $0^0 = 0$ so S_0 is the number of strictly negative eigenvalues. Clearly, Theorem 1.1 implies:

Corollary 1.2. *For any $\gamma > 0$ and $\beta = \beta(V_0)$ given by* (1.8)*,*

$$S_\gamma(0;\beta^{-1}V) \leq S_\gamma(V_0;V) \leq S_\gamma(0;\beta V) \tag{1.11}$$

The standard Lieb–Thirring inequalities (reviewed in [8, 14]) assert

$$S_\gamma(0;\beta V) \leq L_{\gamma,\nu} \int |V(x)|^{\gamma+\frac{\nu}{2}}\, d^\nu x \tag{1.12}$$

for $\gamma \geq \frac{1}{2}$ in $\nu = 1$, $\gamma > 0$ in $\nu = 2$, and $\gamma \geq 0$ in $\nu \geq 3$. In some cases, the optimal constants are known, for example, $L_{\frac{1}{2},1} = \frac{1}{2}$. (These yield good constants in our perturbed estimates but we do not claim optimal constants for our situation!) Clearly, (1.11) implies:

Corollary 1.3.
$$S_\gamma(V_0;V) \leq L_{\gamma,\nu}\beta^{\gamma+\frac{\nu}{2}} \int |V(x)|^{\gamma+\frac{\nu}{2}}\, d^\nu x \tag{1.13}$$
and, in particular,
$$\sum_{j=1}^{\infty} |E_j(V_0;V)|^{\frac{1}{2}} \leq \tfrac{1}{2}\,\beta \int |V(x)|\, dx \tag{1.14}$$
in $\nu = 1$ dimension.

One can also obtain logarithmic estimates as in [13] and Hardy–Lieb–Thirring bounds as in [5]. Since it is known [20, 17, 14] that for $\nu = 1$ and $V \leq 0$,
$$S_{\frac{1}{2}}(0;V) \geq \tfrac{1}{4} \int |V(x)|\, dx \tag{1.15}$$
we conclude that
$$S_{\frac{1}{2}}(V_0;V) \geq \frac{1}{4\beta} \int |V(x)|\, dx \tag{1.16}$$

These results are of interest because there are many cases which are known to have regular ground states.

Example 1.4. If V_0 is periodic, then there is a positive periodic ground state. If V_0 is locally $L^{\frac{\nu}{2}}$ (if $\nu \geq 3$, locally L^1 if $\nu = 1$, and locally L^p with $p > 1$ if $\nu = 2$), then it is known that eigenfunctions are continuous (see [18]) and thus, H_0 has a regular ground state. □

Example 1.5. We will discuss Jacobi matrices in Sections 3 and 4. It is known (see [19, 15, 3]) that elements in the isospectral torus of finite gap Jacobi matrices have regular ground states. □

Example 1.6. If u is any function obeying (1.7), then $V_0 = \frac{\Delta u}{u}$ has a regular ground state. □

In Section 2, we will review the ground state representation and prove a stronger theorem than Theorem 1.1. As hinted, it is the ground state representation that is critical. In this regard, we should emphasize that the variational argument we use in Section 2 has appeared earlier in work of the Birman school—we would mention, in particular, Lemma 6.1 of Birman, Laptev, and Suslina [2], although it may have appeared earlier in their work. Our novelty here is the wide applicability, the use in CLR and Lieb–Thirring bounds, and the applicability to the discrete case and Szegő estimates.

As we will explain in Section 4, an initial motivation for this work was critical Lieb–Thirring bounds for finite gap almost periodic Jacobi matrices in connection with Szegő's theorem for such situations.

Ground state representations do not seem to be in the literature for Jacobi matrices, so we do this first in Section 3, and then prove an analog of Theorem 1.1 for Jacobi matrices in Section 4. Section 5 discusses some other cases.

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2. Comparison for Schrödinger Operators

Fundamental to our results is the ground state representation that if (1.6) holds for u_0, continuous and strictly positive on $\mathbb{R}^\nu$, then

$$\langle gu_0, H_0 gu_0\rangle = \int |\nabla g|^2 u_0^2\, d^\nu x \tag{2.1}$$

Ground state representations go back to Jacobi [11]. For Schrödinger operators, it appears at least as far back as Birman [1] and it was used extensively in constructive quantum field theory (especially by Segal, Nelson, Gross, and Glimm–Jaffe; see Glimm–Jaffe [7]). As a basis for comparison theorems, it was used by Kirsch–Simon [12] and, as noted above, in a similar context by Birman–Laptev–Suslina [2].

We will be cavalier about technical assumptions needed for (2.1). From one point of view, we can use (2.1) as a definition of H_0! Namely, the right side of (2.1) defined for g's with distributional derivative making the right side finite is easily seen to be a closed quadratic form on $\mathcal{H}_{u_0} \equiv L^2(\mathbb{R}^\nu, u_0^2 d^\nu x)$ defining a positive selfadjoint operator $\widetilde{H}_0$ on $\mathcal{H}_{u_0}$. The unitary operator $W\colon L^2(\mathbb{R}^\nu, d^\nu x) \to \mathcal{H}_{u_0}$ by $Wg = u_0^{-1} g$ lets us define $H_0 = W^{-1}\widetilde{H}_0 W$ and our results hold for perturbations of that.

It is not hard to prove that if $V_0 = V_{0,+} + V_{0,-}$ with $V_{0,+} \in L^1_{\text{loc}}(\mathbb{R}^\nu, d^\nu x)$ and $V_{0,-} \in K_\nu$, the Kato class, then the selfadjoint operator H_0 defined as the form closure of $-\Delta + V_0$ on C_0^∞ obeys (2.1) if u_0 is a positive distributional solution of (1.6).

Notice that we do not need (1.2), but only $\inf\operatorname{spec}(H_0) \ge 0$ for this to work, and Theorem 2.1 below holds in that case (although $\inf\operatorname{spec}(H_0) = \inf\operatorname{spec}(H_1)$) under the hypothesis of the theorem. Here is our main result:

Theorem 2.1. *Let H_0, H_1 have the form* (2.1) *for positive continuous functions u_0, u_1. Suppose*

$$0 < \inf\left(\frac{u_0}{u_1}\right) \le \sup\left(\frac{u_0}{u_1}\right) < \infty \tag{2.2}$$

and let

$$\beta \equiv \left[\frac{\sup(\frac{u_0}{u_1})}{\inf(\frac{u_0}{u_1})} \right]^2 \tag{2.3}$$

For any $V \leq 0$, let E_j be given by (1.5). *Then*

$$|E_j(V_0; V)| \leq |E_j(V_1; \beta V)| \tag{2.4}$$

Remark. By interchanging V_0 and V_1 and replacing V by $\beta^{-1}V$, we get the complementary inequality

$$|E_j(V_1; \beta^{-1}V)| \leq |E_j(V_0; V)| \tag{2.5}$$

Lemma 2.2. *Let $V \leq 0$. Let $\tau > 0$. If for some g,*

$$\langle gu_0, (H_0 + V)gu_0 \rangle \leq -\tau \langle gu_0, gu_0 \rangle \tag{2.6}$$

then

$$\langle gu_1, (H_1 + \beta V)gu_1 \rangle \leq -\tau \langle gu_1, gu_1 \rangle \tag{2.7}$$

Proof. Let

$$\beta_+ = \sup\left(\frac{u_0}{u_1}\right)^2 = \left[\inf\left(\frac{u_1}{u_0}\right)\right]^{-2} \tag{2.8}$$

$$\beta_- = \inf\left(\frac{u_0}{u_1}\right)^2 = \left[\sup\left(\frac{u_1}{u_0}\right)\right]^{-2} \tag{2.9}$$

so

$$\beta_+ = \beta\beta_- \Rightarrow \beta\beta_+^{-1} = \beta_-^{-1} \tag{2.10}$$

Since $-Vg^2 \geq 0$ and $|\nabla g|^2 \geq 0$, we have

$$\int (\nabla g)^2 u_1^2 \, dx \leq \beta_-^{-1} \int (\nabla g)^2 u_0^2 \, dx \tag{2.11}$$

$$-\int V g^2 u_1^2 \, dx \geq -\beta_+^{-1} \int V g^2 u_0^2 \, dx \tag{2.12}$$

so, by (2.10) and (2.1),

$$\begin{aligned} \langle gu_1, (H_1 + \beta V)gu_1 \rangle &\leq \beta_-^{-1} \langle gu_0, (H_0 + V)gu_1 \rangle \\ &\leq -\tau\beta_-^{-1} \langle gu_0, gu_0 \rangle \end{aligned} \tag{2.13}$$

But

$$\langle gu_0, gu_0 \rangle \geq \beta_- \langle gu_1, gu_1 \rangle \tag{2.14}$$

and $\tau > 0$, so

$$\text{RHS of (2.13)} \leq -\tau \langle gu_1, gu_1 \rangle$$

proving (2.7). $\square$

Proof of Theorem 2.1. If $E_j(V_0;V)=0$, there is nothing to prove. If $\tau\equiv|E_j(V_0;V)|>0$, there is a space, $\mathcal{H}_j$, of dimension at least j so

$$\langle\psi,(H_0+V)\psi\rangle\leq-\tau\langle\psi,\psi\rangle \tag{2.15}$$

for $\psi\in\mathcal{H}_j$. By the lemma, if $\varphi=\frac{u_1}{u_0}\psi$, we have

$$\langle\varphi,(H_1+\beta V)\varphi\rangle\leq-\tau\langle\varphi,\varphi\rangle \tag{2.16}$$

Thus, there is a space of dimension at least j where (2.16) holds. By the min-max principle (see [16]),

$$E_j(V_1;\beta V)\leq-\tau \tag{2.17}$$

which is (2.4). □

3. Ground State Representation for Jacobi Matrices

While we are interested mainly in semi-infinite one dimension Jacobi matrices, that is, tridiagonal semi-infinite matrices, we can consider the higher-dimensional case as well, so we will. So far as we know, there is no prior literature on the ground state representation for discrete operators, so we start with that in this section.

In $\mathbb{Z}^\nu$, we let δ_j, $j=1,\dots,\nu$, be the ν component vectors with 1 in the jth place and 0 elsewhere. So $k\pm\delta_j$ are the 2ν neighbors of $k\in\mathbb{Z}^\nu$. A Jacobi operator is parametrized by a symmetric $a_{j\ell}>0$ for each j, $\ell\in\mathbb{Z}_\nu$ with $|j-\ell|=1$ and a real number b_k for each $k\in\mathbb{Z}^\nu$. We will suppose

$$\sup_k|b_k|+\sup_\ell|a_{k\ell}|<\infty$$

The Jacobi operator associated with these parameters is the operator J on $\ell^2(\mathbb{Z}^\nu)$ with

$$(J\varphi)_\ell=\sum_{\pm,j=1,\dots,\nu}a_{\ell\,\ell\pm\delta_j}\varphi_{\ell\pm\delta_j}+b_\ell\varphi_\ell \tag{3.1}$$

We will use $J(\{a_{\ell m}\},\{b_\ell\})$ if we want to make the dependence on a and b explicit.

Lemma 3.1. *Let f be a bounded real-valued function on $\mathbb{Z}^\nu$ and M_f the diagonal matrix on $\ell^2(\mathbb{Z}^\nu)$ which is multiplication by f. Then*

$$[M_f,J(\{a_{\ell m}\},\{b_\ell\})]=J(\{a_{\ell m}(f_\ell-f_m),b_\ell\equiv0\}) \tag{3.2}$$

$$[M_f,[M_f,J(\{a_{\ell m}\},\{b_\ell\})]]=J(\{a_{\ell m}(f_\ell-f_m)^2,b_\ell\equiv0\}) \tag{3.3}$$

Proof. (3.2) is an elementary calculation and it implies (3.3). □

Theorem 3.2. *Let J be a Jacobi operator on $\ell^2(\mathbb{Z}^\nu)$ with parameters $\{a_{\ell m}\}, \{b_\ell\}$. Suppose u is a positive "sequence" parametrized by $\mathbb{Z}^\nu$ so that*

$$\sum_{|m-\ell|=1} a_{\ell m} u_m + b_\ell u_\ell = 0 \tag{3.4}$$

for all $\ell \in \mathbb{Z}^\nu$. Then for any f with $fu \in \ell^2(\mathbb{Z}^\nu)$, we have

$$\langle fu, (-J)fu \rangle = \sum_{\substack{m,\ell \\ |m-\ell|=1}} a_{\ell m} u_\ell u_m (f_\ell - f_m)^2 \tag{3.5}$$

Remark. In particular, this shows $-J \geq 0$.

Proof. It suffices to prove (3.5) for f of finite support and take limits. For the left side converges since $-J$ is a bounded operator, and by positivity, the right side converges (a priori perhaps to ∞, but by the equality to a finite limit; it is only here that positivity of u is used).

For finite sequences, f, use (3.3), taking expectations in a vector $\tilde{u}$ which equals u on $\{\ell \mid f(m)$ is non-zero for some m with $|m - \ell| \leq 1\}$. Then $M_f J\tilde{u} = 0$, so $\langle \tilde{u}, [M_f, [M_f J]]\tilde{u}\rangle = -2\langle fu, Jfu\rangle$. □

4. Comparison for Jacobi Operators

In this section, we will prove an analog of Theorem 1.1 for Jacobi operators. One difference that we have to expect is that of $\sup \operatorname{spec}(J) = 0$; the same is true of λJ for any $\lambda > 0$, but the ground states are the same. Thus, comparison of the two J's cannot involve only the ground state ratio but also a setting of scales which will enter as a ratio of a's. In the Schrödinger case, the scale is set by the $-\Delta$ rather than $-\lambda\Delta$.

For notation, we let $E_j(\{a_{\ell m}, b_\ell\})$ be the max of zero and the jth eigenvalue of $J(\{a_{\ell m}, b_\ell\})$ counting from the top. Here is our main result:

Theorem 4.1. *Let $\{a^{(0)}_{\ell m}, b^{(0)}_\ell\}$ and $\{a^{(1)}_{\ell m}, b^{(1)}_\ell\}$ be two sets of bounded Jacobi parameters with positive sequences $u^{(0)}, u^{(1)}$ obeying* (3.4) *for $\{a^{(0)}, b^{(0)}\}, \{a^{(1)}, b^{(1)}\}$, respectively. Let*

$$\beta_+ = \sup_\ell \left(\frac{u^{(0)}_\ell}{u^{(1)}_\ell}\right)^2 \tag{4.1}$$

$$\beta_- = \inf_\ell \left(\frac{u^{(0)}_\ell}{u^{(1)}_\ell}\right)^2 \tag{4.2}$$

$$\gamma_- = \inf\left(\frac{a_{j\ell}^{(0)} u_j^{(0)} u_\ell^{(0)}}{a_{j\ell}^{(1)} u_j^{(1)} u_\ell^{(1)}}\right) \tag{4.3}$$

Suppose $\beta_+ < \infty$ and $\beta_- > 0$. Then for perturbations $\{\delta a_{\ell m}, \delta b_\ell\}$ with $a_{\ell m}^{(0)} + \delta a_{\ell m} > 0$,

$$E_j(\{a_{\ell m}^{(0)}+\delta a_{\ell m}, b_\ell^{(0)}+\delta b_\ell\}) \leq E_j\left(\left\{\eta a_{\ell m}^{(1)}, \eta b_\ell^{(1)}+\beta\left[|\delta b_\ell|+\sum_{|m-\ell|=1}|\delta a_{\ell m}|\right]\right\}\right) \tag{4.4}$$

where

$$\eta = \frac{\gamma_-}{\beta_-} \qquad \beta = \frac{\beta_+}{\beta_-} \tag{4.5}$$

Remarks. 1. We only have a one-sided inequality as we would in the Schrödinger case if we did not demand $V \leq 0$. Since δa terms are never of a definite sign, we cannot have them in a two-sided comparison. But there is clearly a two-sided comparison if $\delta a = 0$ and $\delta b > 0$.

2. Note that rescaling $u^{(0)}$ or $u^{(1)}$ which changes $\beta_+, \beta_-, \gamma_-$ does not change η or β. Similarly, η scales properly under changes of the scale of a.

Following Hundertmark–Simon [9], we begin with a reduction to the case $\delta a = 0$, $\delta b \geq 0$:

Lemma 4.2. *One has*

$$E_j(\{a_{\ell m}^{(0)} + \delta a_{\ell m}, b_\ell^{(0)} + \delta b_\ell\}) \leq E_j\left(\left\{a_{\ell m}^{(0)}, b_\ell^{(0)} + |\delta b_\ell| + \sum_{|m-\ell|=1}|\delta a_{\ell m}|\right\}\right) \tag{4.6}$$

Thus, it suffices to prove (4.4) *when $\delta a_{\ell m} = 0$ and $\delta b_\ell \geq 0$.*

As in [9], this follows from

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \leq \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

and $b \leq |b|$.

Proof of Theorem 4.1. The proof is identical to the proof in Section 2, the sole change being that (2.11) needs to be replaced by

$$\sum_{\substack{\ell,m \\ |\ell-m|=1}} a_{\ell m}^{(1)} u_\ell^{(1)} u_m^{(1)} |f_\ell - f_m|^2 \leq \gamma_-^{-1} \sum_{\substack{\ell,m \\ |\ell-m|=1}} a_{\ell m}^{(0)} u_\ell^{(0)} u_m^{(0)} |f_\ell - f_m|^2 \tag{4.7}$$

So instead of a $\frac{\beta_-}{\beta_-} = 1$, we get $\eta = \frac{\gamma_-}{\beta_-}$. □

Let $W\colon \ell^2(\mathbb{Z}^\nu)\to\ell^2(\mathbb{Z}^\nu)$ by

$$(Wf)_n = (-1)^{|n|} f_n \tag{4.8}$$

with $|n| = |n_1| + \cdots + |n_\nu|$. Then

$$WJ(\{a_{\ell m}, b_\ell\})W^{-1} = -J(\{a_{\ell m}, -b_\ell\}) \tag{4.9}$$

This allows one to control eigenvalues below $\inf\operatorname{spec}(J)$ in the same way. We define E_j^- to be the min of 0 and the jth eigenvalue of J counting from the bottom. We call a sequence u on $\mathbb{Z}^\nu$ alternating positive if $(-1)^{|m|}u_m > 0$ for all m. The result is:

Theorem 4.3. *Let $\{a_{\ell m}^{(0)}, b_\ell^{(0)}\}$ and $\{a_{\ell m}^{(1)}, b_\ell^{(1)}\}$ be two sets of bounded Jacobi parameters with alternating positive sequences $u_m^{(0)}, u_m^{(1)}$ obeying* (3.4) *for $\{a^{(0)}, b^{(0)}\}$, $\{a^{(1)}, b^{(1)}\}$, respectively. Define $\beta_+, \beta_-, \gamma_-, \eta, \beta$ as in* (4.1)–(4.3) *and* (4.5)*. Then* (4.4) *holds with E_j replaced by $|E_j^-|$.*

Corollary 4.4. *Let $\{a_{\ell m}^{(0)}, b_\ell^{(0)}\}$ be a one dimension periodic set of Jacobi parameters or the almost periodic parameters associated with a finite gap spectrum* (*see* [19])*. Let J_0 be the associated half-line Jacobi matrix. Let J be the Jacobi matrix associated with $\{a_{\ell m}^{(0)} + \delta a_{\ell m}, b_\ell^{(0)} + \delta b_\ell\}$ where*

$$\sum_{\substack{|\ell-m|=1\\ \ell=1,2,3,\dots}} |\delta a_{\ell m}| + \sum_{\ell=1}^\infty |\delta b_\ell| < \infty \tag{4.10}$$

Let $E_1^- < E_2^- < \cdots < \inf\operatorname{spec}(J_0) < \sup\operatorname{spec}(J_0) < \cdots < E_2^+ < E_1^+$ be the eigenvalues of J outside the convex hull of $\operatorname{spec}(J_0)$*. Then*

$$\sum_{k,\pm} \operatorname{dist}(E_k^\pm, \sigma(J_0))^{\frac12} \le C \sum (|\delta a_{\ell m}| + |\delta b_\ell|) \tag{4.11}$$

for a constant C depending only on J_0.

Proof. We only need that J_0 and WJ_0W^{-1} have regular ground states. This follows from Floquet theory for the periodic case and from the detailed analysis of Jost solutions for the almost periodic case; see [19, 15, 3]. Then compare to the free J_0 ($a_{\ell m} = 1$ if $|\ell - m| = 1$, $b_\ell = 0$) and use the bound of [9]. □

For the periodic case, this is proven by Damanik, Killip, and Simon [4], who also prove this where the sum in (4.11) is over all eigenvalues including the ones in gaps.

It remains an interesting question relevant to the study of the Szegő condition (see [10, 3]) to get the bound in the almost periodic case. In [10], Hundertmark and Simon prove the weaker bounds where the $\frac12$

power is replaced by any $p > \frac{1}{2}$ or where $p = \frac{1}{2}$ but there is an ℓ^ε added to the sum.

5. Some Final Remarks

All we needed for our arguments is some kind of ground state representation. That means we can replace $-\Delta$ by

$$L_0 = -\sum_{j,k=1}^{\nu} \partial_j A_{jk}(x)\, \partial_k \tag{5.1}$$

with $\{A_{jk}(x)\}_{1\leq j,k\leq \nu}$ a strictly positive matrix. Hence, if $(L_0+V)u_0 = 0$, then

$$\langle fu_0, (L_0+V)fu_0\rangle = \int \langle \nabla f, A\nabla f\rangle |u_0(x)|^2\, d^\nu x$$

and we can still compare to $-\Delta$ although $\sup \|A\|$ and $\inf \|A\|$ will enter.

We can compare magnetic field operators where the magnetic field is fixed but V_0, V vary. For if $H_0 u \equiv (-(\nabla - ia)^2 + V_0)u_0 = 0$, then, since $[f^*, [f, H_0]] = |(\nabla - ia)f|^2$, we have that

$$\langle fu_0, H_0 fu_0\rangle = \int |(\nabla - ia)f|^2 |u_0|^2\, d^\nu x$$

For the discrete case, the key was not tridiagonal matrices, but ones with a ground state representation. For example, non-negative off-diagonal will do.

Using the representation of Frank, Lieb, and Seiringer [6], one can treat some perturbations of $(-\Delta)^\alpha$, $0<\alpha<1$.

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