RANK ONE PERTURBATIONS AND THE ZEROS OF PARAORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. We prove several results about zeros of paraorthogonal polynomials using the theory of rank one perturbations of unitary operators. In particular, we obtain new details on the interlacing of zeros for successive POPUC.

1. Introduction

This note concerns an aspect of the theory of orthogonal polynomials on the unit circle (OPUC); for background, see [11, 4, 7, 8, 9]. Given a nontrivial probability measure, $d\mu$, on $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$, we let $\Phi_n(z)$ (we use $\Phi_n(z, d\mu)$ when $d\mu$ needs to be explicit) be the monic orthogonal polynomials. They obey the Szegő recursion relations

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z) \tag{1.1}$$

$$\Phi_n^*(z) = z^n\, \overline{\Phi_n(1/\bar{z})} \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^\infty \in \mathbb{D}^\infty$ are the Verblunsky coefficients. $d\mu \leftrightarrow \{\alpha_n\}_{n=0}^\infty$ sets up a one-one correspondence between $\mathbb{D}^\infty$ and nontrivial probability measures (Verblunsky's theorem).

Given $\beta \in \partial\mathbb{D}$, the paraorthogonal polynomials (POPUC) are defined by (*Note*: [7] uses β where (1.3) uses $-\bar{\beta}$; (1.3) is the right convention.)

$$\Phi_n(z, d\mu; \beta) = z\Phi_{n-1}(z, d\mu) - \bar{\beta}\Phi_{n-1}^*(z, d\mu) \tag{1.3}$$

More generally, we will consider a sequence $\{\beta_n\}_{n=1}^\infty \in \partial\mathbb{D}$ and

$$\tilde{\Phi}_n(z) = \Phi_n(z, d\mu; \beta_n) \tag{1.4}$$

POPUC were introduced at least as early as Jones, Njåstad, and Thron [6]. About five years ago, Cantero, Moral, and Velázquez [1] and Golinskii [5] realized that zeros of POPUC shared many properties of zeros of OPRL and independently proved a number of basic results about these zeros. Cantero et al. [3] recently proved additional results. The basic tool in [1, 5] is the Christoffel–Darboux formula; [3] also exploits the CMV matrix. Our goal in this paper is to use the theory of rank one perturbations of unitary matrices to recover many of the basic results about zeros of POPUC and prove some new results. In particular, we will illuminate the issue of interlacing of the zeros of successive POPUC.

Date: June 29, 2006.

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First, some notation. Given distinct $z, w \in \partial\mathbb{D}$, (z, w) is the set of points, ζ, in $\partial\mathbb{D}$ with

$$\mathrm{Arg}(z) < \mathrm{Arg}(\zeta) < \mathrm{Arg}(w) \tag{1.5}$$

where a branch of Arg is chosen so $0 < \mathrm{Arg}(w) - \mathrm{Arg}(z) < 2\pi$. An ordered set of points $(z_1, \dots, z_\ell) \in \partial\mathbb{D}^\ell$ is called *cyclicly ordered* if each $(z_j, z_{j+1})_{j=1}^{\ell}$ and (z_ℓ, z_1) contain no other z_j's. The ordering is fixed by such cyclicity up to a single choice. We will always assume zeros of POPUC are cyclicly ordered.

Two cyclicly ordered sets $(z_1, \dots, z_\ell)$ and $(w_1, \dots, w_\ell)$ in $\partial\mathbb{D}^\ell$ are said to *strictly interlace* if after a cyclic permutation of the w's, we have $w_j \in (z_j, z_{j+1})$, $j = 1, 2, \dots, \ell - 1$, $w_\ell \in (z_\ell, z_1)$. This, of course, implies $z_j \in (w_{j-1}, w_j)$, $j = 2, 3, \dots, \ell$ and $z_1 \in (w_\ell, w_1)$.

For $\{\alpha_j\}_{j=0}^\infty$, the second kind polynomials, $\Psi_n(z, d\mu)$ are defined, as usual, to be the Φ_n's associated to $\tilde\alpha_j = -\alpha_j(d\mu)$. We define

$$\Psi_n(z, d\mu; \beta) = z\Psi_{n-1}(z, d\mu) - \bar\beta \Psi_{n-1}^*(z, d\mu) \tag{1.6}$$

We can now state our main results:

Theorem 1.1 ([1, 5])**.** *If (w_0, w_1) is an interval disjoint from* $\mathrm{supp}(d\mu)$*, then for any choice of β and any n, $\Phi_n(z, d\mu; \beta)$ has at most one zero in (w_0, w_1).*

The following has also been proven by Wong [12]:

Theorem 1.2. *Let $(z_1, \dots, z_n)$ be the zeros of some $\Phi_n(z, d\mu; \beta)$ and $(w_1, \dots, w_n)$ of $\Psi_n(z, d\mu; -\beta)$. (Note: $-\beta$, not β.) Then these zeros strictly interlace.*

Theorem 1.3 ([1, 5])**.** *Fix $d\mu$ and n and distinct β, β' in $\partial\mathbb{D}$. Then the zeros of $\Phi_n(z, d\mu; \beta)$ and $\Phi_n(z, d\mu; \beta')$ strictly interlace.*

The power of our approach is shown by the refined version we obtain relating zeros of $\tilde\Phi_{n+1}$ and $\tilde\Phi_n$. We will need the following computed sequence in $\partial\mathbb{D}$:

$$\lambda_n = \bar\beta_{n+1}\bar\beta_n \left(\frac{\beta_n \alpha_n - 1}{\bar\beta_n \bar\alpha_n - 1} \right) \tag{1.7}$$

Theorem 1.4. *For each n, one of two possibilities holds:*

(i) *$\tilde\Phi_n$ and $\tilde\Phi_{n+1}$ have no zeros in common. In that case, λ_n is not a zero of either, and {zeros of $\tilde\Phi_n$} $\cup \{\lambda_n\}$ strictly interlace {zeros of $\tilde\Phi_{n+1}$}.*

(ii) *$\tilde\Phi_n$ and $\tilde\Phi_{n+1}$ have a single zero in common. In that case, λ_n is that zero and {zeros of $\tilde\Phi_n$} strictly interlace {zeros of $\tilde\Phi_{n+1}$}$\backslash\{\lambda_n\}$.*

Corollary 1.5. *If $\tilde\Phi_1, \tilde\Phi_2, \tilde\Phi_3, \dots$ have a common zero at λ, then β_n are given inductively by*

$$\beta_1 = \bar\lambda \tag{1.8}$$

$$\beta_{n+1} = \bar\lambda \bar\beta_n \left(\frac{\beta_n \alpha_n - 1}{\bar\beta_n \bar\alpha_n - 1} \right) \tag{1.9}$$

Example 1.6. $\alpha \equiv 0$. Then $\beta_n = \bar\lambda^n$ and $\tilde\Phi_n(z) = z^n - \lambda^n$ precisely the POPs with a zero at λ for all n. □

The key to our proofs is the connection of $\tilde\Phi_n$ to CMV matrices [2, 7, 10]. $\tilde\Phi_n$ is the determinant of a suitable finite CMV matrix, and so its zeros are the eigenvalues. All our results concern what happens to eigenvalues of unitary matrices under rank one perturbations. Section 2 discusses general rank one perturbations, and Section 3 the application to POPUC.

It is a pleasure to thank María José Cantero and Lilian Wong for useful discussions.

2. Rank One Perturbations

Rank one perturbations of unitaries are discussed in Sections 1.3.9, 1.4.16, 3.2, 4.5, and 10.16 of [7, 8], and some of the results in this section are spread through that material.

If U and V are two unitaries on a finite- or infinite-dimensional Hilbert space and $U-V$ is rank one, we pick a unit vector $\varphi\in\ker(U-V)^\perp$ and note there must be a $\lambda\in\partial\mathbb{D}$ with

$$V\varphi=\lambda U\varphi \tag{2.1}$$

and thus

$$V-U=(\lambda-1)\langle\varphi,\,\cdot\,\rangle U\varphi \tag{2.2}$$

It is convenient to define for $z\in\mathbb{D}$ and A unitary:

$$F_{A,\varphi}(z)=\left\langle\varphi,\,\frac{A+z}{A-z}\,\varphi\right\rangle \tag{2.3}$$

$$f_{A,\varphi}(z)=z^{-1}(1-F(z))(1+F(z))^{-1} \tag{2.4}$$

F is a Carathéodory function (Re $F(z)>0$ on $\mathbb{D}$, $F(0)=1$) and f a Schur function ($|f(z)|<1$ on $\mathbb{D}$). The spectral measure for A,φ is given by

$$F_{A,\varphi}(z)=\int\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_{A,\varphi}(z) \tag{2.5}$$

It is not hard to see that

Proposition 2.1. *Let φ be a cyclic vector for a unitary A (i.e., $\{A^k\varphi\}_{k=-\infty}^{\infty}$ is total). An interval (w_0,w_1) in $\partial\mathbb{D}$ is disjoint from $\sigma_{\text{ess}}(A)$ if and only if f has an analytic continuation through (w_0,w_1) with $|f(z)|=1$ on that interval. In that case,*

(a) Arg(f) *is strictly monotone increasing on (w_0,w_1).*

(b) *The only spectra of A on (w_0,w_1) are simple eigenvalues precisely at the points z where*

$$zf(z)=1 \tag{2.6}$$

Arg(f) is increasing since $|f(z)|<1$ in $\mathbb{D}$ and $|f(z)|=1$ on (w_0,w_1) implies $\partial|f(re^{i\theta})|/\partial r\geq 0$ on (w_0,w_1). So by the Cauchy–Riemann equations, $\partial\text{Arg}(f(e^{i\theta}))/\partial\theta\geq 0$. (b) holds since

$$F(z)=\frac{1+zf(z)}{1-zf(z)} \tag{2.7}$$

has poles at points where (2.6) holds.

When (2.2) holds, a direct calculation (see (1.4.90) and the end of Section 3.2 in [7]) shows that

Proposition 2.2. *If* (2.2) *holds, then*

$$f_{V,\varphi}(z)=\lambda^{-1}f_{U,\varphi}(z) \tag{2.8}$$

We immediately have

Theorem 2.3. *Let* (2.2) *hold. If* $(w_0, w_1) \cap \sigma(U) = \emptyset$*, then* V *has at most one eigenvalue in* $[w_0, w_1]$ *and no other spectrum there.*

Proof. Let $\mathcal{K}$ be the cyclic subspace for U and φ. Since $U = V$ on $\mathcal{K}^\perp$ which is invariant for both, we can suppose φ is cyclic. In that case, picking $z_0 \in (w_0, w_1)$ and then $\text{Arg}(z(f(z)))$ so $\text{Arg}(z_0 f_0(z_0)) \in (0, 2\pi)$, we see $\text{Arg}(z(f(z)) \in (0, 2\pi)$ on all of (w_0, w_1) since (2.6) has no solution there. By the strict monotonicity of $\text{Arg}(f)$, $zf(z) = \lambda$ has at most one solution in $[w_0, w_1]$, so by Propositions 2.1 and 2.2, V that at most one eigenvalue there. □

Proposition 2.4. *Let* U, V *be unitaries on* $\mathbb{C}^n$ *so* (2.2) *holds for* $\lambda \neq 1$ *and for* φ *cyclic for* U*. Then the eigenvalues of* U *and* V *strictly interlace.*

Proof. Since U has a cyclic vector, its spectrum is simple so $zf(z) = 1$ has n solutions. Since $\text{Arg}(z(f))$ is strictly monotone, $zf(z) = \lambda$ has n solutions which interlace the solutions of $zf(z) = 1$. □

One can say something about the case where φ is not cyclic.

Proposition 2.5. *Let* U, V *be unitaries on* $\mathbb{C}^n$ *so* (2.2) *holds. Let* z_0, z_1 *be two eigenvalues of* U*. Then* V *has an eigenvalue in* $[z_0, z_1]$ $(= (z_0, z_1) \cup \{z_0, z_1\})$.

Proof. Let $\mathcal{K}$ be the cyclic subspace of (U, φ) which is invariant for U. If z_0 and z_1 are eigenvalues of $U \restriction \mathcal{K}$, V has an eigenvalue in (z_0, z_1) by Proposition 2.4. If not, since $U \restriction \mathcal{K}^\perp = V \restriction \mathcal{K}^\perp$, either z_0 or z_1 is an eigenvalue of V. □

Finally, we have a specialized result that is precisely what we need to prove Theorem 1.4:

Proposition 2.6. *Let* $U = U_1 \oplus U_2$ *on* $\mathcal{K}_1 \oplus \mathcal{K}_2$*, two finite-dimensional subspaces of* $\mathcal{H}$*, a space of dimension* n*. Let* φ_j $(j = 1, 2)$ *be cyclic vectors for* U_j *on* K_j*. Let* $\varphi = a\varphi_1 \oplus b\varphi_2$ *where* $(a, b) \neq (0, 0)$ *and* $|a|^2 + |b|^2 = 1$*. Let* V *be given by* (2.2) *with* $\lambda \neq 1$*. If* U_1 *and* U_2 *have* ℓ *eigenvalues in common, then* V *has these* ℓ *common values as eigenvalues and its other* $n - \ell$ *eigenvalues strictly interlace those of* U*.*

Proof. Since φ_j is cyclic for U_j, any simple eigenvalue of U is in the cyclic subspace generated by U, φ. Moreover, any common eigenvalue is a simple eigenvalue for $U \restriction \mathcal{K}$ where $\mathcal{K} =$ cyclic subspace of φ. Thus $U \restriction \mathcal{K}$ has all the eigenvalues of U but with multiplicity 1. The eigenvalues of $V \restriction \mathcal{K}$ strictly interlace by Proposition 2.4. The eigenvalues of $V \restriction \mathcal{K}^\perp = U \restriction \mathcal{K}^\perp$ are exactly the common eigenvalues. □

Remark. φ is cyclic if and only if $\ell = 0$.

3. Zeros of POPUC and Finite CMV Matrices

Given a sequence $\{\gamma_n\}_{n=0}^\infty$ of elements in $\overline{\mathbb{D}}$, one defines the CMV matrix $\mathcal{C}(\{\gamma_n\}_{n=0}^\infty)$ on ℓ^2 by

$$\mathcal{C} = \mathcal{L}\mathcal{M} \tag{3.1}$$

$$\mathcal{L} = \Theta(\gamma_0) \oplus \Theta(\gamma_2) \oplus \cdots \tag{3.2}$$

$$\mathcal{M} = \mathbf{1}_{1\times 1} \oplus \Theta(\gamma_1) \oplus \Theta(\gamma_3) \oplus \cdots \tag{3.3}$$

where $\mathbf{1}_{1\times 1}$ is the one-dimensional identity matrix, and Θ is given by

$$\Theta(\gamma) = \begin{pmatrix} \bar{\gamma} & \tau \\ \tau & -\gamma \end{pmatrix} \tag{3.4}$$

$$\tau = (1 - |\gamma|^2)^{1/2} \tag{3.5}$$

It is a fundamental result of Cantero, Moral, and Velázquez [2] (see also [7, Section 4.2]) and see [10] for other references) that if $d\mu$ is a nontrivial probability measure on $\partial\mathbb{D}$, χ_n is the basis of $L^2(\partial\mathbb{D}, d\mu)$ obtained by applying Gram–Schmidt to $1, z, z^{-1}, z^2, z^{-2}, \dots,$ and $\alpha_n(d\mu)$ are the Verblunsky coefficients of $d\mu$, then $\mathcal{C}(\{\alpha_n(d\mu)\}_{n=0}^\infty)$ is the matrix of multiplication by z on $L^2(\partial\mathbb{D}, d\mu)$ in χ_n basis. Note in this case that $\gamma_n \in \mathbb{D}$ (rather than some $\gamma_n \in \partial\mathbb{D}$), in which case we call $\mathcal{C}$ a proper CMV matrix.

If $|\gamma| = 1$, then $\tau = 0$, and $\Theta(\gamma)$ is a direct sum of two 1×1 matrices, and so, if $|\gamma_{n-1}| = 1$, $\mathcal{C}$ breaks into a direct sum of an $n \times n$ matrix and an infinite piece. The finite piece, $\mathcal{C}_n(\{\gamma_0, \dots, \gamma_{n-1}\})$, is called a finite CMV matrix. It is not hard to show that (see, e.g., [10]):

Proposition 3.1. *If $\gamma_n \in \mathbb{D}$ for all n, then $\delta_0 \equiv (1, 0, \dots)^t$ is a cyclic vector for $\mathcal{C}(\{\gamma_n\}_{n=0}^\infty)$. If $\gamma_0, \dots, \gamma_{n-2} \in \mathbb{D}$, $\gamma_{n-1} \in \partial\mathbb{D}$, then δ_0 is a cyclic vector for $\mathcal{C}_n(\{\gamma_m\}_{m=0}^{n-1})$.*

Moreover (see [7, Section 4.2]),

Proposition 3.2. *If $\alpha_0, \dots, \alpha_{n-2} \in \mathbb{D}$ and $\beta \equiv \alpha_{n-1} \in \partial\mathbb{D}$, then*

$$\Phi_n(z, d\mu_\alpha; \beta) = \det(z - \mathcal{C}_n(\{\alpha_j\}_{j=0}^{n-1})) \tag{3.6}$$

In particular, the zeros of $\tilde{\Phi}_n$ are the eigenvalues of a finite CMV matrix.

Finally, we need the following, which generalizes Lemma 4.5.1 of [7]:

Lemma 3.3. *Let $\alpha \in \mathbb{D}$ and $\beta \in \partial\mathbb{D}$. Then*

$$\Theta(\alpha) - \begin{pmatrix} \beta & 0 \\ 0 & x \end{pmatrix} \tag{3.7}$$

is rank one if and only if

$$x = \bar{\beta}\left(\frac{\beta\alpha - 1}{\bar{\beta}\bar{\alpha} - 1}\right) \tag{3.8}$$

Proof. A 2×2 matrix is rank one if and only if $\det(A) = 0$. Since

$$\det\left(\Theta(\alpha) - \begin{pmatrix} \beta & 0 \\ 0 & x \end{pmatrix}\right) = (\bar{\alpha} - \beta)(-\alpha - x) - (1 - |\alpha|^2) \tag{3.9}$$

we see (3.7) is rank one if and only if RHS of (3.9) $= 0$, which is solved by (3.8). □

Note: $|x| = 1$, so $\left(\begin{smallmatrix} \beta & 0 \\ 0 & x \end{smallmatrix}\right)$ is unitary.

Proof of Theorem 1.1. Let $\mathcal{C}$ be the CMV matrix, $\mathcal{C}(\{\alpha_n(d\mu)\}_{n=0}^\infty)$, of $d\mu$. Given n and β, pick x so $\Theta(\alpha_{n-1}) - \left(\begin{smallmatrix} \beta & 0 \\ 0 & x \end{smallmatrix}\right)$ is rank one, and let $\tilde{\mathcal{C}}$ be the matrix obtained from $\mathcal{C}$ by replacing $\Theta(\alpha_{n-1})$ by $\left(\begin{smallmatrix} \beta & 0 \\ 0 & x \end{smallmatrix}\right)$. Then $\tilde{\mathcal{C}}$ is unitary (by the note after Lemma 3.3) and $\mathcal{C} - \tilde{\mathcal{C}}$ is rank one. Thus, by Theorem 2.3, $\tilde{\mathcal{C}}$ has at most one eigenvalue in (w_0, w_1). But $\tilde{\mathcal{C}}$ is a direct sum of $\mathcal{C}_n(\{\alpha_0, \dots, \alpha_{n-2}, \beta\})$ and another matrix, so $\mathcal{C}_n$ has at most one eigenvalue in (w_0, w_1). By Proposition 3.2, zeros of $\tilde{\Phi}_n$ are eigenvalues of $\mathcal{C}_n$. □

Proof of Theorem 1.2. Let $\tilde{\alpha}_{n-1} \equiv \beta$. By Theorem 5.2 of [10] (see also [7, Theorem 4.2.9]), $\mathcal{C}_n(\{-\tilde{\alpha}_m\}_{m=0}^{n-1})$ is unitarily equivalent to $\tilde{\mathcal{C}}_n \equiv \mathcal{L}(\{\tilde{\alpha}_m\}_{m=0}^{n-1})\widetilde{\mathcal{M}}(\{\tilde{\alpha}_m\}_{m=0}^{n-1})$ where $\widetilde{\mathcal{M}}$ differs from $\mathcal{M}$ by having $-\mathbf{1}_{1\times 1}$ in place of $\mathbf{1}_{1\times 1}$. Thus, $\mathcal{C}_n(\{\alpha_m\}_{m=0}^{n-1}) - \tilde{\mathcal{C}}$ is rank one, and Theorem 1.2 follows from Propositions 2.4 and 3.1. $\square$

Proof of Theorem 1.3. If

$$\alpha_j = \alpha_j' = \alpha_j(d\mu) \qquad j = 0, \dots, n-2 \tag{3.10}$$

$$\alpha_{n-1} = \beta \qquad \alpha_{n-1}' = \beta' \tag{3.11}$$

then $\mathcal{C}(\{\alpha_m\}_{m=0}^{n-1}) - \mathcal{C}(\{\alpha_m'\}_{m=0}^{n-1})$ is obviously rank one. Moreover, δ_{n-1} is a cyclic vector since $\mathcal{C}_n$ run backwards is essentially another $\mathcal{C}_n$ (with the initial $\mathbf{1}_{1\times 1}$ replaced by $\bar{\alpha}_{n-1}\mathbf{1}_{1\times 1}$) or $\mathcal{C}_n^t$. Thus, Theorem 1.3 follows from Proposition 2.4. $\square$

Proof of Theorem 1.4. Let $\mathcal{C}_{n+1}$ (resp. $\mathcal{C}_n$) be the $(n+1)\times(n+1)$ (resp. $n \times n$) finite CMV matrix whose characteristic polynomial is $\tilde{\Phi}_{n+1}$ (resp. $\tilde{\Phi}_n$). By Lemma 3.3, a rank one perturbation turns $\mathcal{C}_{n+1}$ into $\mathcal{C}_n \oplus \lambda_n \mathbf{1}_{1\times 1}$ where λ_n is given by (1.7). The vector in the perturbation is $a\delta_{n-1} + b\delta_n$, so Proposition 2.6 applies and proves Theorem 1.4. $\square$

As a final result:

Theorem 3.4. *Let $m > n$. Then strictly between any pair of zeros of $\tilde{\Phi}_n$ is a zero of $\tilde{\Phi}_m$.*

Proof. Let $\mathcal{C}_n, \mathcal{C}_m$ be as in the last proof. By a rank one perturbation, $\mathcal{C}_m$ can be changed to $\mathcal{C}_n \oplus Q_{m-n}$. Now apply Proposition 2.6. $\square$

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