MEROMORPHIC JOST FUNCTIONS AND ASYMPTOTIC EXPANSION FOR JACOBI PARAMETERS

BARRY SIMON*

In memory and appreciation of Mark Krein

Abstract. We show that the parameters a_n, b_n of a Jacobi matrix have a complete asymptotic expansion

$$a_n^2 - 1 = \sum_{k=1}^{K(R)} p_k(n)\mu_k^{-2n} + O(R^{-2n})$$

$$b_n = \sum_{k=1}^{K(R)} p_k(n)\mu_k^{-2n+1} + O(R^{-2n})$$

where $1 < |\mu_j| < R$ for $j \le K(R)$ and all R if and only if the Jost function, u, written in terms of z (where $E = z + z^{-1}$) is an entire meromorphic function. We relate the poles of u to the μ_j's.

1. Introduction

In this paper, we are going to consider semi-infinite Jacobi matrices

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

whose Jacobi parameters have exponential decay (i.e., $\limsup_{n\to\infty}(|b_n| + |a_n - 1|)^{1/n} < 1$. As explained in the first two papers of this series [2, 3] (and well-known earlier), such a J has an associated Jost function, u, defined and analytic in a neighborhood of $\overline{\mathbb{D}}$ where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

As is standard, J describes the recursion relations for orthogonal polynomials on the real line (OPRL). There is a probability measure, γ, so that the orthonormal polynomials, $p_n(x)$ [14, 10, 11], defined by γ obey

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x) \tag{1.2}$$

γ is the spectral measure for J and vector $(1\,0\,0\dots)^t$ and the a's and b's can be obtained from γ by Gram-Schmidt on the moments.

u is defined by γ via the following three facts:

Date: April 12, 2007.

2000 *Mathematics Subject Classification.* 47B36, 81U40, 47A40.

Key words and phrases. Jost function, Jacobi matrix, exponential decay.

* Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592 and in part by Grant No. 2002068 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

(i) $u(z) = 0$ for $z \in \mathbb{D}$ if and only if $z + z^{-1}$ is an eigenvalue of J.

(ii) The support of $d\gamma_{\rm s}$, the singular part of $d\gamma$, is a finite set of eigenvalues in $\mathbb{R}\backslash[-2,2]$ and

$$d\gamma \restriction [-2,2] = f(x)\,dx \tag{1.3}$$

where for any $\theta \in [0, 2\pi)$,

$$f(2\cos\theta) = \frac{1}{\pi}\left[\frac{\lvert\sin\theta\rvert}{\lvert u(e^{i\theta})\rvert^2}\right] \tag{1.4}$$

(iii) $u(0) > 0$.

These determine u by standard theory of nice analytic functions on $\mathbb{D}$. u does not determine γ in many cases. For by (1.3)/(1.4), u determines the a.c. part of γ and the positions of the pure points but not their weights. We prefer to normalize the weights by looking at

$$M(z) = -\int \frac{d\gamma(x)}{x - (z + z^{-1})} \tag{1.5}$$

and looking at the residues of M at the points where $u(z) = 0$. Initially, M is defined by (1.5) for $z \in \mathbb{D}$ only. It is the usual m-function moved to $\mathbb{D}$ via the well-known map.

In any event, u plus the weights are spectral data, and our goal here is to produce equivalences between this spectral data side and the recursion coefficient side.

To state our main theorems, we define

Definition. A sequence, $(x_0, \dots, x_n, \dots)$, of complex numbers is said to have an asymptotic expansion up to $R > 1$ if and only if there exists $\mu_1, \dots, \mu_{K(R)}$ in $\{z \mid 1 < |z| < R\}$ and polynomials $p_1, \dots, p_{K(R)}$ so that

$$\limsup_{n\to\infty} \left| x_n - \sum_{j=1}^{K(R)} p_j(n)\mu_j^{-n} \right|^{1/n} \leq R^{-1} \tag{1.6}$$

We say $(x_0, \dots)$ has a complete asymptotic expansion if it has one for each $R > 1$.

It is easy to see that the x's uniquely determine the p's and μ's and that

Theorem 1.1. *$\{x_n\}_{n=0}^\infty$ has an asymptotic expansion up to R if and only if*

$$f(z) \equiv \sum_{n=0}^\infty x_n z^n \tag{1.7}$$

is meromorphic in $\{z \mid |z| < R\}$ with no singularities in a neighborhood of $\overline{\mathbb{D}}$ and finitely many poles in the region. $\{x_n\}_{n=1}^\infty$ has a complete asymptotic expansion if and only if f is entire meromorphic.

Indeed, the poles are at the μ_j and their orders are one plus the degrees of the p_j.

We say a set of Jacobi parameters has an asymptotic expansion up to R if and only if the sequence

$$(1, -b_1, 1 - a_1^2, -b_2, 1 - a_2^2, \dots) \tag{1.8}$$

has an asymptotic expansion up to R. Thus, the function f is

$$B(z) = 1 - \sum_{n=0}^\infty [b_{n+1}z^{2n+1} + (a_{n+1}^2 - 1)z^{2n+2}] \tag{1.9}$$

B will enter naturally below, but we note the following interpretation: If J_0 is the Jacobi matrix with $a_n = 1$, $b_n \equiv 0$, and $\delta J = J - J_0$, then (see Lemma 6.2 of [2]):

$$\operatorname{Tr}(\delta J(J_0-(z+z^{-1}))^{-1}) = -(z^{-1}-z)^{-1}\biggl\{\sum_{n=1}^\infty b_n(1-z^{2n})+2\sum_{n=1}^\infty (a_n-1)(z-z^{2n+1})\biggr\} \tag{1.10}$$

Moreover (see Theorem 2.16 of [9]),

$$u(z) = \biggl(\prod_{j=1}^\infty a_j\biggr)^{-1} \det(1+\delta J(J_0-(z+z^{-1}))^{-1}) \tag{1.11}$$

Taking into account that $a_n^2 - 1 = 2(a_n-1) + O((a_n-1)^2)$ and $\det(1+A) = 1 + \operatorname{Tr}(A) + O(\|A\|_1^2)$, we see that if δJ is trace class, then

$$-(z^{-1}-z)\biggl(\prod_{j=1}^\infty a_j\biggr)u(z) = c(z) + zB(z) + O(\|\delta J\|_1^2) \tag{1.12}$$

for an affine function $c(z)$. Thus, $B(z)$ is a kind of first-order (Born) approximation to u.

In some ways, our main result in this paper is

Theorem 1.2. *The Jacobi parameters have a complete asymptotic expansion if and only if u is an entire meromorphic function. Equivalently, $B(z)$ is entire meromorphic if and only if $u(z)$ is.*

Of course, one wants to understand the relation between the poles of u and those of B. Both for that understanding and because we will actually use them in our proofs in Section 3, it pays to review our recent results [12] on the analogous problem for orthogonal polynomials on the unit circle (OPUC). The basics (see [10, 11] for background) associate to a nontrivial probability measure, μ, on $\partial\mathbb{D}$ a sequence of Verblunsky coefficients defined by

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar\alpha_n \Phi_n^*(z) \tag{1.13}$$

where Φ_n are the monic orthogonal polynomials for μ and

$$\Phi_n^*(z) = z^n\, \overline{\Phi_n(1/\bar z)} \tag{1.14}$$

In place of B, [12] uses

$$S(z) = 1 - \sum_{j=1}^\infty \alpha_{j-1} z^j \tag{1.15}$$

and, in place of u, the Szegő function

$$D(z) = \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, \log(w(\theta))\, \frac{d\theta}{4\pi}\biggr) \tag{1.16}$$

where $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\rm s}$. One also defines

$$r(z) = \frac{D^{-1}(z)}{\overline{D^{-1}(1/\bar z)}} \tag{1.17}$$

The main theorems of [12] are:

Theorem 1.3 ([4])**.** *If $\limsup|\alpha_n|^{1/n} = R^{-1} < 1$, then $r(z) - S(z)$ is analytic in $\{z \mid 1-\delta < |z| < R^3\}$ for some $\delta > 0$.*

Remarks. 1. This result is due to Deift-Ostensson [4], but [12] has a new proof. Earlier, [10] proved the weaker result when R^3 is replaced by R^2.

2. The point is that r and S both have singularities on $|z| = R$. This theorem says they cancel, as do other singularities in $\{z \mid R < |z| < R^3\}$.

3. [12] has explicit examples where $r(z) - S(z)$ has singularities on $\{z \mid |z| = R^3\}$ and shows that this is the case generically. So R^3 is best possible.

Given a discrete set, $\Omega \subset \{z \mid |z| > 1\}$, with limit points only at ∞, we define

$$\mathbb{G}^{2j-1}(\Omega) = \{\mu_1 \dots \mu_j \bar{\mu}_{j+1} \dots \bar{\mu}_{2j-1} \mid \mu_k \in \Omega\} \tag{1.18}$$

$$\mathbb{G}(\Omega) = \bigcup_{j=1}^{\infty} \mathbb{G}^{2j-1}(\Omega) \tag{1.19}$$

Theorem 1.4 ([12])**.** *S is entire analytic if and only if D^{-1} is. If T is the set of poles of $S(z)$ and P the poles of $D^{-1}(z)$, then*

$$T \subset \mathbb{G}(P) \qquad P \subset \mathbb{G}(T) \tag{1.20}$$

Analogously to Theorem 1.3, we will prove in Section 2 that

Theorem 1.5. *Suppose*

$$\limsup_{n\to\infty} \, (|a_n^2 - 1| + |b_n|)^{1/2n} = R^{-1} < 1 \tag{1.21}$$

Then

$$(1 - z^2)u(z) + z^2 \, \overline{u(1/\bar{z})} \, B(z) \tag{1.22}$$

is analytic in $\{z \mid R^{-1} < |z| < R^2\}$.

Remarks. 1. [3] has necessary and sufficient conditions on $\{u, \text{weights}\}$ for (1.21) to hold. If there are no eigenvalues of J outside $[-2, 2]$, the condition is that u is analytic in $\{z \mid |z| < R\}$.

2. u is real on $\mathbb{R}$ so $\overline{u(\bar{z})} = u(z)$ and thus, (1.22) could be written $(1 - z^2)u(z) + u(1/z)B(z)$; we write it as we do for analogy with the OPUC case.

3. The point, of course, is that B has singularities on $\{z \mid |z| = R\}$, so this theorem implies a cancellation either via zeros of $\overline{u(1/\bar{z})}$ or singularities of u. Since $\overline{u(1/\bar{z})}$ can have zeros in $|z| > 1$ (while $\overline{D(\bar{z})^{-1}}$ cannot), the situation is somewhat different from OPUC. We will discuss this further in Section 2.

4. As we will show in Section 3, the function in (1.22) often has a singularity at $z = R^2$, so one cannot increase the R^2 to R^3 as one can in the OPUC case. The reason for this difference will become clear in Section 3.

For the analog of Theorem 1.4, we need to define a larger set than $\mathbb{G}$. In our situation, u and B are real on $\mathbb{R}$ so their poles are symmetric about $\mathbb{R}$. So for this, we will suppose $\Omega \subset \{z \mid |z| > 1\}$ with limit point only at infinity, and

$$\overline{\Omega} = \Omega \tag{1.23}$$

In that case, for any m, we define

$$\mathbb{G}^{(m)}(\Omega) = \{\mu_1 \dots \mu_m \mid \mu_k \in \Omega\} \tag{1.24}$$

When (1.23) holds, this agrees with the previous definition if $m = 2k - 1$,

$$\widetilde{\mathbb{G}}(\Omega) = \biggl[\, \bigcup_{m=1}^{\infty} \mathbb{G}^{(m)}(\Omega) \biggr] \cup \biggl[- \bigcup_{m=1}^{\infty} \mathbb{G}^{(m)}(\Omega) \biggr] \tag{1.25}$$

Our main results refine Theorem 1.2:

Theorem 1.6. *Let J have no spectrum outside $[-2,2]$ and let u be entire meromorphic and nonvanishing at $z=\pm 1$. Let P be the poles of u and T the poles of B. Then*

$$P \subset \widetilde{\mathbb{G}}(T) \qquad T \subset \widetilde{\mathbb{G}}(P) \tag{1.26}$$

To state the result when there are bound states, we recall and extend a notion from [3].

Definition. Let u be a meromorphic function and $z_0 \in \mathbb{D}$ a point with $u(z_0)=0$ (so z_0 is real and $z_0 + z_0^{-1} \in \sigma(J)$). z_0 is called a noncanonical zero for J if and only if $1/z_0$ is not a pole of u and

$$\lim_{z\to z_0} (z-z_0)M(z) \neq -(z_0 - z_0^{-1}) \biggl[u'(z_0)u\biggl(\frac{1}{z_0}\biggr)\biggr]^{-1} \tag{1.27}$$

Thus, z_0 is not noncanonical (which we will call canonical) if u is regular at $1/z_0$ and equality holds in (1.27). Here is what we will prove in case there are bound states or $u(\pm 1)=0$:

Theorem 1.7. *Suppose u is entire meromorphic. Let T be the poles of B. Let P_1 be the poles of u and P_2 the $\{z^{-1} \mid z \text{ is a noncanonical zero for } J\}$. Let $P = P_1 \cup P_2$. Then (1.26) holds.*

As in [12], one can easily prove results relating meromorphicity of u in $\{z \mid |z| < R^{2\ell-1}\}$ to meromorphicity of B there.

In Section 2, we use the Geronimo-Case equations to prove Theorem 1.5. In Section 3, we use the second Szegő map from OPRL to OPUC to prove Theorem 1.6. In Section 4, we extend the analysis of [3] to obtain Theorem 1.7 from Theorem 1.6.

This research was completed during my stay as a Lady Davis Visiting Professor at Hebrew University, Jerusalem. I'd like to thank H. Farkas and Y. Last for the hospitality of the Mathematics Institute at Hebrew University.

It is a great pleasure to dedicate this paper in honor of the 100th anniversary of the birth of Mark Krein. As a student, I learned of the Krein-Millman theorem but it was only with Krein's impact on my own work that I appreciated the tremendous breadth of his accomplishments: trace ideals, self-adjoint extensions (especially for quadratic forms), the spectral shift function, and his manifold contributions to the spectral theory of orthogonal polynomials have had a profound influence on me.

2. The Geronimo-Case Equations and the R^{-2} Result

In this section, we will prove Theorem 1.5 using a strategy similar to that used in [12] to prove Theorem 1.3. There the critical element was the use of Szegő recursion (1.13) and its *, that is,

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \tag{2.1}$$

at z and $1/\bar{z}$.

Here we will instead use the Geronimo-Case equations [5] in the form introduced in [3]. Define

$$C_n(z) = z^n P_n\biggl(z + \frac{1}{z}\biggr) \tag{2.2}$$

where $P_n(x) = (\prod_{j=1}^n a_j)p_n(x)$ is the monic orthogonal polynomial. The equations

$$C_n(z) = (z^2 - b_n z)C_{n-1}(z) + G_{n-1}(z) \tag{2.3}$$

$$G_n(z) = G_{n-1}(z) + [(1-a_n^2)z^2 - b_n z]C_{n-1}(z) \tag{2.4}$$

are the unnormalized GC equations. With initial condition $G_0(z) = C_0(z) = 1$, they define monic polynomials of degree at most $2n$. C_n has the form (2.2), and if

$$\sum_{n=1}^\infty |a_n^2 - 1| + |b_n| < \infty \tag{2.5}$$

then for $|z| < 1$,

$$\lim_{n\to\infty} G_n(z) = \biggl(\prod_{j=1}^\infty a_j\biggr) u(z) \tag{2.6}$$

(see Theorem A.3 of [3]). We set the right side of (2.6) to be $\tilde u(z)$.

(2.3)/(2.4) have a structure somewhat like (1.13)/(2.1). The difference is that (1.14) is replaced by

$$C_n(z) = z^{2n} C_n\biggl(\frac{1}{z}\biggr) \tag{2.7}$$

as is obvious from (2.2). We introduce $f = \widetilde{O}(g)$ where $g \to 0$ if and only if for all $\varepsilon > 0$, $|f|/|g|^{1-\varepsilon} \to 0$.

Lemma 2.1. *If* (1.21) *holds, then for $z \in \mathbb{D}$,*

(i) $$|G_n(z) - \tilde u(z)| \le \widetilde{O}(R^{-2n}) \tag{2.8}$$

(ii) $$\biggl|C_n(z) - \frac{\tilde u(z)}{1-z^2}\biggr| \le \widetilde{O}([\max(|z|, R^{-1})]^{2n}) \tag{2.9}$$

Proof. (i) By Theorem A.3 of [3],

$$\lim_{n\to\infty} C_n(z) = \frac{\tilde u(z)}{1-z^2} \tag{2.10}$$

By (2.4) and $\sup_n |C_n(z)| < \infty$, we see

$$\begin{aligned} |G_n(z) - \tilde u(z)| &\le \sum_{m=n}^\infty |G_{m+1}(z) - G_m(z)| \\ &\le \bigl(\sup_n |C_n(z)|\bigr) \sum_{m=1}^\infty (|1 - a_{n+m}^2| + |b_{n+m}|) \\ &= \widetilde{O}(R^{-2n}) \end{aligned}$$

since the series of bounds converges exponentially.

(ii) By (2.3),

$$|C_n - G_{n-1} - z^2 C_{n-1}| \le \sup_n |C_n(z)|\, |b_n|$$

so iterating,

$$\begin{aligned} \biggl|C_n - \sum_{j=0}^{n-1} G_{n-j-1} z^{2j}\biggr| &\le |z|^{2n} + \sup_n |C_n(z)| \sum_{j=0}^{n-1} |b_{n-j}||z^{2j}| \\ &\le \widetilde{O}(\max(|z|, R^{-1})^{2n}) \end{aligned}$$

By (2.8),

$$\left|\sum_{j=0}^{n-1}(G_{n-j-1}-\tilde{u})z^{2j}\right|\leq\widetilde{O}(\max(|z|,R^{-1})^{2n})$$

Since $\sum_j z^{2j}u=(1-z^2)^{-1}u$, we have (2.9). $\square$

Proof of Theorem 1.5. By (2.4) and (2.7) for $|z|>1$,

$$|G_{n+1}-G_n|\leq\left[\sup_n\left|C_n\left(\frac{1}{z}\right)\right|\right]|z|^{2n+2}[|1-a_n^2|+|b_n|] \tag{2.11}$$

which proves that for $1<|z|<R$, G_n converges uniformly, so by the maximum principle, we have convergence for $|z|<R$, so u has an analytic continuation to that region. In that region,

$$\begin{aligned}\tilde{u}(z)&=1+\sum_{n=0}^{\infty}(G_{n+1}(z)-G_n(z))\\&=1+\sum_{n=0}^{\infty}((1-a_{n+1}^2)z^2-b_{n+1}z)C_n(z) &(2.12)\\&=\left(\frac{\tilde{u}(\frac{1}{z})}{1-\frac{1}{z^2}}\right)(B(z)-1)+1+\sum_{n=0}^{\infty}f_n(z) &(2.13)\end{aligned}$$

where

$$f_n(z)=((1-a_{n+1}^2)z^2-b_{n+1}z)z^{2n}\left(C_n\left(\frac{1}{z}\right)-\frac{\tilde{u}(\frac{1}{z})}{1-\frac{1}{z^2}}\right) \tag{2.14}$$

Thus

$$(1-z^2)\tilde{u}(z)+\tilde{u}\left(\frac{1}{z}\right)z^2B(z)=\tilde{u}\left(\frac{1}{z}\right)z^2+(1-z^2)+\sum_{n=0}^{\infty}(1-z^2)f_n(z) \tag{2.15}$$

Each function f_n is analytic in $\{z\mid|z|>1\}$, so if we can prove that the sum converges uniformly in $\{z\mid 1<|z|<R^2\}$, we know the left side of (2.15) has an analytic continuation in that region.

By (2.9), for $|z|>1$,

$$\left|C_n\left(\frac{1}{z}\right)-\frac{\tilde{u}(\frac{1}{z})}{1-\frac{1}{z^2}}\right|\leq\widetilde{O}\left(\max\left(\frac{1}{|z|},R^{-1}\right)^{2n}\right)$$

so

$$\left|z^{2n}\left[C_n\left(\frac{1}{z}\right)-\frac{\tilde{u}(\frac{1}{z})}{1-\frac{1}{z^2}}\right]\right|\leq\widetilde{O}(\max(1,|z|R^{-1})^{2n})$$

and thus,

$$|f_n(z)|\leq\widetilde{O}(R^{-2n})\widetilde{O}(\max(1,|z|R^{-1})^{2n})$$

For $1<|z|<R$, this is $\widetilde{O}(R^{-2n})$ and so summable. For $R\leq|z|<R^2$, it is $\widetilde{O}((|z|R^{-2})^{2n})$ and so also summable. $\square$

If $u(\pm R^{-1})\neq 0$, (1.22) tells us that since B has a singularity on the circle of radius R, so must u. However, if $u(R^{-1})=0$ and/or $u(-R^{-1})=0$, that zero can compensate for a pole in B and u can have a larger region of analyticity than B. This is exactly what happens in the case of noncanonical weights, as explained in [3].

3. The Second Szegő Map and Jost Functions With No Bound States

In [13, 14], Szegő defined two maps from the probability measures on $\partial\mathbb{D}$ invariant under $z \to \bar{z}$ to the probability measures on $[-2,2]$; let us call them Sz_1 and Sz_2. Both are injective, but only Sz_1 is surjective — and for this reason, Sz_1 is the one most often used and studied (see [11, Section 13.1]). Here we will see that Sz_2 is also exceedingly useful, especially for studying Jost functions analytic in a neighborhood of $\overline{\mathbb{D}}$ and nonvanishing on $\overline{\mathbb{D}}$ (i.e., J has no bound states and no resonance at ± 2).

For a.c. measures, the relations are

$$d\mu = w(\theta)\,\frac{d\theta}{2\pi} \qquad \mathrm{Sz}_1(d\mu) = f_1(x)\,dx \qquad \mathrm{Sz}_2(d\mu) = f_2(x)\,dx \tag{3.1}$$

where $w(\theta) = w(-\theta)$ and (formulae (13.1.6) and (13.2.22) of [11])

$$f_1(x) = \pi^{-1}(4-x^2)^{-1/2}\,w\biggl(\arccos\biggl(\frac{x}{2}\biggr)\biggr) \tag{3.2}$$

$$f_2(x) = \pi^{-1}c^2(4-x^2)^{1/2}\,w\biggl(\arccos\biggl(\frac{x}{2}\biggr)\biggr) \tag{3.3}$$

where

$$c = [2(1-\lvert\alpha_0\rvert^2)(1-\alpha_1)]^{-1/2} \tag{3.4}$$

Taking into account that Sz_1 is a bijection of even measures on $\partial\mathbb{D}$ and all measures on $[-2,2]$, we see that

$$d\gamma \in \mathrm{ran}(\mathrm{Sz}_2) \Leftrightarrow \int_{-2}^{2} (4-x^2)^{-1}\,d\gamma(x) < \infty \tag{3.5}$$

Proposition 3.1. *If $d\gamma$ has a Jost function u analytic in a neighborhood of $\overline{\mathbb{D}}$ and nonvanishing on $\overline{\mathbb{D}}$, then $d\gamma \in \mathrm{ran}(\mathrm{Sz}_2)$.*

Proof. Since $d(2\cos\theta) = -2\sin\theta\,d\theta$ and $(4-4\cos^2\theta) = 4\sin^2\theta$, by (1.4) the right side of (3.5) is equivalent to

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \frac{d\theta}{\lvert u(e^{i\theta})\rvert^2} &= \int_0^\pi (\sin^2\theta)^{-1} f(2\cos\theta)\sin\theta\,d\theta \\ &= 2\int_{-2}^{2} (4-x^2)^{-1} f(x)\,dx < \infty\end{aligned}$$

which is true if $\lvert u\rvert$ is bounded away from zero. $\square$

For our purposes, what is critical is:

Theorem 3.2. *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ obeying the Szegő condition with Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$ and Szegő function $D(z_0)$. Let $d\gamma = \mathrm{Sz}_2(d\mu)$ have Jost function u and Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$. Then*

$$b_{n+1} = \alpha_{2n} - \alpha_{2n+2} - \alpha_{2n+1}(\alpha_{2n} + \alpha_{2n+2}) \tag{3.6}$$

$$a_{n+1}^2 - 1 = \alpha_{2n+1} - \alpha_{2n+3} - \alpha_{2n+2}^2(1-\alpha_{2n+3})(1+\alpha_{2n+1}) - \alpha_{2n+3}\alpha_{2n+1} \tag{3.7}$$

$$u(z) = (1-\lvert\alpha_0\rvert^2)(1-\alpha_1)D(z)^{-1} \tag{3.8}$$

Remark. Formulae of the form (3.6)/(3.7) for Sz_1 go back to Geronimus [6, 7]. For Sz_2, the earliest reference I am aware of is Berriochoa, Cachafeiro, and García-Amor [1]; see also [8].

Proof. (3.6)/(3.7) are (13.2.20)/(13.2.21) of [11]. To see (3.8), note that, by (1.4) and (3.3),

$$\begin{aligned}|u(e^{i\theta})|^{-2} &= \pi f_2(2\cos\theta)(\sin\theta)^{-1}\\ &= 2c^2 w\\ &= 2c^2|D|^2\end{aligned}$$

Thus, the absolute value of (3.8) holds if $z = e^{i\theta}$. Since both sides are analytic, nonvanishing on $\mathbb{D}$, and positive at $z = 0$, (3.8) holds for all z. □

(3.6)/(3.7) first of all provide a second proof of Theorem 1.5 in case u is nonvanishing on $\overline{\mathbb{D}}$ and, more importantly, show generically that R^2 is optimal. We note first:

Proposition 3.3. *We have that*

$$B(z) = \alpha_0 z^{-1} + \alpha_1 + 1 + (S(z) - 1)(1 - z^{-2}) + Q(z) \tag{3.9}$$

where, if

$$\limsup|\alpha_n|^{1/n} = R^{-1} \tag{3.10}$$

then Q is analytic in $\{z \mid |z| < R^2\}$.

Proof. By (1.9), (1.15), and (3.6)/(3.7), we have (3.9) where

$$\begin{aligned}Q(z) = \sum_{n=0}^{\infty} &\alpha_{2n+1}(\alpha_{2n} + \alpha_{2n+2})z^{2n+1}\\ &+ \{\alpha_{2n+2}^2(1 - \alpha_{2n+3})(1 + \alpha_{2n+1}) + \alpha_{2n+3}\alpha_{2n+1}\}z^{2n+2}\end{aligned} \tag{3.11}$$

By (3.10), $Q(z)$ is analytic in $\{z \mid |z| < R^2\}$. □

Second proof of Theorem 1.5 when u is nonvanishing on $\overline{\mathbb{D}}$. As we will show below (see Lemma 3.5), (1.26) implies (3.10). By Theorem 1.3 and (3.8), we conclude that

$$(z^2 - 1)\big[u(z) - \overline{u(1/\bar{z})}\, S(z)\big] \tag{3.12}$$

is analytic in $\{z \mid R^{-1} < |z| < R^3\}$. By Proposition 3.3,

$$z^2 B(z) - (z^2 - 1)S(z) \tag{3.13}$$

is analytic in $\{z \mid |z| < R^2\}$, so by (3.12), the function in (1.22) is analytic in $\{z \mid R^{-1} < |z| < R^2\}$. □

Example 3.4. Suppose $\alpha_{2n} \equiv 0$ (true if and only if $b_n \equiv 0$) and $\alpha_{2n+1} = R^{-(2n+1)}$. Then, by (3.10),

$$\begin{aligned}Q(z) &= \sum_{n=0}^{\infty} z^{2n+2} R^{-4n-4}\\ &= z^2 R^{-4}(1 - z^2 R^{-4})^{-1}\end{aligned}$$

has poles at $z = \pm R^2$. This shows that (1.22) may not be analytic in any larger annulus than $\{z \mid R^{-1} < |z| < R^2\}$. It is also clear that by a similar analysis, if B is meromorphic in $\{z \mid |z| < R^{1+\varepsilon}\}$, then generically (1.22) will have singularities on the circle of radius R^2. The change from R^3 to R^2 in going from Theorem 1.3 to Theorem 1.5 is due to the quadratic terms in (3.6)/(3.7). □

Above we used and below we will need:

Lemma 3.5. *Suppose* $\{\alpha_n\}_{n=0}^\infty$ *and* $\{a_n, b_n\}_{n=1}^\infty$ *are related by* (3.6)/(3.7)*, and*

$$\limsup|\alpha_n|^{1/n} \equiv R_1^{-1} < 1 \qquad \limsup_{n\to\infty} (|a_n^2 - 1| + |b_n|)^{1/2n} \equiv R_2^{-1} < 1 \tag{3.14}$$

Then $R_1 = R_2$. *Moreover,* α_n *has a complete asymptotic expansion if and only if* $\{a_n, b_n\}$ *do, and if* T *is the set of powers that enter for* $\{a_n, b_n\}_{n=1}^\infty$ *(i.e.,* T *are the poles of* B*) and* $\widetilde{T}$ *for* $\{\alpha_n\}_{n=0}^\infty$ *(i.e.,* $\widetilde{T}$ *are the poles of* S*), then*

$$T \subset \widetilde{\mathbb{G}}(\widetilde{T}) \qquad \widetilde{T} \subset \widetilde{\mathbb{G}}(T) \tag{3.15}$$

Remark. For Sz_1, there are equations similar to (3.6)/(3.7) which have solutions where $\{a_n, b_n\}_{n=0}^\infty$ has rapid decay while $\alpha_{2n+1} \sim n^{-1}$ at infinity. (Indeed, for Sz_1 but *not* Sz_2, this happens for J_0; see Example 13.1.3 revisited in [11].) In fact, the results in this paper plus [12] imply that $R_1^{-1} < 1$ if and only if $R_2^{-1} < 1$.

Proof. It follows from (3.6) that if $R_1^{-1}, R_2^{-1} < 1$, then $R_2 = R_1$ since the nonleading terms are exponentially small. In addition, if α_n has a complete asymptotic expansion, one gets that b_{n+1} and $a_{n+1}^2 - 1$ individually have asymptotic expansion in μ_k^{-2n} with $\mu_k \in \cup_{j=1}^\infty \mathbb{G}^{(j)}(\widetilde{T})$. Since

$$c_1\mu_k^{-2n} = \tfrac{1}{2}\,(c_1 + c_2)\mu_k^{-2n} + \tfrac{1}{2}\,(c_1 - c_2)(-\mu_k)^{-2n} \tag{3.16}$$

$$c_2\mu_k^{-2n-1} = \tfrac{1}{2}\,(c_1 + c_2)\mu_k^{-2n-1} + \tfrac{1}{2}\,(c_1 - c_2)(-\mu_k)^{-2n-1} \tag{3.17}$$

we can combine into a single expansion by taking $-\mu$'s as well as μ's.

For the converse, note that since the α's decay exponentially,

$$\alpha_{2n} = \sum_{m=0}^\infty b_{n+m+1} + O(R^{-2n})$$

and similarly for α_{2n+1} and $\sum_{m=0}^\infty (a_{n+m+1}^2 - 1)$. Plugging this into (3.6) and summing yields α_{2n} and α_{2n+1} as explicit sums of products of four or fewer b's and $(1 - a^2)$'s plus an error of $O(R^{-3n})$. Iterating gives explicit formulae for α's as "polynomials" in b and $1 - a^2$ of degree k plus an error of order $O(R^{-(k+2)n})$. This shows that if a and b have asymptotic expansion to order $R^{-(k+1)n}$, so do α_{2n} and α_{2n-1} with rates in $\cup_{j=1}^\infty \mathbb{G}^{(j)}(T)$. Using formulae like (3.16)/(3.17), we can combine to a single series by using $-\mu$'s, so $\widetilde{T} \subset \widetilde{\mathbb{G}}(T)$. □

Proof of Theorem 1.6 and Theorem 1.2 when u is nonvanishing on $\overline{\mathbb{D}}$. Since u is nonvanishing on $\overline{\mathbb{D}}$, $\gamma \in \mathrm{ran}(\mathrm{Sz}_2)$, so we can define S, α_n, etc. If u is entire meromorphic, by (3.8), so is D^{-1}. Thus, by Theorem 1.4, S is entire meromorphic, and if $\widetilde{T}$ are the poles of S, then

$$\widetilde{T} \subset \mathbb{G}(P)$$

By Lemma 3.5, B is meromorphic and

$$T \subset \widetilde{\mathbb{G}}(\widetilde{T}) \subset \widetilde{\mathbb{G}}(\mathbb{G}(P)) = \widetilde{\mathbb{G}}(P)$$

Conversely, if B is entire meromorphic, by Lemma 3.5, so is S, and if $\widetilde{T}$ are the poles of S, then

$$\widetilde{T} \subset \widetilde{\mathbb{G}}(T)$$

By Theorem 1.4, D^{-1}, and so u, is entire meromorphic and

$$P \subset \mathbb{G}(\widetilde{T}) \subset \mathbb{G}(\widetilde{\mathbb{G}}(T)) = \widetilde{\mathbb{G}}(T) \qquad \square$$

4. Coefficient Stripping and Jost Functions With Bound States

As in [3], we will go from the no bound state theorem to the general case (i.e., in our situation, Theorem 1.6 to Theorem 1.7) by coefficient stripping, that is, pass from J to the Jacobi matrix $J^{(m)}$ with Jacobi parameters $\{a_{n+m}, b_{n+m}\}_{n=1}^{\infty}$. By Theorem 3.1 of [3], if J has a Jost function analytic in a neighborhood of $\overline{\mathbb{D}}$, there exists a k with $\sigma(J^{(k)}) = [-2, 2]$, and by a slight extension of the argument, we can also suppose its Jacobi function obeys $u^{(k)}(\pm 1) \neq 0$ (for if $\sigma(J^{(k-1)}) = [-2, 2]$, if $u^{(k+1)}$ vanishes at ± 1, $M^{(k-1)}(z)$ has a pole there and $u^{(k)} = u^{(k-1)} M^{(k-1)}$ is nonvanishing). Thus, we claim that we need only prove (as we shall do below) that

Theorem 4.1. *If $P = P_1 \cup P_2$ as in Theorem 1.7 and we make the J-dependence explicit, then*

$$P(J) = P(J^{(1)}) \tag{4.1}$$

Proof of Theorems 1.1 and 1.7 given Theorems 4.1 and 1.6. Theorem 1.7 implies Theorem 1.1. By (4.1) and induction, $P(J) = P(J^{(k)})$ where k is chosen as above so Theorem 1.6 is applicable. (1.26) for $J^{(k)}$ implies it for J. □

As in [3], we will make use of the M-function, its connection to u, and the update relations. We define (consistently with (1.5))

$$M^{(k)}(z) = \langle \delta_1, (z + z^{-1} - J^{(k)})^{-1} \delta_1 \rangle \tag{4.2}$$

for $z \in \mathbb{D} \backslash \{w \mid w + w^{-1} \in \sigma(J^{(k)})\}$. $M^{(k)}$ has poles at the set in $\mathbb{D}$ with $w + w^{-1} \in \sigma(J^{(k)})$, and $u^{(k)}$ has zeros there. The update equations ((2.4)/(2.5) of [3]) are (initially for $z \in \mathbb{D}$)

$$u^{(k+1)}(z) = a_{k+1} z^{-1} u^{(k)}(z) M^{(k)}(z) \tag{4.3}$$

$$M^{(k)}(z)^{-1} = z + z^{-1} - b_{k+1} - a_{k+1}^2 M^{(k+1)}(z) \tag{4.4}$$

Moreover, we have the analytic continuation of (1.4) plus $\pi f(2\cos\theta) = \operatorname{Im} M(e^{i\theta})$ for $\theta \in [0, \pi]$,

$$[M(z) - \overline{M}(1/\bar{z})]\, \overline{u(1/\bar{z})}\, u(z) = z - z^{-1} \tag{4.5}$$

(4.5) can be used to meromorphically continue M from $\mathbb{D}$ to $\mathbb{C}$ if u is entire meromorphic. Once one makes these continuations, (4.3)/(4.4) extend to all $z \in \mathbb{C}$ (as meromorphic relations including possible cancellations of poles and zeros). (4.3)/(4.4) also show that if u is entire meromorphic, so is $u^{(1)}$.

We begin by rephrasing the set P_2:

Proposition 4.2. *$z_0 \in \mathbb{C} \backslash \overline{\mathbb{D}}$ is in P_2 if and only if*

(i) *z_0 is not a pole of u.*

(ii) *Both z_0 and z_0^{-1} are poles of M.*

Remark. In $\mathbb{D}$, all poles of M are real, so (ii) implies that z_0 is real.

Proof. By definition, $z_0 \in P_2$ if and only if $u(z_0^{-1}) = 0$, z_0 is not a pole of u, and (1.27) holds. Since $z_0^{-1} \in \mathbb{D}$, $u(z_0^{-1}) = 0 \Leftrightarrow z_0 + z_0^{-1} \in \sigma(J) \Leftrightarrow z_0^{-1}$ is a pole of $M(z)$.

As shown in [3], by (4.5), if $u(z_0) = 0$, z_0 has a pole of M of order two or more and, of course, (1.27) holds at z_0^{-1} since the left side is infinite. If $u(z_0) \neq 0$, (1.27) is precisely the condition, via (4.5), that $M(z)$ has a pole at z_0. □

We have been careful in considering situations where z_0 is a pole of u and z_0^{-1} is a zero of u. We need to consider that case separately:

Proposition 4.3. *If $z_0 \in \mathbb{C}\backslash\mathbb{D}$ is a pole of u and z_0^{-1} is a zero of u, then z_0 is a pole of $u^{(1)}$.*

Proof. Consider (4.5) near $z = z_0$. Zeros of u in $\mathbb{D}$ are simple, so $u(z)\,\overline{u(1/\bar{z})}$ either has a pole at z_0 or a finite nonzero limit. Thus, (4.5) shows $M(z) - \overline{M(1/\bar{z})}$ must be regular (perhaps even zero) at z_0. Since $\overline{M(1/\bar{z})}$ has a pole at z_0, $M(z)$ must have a pole there also. It follows that $u^{(1)} = a_1 z^{-1} uM$ has a pole (indeed, at least a second-order pole) at z_0. $\square$

Proposition 4.4. *If $z_0 \in P_2(J)$, then $z_0 \in P_1(J^{(1)})$.*

Remarks. 1. For z_0 within the disk of analyticity of u, this result is in [3]. The proof here is essentially identical.

2. $z_0 \in P_2(J)$ is essentially a statement of the vanishing of a "resonance eigenfunction," so this says that such eigenfunctions cannot have successive zeros because of a second-order difference equation.

Proof. Suppose first that $u(z_0) \neq 0$. By Proposition 4.2, M has a pole at z_0, so $u^{(1)} = a_1 z^{-1} uM$ has a pole at z_0.

If u has a kth-order zero, $k \geq 1$, at z_0, $\overline{u(1/\bar{z})}\,u(z)$ has a $(k+1)$st-order zero, so $M(z) - \overline{M(1/\bar{z})}$ has a $(k+1)$st-order pole z_0 by (4.5). Since M has simple poles at points in $\mathbb{D}$ like $1/z_0$, M has to have a $(k+1)$st-order pole at z_0. Thus, $u^{(1)} = a_1 z^{-1} uM$ has a pole at z_0. $\square$

Proposition 4.5. *If $z_0 \in P_1(J)$ and $z_0 \notin P_1(J^{(1)})$, then $z_0 \in P_2(J^{(1)})$.*

Proof. By (4.4), poles of $M^{(1)}(z)$ are precisely at zeros of $M(z)$. Thus, by Proposition 4.2, we need to prove that

$$z_0 \in P_1(J) + z_0 \notin P_1(J^{(1)}) \Rightarrow M(z_0) = M\left(\frac{1}{z_0}\right) = 0 \tag{4.6}$$

Since u has a pole at z_0 and $u^{(1)} = a_1 z^{-1} uM$ does not, $M(z_0) = 0$. By Proposition 4.3, $z_0 \notin P_1(J^{(1)})$ implies $\underline{u(1/\bar{z})} \neq 0$. Thus, $u(z)\,\overline{u(1/\bar{z})}$ has a pole at z_0. (4.5) then implies that $M(z) - \overline{M(1/\bar{z})}|_{z=z_0} = 0$. Since $M(z_0) = 0$, we conclude $M(1/\bar{z}_0) = 0$. This proves (4.6). $\square$

We also need some results that go back from $z_0 \in P(J^{(1)})$.

Proposition 4.6. *If $z_0 \in P_2(J^{(1)})$, then $z_0 \in P_1(J)$.*

Proof. By Proposition 4.2, z_0 and z_0^{-1} are poles of $M^{(1)}$, so by (4.4), they are zeros of M. As in the proof of Proposition 4.4, if $u^{(1)}$ has a kth-order zero (including $k = 0$, i.e., $u^{(1)}(z_0) \neq 0$), then $M^{(1)}(z)$ has a $(k+1)$st-order pole there, and so $M(z)$ has a $(k+1)$st-order zero. This is only consistent with $u^{(1)} = uM$ if u has a pole at z_0. $\square$

Proposition 4.7. *If $z_0 \in P_1(J^{(1)})$ and $z_0 \notin P_1(J)$, then $z_0 \in P_2(J)$.*

Proof. By hypothesis, z_0 is not a pole of u, and so (i) of Proposition 4.2 holds. So we need only show that $M(z)$ has poles at z_0 and z_0^{-1}. Suppose u has a kth-order zero at z_0 (including $k = 0$, i.e., $u(z_0) \neq 0$). By (4.3) and the fact that z_0 is a pole of $u^{(1)}$, we see that z_0 is a $(k+1)$st-order pole of $M(z)$ and, in particular, a pole of $M(z)$ (since $k+1 \geq 1$).

If $k \geq 1$, this is only consistent with (4.5) if $\overline{u(1/\bar{z})}$ has a zero at z_0 since the possible pole of M at z_0^{-1} is of order 1 and cannot cancel the $(k+1)$st-order pole at z_0. Thus, $\overline{u(1/\bar{z}_0)} = 0$, so z_0^{-1} is real and a pole of $M(z)$, that is, (ii) of Proposition 4.2 holds and $z_0 \in P_2(J)$.

If $k = 0$ and $M(z)$ does not have a pole at $\bar{z}_0^{-1}$, then $M(z) - \overline{M}(1/\bar{z})$ has a pole at z_0, while $\overline{u(1/\bar{z}_0)} \neq 0 \neq u(z_0)$ (since $k = 0$ and $1/\bar{z}_0$ is not a pole), violating (4.5). Thus, M must have a pole at $\bar{z}_0^{-1}$ and $z_0 \in P_2(J)$ by Proposition 4.2. $\square$

Proof of Theorem 4.1. If $z_0 \in P(J^{(1)})$, either $z_0 \in P_2(J^{(1)}) \Rightarrow z_0 \in P_1(J)$ (by Proposition 4.6) or $z_0 \in P_1(J^{(1)}) \Rightarrow z_0 \in P_1(J) \cup P_2(J)$ (by Proposition 4.7). Thus, $P(J^{(1)}) \subset P(J)$.

If $z_0 \in P(J)$, either $z_0 \in P_2(J) \Rightarrow z_0 \in P_1(J^{(1)})$ (by Proposition 4.4) or $z_0 \in P_1(J) \Rightarrow z_0 \in P_1(J^{(1)}) \cup P_2(J^{(1)})$ (by Proposition 4.5). Thus, $P(J) \subset P(J^{(1)})$. $\square$

References

[1] E. Berriochoa, A. Cachafeiro, and J. García-Amor, *Generalizations of the Szegő transformation interval-unit circle*, preprint.

[2] D. Damanik and B. Simon, *Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics*, Invent. Math. **165** (2006), 1–50.

[3] D. Damanik and B. Simon, *Jost functions and Jost solutions for Jacobi matrices, II. Decay and analyticity*, to appear in Int. Math. Res. Not.

[4] P. Deift and J. Ostensson, *A Riemann-Hilbert approach to some theorems on Toeplitz operators and orthogonal polynomials*, J. Approx. Theory **139** (2006), 144–171.

[5] J. S. Geronimo and K. M. Case, *Scattering theory and polynomials orthogonal on the unit circle*, J. Math. Phys. **20** (1979), 299–310.

[6] Ya. L. Geronimus, *On the trigonometric moment problem*, Ann. of Math. (2) **47** (1946), 742–761.

[7] Ya. L. Geronimus, *Polynomials Orthogonal on a Circle and Their Applications*, Amer. Math. Soc. Translation **1954** (1954), no. 104, 79 pp.

[8] R. Killip and I. Nenciu, *Matrix models for circular ensembles*, Int. Math. Res. Not. (2004), no. 50, 2665–2701.

[9] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), 253–321.

[10] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, American Mathematical Society, Providence, R.I., 2005.

[11] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, American Mathematical Society, Providence, R.I., 2005.

[12] B. Simon, *Meromorphic Szegő functions and asymptotic series for Verblunsky coefficients*, Acta Math. **195** (2005), 267–285.

[13] G. Szegő, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Math. Ann. **86** (1922), 114–139.

[14] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, R.I., 1939; 3rd edition, 1967.