MEROMORPHIC SZEGŐ FUNCTIONS AND ASYMPTOTIC SERIES FOR VERBLUNSKY COEFFICIENTS

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ABSTRACT. We prove that the Szegő function, $D(z)$, of a measure on the unit circle is entire meromorphic if and only if the Verblunsky coefficients have an asymptotic expansion in exponentials. We relate the positions of the poles of $D(z)^{-1}$ to the exponential rates in the asymptotic expansion. Basically, either set is contained in the sets generated from the other by considering products of the form, $z_1 \dots z_\ell \bar{z}_{\ell-1} \dots \bar{z}_{2\ell-1}$ with z_j in the set. The proofs use nothing more than iterated Szegő recursion at z and $1/\bar{z}$.

1. INTRODUCTION

This paper is concerned with the spectral theory of orthogonal polynomials on the unit circle (OPUC) [14, 15, 21, 7, 8] in the case of particularly regular measures. Throughout, we will consider probability measures on $\partial\mathbb{D} = \{z \mid |z| = 1\}$ of the form

$$d\mu = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\rm s} \tag{1.1}$$

where w obeys the Szegő condition, that is,

$$\int \log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty \tag{1.2}$$

In that case, the Szegő function is defined by

$$D(z) = \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log(w(\theta))\,\frac{d\theta}{4\pi}\biggr) \tag{1.3}$$

Not only does w determine D but D determines w, since $\lim_{r\uparrow 1} D(re^{i\theta}) \equiv D(e^{i\theta})$ exists for a.e. θ and

$$w(\theta) = |D(e^{i\theta})|^2 \tag{1.4}$$

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Indeed, D is the unique function analytic on $\mathbb{D} = \{z \mid |z| < 1\}$ with D nonvanishing on $\mathbb{D}$ so that (1.4) holds.

Given $d\mu$, we let Φ_n be the monic orthogonal polynomial and $\varphi_n = \Phi_n/\|\Phi_n\|_{L^2(d\mu)}$. The Φ_n's obey the Szegő recursion:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z) \tag{1.5}$$

where for P_n a polynomial of degree n,

$$P_n^*(z) = z^n \, \overline{P_n(1/\bar{z})} \tag{1.6}$$

The α_n are called *Verblunsky coefficients*. They lie in $\mathbb{D}$ and $\mu \mapsto \{\alpha_n\}_{n=0}^\infty$ is a bijection of nontrivial measures on $\partial\mathbb{D}$ and $\mathbb{D}^\infty$. Our goal here is to focus on the map and its inverse. Here is the background on our first main result:

A. Nevai-Totik [11] proved that $\limsup_{n\to\infty} |\alpha_n|^{1/n} \leq R^{-1} < 1$ if and only if
   (a) $d\mu$ obeys the Szegő condition and $d\mu_\mathrm{s} = 0$.
   (b) $D(z)^{-1}$ is analytic in $\{z \mid |z| < R\}$.

B. Barrios-López-Saff [1] proved that for $R > 1$,
$$\alpha_n = cR^{-n} + O(((1-\varepsilon)R^{-1})^n) \tag{1.7}$$
if and only if $D(z)^{-1}$ is meromorphic in a circle of radius $R(1+\delta)$ with a single, simple pole at $z = R$.

C. Simon [14] considered the functions
$$S(z) = -\sum_{j=0}^\infty \alpha_{j-1} z^j \tag{1.8}$$
(with $\alpha_{-1} = -1$) and
$$r(z) = \overline{D(1/\bar{z})}\, D(z)^{-1} \tag{1.9}$$
and proved that if $\limsup |\alpha_n|^{1/n} \leq R^{-1} < 1$, then for some $\delta > 0$, $r(z) - S(z)$ is analytic in $\{z \mid 1-\delta < |z| < R^2\}$ so that $S(z)$ and $r(z)$, which will have singularities on $|z| = R$ if $\limsup |\alpha_n|^{1/n} = R^{-1}$, must have the same singularities in $\{z \mid R \leq |z| < R^2\}$. In [14], instead of $S(z)$ as defined by (1.8), one has $S(z)$ defined by $S_\mathrm{book}(z) = \sum_{j=0}^\infty \alpha_j z^j$, and the theorem is stated as analyticity of $z^{-1} r(z) + S_\mathrm{book}(z)$, equivalent to analyticity of $r - S$. But, as we will explain in Section 4, (1.8) is the more natural object. Rather than $1 - \delta < |z| < R^2$, [14] has $R^{-1} < |z| < R^2$, but that is wrong since $\overline{D(1/\bar{z})}$ can have poles at the Nevai-Totik zeros.

D. Using Riemann-Hilbert methods, Deift-Ostensson [4] have extended the result on analyticity of $r(z) - S(z)$ to $\{z \mid 1 - \delta < |z| < R^3\}$.

E. Barrios-López-Saff [2] have proven that if

$$\alpha_n = cR^{-n} + O(((1+\varepsilon)R)^{-n-nm^2}) \tag{1.10}$$

then $D(z)^{-1}$ is meromorphic in $\{z \mid |z| < R^{2m-1} + \delta\}$ with poles precisely at $z_k = R^{2k-1}$, $k = 1, 2, \dots, m$. In particular, if (1.10) holds for all n, then $D(z)^{-1}$ is entire meromorphic except for poles at R^{2k-1}, $k = 1, 2, \dots$.

Our main goal in this paper is to give a complete analysis of what can be said about α_n if $D(z)^{-1}$ is meromorphic in some disk and, contrariwise, about $D(z)^{-1}$ if α_n has an asymptotic expansion as a sum of exponentials. We describe our precise results in Section 4.

Along the way, we found a direct, simple proof of the Deift-Ostensson result that is also simpler than the argument Simon used for his weaker result in [14]. So we will give this proof next, then analyze two simple examples, and return in Section 4 to a general overview and sketch of the rest of the paper.

Of course, included among the entire meromorphic functions are the rational functions, and there is prior literature on this case. Szabados [18] considered the case $D(z)^{-1} = 1/q(z)$ for a polynomial q and Ismail-Ruedemann [9] and Pakula [13] discussed $D(z)^{-1} = p(z)/q(z)$ for polynomials p and q. They have some results on asymptotics of Φ_n but no discussion of links to the $\{\alpha_n\}_{n=1}^{\infty}$. As I was completing this manuscript, I received the latest draft of a paper of Martínez-Finkelshtein, McLaughlin, and Saff [10] that has some overlap with this paper.

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2. The R^3 Result

Our goal in this section is to prove

Theorem 2.1. *Let*

$$\limsup |\alpha_n|^{1/n} = R^{-1} < 1 \tag{2.1}$$

so that $D(z)^{-1}$ and $S(z)$ are analytic in $\{z \mid |z| < R\}$. Then for some $\delta > 0$, $r(z) - S(z)$ is analytic in $\{z \mid 1 - \delta < |z| < R^3\}$.

Remarks. 1. Here D is given by (1.3), $S(z)$ by (1.8), and $r(z)$ by (1.9).

2. The proof will make repeated use of Szegő recursion (1.5).

We introduce the symbol $\widetilde{O}$ by $f = \widetilde{O}(g)$ if and only if for all ε, $|f|/|g|^{1-\varepsilon} \to 0$.

Lemma 2.2. *Let* (2.1) *hold. Then*
(a) *For all* $\varepsilon > 0$,

$$\sup_{n,\,|z|<R-\varepsilon} |\Phi_n^*(z)| < \infty \tag{2.2}$$

(b) *For* $|z| \leq 1$,

$$|\Phi_n(z)| = \widetilde{O}(\max(R^{-1}, |z|)^n) \tag{2.3}$$

(c) *For* $|z| \leq 1$,

$$|\Phi_n^*(z) - D(0)D(z)^{-1}| = \widetilde{O}(R^{-n} \max(R^{-1}, |z|)^n) \tag{2.4}$$

Remarks. 1. There is an implicit uniformity in z in the $\widetilde{O}$ statements (2.3), (2.4).

2. (a) is due to Nevai-Totik; (b) appears in Simon [16, 17].

Proof. (a) From (1.5) and $|\Phi_n(e^{i\theta})| = |\Phi_n^*(e^{i\theta})|$, we see

$$\sup_{|z|=1} |\Phi_n(z)| \leq \prod_{j=0}^{n-1} (1 + |\alpha_j|) \tag{2.5}$$

so, by $\prod_{j=0}^{\infty}(1 + |\alpha_j|) < \infty$ and the maximum principle,

$$\sup_{n,\,|z|\leq 1} |\Phi_n^*(z)| < \infty \tag{2.6}$$

from which we get, by (1.6), that

$$C_1 \equiv \sup_{n,\,|z|\geq 1} |z|^{-n} |\Phi_n(z)| < \infty \tag{2.7}$$

The * of (1.5) is

$$\Phi_{n+1}^*(z) - \Phi_n^*(z) = -\alpha_n z \Phi_n(z) \tag{2.8}$$

so that

$$\Phi_n^*(z) = 1 - \sum_{j=0}^{n-1} \alpha_j z \Phi_j(z) \tag{2.9}$$

Thus, by (2.7),

$$|\Phi_n^*(z)| \leq 1 + C_1 \sum_{j=0}^{n-1} |\alpha_j|\, |z|^{j+1} \tag{2.10}$$

Given (2.1), we see that (2.2) holds.

(b) (2.2) and (1.6) imply that for $|z| > R^{-1} + \varepsilon$,

$$|\Phi_n(z)| \leq C_\varepsilon |z|^n \qquad (|z| > R^{-1} + \varepsilon) \tag{2.11}$$

This plus the maximum principle implies (2.3).

(c) It is a theorem of Szegő [19, 20] (see Theorem 2.4.1 of [14]) that in $|z| < 1$,

$$\lim_{n\to\infty} \Phi_n^*(z) = D(0)D(z)^{-1} \equiv d(z)^{-1} \tag{2.12}$$

Thus, summing (2.8) to infinity,

$$|d(z)^{-1} - \Phi_n^*(z)| \leq \sum_{j=n}^{\infty} |\alpha_j|\,|z|\,|\Phi_j(z)| \tag{2.13}$$

Since $\alpha_j = \widetilde{O}(R^{-j})$ and (2.3) holds, we obtain (2.4). □

Proof of Theorem 2.1. We use the function $d(z)$ of (2.12). Since $\Phi_n^*(z) \to d(z)^{-1}$ for $|z| < 1$ and (1.2) holds, the Vitali theorem implies $d(z)^{-1}$ is analytic in $\{z \mid |z| < R\}$ and $\Phi_n^*(z) \to d(z)^{-1}$ in that region. By summing (2.8) to infinity,

$$d(z)^{-1} = 1 - \sum_{j=1}^{\infty} \alpha_{j-1} z \Phi_{j-1}(z) \tag{2.14}$$

which we write

$$d(z)^{-1} = \overline{d(1/\bar{z})}^{\,-1} S(z) + \bigl[1 - \overline{d(1/\bar{z})}^{\,-1}\bigr] - \sum_{j=1}^{\infty} \alpha_{j-1} z \bigl[\Phi_{j-1}(z) - \overline{d(1/\bar{z})}^{\,-1} z^{j-1}\bigr] \tag{2.15}$$

where this formula is valid in $\{z \mid R^{-1} < |z| < R\}$.

Apply * to (2.4) and see that in $|z| \geq 1$,

$$|\Phi_n(z) - \overline{d(1/\bar{z})}^{\,-1} z^n| \leq \widetilde{O}(R^{-n} \max(|z|R^{-1}, 1)^n) \tag{2.16}$$

Thus the summand in (2.15) is bounded in $\{z \mid |z| \geq 1\}$ by $|z|\widetilde{O}(R^{-2n} \max(|z|R^{-1}, 1)^n)$. In $1 \leq |z| \leq R$, this is bounded by $\widetilde{O}(RR^{-2n})$ and in $R \leq |z|$ by $\widetilde{O}(|z|^{n+1}R^{-3n})$. Thus, the sum in (2.15), which is a sum of functions each analytic in $\{z \mid |z| > R^{-1}\}$, converges uniformly in $\{z \mid 1 \leq |z| < R^3\}$. Multiplying by $\overline{d(1/\bar{z})}$, which is analytic in $\{z \mid |z| > 1 - \delta\}$, implies the result. □

3. Two Examples

We want to analyze two examples from [14] from the point of view of singularities of $D(z)^{-1}$ and asymptotics of α_n. The first is already mentioned in this context in [2].

Example 3.1 (Rogers-Szegő polynomials; Example 1.6.5 of [14])**.** Here $0 < q < 1$,

$$\alpha_n = (-1)^n q^{(n+1)/2} \tag{3.1}$$

and

$$D(z) = \prod_{j=0}^{\infty} (1-q^j)^{1/2}(1+q^{j+1/2}z) \tag{3.2}$$

Let $R = q^{-1/2}$. Then

$$\begin{aligned} S(z) &= -\sum_{j=0}^{\infty} (-1)^{j-1} q^{j/2} z^j \\ &= (1+zR^{-1})^{-1} \end{aligned} \tag{3.3}$$

has a single pole at

$$z_1 = -R \tag{3.4}$$

On the other hand, by (3.2), $D(z)$ has a zero and so $D(z)^{-1}$ a pole at

$$z_\ell = -R^{2\ell-1} \qquad \ell = 1, 2, \dots \tag{3.5}$$

Example 3.2 (Single nontrivial moment; Example 1.6.4 of [14])**.** Fix $0 < a < 1$ and let

$$d\mu_a(\theta) = (1 - a\cos\theta)\,\frac{d\theta}{2\pi} \tag{3.6}$$

Let

$$\mu_\pm = \frac{1}{a} \pm \sqrt{\left(\frac{1}{a}\right)^2 - 1} \tag{3.7}$$

so $\mu_- \mu_+ = 1$ and $\mu_- < 1$. Then

$$D(z) = \sqrt{\frac{a}{2\mu_-}}\,(1 - \mu_- z) \tag{3.8}$$

$$= \sqrt{\frac{a}{2\mu_-}}\left(1 - \frac{z}{\mu_+}\right) \tag{3.9}$$

so $D(z)^{-1}$ has a single pole at

$$z_1 = \mu_+ \tag{3.10}$$

On the other hand,

$$\begin{aligned} \alpha_n &= \frac{-(\mu_+ - \mu_-)}{(\mu_+^{n+2} - \mu_-^{(n+2)})} \\ &= -(\mu_+ - \mu_-)\mu_+^{-n-2}(1 - \mu_+^{-(2n+4)})^{-1} \\ &= -(\mu_+ - \mu_-) \sum_{j=1}^{\infty} (\mu_+^{-n-2})^{2j-1} \end{aligned} \tag{3.11}$$

and so $S(z)$ has poles at

$$z_j = \mu_+^{2j-1} \qquad j = 1, 2, \dots \tag{3.12}$$

In these examples, the set of singularities of S and of $D(z)^{-1}$ are distinct and one or the other might be larger. If $\{z_j\}$ are the singularities, then $\{z_j^{2\ell-1}\}_{j,\ell=1,2,\dots}$ are identical for S and D^{-1}, which motivates the $\mathbb{G}$ construction of the next section.

4. Overview and Discussion of Further Results

Definition. A sequence $\{A_n\}_{n=-1}^\infty$ of complex numbers is said to have an asymptotic series with error R^{-n} for some $R > 1$ if and only if there exists a finite number of points $\{\mu_j\}_{j=1}^J$ in $\{w \mid 1 < |w| < R\}$ and polynomials $\{P_j\}_{j=1}^J$ so that

$$\limsup_{n\to\infty} \left| A_n - \sum_{j=1}^J P_j(n)\mu_j^{-(n+1)} \right|^{1/n} \le R^{-1} \tag{4.1}$$

Equivalently,

$$A_n = \sum_{j=1}^J P_j(n)\mu_j^{-n-1} + \widetilde{O}(R^{-n})$$

We say A_n has a *complete asymptotic series* if it has an asymptotic series with error R^{-n} for all $R > 1$.

In many ways, our main result in this paper is:

Theorem 4.1. *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ with Verblunsky coefficients, α_n. Then α_n has a complete asymptotic series if and only if*
(1) *$d\mu_\mathrm{s} = 0$ and $d\mu$ obeys the Szegő condition.*
(2) *$D(z)^{-1}$ is an entire meromorphic function.*

Of course,

$$\sum_{n=0}^\infty z^n \mu_j^{-n} = \left(1 - \frac{z}{\mu_j}\right)^{-1} \tag{4.2}$$

and so, taking derivatives, for $\ell = 1, 2\dots,$

$$\sum_{n=0}^\infty (n+\ell)(n+\ell-1)\dots(n+1)z^n\mu_j^{-n} = \ell!\left(1 - \frac{z}{\mu_j}\right)^{-\ell-1} \tag{4.3}$$

So (4.1) is equivalent to a sum of explicit pole terms:

Proposition 4.2. *$\{A_n\}_{n=-1}^\infty$ has an asymptotic series with error R^{-n} if and only if*

$$F(z) = \sum_{n=0}^\infty A_{n-1} z^n \tag{4.4}$$

is meromorphic in $\{z \mid |z| < R\}$ *with a finite number of poles, all in* $\{z \mid 1 < |z| < R\}$. *In particular,* $\{A_n\}_{n=-1}^{\infty}$ *has a complete asymptotic series if and only if* $F(z)$ *is an entire meromorphic function.*

Thus, Theorem 4.1 is equivalent to

Theorem 4.3. (1) *and* (2) *of Theorem 4.1 are equivalent to the function* S *of* (1.8) *being an entire meromorphic function.*

Note that the μ_j's and P_j's are determined uniquely by the A_n's.

Both to prove the results and for its intrinsic interest, we are interested in the relation between the poles of $S(z)$ and of $D(z)^{-1}$ and in results in fixed circles. By a *discrete exterior set*, we mean a subset, T, of $\{w \mid 1 < |w| < \infty\}$ so that $\#[\{w \mid 1 < |w| < R\} \cap T]$ is finite for each $R > 1$. Given a discrete exterior set T, define for $k = 1, 2, \dots,$

$$\mathbb{G}^{(2k-1)}(T) = \{\lambda_{i_1} \dots \lambda_{i_k} \bar{\lambda}_{i_{k+1}} \dots \bar{\lambda}_{i_{2k-1}} \mid \lambda_j \in T\} \tag{4.5}$$

$$\mathbb{G}_{2k-1}(T) = \bigcup_{j=k}^{\infty} \mathbb{G}^{(2j-1)}(T) \tag{4.6}$$

$$\mathbb{G}(T) = \mathbb{G}_1(T) \tag{4.7}$$

$\mathbb{G}(T)$ will be called the generated set. Note that

$$\mathbb{G}(\mathbb{G}(T)) = \mathbb{G}(T) \tag{4.8}$$

We will prove the following:

Theorem 4.4. *Let* α_n *be a set of Verblunsky coefficients with complete asymptotic series and let* T *be the set of* λ_j*'s that enter in the series. Let* P *be a set of poles of* $D(z)^{-1}$. *Then*

$$T \subset \mathbb{G}(P) \quad \textit{and} \quad P \subset \mathbb{G}(T) \tag{4.9}$$

This implies the following refined form of Theorem 4.1:

Theorem 4.5. *Let* Q *be an exterior discrete set with* $\mathbb{G}(Q) = Q$. *Then* α_n *is a set of Verblunsky coefficients with* λ_j*'s in* Q *if and only if condition* (1) *of Theorem 4.1 holds and the poles of* $D(z)^{-1}$ *lie in* Q.

Theorems 4.1, 4.4, and 4.5 are equivalence results and thus both a direct (going from α to D) and inverse (going from D to α) aspect. Generally, direct arguments are simpler than inverse. We will actually deduce everything from direct arguments and a bootstrap. An inverse argument is only used to start the analysis, and that was already done by Nevai-Totik. Here is the master stepping stone we will need. Throughout, we suppose there is $R > 1$ so

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = R^{-1} \tag{4.10}$$

Theorem 4.6. *Fix $\ell = 1, 2, \dots$. Suppose $S(z)$ is meromorphic in*

$$\mathcal{R}_\ell = \{z \mid 0 < |z| < R^{2\ell-1}\} \tag{4.11}$$

Then $D(z)^{-1}$ is meromorphic there and the poles of $D(z)^{-1}$ there lie in $\mathbb{G}(T_\ell)$ where T_ℓ is the set of poles of $S(z)$ in $\mathcal{R}_\ell$. Moreover, $r(z) - S(z)$ has a meromorphic continuation to $\mathcal{R}_{\ell+1} \cap \{z \mid 1 > |z| - \delta\}$ and the poles of this difference lie in $\mathbb{G}_3(T_\ell)$.

We are heading towards a proof that Theorem 4.6 implies the earlier Theorems 4.1, 4.4, and 4.5. We need a preliminary notion and fact.

Definition. Let Q be an exterior discrete set with $\mathbb{G}(Q) = Q$. We say that $W \subset Q$ is a set of minimal generators if and only if $\mathbb{G}(W) = Q$ and $\mathbb{G}_3(W) \cap W = \emptyset$.

Proposition 4.7. *Any exterior discrete set Q with $\mathbb{G}(Q) = Q$ has a minimal set of generators.*

Proof. Order the points in Q, $w_1, w_2, \dots$ so $|w_n| \leq |w_{n+1}|$. Define W inductively by putting w_n in W if and only if $w_n \notin \mathbb{G}_3(\{w_1, \dots, w_{n-1}\})$. It is easy to see that W is a set of minimal generators. $\square$

Proof that Theorem 4.6 implies Theorems 4.4, 4.5, and 4.1. It suffices to prove that D^{-1} is entire meromorphic if and only if S is, and to prove Theorem 4.4 since it in turn implies Theorem 4.5, which implies Theorem 4.1. If $S(z)$ is entire meromorphic, it is meromorphic in each $\mathcal{R}_\ell$, so $D(z)^{-1}$ is meromorphic in each $\mathcal{R}_\ell$ and, clearly, $P \subset \mathbb{G}(T)$.

Conversely, if $D(z)^{-1}$ is entire meromorphic, we prove $S(z)$ is entire meromorphic by proving inductively that it is meromorphic in each $\mathcal{R}_\ell$. $S(z)$ is meromorphic in $\mathcal{R}_1$ by the Nevai-Totik theorem. If we know $S(z)$ is meromorphic in $\mathcal{R}_\ell$, then by Theorem 4.6, $r(z) - S(z)$ is meromorphic in $\mathcal{R}_{\ell+1} \backslash \mathcal{R}_\ell$ so, since $r(z)$ is meromorphic on $\mathcal{R}_{\ell+1}$, we conclude that $S(z)$ is meromorphic there also.

Finally, to identify the points of T, as lying in $\mathbb{G}(P)$ with P the poles of $D^{-1}(z)$, suppose W is a set of minimal generators of T. If $w_j \in W$, then $w_j \notin \mathbb{G}_3(T)$, so $S - r$ is regular at w_j by Theorem 4.6. Since w_j is a singularity of S, it must be a singularity of r, that is, $w_j \in P$. Thus, $T = \mathbb{G}(W) \subset \mathbb{G}(P)$. $\square$

Our proof of Theorem 4.6 will also show

Theorem 4.8. *Suppose $z_0 \in \mathbb{G}^3(T)$ has a unique expression as $z_0 = \mu_1^2 \bar{\mu}_2$ with $\mu_1, \mu_2 \in T$. Suppose also that $z_0 \notin T \cup \mathbb{G}_5(T)$. Then $r(z) - S(z)$ has a singularity at z_0.*

Remarks. 1. For example, if $S(z)$ has a single pole, z_0, with $|z_0| = R$, then either S or D^{-1} or both have a pole at $z_0 R^2$.

2. Our proof shows that if the poles of $S(z)$ have $\{\log|z_j|\}$ independent over the rationals, D^{-1} has a pole at every point in $\mathbb{G}(T)$.

3. Our proof also allows the precise calculation of the singularity in $S - r$ at any point in $\mathbb{G}(T)$. There can be cancellations if z_0 can be written as a product in $\mathbb{G}(T)$ in more than one way. So one cannot guarantee a singularity of $r(z) - S(z)$ at every point of $\mathbb{G}_3(T)$, but that will happen in some generic sense.

Note that our results generalize those of Barrios-López-Saff [2] in three ways:

(a) They only have results on the the direct problem, that is, going from α to D^{-1}, while we have results in both directions.

(b) They allow only a single term in the α asymptotics.

(c) Their error assumptions in case of disks are much stronger (R^{-nm^2} vs. $R^{-n(2m+1)}$) than ours.

This concludes the description of our main results — and reduces everything to proving Theorems 4.6 and 4.8. We will do this for $2\ell-1 = 3$ in Section 5 and general $2\ell - 1$ in Section 6.

One could analyze other situations such as where $S(z)$ has a branch cut associated with specific asymptotics for α_n such as $n^\beta R^{-n}$ with β nonintegral.

We close this section, which is the continuation of the introduction, with two remarks. First, there is a scattering theoretic interpretation of S and r. Since $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$, one can define wave operators (see Geronimo-Case [6] and Section 10.7 of [15]), $\Omega^\pm : L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi}) \to L^2(\partial\mathbb{D}, |D|^2 \frac{d\theta}{2\pi})$, which obey

$$(\Omega^+ f)(\theta) = D(e^{i\theta})^{-1} f(\theta) \qquad (\Omega^- f)(\theta) = \overline{D(e^{i\theta})}^{-1} f(\theta) \tag{4.12}$$

Thus, the reflection coefficient is given by

$$((\Omega^-)^{-1}\Omega^+ f)(\theta) = \overline{D(e^{i\theta})}\, D(e^{i\theta})^{-1} f(\theta) \tag{4.13}$$

so $r(z)$ is the analytic continuation of the reflection coefficient. $S(z)$ is the leading Born approximation to r (see Newton [12] and Chadan-Sabatier [3] for background on scattering theory). While we will not study it from this point of view, it is presumably true that the arguments in the next two sections can be interpreted as use of some kind of Born series.

The second issue concerns a comparison between the basic formula used by Nevai-Totik [11] to do the inverse problem and a different, but similar-looking, formula used in our discussion, namely (2.14). The formula they use, where they quote Freud [5], is also in Geronimus [8]:

$$\alpha_n = -\kappa_\infty \int \overline{\Phi_{n+1}(e^{i\theta})}\, D(e^{i\theta})^{-1}\, d\mu(\theta) \tag{4.14}$$

where $\kappa_\infty = \lim_{n\to\infty} \kappa_n$ with $\kappa_n = \|\Phi_n\|^{-1}$, so $\kappa_n = \varphi_n^*(0)$ and

$$\kappa_\infty = D(0)^{-1} \tag{4.15}$$

(4.14) only holds if $d\mu_{\text{sing}} = 0$.

Since $\varphi_n = \kappa_n \Phi_n$, (4.14) can be rewritten as

$$\alpha_n = -\kappa_\infty \kappa_n^{-1} \langle \varphi_{n+1}, D^{-1}\rangle \tag{4.16}$$

Since

$$\langle 1, D^{-1}\rangle = \int \overline{D}(e^{i\theta})\, \frac{d\theta}{2\pi} = \overline{D(0)} = \kappa_\infty^{-1}$$

(4.16) also holds if we interpret $\alpha_{-1} = -1$ and $\kappa_{-1} = 1$. Thus, with $\alpha_{-1} = -1$, (4.16) is equivalent to

$$d(z)^{-1} \equiv D(0)D(z)^{-1} = -\kappa_\infty^{-2} \sum_{n=-1}^\infty \kappa_n \alpha_n \varphi_n(z) \tag{4.17}$$

On the other hand, (2.14) says

$$(d(z)^{-1} - 1)z^{-1} = -\sum_{n=0}^\infty \alpha_n \kappa_n^{-1} \varphi_n(z) \tag{4.18}$$

or equivalently,

$$\alpha_n = -\kappa_\infty^{-1} \kappa_n^2 \int \overline{\Phi_n(e^{i\theta})}\, [D(e^{i\theta})^{-1} - D(0)^{-1}] e^{-i\theta}\, d\mu(\theta) \tag{4.19}$$

These formulae are distinct, and it is striking that both are true and their proofs (see (2.4.35) of [14]) are so different. Where Nevai-Totik [11] use (4.14), one could just as well use (4.19).

5. The R^5 Result

In this section, as a warmup and also as the start of induction for the general case, we consider the case $2\ell - 1 = 3$, that is, $\ell = 2$ where we deal with induced singularities in $\{z \mid R^3 \le |z| < R^5\}$. Thus, we should suppose

$$\alpha_n = \sum_{k=1}^K P_k(n) \mu_k^{-n-1} + \widetilde{O}(R^{-3n}) \tag{5.1}$$

with $R \le |\mu_k| < R^3$. Here, $P_k(n)$ are polynomials. We will instead suppose that

$$\alpha_n = \sum_{k=1}^{K} c_k \mu_k^{-n-1} + \widetilde{O}(R^{-3n}) \tag{5.2}$$

The consideration of general P_k's rather than constants presents no difficulties other than notational ones, so we spare the reader. Our goal is to prove

Theorem 5.1. *If* (5.2) *holds, then* $D(z)^{-1}$ *is meromorphic in* $\{z \mid |z| < R^3\}$ *with poles precisely at* $\{\mu_k\}_{k=1}^{K}$. *In addition,* $S(z) - r(z)$ *is meromorphic in* $\{z \mid 1-\delta < |z| < R^5\}$ *with poles contained in* $\mathbb{G}^{(3)}(\{\mu_k\}_{k=1}^{K})$. *Moreover, if* $z_0 = \mu_{i_1}^2 \bar\mu_{i_2}$ *in precisely one way and* $|z_0| \le R^5$, *then* $S(z) - r(z)$ *has a pole at* z_0.

We note that the first statement is immediate from Theorem 2.1, so we will focus on $R^3 \le |z| < R^5$. We will follow the same three-step strategy used in Section 2:

(i) Estimate Φ_n in $\{z \mid |z| < R^{-3}(1+\delta)\}$.
(ii) Estimate $\Phi_n^* - d(z)^{-1}$ in $\{z \mid |z| < R^{-3}(1+\delta)\}$ using (2.8) and Step (i).
(iii) Estimate $S(z) - r(z)$ in $|z| > R^3/(1+\delta)$ using (2.8), the formula $\Phi_n(z) z^{-n} = \overline{\Phi_n^*(1/\bar z)}$, and the estimate in Step (ii).

What will be different from Section 2 is that we will find the leading asymptotics of Φ_n rather than just use $|\Phi_n(z)| \le \widetilde{O}(R^{-n})$ in $|z| < R$. In essence, this leading asymptotics was discussed in [17] and we will use the techniques from there, although in a slightly more general context.

Theorem 5.2. *Suppose* (5.2) *holds. Choose* δ, *so for all* k, $|\mu_k| < R^3/(1+\delta)$. *Define* $E_n(z)$ *in* $\{z \mid |z| < R^{-3}(1+\delta)\}$ *by*

$$\Phi_n(z) = -d(z)^{-1} \sum_{k=1}^{K} \bar c_k \bar\mu_k^{-n} (1 - z\bar\mu_k)^{-1} + E_n(z) \tag{5.3}$$

Then, for $|z| < R^{-3}(1+\delta)$,

$$|E_n(z)| = \widetilde{O}(\max(R^{-3}, |z|)^n) \tag{5.4}$$

Remark. Since $|\mu_k| < R^3(1+\delta)^{-1}$ and $|z| < R^{-3}(1+\delta)$, $||z|\mu_k| < 1$.

Proof. Iterating (1.5) from $j = n-1$ down to $j = 0$ yields

$$\Phi_n(z) = z^n - \sum_{j=1}^{n} \bar\alpha_{n-j} z^{j-1} \Phi_{n-j}^*(z) \tag{5.5}$$

Write (5.2) as

$$\alpha_n = \sum_{k=1}^{K} c_k \mu_k^{-n-1} + (\delta\alpha)_n \tag{5.6}$$

$$(\delta\alpha)_n = \widetilde{O}(R^{-3n}) \tag{5.7}$$

In (5.5), do the following:

$$E_0^{(n)} = z^n \tag{5.8}$$

$$E_1^{(n)} = -\sum_{j=1}^{n} \bar{\alpha}_{n-j} z^{j-1} [\Phi_{n-j}^*(z) - d(z)^{-1}] \tag{5.9}$$

$$E_2^{(n)} = -\sum_{j=1}^{n} \overline{(\delta\alpha)}_{n-j}\, z^{j-1}\, d(z)^{-1} \tag{5.10}$$

$$E_3^{(n)} = \sum_{k=1}^{K} \bar{c}_k \sum_{j=n+1}^{\infty} \bar{\mu}_k^{-(n+1-j)} z^{j-1}\, d(z)^{-1} \tag{5.11}$$

Since $\sum_{j=1}^{\infty} \bar{\mu}_k^{-(n+1-j)} z^{j-1} = \bar{\mu}_k^{-1}(1 - z\bar{\mu}_k)^{-1}$, (5.3) holds where

$$E_n(z) = E_0^{(n)} + E_1^{(n)} + E_2^{(n)} + E_3^{(n)}$$

We need to show that for $j = 0, 1, 2, 3$,

$$|E_j^{(n)}(z)| = \widetilde{O}(\max(R^{-3}, |z|)^n) \tag{5.12}$$

This is trivial for $j = 0$. For $j = 1$, we use (2.4) and $|\alpha_{n-j}| = \widetilde{O}(R^{-(n-j)})$ to see if $|z| < R^{-1}$:

$$\begin{aligned} |E_1^{(n)}| &\leq \sum_{j=1}^{n} |z|^{j-1} \widetilde{O}(R^{-(n-j)}) \widetilde{O}(R^{-2(n-j)}) \\ &= n\widetilde{O}(\max(|z|, R^{-3})^n) \\ &= \widetilde{O}(\max(|z|, R^{-3})^n) \end{aligned}$$

By (5.7), $d(z)^{-1}$ is bounded in $\{z \mid |z| < R^{-1}\}$ and $n\widetilde{O}(\max(|z|, R^{-3})^n) = \widetilde{O}(\max(|z|, R^{-3})^n)$, we have that (5.12) holds for $j = 2$. For $j = 3$, we note that the sum of the geometric series is $z^n(1 - \bar{\mu}_k z)^{-1}$, so in $|z| < R^{-3}(1 + \delta)$, we get a $|z|^n$ bound. This proves (5.12). □

Theorem 5.3. *Let* (5.2) *hold and let* δ *be as in Theorem 5.2. Define* $\widetilde{E}_n$ *for* $|z| < R^{-3}(1+\delta)$,

$$\Phi_n^*(z)-d(z)^{-1} = -\biggl[d(z)^{-1} \sum_{k,\ell=1}^{K} \bar{c}_k c_\ell \mu_\ell^{-1}(1-z\bar{\mu}_k)^{-1}(1-\bar{\mu}_k^{-1}\mu_\ell^{-1})(\bar{\mu}_k\mu_\ell)^{-n}\biggr]+\widetilde{E}_n(z) \tag{5.13}$$

Then for $|z| < R^{-3}(1+\delta)$,

$$\widetilde{E}_n(z) = \widetilde{O}(R^{-n}\max(|z|, R^{-3})^n) \tag{5.14}$$

Proof. We iterate (2.8) to get

$$\Phi_n^*(z) - d(z)^{-1} = \sum_{j=n}^{\infty} \alpha_j z \Phi_j(z) \tag{5.15}$$

In (5.15), first replace α_j by (5.6) and then, in the main term, replace Φ_j by (5.3). Noting that $\sum_{j=n}^{\infty} \bar{\mu}_k^{-j}\mu_\ell^{-j-1} = \mu_\ell^{-1}(1-\bar{\mu}_k^{-1}\mu_\ell^{-1})^{-1}(\bar{\mu}_k\mu_\ell)^{-n}$, we see that (5.13) holds, where

$$\widetilde{E}_n = \widetilde{E}_1^{(n)} + \widetilde{E}_2^{(n)}$$

with

$$\widetilde{E}_1^{(n)} = \sum_{j=n}^{\infty} (\delta\alpha_j) z \Phi_j(z) \tag{5.16}$$

$$\widetilde{E}_2^{(n)} = \sum_{j=n}^{\infty} \biggl(\sum_{k=1}^{K} c_k \mu_k^{-j-1}\biggr) z E_j(z) \tag{5.17}$$

By (5.7) and (2.3),

$$|E_1^{(n)}| \le c \sum_{j=n}^{\infty} \widetilde{O}(R^{-3n})\widetilde{O}(R^{-n}) = \widetilde{O}(R^{-4n}) \tag{5.18}$$

By (5.4),

$$\begin{aligned} |E_2^{(n)}| &\le c \sum_{j=n}^{\infty} R^{-j-1}\, \widetilde{O}(\max(R^{-3}, |z|)^j) \\ &= \widetilde{O}(R^{-n}\max(|z|, R^{-3})^n) \end{aligned} \tag{5.19}$$

proving (5.14). □

Theorem 5.4. *Suppose* (5.2) *holds and* δ *is chosen as in Theorem 5.2. Then in* $\{z \mid R^3(1+\delta)^{-1} < |z| < R^5\}$, *we have that* $r(z) - S(z) - q_3(z)$

is analytic where

$$q_3(z) = -\sum_{k,\ell,r=1}^{K} c_k\bar{c}_\ell c_r\, z(z-\mu_k)^{-1}(1-\mu_k^{-1}\bar{\mu}_\ell^{-1})^{-1}\mu_k(1-z\mu_k^{-1}\bar{\mu}_\ell^{-1}\mu_r^{-1})^{-1} \tag{5.20}$$

Proof. By (2.15),

$$r(z)-S(z) = \big[\overline{d(1/\bar{z})}-1\big] - \overline{d(1/\bar{z})}\sum_{j=1}^\infty \alpha_{j-1}z[\Phi_{j-1}-\overline{d(1/\bar{z})}^{\,-1}z^{-j-1}\big] \tag{5.21}$$

Because $q_3(z)$ is obtained by summing

$$\sum_{j=1}^\infty (\alpha_{j-1}-\delta\alpha_{j-1})\big[\Phi_{j-1}-\overline{d(1/\bar{z})}\,z^{j-1}-z^{j-1}\,\overline{\widetilde{E}_{j-1}(1/\bar{z})}\,\big] \tag{5.22}$$

we see that

$$r(z)-S(z)-q_3(z) = -\overline{d(1/\bar{z})}\sum_{j=1}^\infty F_{1,j}(z)+F_{2,j}(z) \tag{5.23}$$

where

$$F_{1,j}(z) = \delta\alpha_{j-1}z[\Phi_{j-1}-d(1/\bar{z})z^{j-1}] \tag{5.24}$$

$$F_{2,j}(z) = [\alpha_{j-1}-\delta\alpha_{j-1}]z^{j-1}\,\overline{\widetilde{E}_{j-1}(1/\bar{z})} \tag{5.25}$$

By (2.16) and (5.7),

$$|F_{1,j}(z)| = \widetilde{O}(R^{-3j}R^{-j}\max(|z|R^{-1},1)^j) \tag{5.26}$$

so if $R^3/(1+\delta)<|z|<R^5$,

$$\sum_{j=1}^\infty |F_{1,j}(z)| \le \sum_{j=1}^\infty R^{-5j}|z|^j < \infty \tag{5.27}$$

By (5.14), if $|z|>R^3/(1+\delta)$,

$$z^k\widetilde{E}_k(1/\bar{z}) = \widetilde{O}(R^{-n}\max(1,|z|R^{-3})^n) \tag{5.28}$$

and thus,

$$|F_{2,j}(z)| \le \widetilde{O}(R^{-2j}\max(1,|z|R^{-3})^j) \tag{5.29}$$

$$\le \begin{cases} \widetilde{O}(|z|^jR^{-5j}) & \text{if } |z|\ge R^3 \\ \widetilde{O}(R^{-2j}) & \text{if } |z|\le R^3 \end{cases} \tag{5.30}$$

so $\sum_{j=1}^\infty |F_{2,j}(z)|<\infty$ uniformly on compacts of $\{z \mid R^3/(1+\delta)\le|z|<R^5\}$. This implies $r(z)-S(z)-q_3(z)$ is analytic there. $\square$

Proof of Theorem 5.1. As already noted, Theorem 2.1 proves the results in $\{z \mid 1 < |z| < R^3\}$. In $\{z \mid R^3/(1+\delta) < |z| < R^5\}$, Theorem 5.4 shows $r - S$ is meromorphic with poles contained in $\mathbb{G}^{(3)}(\{\mu_k\}_{k=1}^K)$. The explicit formula shows that there is a pole if there is a single summand contributing to the potential pole. $\square$

6. The $R^{2\ell-1}$ Result

In this section, we will prove the following (again, for simplicity of exposition, we replace general $P_j(n)$ by constants), which clearly implies Theorems 4.6 and 4.8.

Theorem 6.1. *Let $\{\mu_k\}_{k=1}^K$ obey $R \le |\mu_k| < R^{2\ell-1}$ with $\min_k|\mu_k| = R$. Suppose that*

$$\alpha_n = \sum_{k=1}^K c_k \mu_k^{-n-1} + \widetilde{O}(R^{-(2\ell-1)}) \tag{6.1}$$

Then $D(z)^{-1}$ is meromorphic in $\{z \mid |z| < R^{2\ell-1}\}$ with poles contained in $\mathbb{G}(\{\mu_k\}_{k=1}^K)$. In addition, $S(z) - r(z)$ is meromorphic in $\{z \mid 1-\delta_0 < |z| < R^{2\ell+1}\}$ and the only poles in $\{z \mid R^{2\ell-1} \le |z| < R^{2\ell+1}\}$ lie in $\mathbb{G}_3(\{\mu_k\}_{k=1}^K)$. If z_0 obeys $z_0 = \mu_{i_1}^2 \bar{\mu}_{i_2}$ with $|z_0| < R^{2\ell+1}$ and z_0 cannot be written as any other $\mathbb{G}(\{\mu_k\}_{k=1}^K)$ product, then $s(z) - r(z)$ has a pole at z_0.

The strategy is the same as in the last section. Pick $\delta > 0$ so that $|\mu_k| < R^{2\ell-1}/(1+\delta)$ for all k. We will prove the following estimate on the Φ's and Φ^*'s inductively:

Theorem 6.2. *Under the hypothesis of Theorem 6.1, in $\mathcal{Q} \equiv \{z \mid |z| \le R^{-(2\ell-1)}(1+\delta)\}$, we have*

$$\Phi_n(z) = \sum_p f_{p,\ell}^{(\ell)}(z) \bar{w}_{p,\ell}^{-n} + E_{n,\ell}(z) \tag{6.2}$$

where the sum is over all points w in

$$\left[\bigcup_{m=1}^{2\ell-3} \mathbb{G}^{(2m-1)}(\{\mu_k\}_{k=1}^K)\right] \bigcap \left\{z \;\middle|\; R \le |z| < \frac{R^{2\ell-1}}{1+\delta}\right\}$$

each $f_{p,\ell}^{(\ell)}$ is analytic in $\mathcal{Q}$, and on $\mathcal{Q}$,

$$|E_{n,\ell}(z)| = \widetilde{O}(\max(R^{-(2\ell-1)}, |z|)^n) \tag{6.3}$$

In addition, in $\mathcal{Q}$,

$$\Phi_n^*(z) - d(z)^{-1} = \sum_p g_{p,\ell}(z) y_{p,\ell}^{-n} + \widetilde{E}_{n,\ell}(z) \tag{6.4}$$

where the sum is over all products, $p = \mu_{i_1} \dots \mu_{i_m} \bar{\mu}_{i_{m+1}} \dots \bar{\mu}_{i_{2n}}$ *with* $i_1, \dots, i_{2m} \in \{1, \dots, K\}$ *and with the product lying in* $\{z \mid 2R \leq |z| < R^{2\ell}/(1+\delta)\}$ *and on* $\mathcal{Q}$*, each* $g_{p,\ell}$ *is analytic and*

$$|\widetilde{E}_{n,\ell}(z)| = \widetilde{O}(R^{-n}(\max(R^{-(2\ell-1)}, z))^n) \tag{6.5}$$

Proof. The proof is by induction in ℓ. Theorems 5.2 and 5.3 establish the case $\ell = 2$. Suppose that we have the result for $\ell - 1$ with $\ell \geq 3$ and that (6.1) holds. Write α_n as in (5.6) where now

$$(\delta\alpha)_n = \widetilde{O}(R^{-(2\ell-1)n}) \tag{6.6}$$

In (5.5), do the following:

$$E_{0,\ell}^{(n)} = z^n \tag{6.7}$$

$$E_{1,\ell}^{(n)} = -\sum_{j=1}^{n} \bar{\alpha}_{n-j} z^{j-1} \Big[\Phi_{n-j}^* - d(z)^{-1} - \sum_p g_{p,\ell-1}(z) y_{p,\ell-1}^{-(n-j)} \Big] \tag{6.8}$$

$$E_{2,\ell}^{(n)} = -\sum_{j=1}^{n} \overline{(\delta\alpha)}_{n-j} z^{j-1} \Big[d(z)^{-1} + \sum_p g_{p,\ell-1}(z) y_{p,\ell-1}^{(n-j)} \Big] \tag{6.9}$$

Then

$$\Phi_n - \sum_{k=0}^{2} E_{k,\ell}^{(n)} = -\sum_{j=1}^{n} \bigg(\sum_{k=1}^{K} \bar{c}_k \bar{\mu}_k^{-(n-j)-1} \bigg) \Big[d(z)^{-1} + \sum_p g_{p,\ell-1}(z) y_{p,\ell-1}^{-(n-j)} \Big] \tag{6.10}$$

Define $E_{3,\ell}^{(n)}$ to be the summand on the right side of (6.10) with the sum from $n+1$ to ∞.

The infinite sum yields geometric series which precisely have the form $\sum_p g_{p,\ell}(z) y_{p,\ell}^{-n}$, and as in the proof of Theorem 5.1,

$$|E_{0,\ell}^{(n)}| + |E_{1,\ell}^{(n)}| + |E_{2,\ell}^{(n)}| + |E_{3,\ell}^{(n)}| = \widetilde{O}(\max(|z|, R^{-(2\ell-1)}))$$

since $\widetilde{E}_{n,\ell-1}$ by induction has $\widetilde{O}(\max(R^{-2\ell}, |z|)^n)$ decay and other terms are bounded by $\widetilde{O}(\max(R^{-1}, |z|)^n)$. This proves (6.3) for ℓ.

To bound $\Phi_n^*(z) - d(z)^{-1}$, we use (5.15), replace $\Phi_j(z)$ by (6.2), α_n by $(\delta\alpha)_n$ plus the asymptotic exponentials, and obtain (6.4) and (6.5) for ℓ by the same estimate as in Theorem 5.3. □

Basically, using Φ_n^* to order $R^{-2(\ell-1)n}$ in the expansion of Φ_n gets us Φ_n to order $R^{-(2\ell-1)n}$, and then plugging that into the expansion of Φ_n^* gets us Φ_n^* to order $R^{-2\ell n}$. Each full iteration improves by R^{-2n}.

Proof of Theorem 6.1. By induction, $S - r$ is meromorphic in $\{z \mid |z| < R^{2\ell-1}\}$, so knowing S meromorphic implies meromorphicity of r and so D^{-1} there, and the poles of both $S - r$ and S lie in $\mathbb{G}(\{\mu_k\}_{k=1}^K)$.

Using (6.4) in $|z| < R^{-(2\ell-1)}/(1+\delta)$ yields an expansion of $\Phi_n(z)$ in $|z| > R^{2\ell-1}/(1+\delta)$. Plug this into (5.21) and use (5.6). The purely geometric terms sum to poles in $\{z \mid R^{2\ell-1} < |z| < R^{2\ell+1}\}$. The bounds on the errors as in the proof of Theorem 5.4 converge to an analytic function in the annulus. The poles are clearly in $\mathbb{G}_3(\{\mu_k\}_{k=1}^K)$.

Tracking the contribution of a single $\mu_1^2\bar{\mu}_2$ shows that it yields a nonvanishing pole which, by the unique product hypothesis, cannot be cancelled. □

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