AIZENMAN'S THEOREM FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. For suitable classes of random Verblunsky coefficients, including independent, identically distributed, rotationally invariant ones, we prove that if

$$\mathbb{E}\biggl(\int \frac{d\theta}{2\pi}\biggl|\biggl(\frac{\mathcal{C}+e^{i\theta}}{\mathcal{C}-e^{i\theta}}\biggr)_{k\ell}\biggr|^p\biggr) \leq C_1 e^{-\kappa_1|k-\ell|}$$

for some $\kappa_1 > 0$ and $p < 1$, then for suitable C_2 and $\kappa_2 > 0$,

$$\mathbb{E}\bigl(\sup_n |(\mathcal{C}^n)_{k\ell}|\bigr) \leq C_2 e^{-\kappa_2|k-\ell|}$$

Here $\mathcal{C}$ is the CMV matrix.

1. Introduction

This paper is a contribution to the theory of orthogonal polynomials on the unit circle (OPUC); for background on OPUC, see Szegő [19], Geronimus [7], and Simon [15, 16]. Our goal here is to prove an analog of a result of Aizenman [1] for random discrete Schrödinger operators. Aizenman considers operators on $\ell^2(\mathbb{Z}^\nu)$ of the form $h_\omega = h_0 + V_\omega$ where ([1] allows more general h_0 than this!)

$$(h_0 u)(n) = \sum_{|j|=1} u(n+j)$$

and V_ω is the diagonal matrix whose matrix elements are independent identically distributed random variables. Aizenman's theorem states

Theorem (Aizenman [1]). *Under suitable hypotheses on the distribution of V, if for some $[a,b] \subset \mathbb{R}$,*

$$\int_a^b \mathbb{E}\bigl(|[(h_\omega - E - i0)^{-1} P_{[a,b]}(h_\omega)]_{k\ell}|^p\bigr)\, dE \leq C_1 e^{-\kappa_1|k-\ell|} \tag{1.1}$$

for some $0 < p < 1$ and $\kappa_1 > 0$, then for some $\kappa_2 > 0$ and C_2,

$$\mathbb{E}\bigl(\sup_t |[e^{-ith_\omega} P_{[a,b]}(h_\omega)]_{k\ell}|\bigr) \leq C_2 e^{-\kappa_2|k-\ell|} \tag{1.2}$$

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Aizenman's motivation was that Aizenman-Molchanov [2] had proven bounds of the form (1.1) (generally called Aizenman-Molchanov or fractional moment bounds) realizing that the key was to restrict p to be less than 1. From their bounds, they easily obtained spectral localization (i.e., pure point spectrum) by using the Simon-Wolff criterion [17]. Aizenman was interested in (1.2) because it is a form of physical localization. It was used by del Rio et al. [5] to obtain what is now the standard strong form of eigenfunction localization (SUDL) and by Minami [12] to prove Poisson distribution of the eigenvalues of h_ω restricted to a large box. Del Rio et al. also simplified Aizenman's proof and slightly extended the result (so that the theorem we stated above is their form with some extra hypotheses dropped).

To describe precisely the result we want to prove here, we need some preliminaries. Given a set of Verblunsky coefficients, $\{\alpha_j\}_{j=0}^\infty$ (see [15, Section 1.5]), one forms the CMV matrix, $\mathcal{C}$ (see Cantero-Moral-Velázquez [4] or [15, Chapter 4]). We define, for $z \in \mathbb{D}$, the unit disk

$$F_{k\ell}(z) = \left[\frac{\mathcal{C}+z}{\mathcal{C}-z}\right]_{k\ell} \tag{1.3}$$

By Kolmogorov's theorem (see [11] or Duren [6, Section 4.2]), $F_{k\ell}$ lies in the Hardy spaces $H^p(\mathbb{D})$ for $0<p<1$, so

$$F_{k\ell}(e^{i\theta}) \equiv \lim_{r\uparrow 1} F_{k\ell}(re^{i\theta}) \tag{1.4}$$

exists for $\frac{d\theta}{2\pi}$-a.e. θ and has an integrable p-th power over $\partial\mathbb{D}$ for $p\in(0,1)$.

Now let the α's be random variables which define a measure $d\Gamma$ on $\times_{j=0}^\infty \mathbb{D} \equiv \mathbb{D}^\infty$. For each $n=0,1,2,\dots$ and $\lambda\in\partial\mathbb{D}$, define $T_{n,\lambda}:\mathbb{D}^\infty\to\mathbb{D}^\infty$ by

$$\begin{aligned} (T_{n,\lambda}(\alpha))_j &= \alpha_j & \qquad j&=0,1,\dots,n-1 \\ &= \lambda\alpha_j & \qquad j&=n,n+1,\dots \end{aligned} \tag{1.5}$$

and let $d\Gamma_{n,\lambda}(\alpha) = d\Gamma(T_{n,\lambda}(\alpha))$. We say $d\Gamma$ is strongly quasi-invariant if each $d\Gamma_{n,\lambda}$ is $d\Gamma$-absolutely continuous and $\sup_{n,\lambda}\|d\Gamma_{n,\lambda}/d\Gamma\|_\infty<\infty$. Clearly, if $d\Gamma$ is a product of rotation invariant measures (invariant i.i.d.'s), $d\Gamma$ is strongly quasi-invariant. We will discuss other examples in Section 7. Here is our main result:

Theorem 1.1. *Suppose $\{\alpha_j\}_{j=0}^\infty$ are random Verblunsky coefficients which are strongly quasi-invariant, and for some $p<1$ and $\kappa_1>0$,*

$$\mathbb{E}\biggl(\int_0^{2\pi} |F_{k\ell}(e^{i\theta})|^p \,\frac{d\theta}{2\pi}\biggr) \le C_1 e^{-\kappa_1|k-\ell|} \tag{1.6}$$

Then for suitable $\kappa_2>0$ and C_2,

$$\mathbb{E}\bigl(\sup_n |(\mathcal{C}^n)_{k\ell}|\bigr) \le C_2 e^{-\kappa_2|k-\ell|} \tag{1.7}$$

Remarks. 1. This result is interesting only because one can prove (1.6). For certain cases of rotation invariant i.i.d. α's, Stoiciu [18] has proven (1.6). Indeed, I proved Theorem 1.1 precisely to fill in a missing step in his program to prove Poisson distribution for the zeros of paraorthogonal polynomials with random Verblunsky coefficients.

2. Kolmogorov's argument proves for any OPUC, any k, ℓ, and $0 < p < 1$,

$$\int_0^{2\pi} |F_{k\ell}(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \leq 2\cos\left(\frac{p\pi}{2}\right) \tag{1.8}$$

This means that if (1.6) holds for one $p \in (0,1)$, it holds for all such p (with κ_1 dependent on p).

3. The $\sup_n$ is over $n = 0, \pm 1, \pm 2, \dots$.

4. (1.7) is a strong statement about the structure of eigenfunctions of $\mathcal{C}$ and of OPUC to be compared with the case $\alpha \equiv 0$ where $\sup_n |(\mathcal{C}^n)_{k\ell}| = 1$.

5. If (1.6) holds for $p = 1$ (it cannot, as we will see in Remark 6!), (1.7) would be immediate since $2(\mathcal{C}^n)_{k\ell}$ are the Taylor coefficients of $F_{k\ell}$. Thus, (1.6) $\Rightarrow$ (1.7) without recourse to the expectations. But, in general, for $p < 1$, H^p functions have Taylor coefficients that can grow as $o(n^{1/p-1})$ and no better (see Duren [6, Chapter 6]), so (1.6) $\Rightarrow$ (1.7) only holds because the $\sup|(\mathcal{C}^n)_{k\ell}|$ is averaged over a set of rank one perturbations. This is Aizenman's key discovery in [1].

6. (1.6) cannot hold for $p = 1$. Indeed, if $\sum_\ell (\int_0^{2\pi} |F_{k\ell}(e^{i\theta})| \frac{d\theta}{2\pi})^2 < \infty$ for fixed k, then $\sum_\ell |\mathcal{C}_{k\ell}^n|^2 \to 0$ by appealing to the dominated convergence theorem for sums and the Riemann-Lebesgue lemma which implies $(\mathcal{C}^n)_{k\ell} \to 0$ if $\int |F_{k\ell}(e^{i\theta})| \frac{d\theta}{2\pi} < \infty$. But since $\mathcal{C}^n$ is unitary, $\sum_\ell |\mathcal{C}_{k\ell}^n|^2 = 1$.

While part of our proof of Theorem 1.1 follows the arguments in del Rio et al. [5], there are two novel aspects that prompted me to write this separate note. The first involves the theory of rank one perturbations. This theory is well-developed for selfadjoint operators (see [3, 14]), but I could not find any extensive theory for the unitary case when I wrote [15, 16], so I developed the theory there (see [15, Subsections 1.3.9, 1.4.16, and Section 4.5]). It turns out that a key formula needed here (see (2.14) below) is not in that presentation.

Secondly, OPUC has a subtlety missing from the Schrödinger case. Namely, the relevant rank one perturbations of the Schrödinger operators are also Schrödinger operators, but the rank one perturbations of CMV matrices are not CMV matrices. Of course, as unitary matrices with a cyclic vector, these are unitarily equivalent to CMV matrices and, as we will see, a formula of Khrushchev [10] even implies what the Verblunsky coefficients are for the new matrices. But we will need to know the form of the unitary, and this will require an illuminating calculation that should be useful in other contexts.

In Section 2, we discuss some aspects of the theory of rank one perturbations of unitaries, which we use in Section 3, following [1, 5], to compute explicit spectral representations. In Section 4, we use this to obtain a deterministic form of Aizenman's theorem that involves averaging under rank one perturbations, and in Section 5, we write these rank one perturbations in terms of CMV matrices. Section 6 puts everything together to get Theorem 1.1 and Section 7 has some comments.

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2. Rank One Perturbations of Unitaries Revisited

Rank one perturbations of unitaries are best understood multiplicatively. Let U be a unitary operator on a Hilbert space, $\mathcal{H}$, and $\varphi \in \mathcal{H}$ a unit vector. Let $P = \langle \varphi, \cdot \rangle \varphi$ be the projection onto the multiples of φ. One defines for $\lambda \in \partial\mathbb{D}$,

$$U_\lambda = U(1-P) + \lambda UP = U[(1-P) + \lambda P] \tag{2.1}$$

(if φ is part of an orthonormal basis, $(1-P)+\lambda P$ is the diagonal matrix with λ in φ position and 1 in all others). Thus

$$U_\lambda \varphi = \lambda U \varphi \tag{2.2}$$

$$U_\lambda \psi = U\psi \qquad \text{if } \psi \perp \varphi \tag{2.3}$$

which also defines U_λ. Note also that

$$U_\lambda - U = (\lambda - 1)UP \tag{2.4}$$

For $z \in \mathbb{D}$ and $\|\varphi\| = 1$, define

$$F_\varphi(z) = \left\langle \varphi, \frac{U+z}{U-z}\,\varphi \right\rangle \tag{2.5}$$

which is a Carathéodory function (i.e., $F(0) = 1$ and $\operatorname{Re} F > 0$ on $\mathbb{D}$). Since

$$\left\langle \varphi, \frac{U-z}{U-z}\,\varphi \right\rangle = 1 \tag{2.6}$$

we can solve for

$$\langle \varphi, U(U-z)^{-1}\varphi \rangle = \tfrac{1}{2}\,[F_\varphi(z) + 1] \tag{2.7}$$

$$\langle \varphi, (U-z)^{-1}\varphi \rangle = \tfrac{1}{2z}\,[F_\varphi(z) - 1] \tag{2.8}$$

Notice (2.7) and (2.8) hold if U is replaced by U_λ and F_φ by $F_\varphi^\lambda(z)$ given by (2.5) with U replaced by U_λ. By the second resolvent equation,

$$\begin{aligned}(z-U_\lambda)^{-1}U\varphi &= (z-U)^{-1}U\varphi + (z-U)^{-1}(U_\lambda - U)(z-U_\lambda)^{-1}U\varphi \\ &= [1 + (\lambda-1)(\varphi, (z-U_\lambda)^{-1}U\varphi)](z-U)^{-1}U\varphi \end{aligned} \tag{2.9}$$

on account of (2.4).

Taking an inner product of (2.9) with φ and using (2.7)/(2.8) for U and U_λ let us solve for $F_\varphi^\lambda(z)$ in terms of $F_\varphi(z)$. The result (see [15, Subsection 1.4.16]) is expressed most succinctly via the function f defined by

$$F_\varphi(z) = \frac{1 + zf(z)}{1 - zf(z)} \tag{2.10}$$

for then the inner product of φ with (2.9) implies

$$F_\varphi^\lambda(z) = \frac{1+\lambda^{-1}zf(z)}{1-\lambda^{-1}zf(z)} \tag{2.11}$$

Notice, by (2.7)/(2.8), that

$$\langle \varphi, U_\lambda(U_\lambda - z)^{-1}\varphi\rangle = \frac{1}{1-\lambda^{-1}zf(z)} \tag{2.12}$$

$$\langle \varphi, (U_\lambda - z)^{-1}\varphi\rangle = \frac{\lambda^{-1}f(z)}{1-\lambda^{-1}zf(z)} \tag{2.13}$$

Thus far, the formulae are identical to what is in [15]. What is new here is to note that (2.9) says that as a vector in $\mathcal{H}$, for each $z \in \mathbb{D}$, $\lambda \in \partial\mathbb{D}$, $(U_\lambda - z)^{-1}U_\lambda\varphi$ is a multiple of $(U-z)^{-1}U\varphi$, so for any $\psi \in \mathcal{H}$,

$$\frac{\langle \psi, (U_\lambda - z)^{-1}U_\lambda\varphi\rangle}{\langle \varphi, (U_\lambda - z)^{-1}U_\lambda\varphi\rangle} = \frac{\langle \psi, (U - z)^{-1}U\varphi\rangle}{\langle \varphi, (U - z)^{-1}U\varphi\rangle} \tag{2.14}$$

In particular, by (2.12) and $\langle \psi, (U_\lambda - z)(U_\lambda - z)^{-1}\varphi\rangle = \langle \psi, \varphi\rangle$, we see

Proposition 2.1. *If $\psi \perp \varphi$, then*

$$\left\langle \psi, \frac{U_\lambda + z}{U_\lambda - z}\,\varphi\right\rangle = \frac{1-zf(z)}{1-\lambda^{-1}zf(z)}\left\langle \psi, \frac{U + z}{U - z}\,\varphi\right\rangle \tag{2.15}$$

Remark. (2.14) is an analog of (3.2) of Aizenman [1].

3. The Spectral Representation

We add two extra assumptions to our analysis of rank one perturbations of unitaries. First, we suppose φ is cyclic for U, that is, $\{U^k\varphi\}_{k=-\infty}^{\infty}$ spans $\mathcal{H}$, in which case it is easy to see that φ is cyclic for U_λ. The spectral theorem then implies there are spectral measures, $d\mu_\lambda$, on $\partial\mathbb{D}$ defined by

$$\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_\lambda(\theta) = F_\varphi^\lambda(z) \tag{3.1}$$

for $z \in \mathbb{D}$ and unique unitary maps $\mathcal{F}_\lambda : \mathcal{H} \to L^2(\partial\mathbb{D}, d\mu_\lambda)$, so that

$$(\mathcal{F}_\lambda U_\lambda \psi)(z) = z(\mathcal{F}_\lambda\psi)(z) \qquad \mathcal{F}_\lambda\varphi \equiv 1 \tag{3.2}$$

In particular, if $U_\lambda\eta_{z_0} = z_0\eta_{z_0}$ for some $z_0 \in \partial\mathbb{D}$ and $0 \neq \eta \in \mathcal{H}$, then, by (3.2),

$$(\mathcal{F}_\lambda\psi)(z_0) = \frac{\langle \eta_{z_0}, \psi\rangle}{\langle \eta_{z_0}, \varphi\rangle} \tag{3.3}$$

Our second assumption, following [3, 17], is that for a.e. $e^{i\theta_0} \in \partial\mathbb{D}$,

$$G(e^{i\theta_0}) \equiv \int \frac{d\mu(\theta)}{|e^{i\theta_0} - e^{i\theta}|^2} < \infty \tag{3.4}$$

By arguments in [17], it is easy to see that if $\{\psi_j\}_{j=0}^\infty$ is a basis for $\mathcal{H}$, then (3.4) at θ_0 is equivalent to

$$\sum_{j=0}^\infty \lim_{r\uparrow 1} \left| \left\langle \psi_j, \left(\frac{U + re^{i\theta_0}}{U - re^{i\theta_0}} \right) \varphi \right\rangle \right|^2 < \infty \tag{3.5}$$

Moreover, if (3.4) holds, then

$$\lim_{r\uparrow 1} e^{i\theta_0} f(re^{i\theta_0}) = \lambda_0 \in \partial\mathbb{D} \tag{3.6}$$

and z_0 is an eigenvalue of U_{λ_0}. Since (spectral averaging, due to Golinskii-Nevai [8] in this setting)

$$\int_0^{2\pi} (d\mu_{e^{i\varphi}}(\theta)) \, \frac{d\varphi}{2\pi} = \frac{d\theta}{2\pi} \tag{3.7}$$

(3.4) for a.e. $e^{i\theta_0}$ implies U_λ has pure point spectrum for a.e. λ. (These facts are all explained in [16, Sections 10.1 and 10.2].)

Proposition 3.1. *If φ is cyclic and* (3.4) *holds at θ_0, let λ_0 be given by* (3.6)*. Then for any $\psi \perp \varphi$,*

$$(\mathcal{F}_{\lambda_0}\psi)(z_0) = \overline{(1-\lambda_0)z_0} \; \overline{\langle \psi, (U - z_0)^{-1}\varphi\rangle} \tag{3.8}$$

where $z_0 = e^{i\theta_0}$ and (3.8) *is shorthand for* $\lim_{r\uparrow 1} \overline{\langle \psi, (U - rz_0)^{-1}\varphi\rangle}$*, which it is asserted exists.*

Proof. We use (2.14) for $z = rz_0$ and $\lambda = \lambda_0$. By the spectral theorem since z_0 is an eigenvalue of U_{λ_0},

$$\lim_{r\uparrow 1} (1-r)(U_{\lambda_0} - rz_0)^{-1} U_{\lambda_0}\varphi \equiv \eta_{z_0}$$

is an eigenvector for U_{λ_0} with eigenvalue z_0 or it is zero. Since (3.6) holds, we have that

$$\begin{aligned}
\langle \varphi, \eta_{z_0}\rangle &= \lim_{r\uparrow 1} \frac{1-r}{2} \, [F_\varphi^{\lambda_0}(rz_0) + 1] \\
&= \lambda_0 \left[\frac{1-r}{\lambda_0 - [zf(z)]|_{z=rz_0}} \right] \qquad \text{by (2.11)} \\
&= \frac{\lambda_0 z_0}{(1-\lambda_0 z_0)^2} \, (-G(z_0))^{-1}
\end{aligned}$$

by (10.1.7) of [16]. This is nonzero by the assumption $G(z_0) < \infty$. One can also use cyclicity of φ to conclude that $\eta_{z_0} \neq 0$. Thus $\eta_{z_0} \neq 0$ and so, by (3.3) and (2.14),

$$(\mathcal{F}_\lambda\psi)(z_0) = \lim_{r\uparrow 1} \frac{\overline{\langle \psi, (U - rz_0)^{-1}U\varphi\rangle}}{\overline{\langle \varphi, (U - rz_0)^{-1}U\varphi\rangle}} \tag{3.9}$$

By (2.12) and $rz_0 f(rz_0) \to \lambda_0$,

$$\lim_{r\uparrow 1} \langle \varphi, (U - rz_0)^{-1}U\varphi\rangle = (1-\lambda_0)^{-1} \tag{3.10}$$

and since $(\psi, (U-z)^{-1}(U-z)\varphi) = 0$, we have

$$\langle \psi, (U - rz_0)^{-1} U\varphi\rangle = rz_0 \langle \psi, (U - rz_0)^{-1}\varphi\rangle \tag{3.11}$$

Thus, we see that (3.9)–(3.11) implies (3.8). □

Remark. In fact, (3.8) holds whenever $d\mu$ is purely singular and for all $\lambda_0 \neq 1$. This is because $d\mu$ purely singular implies $d\mu_\lambda$ is purely singular and (3.9) can be replaced by Poltoratskii's theorem [13, 9], which says that for any complex Borel measure η on $\partial\mathbb{D}$ and any $g \in L^1(\partial\mathbb{D}, d\eta)$, we have, for almost any $e^{i\theta_0}$ with respect to $d\eta_{\mathrm{s}}$ (but not for $d\eta_{\mathrm{ac}}$), that

$$\lim_{r\uparrow 1} \frac{[\int \frac{e^{i\theta}+re^{i\theta_0}}{e^{i\theta}-re^{i\theta_0}} f(\theta)\, d\eta(\theta)]}{[\int \frac{e^{i\theta}+re^{i\theta_0}}{e^{i\theta}-re^{i\theta_0}}\, d\eta(\theta)]} = f(\theta_0)$$

4. Deterministic Form of Aizenman's Theorem

We now follow Aizenman [1] and del Rio et al. [5]. Under the assumption that φ is cyclic and $G(z_0) < \infty$ for a.e. z_0 in $\partial\mathbb{D}$, we have for $\psi \perp \varphi$ that for a.e. λ_0 that

$$|(\mathcal{F}_{\lambda_0}\psi)(z_0)| \leq \big|\lim_{r\uparrow 1} \langle \varphi, (U + rz_0)(U - rz_0)^{-1}\psi\rangle\big| \tag{4.1}$$

since $\psi \perp \varphi$ implies $\langle \varphi, (U+z)(U-z)^{-1}\psi\rangle = 2z\langle \varphi, (U-z)^{-1}\psi\rangle$. (4.1) holds for all eigenvalues of U_{λ_0} and so, for a.e. z_0 w.r.t. $d\mu_{\lambda_0}$ if U_{λ_0} is pure point.

Since $\mathcal{F}_\lambda$ is a unitary operator, we have

$$\int |(\mathcal{F}_{\lambda_0}\psi)(z)|^2\, d\mu_{\lambda_0}(z) = \|\psi\|^2 \tag{4.2}$$

Moreover, since $\mathcal{F}_{\lambda_0} U \mathcal{F}_{\lambda_0}^{-1} = z$ and $\mathcal{F}_{\lambda_0}\varphi \equiv 1$,

$$\begin{aligned} |\langle \varphi, U_{\lambda_0}^n \psi\rangle| &= \left| \int z^n \mathcal{F}_{\lambda_0}\psi(z)\, d\mu_{\lambda_0}(z)\right| \\ &\leq \int |\mathcal{F}_{\lambda_0}\psi(z_0)|\, d\mu_{\lambda_0}(z) \end{aligned} \tag{4.3}$$

We conclude, using Hölder's inequality and (4.2):

Proposition 4.1. *If φ is cyclic and $G(z_0) < \infty$ for a.e. z_0, then for a.e. λ_0 and any $0 < p < 1$, we have*

$$\sup_n |\langle \varphi, U_{\lambda_0}^n \psi\rangle| \leq \left[\int \lim_{r\uparrow 1} |\langle \varphi, (U + rz)(U - rz)^{-1}\psi\rangle|^p\, d\mu_{\lambda_0}(z)\right]^{1/(2-p)} \tag{4.4}$$

Proof. We have

$$\int |g|\, d\mu_\lambda \leq \left(\int |g|^2\, d\mu_\lambda\right)^{(1-p)/(2-p)} \left(\int |g|^p\, d\mu_\lambda\right)^{1/(2-p)}$$

since $(1-p)/(2-p) + 1/(2-p) = 1$ and $2(1-p)/(2-p) + p/(2-p) = 1$, and Hölder's inequality says $q \to \log(\int |g|^q\, d\mu)^{1/q}$ is convex. (4.4) follows by taking $g = \mathcal{F}_{\lambda_0}\psi$ and using (4.1), (4.2), and (4.3). □

Since $2-p>1$, for any probability measure $d\nu$, $\int h^{1/(2-p)}\,d\nu \leq (\int h\,d\nu)^{1/(2-p)}$ by Hölder's inequality. Thus writing $\lambda_0 = e^{i\eta_0}$ and integrating (4.4) with $d\eta_0/2\pi$, we find, using (3.7), that

Theorem 4.2 (Deterministic Aizenman's Theorem)**.** *If φ is cyclic, $\psi \perp \varphi$, and $G(z_0) < \infty$ for a.e. $z_0 \in \partial\mathbb{D}$, then for any $0 < p < 1$,*

$$\int \frac{d\eta}{2\pi} \sup_n |\langle \varphi, U^n_{e^{i\eta}}\psi\rangle| \leq \left(\int \lim_{r\uparrow 1} |\langle \varphi, (U+re^{i\theta})(U-re^{i\theta})^{-1}\psi\rangle|^p \,\frac{d\theta}{2\pi}\right)^{1/(2-p)} \tag{4.5}$$

5. Rank One Perturbations of CMV Matrices

We now specialize to $U = \mathcal{C}(\{\alpha_j\}_{j=0}^\infty)$, a CMV matrix (see [4] or [15, Chapter 4]), and $\varphi = \delta_n$, the vector with 1 in the n-th position ($n = 0, 1, 2, \dots$). We need a notation for diagonal matrices with diagonal matrix elements λ, λ^{-1}, or 1. $D(\lambda^k (\lambda^{-1})^\ell (1\lambda)^\infty)$ will denote the diagonal matrix with k λ's, $\ell(\lambda^{-1})$'s, and then alternating 1 and λ. We will also use the maps $T_{n,\lambda}$ of (1.5). Here is the main result:

Theorem 5.1. *Define U_n for $n = 0, 1, 2, \dots$ by*

$$U_{2k-1} = D(1^{2k}(1\lambda)^\infty) \tag{5.1}$$

$$U_{2k} = D(\lambda^{2k}(1\lambda)^\infty) \tag{5.2}$$

Then

$$U_n \mathcal{C}(T_{n,\lambda^{-1}}(\alpha)) U_n^{-1} = \mathcal{C}(\alpha)\Delta_n(\lambda) \tag{5.3}$$

where

$$\Delta_n(\lambda) = D(1^n \lambda 1^\infty) \tag{5.4}$$

Remarks. 1. Since $\mathcal{C}(\alpha)\Delta_n(\lambda)$ is unitary with δ_0 cyclic, it is unitarily equivalent to some $\mathcal{C}(\tilde\alpha)$. Since $\mathcal{C}(\alpha) \restriction \{\delta_j\}_{j=0}^{n-1} = \mathcal{C}(\tilde\alpha) \restriction \{\delta_j\}_{j=0}^{n-1}$, it is easy to see that $\tilde\alpha_k = \alpha_k$ for $k = 0, 1, \dots, n-1$. By Khrushchev's formula ([16, Theorem 9.2.4]), the spectral measure for $\mathcal{C}(\alpha)$ and vector δ_n has Schur function $\Phi_n(z;\alpha_0,\dots,\alpha_{n-1})\Phi_n^*(z;\alpha_0,\dots,\alpha_{n-1})^{-1} f(z;\alpha_n,\alpha_{n+1},\dots)$. By (2.11), $\tilde{\mathcal{C}}(\alpha)$ thus has Schur function which is this times λ^{-1}, so by Khrushchev's formula again, $f(z;\tilde\alpha_n,\tilde\alpha_{n+1},\dots) = \lambda^{-1} f(z;\alpha_n,\alpha_{n+1},\dots) = f(z;\lambda^{-1}\alpha_n,\lambda^{-1}\alpha_{n+1},\dots)$. We conclude $\tilde\alpha_j = \lambda^{-1}\alpha_j$ for $j \geq n$. (5.3) goes beyond this by making the unitary equivalence explicit.

2. The case $n = 0$ is essentially Theorem 4.2.4 of [15].

Proof. First, some preliminaries. We use $\oplus$ for direct sum, normally of 2×2 matrices but sometimes of a 1×1 followed by 2×2, in which case we write $\mathbf{1}_1$ or $\lambda\mathbf{1}_1$ so that, for example, if

$$v(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad \tilde v(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{5.5}$$

then

$$U_{2k-1} = \underbrace{\mathbf{1}_2 \oplus \mathbf{1}_2 \oplus \cdots \oplus \mathbf{1}_2}_{k \text{ times}} \oplus v(\lambda) \oplus v(\lambda) \oplus \cdots \tag{5.6}$$

$$= \mathbf{1}_1 \oplus \underbrace{\mathbf{1}_2 \oplus \cdots \oplus \mathbf{1}_2}_{k \text{ times}} \oplus \tilde{v}(\lambda) \oplus \tilde{v}(\lambda) \oplus \cdots \tag{5.7}$$

If

$$\Theta(\alpha) = \begin{pmatrix} \bar{\alpha} & \rho \\ \rho & -\alpha \end{pmatrix}$$

with $\rho = (1-|\alpha|^2)^{1/2}$, then (see [15, Theorem 4.2.5])

$$\mathcal{C}(\alpha) = \mathcal{L}(\alpha)\mathcal{M}(\alpha) \tag{5.8}$$

$$\mathcal{L}(\alpha) = \Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \Theta(\alpha_4) \oplus \cdots \tag{5.9}$$

$$\mathcal{M}(\alpha) = \mathbf{1}_1 \oplus \Theta(\alpha_1) \oplus \Theta(\alpha_2) \oplus \cdots \tag{5.10}$$

Notice next that (note $v(\lambda)$ not $v(\lambda)^{-1}$ in both places!)

$$v(\lambda)\Theta(\lambda^{-1}\alpha)v(\lambda) = \lambda\Theta(\alpha) \tag{5.11}$$

$$\tilde{v}(\lambda)^{-1}\Theta(\lambda^{-1}\alpha)\tilde{v}(\lambda)^{-1} = \lambda^{-1}\Theta(\alpha) \tag{5.12}$$

We now turn to the proof of (5.3) for $n = 2k-1$. By (5.6), (5.9), and (5.11),

$$U_{2k-1}\mathcal{L}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1} = \lambda\mathcal{L}(\alpha)W \tag{5.13}$$

where

$$W = D((\lambda^{-1})^{2k}1^\infty) \tag{5.14}$$

Now

$$\begin{aligned} WU_{2k-1}^{-1} &= D((\lambda^{-1})^{2k}(1\lambda^{-1})^\infty) \\ &= \lambda^{-1}\mathbf{1}_1 \oplus \underbrace{\lambda^{-1}\mathbf{1}_2 \oplus \cdots \oplus \lambda^{-1}\mathbf{1}_2}_{(k-1) \text{ times}} \oplus \tilde{v}(\lambda)^{-1} \oplus \tilde{v}(\lambda)^{-1} \oplus \cdots \end{aligned} \tag{5.15}$$

since $\lambda^{-1}(1\lambda^{-1})^\infty = (\lambda^{-1}1)^\infty$. Thus, by (5.12) and (5.10),

$$WU_{2k-1}^{-1}\mathcal{M}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1}^{-1}W = \lambda^{-1}\mathcal{M}(\alpha)D((\lambda^{-1})^{2k-1}1^\infty) \tag{5.16}$$

We conclude

$$\begin{aligned} &U_{2k-1}\mathcal{C}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1}^{-1} \\ &\quad = U_{2k-1}\mathcal{L}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1}U_{2k-1}^{-1}\mathcal{M}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1}^{-1} \\ &\quad = \lambda\mathcal{L}(\alpha)WU_{2k-1}^{-1}\mathcal{M}(T_{2k-1,\lambda^{-1}}(\alpha))U_{2k-1}^{-1}WW^{-1} \quad \text{(by (5.13))} \\ &\quad = \mathcal{L}(\alpha)\mathcal{M}(\alpha)D((\lambda^{-1})^{2k-1}1^\infty)D((\lambda)^{2k}1^\infty) \quad \text{(by (5.16))} \\ &\quad = \mathcal{C}(\alpha)\Delta_n(\lambda) \end{aligned}$$

since $\Delta_n(\lambda) = (1^n\lambda 1^\infty)$. This proves (5.3) for $n = 2k-1$.

Now suppose $n = 2k$. Then, by (5.11),

$$U_{2k}\mathcal{L}(T_{2k,\lambda^{-1}}(\alpha))U_{2k} = \lambda\mathcal{L}(\alpha)\widetilde{W} \tag{5.17}$$

where

$$\widetilde{W} = D((\lambda)^{2k} 1^\infty) \tag{5.18}$$

Thus

$$\begin{aligned} \widetilde{W} U_{2k}^{-1} &= D(1^{2k}(1\lambda^{-1})^\infty) \\ &= \mathbf{1}_1 \oplus \underbrace{\mathbf{1}_2 \oplus \cdots \oplus \mathbf{1}_2}_{k \text{ times}} \oplus \tilde{v}(\lambda)^{-1} \oplus \tilde{v}(\lambda)^{-1} \oplus \cdots \end{aligned} \tag{5.19}$$

This is the reason odd and even n differ. (5.15) has $k-1$ $\mathbf{1}_2$'s and (5.19) has k of them. By (5.12) and (5.10),

$$\widetilde{W} U_{2k}^{-1} \mathcal{M}(T_{2k,\lambda^{-1}}(\alpha)) U_{2k}^{-1} \widetilde{W} = \lambda^{-1} \mathcal{M}(\alpha) D(\lambda^{2k-1} 1^\infty) \tag{5.20}$$

It follows that

$$\begin{aligned} U_{2k} \mathcal{C}(T_{2k,\lambda^{-1}}(\alpha)) U_{2k} &= U_{2k} \mathcal{L}(T_{2k,\lambda^{-1}}(\alpha)) U_{2k} U_{2k}^{-1} \mathcal{M}(T_{2k,\lambda^{-1}}(\alpha)) U_{2k}^{-1} \\ &= \lambda \mathcal{L}(\alpha) \widetilde{W} U_{2k}^{-1} \mathcal{M}(T_{2k,\lambda^{-1}}(\alpha)) U_{2k}^{-1} \widetilde{W} \widetilde{W}^{-1} \\ &= \mathcal{L}(\alpha) \mathcal{M}(\alpha) D(\lambda^{2k+1} 1^\infty) D((\lambda^{-1})^{2k} 1^\infty) \\ &= \mathcal{C}(\alpha) \Delta_n(\lambda) \end{aligned}$$

□

6. Proof of Theorem 1.1

We are now ready to put it all together:

Proof of Theorem 1.1. By (3.5) and the exponential decay in (1.6), $G(z_0) < \infty$ for a.e. α, so Theorem 4.2 applies for a.e. α. By (4.5) with $\varphi = \delta_k$ and $\psi = \delta_m$ and $U = \mathcal{C}(\alpha)$,

$$\int \frac{d\varphi}{2\pi} \sup_n |[(\mathcal{C}(\alpha)\Delta_k(e^{i\varphi}))^n]_{km}| \le \biggl(\int |F_{km}(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \biggr)^{1/2} \tag{6.1}$$

Now use Theorem 5.1 and the fact that U is diagonal and unitary to see

$$|[(\mathcal{C}(\alpha)\Delta_k(\lambda))^n]_{km}| = |[\mathcal{C}(T_{k,\lambda^{-1}}(\alpha))^n]_{km}|$$

so that

$$\int \frac{d\varphi}{2\pi} \sup_n |[\mathcal{C}(T_{k,e^{-i\varphi}}(\alpha))^n]_{km}| \le \biggl(\int |F_{km}(e^{i\theta})|^p \frac{d\theta}{2\pi} \biggr)^{1/2} \tag{6.2}$$

Take expectations of both sides. Use the quasi-invariance to write

$$\mathbb{E}\bigl(\sup_n |[\mathcal{C}(T_{k,e^{-i\varphi}}(\alpha))^n]_{k\ell}|\bigr) \ge C^{-1} \mathbb{E}\bigl(\sup_n |[C(\alpha)^n]_{k\ell}|\bigr)$$

for a constant C independent of k, ℓ, φ. Use Hölder's inequality to bring $\mathbb{E}$ inside $(\cdot)^{1/(2-p)}$. The result is

$$\mathbb{E}\bigl(\sup_n |(\mathcal{C}^n)_{k\ell}|\bigr) \le C \biggl(\int \mathbb{E}(|F_{k\ell}(e^{i\theta})|^p) \, \frac{d\theta}{2\pi} \biggr)^{1/2}$$

which shows (1.6) implies (1.7). □

7. Remarks

Some closing remarks:

1. It is not hard to prove a local version of this theorem where $\int_0^{2\pi}$ in (1.6) is replaced by $\int_a^b$ and an extra $P_{(a,b)}(\mathcal{C})$ is added in (1.7). This might be useful in the quasi-invariant case, but in the i.i.d. rotation invariant case, $\mathbb{E}(|F_{k\ell}(e^{i\theta})|^p)$ is θ-invariant and so the integral has exponential decay for (a,b) if and only if it does for $(0,2\pi)$.

2. If $d\rho$ is a rotation quasi-invariant measure on $\mathbb{D}$ and $d\gamma$ one on $\partial\mathbb{D}$, and if $\alpha_0, \bar\alpha_0\alpha_1, \bar\alpha_1\alpha_2, \dots$ are independent random variables with α_0 $d\rho$-distributed and each $\bar\alpha_j\alpha_{j+1}$, $d\rho$-distributed, then this measure is quasi-invariant. It would be interesting to do localization theory (both spectral and dynamic) for this model.

3. It would be interesting to know if (1.7) implies (1.6).

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