BOUND STATES AND THE SZEGŐ CONDITION FOR JACOBI MATRICES AND SCHRÖDINGER OPERATORS

DAVID DAMANIK[1,2], DIRK HUNDERTMARK[1], AND BARRY SIMON[1,3]

ABSTRACT. For Jacobi matrices with $a_n = 1 + (-1)^n \alpha n^{-\gamma}$, $b_n = (-1)^n \beta n^{-\gamma}$, we study bound states and the Szegő condition. We provide a new proof of Nevai's result that if $\gamma > \frac{1}{2}$, the Szegő condition holds, which works also if one replaces $(-1)^n$ by $\cos(\mu n)$. We show that if $\alpha = 0$, $\beta \neq 0$, and $\gamma < \frac{1}{2}$, the Szegő condition fails. We also show that if $\gamma = 1$, α and β are small enough ($\beta^2 + 8\alpha^2 < \frac{1}{24}$ will do), then the Jacobi matrix has finitely many bound states (for $\alpha = 0$, β large, it has infinitely many).

1. Introduction

This paper focuses on Jacobi matrices, that is, operators J on $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, \dots\}$, given by ($b_n$ real, $a_n > 0$)

$$(Ju)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1) \tag{1.1}$$

where the $a_{n-1}u(n-1)$ term is dropped if $n = 1$. We define J_0 by $a_n \equiv 1$, $b_n \equiv 0$, and will suppose $J - J_0$ is compact, so $\sigma_{\text{ess}}(J) = [-2, 2]$. We are interested especially in the Szegő condition,

$$Z(J) \equiv \frac{1}{2\pi} \int_{-2}^{2} \log\left(\frac{\sqrt{4-E^2}}{2\pi \frac{d\nu_{\text{ac}}}{dE}} \right) \frac{dE}{\sqrt{4-E^2}} < \infty \tag{1.2}$$

where ν is the spectral measure for J and the vector δ_1. We will also consider some aspects of Schrödinger operators $-\Delta + V$.

In 1979, Nevai [24] proved a conjecture of Askey that if

$$a_n = 1 + \frac{(-1)^n \alpha}{n} + O(n^{-2}) \qquad b_n = \frac{(-1)^n \beta}{n} + O(n^{-2}) \tag{1.3}$$

then the Szegő condition holds. Our goal here is to understand this result from the point of view of sum rules recently used to study the Szegő

Date: August 19, 2002.

[1] Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: damanik@its.caltech.edu; dirkh@its.caltech.edu; bsimon@caltech.edu.

[2] Supported in part by NSF grant DMS-0227289.

[3] Supported in part by NSF grants DMS-9707661, DMS-0140592.

condition by Killip-Simon [17] and Simon-Zlatoš [31], and to consider various extensions and borderline cases, in particular, the following four questions:

(1) Nevai [24] allows replacement of $\frac{(-1)^n}{n}$ by $\frac{(-1)^n}{n^\gamma}$ with $\gamma > \frac{1}{2}$ and still gets (1.2). Is $\gamma = \frac{1}{2}$ a borderline or just where Nevai's method fails? We will see that $\gamma = \frac{1}{2}$ is indeed a borderline and that $a_n = 1$, $b_n = \frac{(-1)^n}{n^\gamma}$ obeys (1.2) if and only if $\gamma > \frac{1}{2}$. This is a subtle issue: one might think the key is that $b_{n+1} - b_n$ decay faster than n^{-1}, in which case $\gamma = \frac{1}{2}$ is not special but, as is the case in many other situations [18], $b_n \in \ell_2$ is critical; see Theorem 2 below.
(2) What is the condition on the errors $O(n^{-2})$ in (1.3)? Nevai actually shows if those errors, $e_a(n), e_b(n)$, obey $\sum_{n=2}^\infty (\log n)|e_j(n)| < \infty$ for $j \in \{a, b\}$, then (1.2) still holds. In line with the advances in [17, 31], we will only require $\sum_{n=2}^\infty |e_j(n)| < \infty$, for $j \in \{a, b\}$. Indeed, our results are logarithmically better than Nevai's in the leading term. If $\frac{(-1)^n}{n}$ in (1.3) is replaced by $\frac{(-1)^n}{n^{1/2}}[\log n]^{-\gamma}$, then Nevai's method requires $\gamma > 1$, while we require only $\gamma > \frac{1}{2}$.
(3) What about other oscillatory potentials like $\frac{\cos(\eta n)}{n^\gamma}$ for $\eta \in (0, 2\pi)$? (1.3) is the case $\eta = \pi$. Although it is possible his methods extend to this case, the conditions in Nevai's paper require cancellations in $b_n + b_{n+1}$ and do not work for $\eta \neq \pi$. We will accommodate general η.
(4) Nevai's work suggests that $\frac{(-1)^n}{n}$ is akin to n^{-2} potentials, which suggests that for $|\alpha| + |\beta|$ small, (1.3) has finitely many eigenvalues outside $[-2, 2]$ while for $|\alpha| + |\beta|$ large, it has infinitely many. We will prove the finiteness result below. We note that while he does not discuss this case explicitly, Chihara's conditions in [7] imply finitely many bound states if $a_n = 1$ and $|\beta|$ is small.

For Jacobi matrices, our main results are:

Theorem 1. *Suppose*

$$a_n = 1 + c_n + d_{n+1} - d_n \tag{1.4}$$

$$b_n = e_n + f_{n+1} - f_n \tag{1.5}$$

with

$$\sum_{n=1}^\infty |c_n| + |e_n| + |d_n|^2 + |f_n|^2 < \infty \tag{1.6}$$

Then $Z(J) < \infty$ and

$$\sum_{j,\pm} \sqrt{E_j^{\pm}(J)^2 - 4} < \infty \tag{1.7}$$

where $E_j^{\pm}(J)$ are the eigenvalues of J in $\pm(2,\infty)$.

Remarks. 1. In case $a_n - 1 = \frac{\alpha(-1)^n}{n^\gamma} + e_a(n)$, $b_n = \frac{\beta(-1)^n}{n^\gamma} + e_b(n)$, we define

$$d_n = -\sum_{j=n}^{\infty} \frac{\alpha(-1)^n}{n^\gamma} \qquad f_n = -\sum_{j=n}^{\infty} \frac{\beta(-1)^n}{n^\gamma} \tag{1.8}$$

Since the sums are $O(n^{-\gamma})$, (1.6) is then true if $\sum |e_a(n)| + |e_b(n)| < \infty$ and $\gamma > \frac{1}{2}$.

2. If b_n is instead $\frac{\cos(\eta n)}{n}$, it is still true that $f_n \equiv -\sum_{j=n}^{\infty} \frac{\cos(\eta n)}{n}$ is $O(n^{-1})$, and so in ℓ^2, and thus this theorem also includes cases like $\frac{\cos(\eta n)}{n}$ where $b_n + b_{n+1}$ does not have cancellations.

3. By mimicking the construction of Wigner and von Neumann (see, e.g., [27, Example 1 in Chapter XIII.13], one can construct Jacobi matrices J with $a_n \sim 1 + \frac{(-1)^n}{n}$ and $b_n \sim \frac{(-1)^n}{n}$ as $n \to \infty$ which have 0 as an eigenvalue embedded in the essential spectrum.

As a converse to Theorem 1, we note

Theorem 2. *Suppose*
(i) $\limsup[-\sum_{j=1}^{n} \log(a_j)] > -\infty$
(ii) $\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 = \infty$
Then $Z(J) = \infty$.

Remark. If $a_n = 1$ (or $a_n = \exp(\frac{\alpha(-1)^n}{n^\gamma})$), $b_n = \frac{\beta(-1)^n}{n^\gamma}$, and $\gamma \leq \frac{1}{2}$, then this implies $Z(J) = \infty$, showing $\gamma = \frac{1}{2}$ is the borderline.

Proof. Suppose $Z(J) < \infty$ and (i) holds. Then, by Theorem 1 of Simon-Zlatoš [31], (1.7) holds. A fortiori, the quasi-Szegő condition, (1.8) of Killip-Simon [17], and the $\frac{3}{2}$ Lieb-Thirring bound hold. So, by Theorem 1 of Killip-Simon [17], (ii) fails. Thus, (i) + $Z(J) < \infty \Rightarrow$ not (ii). So (i) + (ii) $\Rightarrow Z(J) = \infty$. $\square$

Theorem 3. *Suppose $a_n = 1$ and (1.5) holds with*

$$\limsup_n n^2[|e_n| + |f_n|^2 + |f_{n+1}|^2] < \tfrac{1}{8} \tag{1.9}$$

Then J has only finitely many bound states. If (1.4), (1.5) hold and

$$\limsup_n n^2 \big[|c_n| + |c_{n-1}| + 24|d_{n-1}|^2 + 48|d_n|^2 + 24|d_{n+1}|^2 + |e_n| + 6|f_n|^2 + 6|f_{n+1}|^2 \big] < \tfrac{1}{8} \tag{1.10}$$

then J has finitely many bound states.

Remark. In particular, if $a_n = 1$, $b_n = \frac{\beta(-1)^n}{n}$, and $|\beta| < \frac{1}{2}$, then J has only finitely many bound states. If $|\beta| > 1$, it is proven in [11] that J has infinitely many bound states. Also, if $a_n = 1 + \frac{\alpha(-1)^n}{n}$ and b_n as before, then for $\beta^2 + 8\alpha^2 < \frac{1}{24}$, J has also only finitely many bound states, but this bound seems to be far from optimal.

The techniques we will use are two-fold: First, we will use the result of Simon-Zlatoš [31] that if $-\sum_{j=1}^{\infty} \log(a_j)$ is conditionally convergent, then (1.2) holds if and only if (1.7) holds (by a Case-type sum rule). This means that all the results on finiteness on $Z(J)$ which we are discussing are equivalent to suitable bounds on eigenvalues. Second, to bound eigenvalues, we will use ideas developed in the 1970's to discuss Schrödinger operators with oscillatory potentials [1, 5, 6, 8, 9, 15, 16, 23, 29, 30, 32]. Interestingly, the focus of that work was to handle wild, pathological cases like $V(r) = (1+r)^{-2}e^{1/r}\sin(e^{1/r})$ or $V(r) = (1+r)^{-2}e^{r}\sin(e^{r})$, which are extremely unbounded near $r = 0$ or $r = \infty$ but whose oscillators cause $-\Delta + V$, defined by quadratic form methods, to be well behaved. In fact, we believe that the most interesting examples are ones like $r^{-1}\sin(r)$ which are not unbounded at all, but oscillatory and slowly decaying.

Most of the 1970's papers discuss scattering or selfadjointness results, although Combescure-Ginibre [9] and Chadan-Martin [6] do discuss bounds on the number of bound states. Since they were not as efficient in using operator bounds, we begin in Section 2 with the continuum Schrödinger operator case. In Section 3, we discuss the growth of $N(\lambda V)$ as $\lambda \to \infty$ for long-range oscillatory potentials. We will prove

Theorem 4. *Let $V_\beta(x) = (1+|x|)^{-\beta}\sin(|x|)$ for $2 > \beta > 1$. On $\mathbb{R}^\nu$, we have*
$$-a_- + b_-\lambda^{\nu/\beta} \le N(\lambda V) \le a_+ + b_+\lambda^{\nu/\beta}$$
for suitable (β-dependent) $a_\pm, b_\pm > 0$.

We note that if $\beta > 2$ so $V_\beta \in L^{\nu/2}$, then it is known (see [27, Theorem XIII.80]) that
$$\lim_{\lambda\to\infty} \lambda^{-\nu/2} N(\lambda V) = \frac{\tau_\nu}{(2\pi)^\nu} \int_{V_\beta(x)\le 0} (-V_\beta(x))^{\nu/2}\, d^\nu x$$
with τ_ν the volume of the unit ball in $\mathbb{R}^\nu$.

In Section 4, we discuss the discrete Schrödinger case, that is, Jacobi matrices with $a_n \equiv 1$. In Section 5, we discuss the general Jacobi case. The appendix contains bounds on the $O(n^{-2})$ situation that we will need in the body of the paper. Since these have not been proven in the Jacobi case with optimal constants, it was necessary to include this appendix. In particular, in Theorems A.6 and A.7, we study Jacobi matrices J with $|a_n - 1| \sim \frac{\gamma_a}{n^2}$ and $|b_n| \sim \frac{\gamma_b}{n^2}$ and discuss finiteness (resp. infinitude) of the discrete spectrum of J in $[-2,2]^{\mathrm{c}}$, depending on whether $2\gamma_a + \gamma_b < \frac{1}{4}$, (resp. $2\gamma_a + \gamma_b > \frac{1}{4}$), thereby extending results of Chihara in [7].

We would like to thank Rowan Killip, Paul Nevai, Mihai Stoiciu, and Andrej Zlatoš for valuable communications.

2. The Continuum Schrödinger Case

Let W be an $\mathbb{R}^\nu$-valued C^1 function on $\mathbb{R}^\nu$ or a piecewise C^1 continuous function on $\mathbb{R}$ so that $\nabla \cdot W$ is also bounded. In fact, once one has the bounds below, it is easy to accommodate arbitrary distributions W with $W \in L^\nu + L^\infty$ (when $\nu \geq 3$) even if $\nabla \cdot W$ is not bounded. For our applications of interest, we make these simplifying assumptions.

Proposition 2.1 (Combescure-Ginibre Lemma [9])**.** *If* $\varphi \in C_0^\infty$,

$$|\langle \varphi, \nabla W \varphi \rangle| \leq 2\|W\varphi\|\,\|\nabla\varphi\| \tag{2.1}$$

Proof. First, integrate by parts, $\langle \varphi, \nabla W \varphi \rangle = 2\,\mathrm{Re}\langle W\varphi, \nabla\varphi \rangle$. Then use the Schwarz inequality. □

Theorem 2.2. *Let* $H = -\Delta + V_1 + \nabla \cdot W$ *and* $H_1 = -\Delta + 2V_1 - 4W^2$. *Then*

$$H \geq \tfrac{1}{2}\, H_1 \tag{2.2}$$

In particular, if $N(V)$ *is the number of negative eigenvalues of* $-\Delta + V$, *then*

$$N(V_1 + \nabla \cdot W) \leq N(2V_1 - 4W^2) \tag{2.3}$$

and if $E_p(V) = \mathrm{Tr}([-HE_{(-\infty,0]}(H)]^p)$ *for* $H = -\Delta + V$ *is the Lieb-Thirring sum of eigenvalue powers, then*

$$E_p(V_1 + \nabla \cdot W) \leq 2^{-p} E_p(2V_1 - 4W^2) \tag{2.4}$$

Proof. By (2.1),

$$\begin{aligned}\langle \varphi, (-\Delta + V_1 + \nabla W)\varphi \rangle &\geq \langle \varphi, (-\Delta + V_1)\varphi \rangle - \varepsilon \langle \varphi, -\Delta\varphi \rangle - \varepsilon^{-1}\langle \varphi, W^2\varphi \rangle \\ &= \left\langle \varphi, (1-\varepsilon)\left(-\Delta + \frac{1}{1-\varepsilon}\, V_1 - \frac{1}{\varepsilon(1-\varepsilon)}\, W^2\right)\varphi \right\rangle\end{aligned}$$

In the absence of a V_1 term, the optimal choice of ε is $\varepsilon = \frac{1}{2}$ (to minimize $\frac{1}{\varepsilon(1-\varepsilon)}$), so we make that choice in general. It yields (2.2), which in turn immediately implies (2.3) and (2.4). □

The bound (2.1) and its proof are taken from Combescure-Ginibre [9]. While they use the Schwarz inequality, they do not explicitly note (2.2), which causes them to make extra arguments that can be less efficient than using (2.2). For example, if $V_1 = 0$, $\nu = 3$, and

$$w^2 = (4\pi)^{-2} \int d^3x\, dy^3\, \frac{W^2(x)W^2(y)}{|x-y|^2}$$

then (2.3) and the Birman-Schwinger principle immediately imply that

$$N(\nabla \cdot W) \leq 16w^2$$

while Combescure-Ginibre [9] only claim

$$N(\nabla \cdot W) \leq 16w^2(1+w)^2$$

which is much worse for large w.

For $\nu = 1$, $V_1 = 0$, (2.3) is a result of Chadan-Martin [6] who use Sturm comparison methods rather than the Schwarz inequality and the Combescure-Ginibre lemma. Theorem 2.2 has some immediate consequences:

Corollary 2.3. *For $\nu \geq 3$,*

$$N(V_1 + \nabla \cdot W) \leq c_\nu(\|V_1\|_{\nu/2}^{\nu/2} + \|W\|_\nu^\nu) \tag{2.5}$$

For general ν and $p \geq \frac{1}{2}$ if $\nu = 1$, $p > 0$ if $\nu = 2$, and $p \geq 0$ if $\nu \geq 3$,

$$E_p(V_1 + \nabla W) \leq c_{\nu,p}(\|V_1\|_{p+\nu/2}^{p+\nu/2} + \|W\|_{2p+\nu}^{2p+\nu}) \tag{2.6}$$

Proof. (2.5) is just the Cwikel-Lieb-Rozenblum [10, 19, 28] bound, given (2.3). (2.6) is the Lieb-Thirring bound [21, 22] when p is strictly larger than the minimal value. $p = 0$ for $\nu \geq 3$ is (2.5) while $p = \frac{1}{2}$, $\nu = 1$ is due to Weidl [34] (see also Hundertmark-Lieb-Thomas [13]). □

If $\nu \geq 3$ and $V \in L^{\nu/2}$, we have $N(\lambda V) \leq c\lambda^{\nu/2}$, but (2.5) only implies that

$$N(\lambda(V_1 + \nabla W)) \leq c_1\lambda^{\nu/2} + c_2\lambda^\nu$$

In the next section, we see that in some specific cases, $N(\lambda V)$ really does grow at rates arbitrarily close to λ^ν.

Corollary 2.4. *If $\nu \neq 2$ and $4W^2 - 2V_1 \leq \frac{(\nu-2)^2}{4}|x|^{-2}$, then $-\Delta + V_1 + \nabla W_1$ has no bound states. If $4W^2 < \frac{(\nu-2)^2}{4}|x|^{-2}$ and $V_1 \in L^{\nu/2}$ (if $\nu \geq 3$) or $\int (1+|x|)|V_1(x)|\, dx < \infty$ ($\nu = 1$), then $-\Delta + V_1 + \nabla \cdot W_1$ has finitely many bound states.*

Proof. The first statement is immediate from (2.3) and Theorem A.3. The second follows from

$$-\Delta + 2V_1 - 4W^2 = -(1-\varepsilon)\Delta - 4W^2 - \varepsilon\Delta + 2V_1$$

and the Cwikel-Lieb-Rozenblum estimate if $\nu \geq 3$ and Bargmann's bound if $\nu = 1$. $\square$

Example 2.5. If $V = \frac{\sin(r)}{r^\alpha}$, we can write $V = V_1 + \nabla \cdot W$ where $W = \hat{r}(-\frac{\cos(r)}{r^\alpha} - \frac{\alpha \sin(r)}{r^{\alpha+1}})f(r)$ where $f \in C^\infty$ vanishes near 0 and is 1 near ∞. This shows that for $\alpha < 1$, $-\Delta + \lambda V_0$ has finitely many bound states for all λ, and when $\alpha = 1$, it has finitely many bound states if $|\lambda|$ is small. An argument similar to that in [11] shows that if $\alpha = 1$ and λ is large, $-\Delta + \lambda V$ has an infinity of negative eigenvalues. $\square$

3. Schrödinger Operators at Large Coupling

Our purpose here is to prove Theorem 4 that $V_\beta = \frac{\sin(r)}{(1+r)^\beta}$ (with $2 > \beta > 1$) has $N(\lambda V_\beta)$ growing as $\lambda^{\nu/\beta}$ for λ large. We give the details when $\nu = 1$ on a half-line and then discuss the case when $\nu \geq 2$.

Proof of Theorem 4. Half-line case: We begin with the upper bound. Let $\varphi_R(r)$ be a C^∞ function with $\|\varphi_R\|_\infty = 1$, which is 0 if $r < R$ and 1 if $r > R + 1$. By translation, we may assume the derivatives $\frac{d^\alpha \varphi_R}{dx^\alpha}$ are uniformly bounded in x and R (for fixed α). Let $W_R(r) = -\int_r^\infty \varphi_R(s) V_\beta(s)\, ds$ and $V_{1,R} = V_\beta - \frac{d}{dr} W_R$. Define $V_{2,R}(r) = \max_{s \geq r} |V_{1,R}(s)|$. Then

$$\begin{aligned} -\frac{d^2}{dx^2} + \lambda V_\beta &= \left(-\frac{1}{2}\frac{d^2}{dx^2} + \lambda V_{1,R}\right) + \left(-\frac{1}{2}\frac{d^2}{dx^2} + \lambda \frac{dW_R}{dr}\right) \\ &\geq \left(-\frac{1}{2}\frac{d^2}{dx^2} - \lambda V_{2,R}\right) + \frac{1}{2}\left(-\frac{1}{2}\frac{d^2}{dx^2} - 8\lambda^2 W_R^2\right) \end{aligned} \tag{3.1}$$

Next, note that

$$W_R(r) \leq C(\max(r, R))^{-\beta} \tag{3.2}$$

We also have

$$|V_{2,R}(r)| \leq r^{-\beta} \tag{3.3}$$

and is zero if $r > R + 1$.

Calogero [3] has proven that if V is monotone decreasing and non-negative, then $N(-V) \leq 2\pi^{-1} \int_0^\infty |V(s)|^{1/2}\, ds$. This bound, (3.1)–(3.3), and the fact that $\dim(E_{(-\infty,0)}(A + B)) \leq \dim E_{(-\infty,0)}(A) +$

$\dim E_{(-\infty,0)}(B)$ (by the variational principle) imply that for any R,

$$N(\lambda V_\beta) \leq C_1 \left[\lambda^{1/2} \int_0^{R+1} r^{-\beta/2}\, dr + \lambda \int_0^\infty \max(R,r)^{-\beta}\, dr \right]$$
$$= C_2[\lambda^{1/2} R^{1-\beta/2} + \lambda R^{1-\beta}]$$

since $1 < \beta < 2$. Pick $R = \lambda^{1/\beta}$ and get

$$N(\lambda V_\beta) \leq 2C_2 \lambda^{1/\beta}$$

On the other side, consider the operator $\tilde{H}(\lambda)$, which is $-\frac{d^2}{dx^2} + \lambda V_\beta$ with Dirichlet boundary conditions added at the points $(2n + \frac{3}{2})\pi \pm \frac{\pi}{3}$. Adding such boundary conditions only increases the operator, so $N(\lambda V_\beta) \geq \#$ of negative eigenvalues of $\tilde{H}(\lambda)$. In each interval of the form $[(\frac{2n+3}{2})\pi - \frac{\pi}{3}, (\frac{2n+3}{2})\pi + \frac{\pi}{3}]$, $\sin(r)$ is less than $-\frac{1}{2}$, so $V_\beta \leq -\frac{\lambda}{2[(2n+3)\pi]^\beta}$ on the entire interval. The lowest Dirichlet eigenvalue of $-\frac{d^2}{dx^2}$ on such an interval is $\frac{9}{4}$, so each interval with

$$\frac{\lambda}{2[(2n+3)\pi]^\beta} > \frac{9}{4}$$

contributes an eigenvalue so

$$N(\lambda V_\beta) \geq C_3 \lambda^{1/\beta}$$

This completes the proof of Theorem 4 in the half-line case.

One might think that it would help to use the fact that small n intervals provide $O(\lambda^{1/2})$ eigenvalues rather than just the 1 we use, but a detailed analysis shows it improves the constant in front of $\lambda^{1/\beta}$ but not the power.

Higher dimensions: The lower bound is similar to the half-line case. We have $\sin(r) < -\frac{1}{2}$ on annuli which we can partially cover with suitable disjoint cubes of fixed size, finding cubes where V is deep enough when the distance of the cube from the origin is no more than $C\lambda^{1/\beta}$. The number of such cubes is $O(\lambda^{\nu/\beta})$ so we get an $O(\lambda^{\nu/\beta})$ lower bound.

For the upper bound when $\nu \geq 3$, we can replace Calogero's bound with the Cwikel-Lieb-Rozenblum bound. Since $\int_0^R r^{-\beta\nu/2}\, d^\nu r = C_4 R^{\nu(1-\beta/2)}$ and $\int_R^\infty r^{-\beta\nu}\, d^\nu r = C_5 R^{\nu(1-\beta)}$, we find

$$N(\lambda V_\beta) \leq C_6[\lambda^{\nu/2} R^{\nu(1-\beta/2)} + \lambda^\nu R^{\nu(1-\beta)}]$$

so picking $R = \lambda^{1/\beta}$, we get $N(\lambda V_\beta) \leq C_7 \lambda^{\nu/\beta}$.

$\nu = 2$ is messier. We will sketch the idea, but omit the details. One needs to use the spherical symmetry and consider each partial wave separately. By using the analog of (3.1), we see, on functions of angular momentum ℓ, there is an effective potential which bounds $-\Delta + \lambda V_\beta$ from below, viz,

$$V_{\ell,\mathrm{eff}} = -\frac{1}{4r^2} + \frac{\ell^2}{4r^2} - \lambda r^{-\beta}\chi_{(0,R+1)} - \lambda^2 \max(r,R)^{-2\beta}$$

We need to consider three regions:

(i) $\ell \geq C_8\lambda^{1/\beta}$: Take $R = \lambda^{1/\beta}$ and find $V_{\ell,\mathrm{eff}} \geq 0$ so there are no bound states.

(ii) $1 \leq \ell \leq C_8\lambda^{1/\beta}$: We take $R = \lambda^{1/\beta}$, drop the $\frac{\ell^2-4}{4r^2}$ term, and use Calogero's bound to get a bound per partial wave of $C_9\lambda^{1/\beta}$ as in the one-dimensional case.

(iii) $\ell = 0$: The singularity of $-r^{-2}$ at both 0 and infinity requires us to place Dirichlet boundary conditions at 1 and a point $R_2 = \lambda^{2/\beta-1}$, which for large λ is much larger than $R_1 = \lambda^{1/\beta}$ (since $\frac{1}{\beta} < 1 < \frac{1}{\beta-1} < \frac{2}{\beta-1}$). On (R_2,∞), we can use the fact that $\lambda V_\beta \geq -\frac{1}{4}\frac{1}{r^2\log r}$ and Theorem A.2 to see the Dirichlet operator has no bound states. On $(0,1)$, we can bound the $\ell = 0$ states by all states for the Dirichlet Laplacian in $L^2(\{|x|<1\}, d^2x)$ with Dirichlet boundary conditions with energy below $c_a\lambda$ (where $c = \max_{|r|\leq 1} -\frac{\sin(r)}{(1+r)^\beta}$). It is known (by Weyl's theorem, see [27, p. 271]) that this is asymptotically $c_{10}\lambda$ since $\nu = 2$. In $(1,R_2)$, we can use Calogero's bound where now "V" is $-\frac{1}{4r^2} - \lambda V_{2,R} - 4\lambda^2 W_R^2$. We get a bound by $c_{11}\int_1^{R_2}\frac{dr}{r} + c_{12}\lambda^{1/\beta}$. Taking into account the possible two states lost by adding the Dirichlet boundary conditions in the $\ell = 0$ space, we get

$$N(\lambda V_\beta) \leq c_{13}\lambda^{1/\beta}(\lambda^{1/\beta}) + c_{14}(\lambda + \lambda^{1/\beta} + \log(|\lambda|+1))$$

which is the required large $\lambda^{2/\beta}$ bound.

□

4. Discrete Schrödinger Operators

Our main goal in this section is to extend (2.1) and Theorem 2.2 to the discrete case. It will be convenient to consider operators on all of $\mathbb{Z}$ and get bounds on Jacobi matrices by restriction. We will also restrict to eigenvalues above energy 2. One can then control energies below -2 by using

$$U_0 J(\{a_n\},\{b_n\})U_0^{-1} = -J(\{a_n\},\{-b_n\}) \tag{4.1}$$

where $J(\{a_n\},\{b_n\})$ is the Jacobi matrix (1.1) with parameters a_n, b_n, and

$$(U_0 u)(n) = (-1)^n u(n) \tag{4.2}$$

On $\ell^2(\mathbb{Z})$, define two operators H_0 and δ_+ as

$$(H_0 u)(n) = u(n+1) + u(n-1) \tag{4.3}$$

$$(\delta_+ u)(n) = u(n+1) - u(n) \tag{4.4}$$

Then $\delta_- \equiv \delta_+^*$ is given by

$$(\delta_- u)(n) = u(n-1) - u(n)$$

and

$$\delta_+\delta_- = \delta_-\delta_+ = 2 - H_0 \tag{4.5}$$

(for if $\delta_+ = R-1$ and $\delta_- = L-1$, then $RL = LR = 1$ and $H_0 = L+R$). Let b_n and f_n be sequences on $\mathbb{Z}$ and suppose

$$b_n = f_{n+1} - f_n = (\delta_+ f)_n \tag{4.6}$$

Then in $\ell^2(\mathbb{Z})$, for u real and of finite support,

$$\begin{aligned}\langle u, bu\rangle &= \sum_n b_n |u(n)|^2 \\ &= \sum_n (f_{n+1} - f_n)|u(n)|^2 \\ &= \sum_n f_n(|u(n-1)|^2 - |u(n)|^2) \\ &= \langle \delta_- u, f(1+L)u\rangle \end{aligned} \tag{4.7}$$

Since $\|\delta_- u\|^2 = \langle u, \delta_+\delta_- u\rangle = \langle u, (2-H_0)u\rangle$ by (4.5), we see that

Lemma 4.1. *If b is given by* (4.6), *then*

$$|\langle u, bu\rangle| \leq \langle u, (2-H_0)u\rangle^{1/2}[2\langle u, (f^2 + \tilde{f}^2)u\rangle]^{1/2} \tag{4.8}$$

where

$$\tilde{f}_n = f_{n+1} \tag{4.9}$$

Proof. In getting (4.8), we used (4.5), (4.7),

$$\|f(1+L)u\|^2 \leq 2\|fu\|^2 + 2\|fLu\|^2 = 2\|fu\|^2 + 2\|\tilde{f}u\|^2$$

and the fact that, because of $||x| - |y|| \leq |x - y|$, we also have

$$\langle |u|, (2-H_0)|u|\rangle = \tfrac{1}{2}\sum_n ||u(n+1)| - |u(n)||^2 \leq \langle u, (2-H_0)u\rangle$$

so it suffices to prove the result for real valued sequences u. $\square$

We will later need the following estimate that was proven along the way (we get J_0 by restricting to u's of support on $\mathbb{Z}_+$):

$$|\langle u, \delta_+ f\, u\rangle| \leq 2|\langle u, (2-H_0)u\rangle|^{1/2}|\langle u, \tfrac{1}{2}\,(f^2 + \tilde{f}^2)u\rangle|^{1/2} \tag{4.10}$$

$$\leq \varepsilon\langle u, (2-J_0)u\rangle + \varepsilon^{-1}\langle u, \tfrac{1}{2}\,(f^2 + \tilde{f}^2)u\rangle \tag{4.11}$$

Theorem 4.2. *Let b_n be a sequence on $\mathbb{Z}_+$ so that* $\lim_{n\to\infty}\sum_{j=1}^n b_j$ *exists, and let*

$$f_n = -\sum_{j=n}^{\infty} b_j \tag{4.12}$$

Let J be the Jacobi matrix with $a_n \equiv 1$ and b's given by b_n. Let $J^\pm$ be the Jacobi matrix with $a_n \equiv 1$ and b's given by

$$\pm 2(f^2 + \tilde{f}^2) \tag{4.13}$$

Then
(i)

$$\dim E_{(2,\infty)}(J) \leq \dim E_{(2,\infty)}(J^+) \tag{4.14}$$

(ii)

$$\dim E_{(-\infty,-2)}(J) \leq \dim E_{(-\infty,-2)}(J^-) \tag{4.15}$$

(iii) *If the eigenvalues E of $J^\pm$ outside* $[-2,2]$ *obey*

$$\sum_j (|E_j(J^\pm)| - 2)^\alpha < \infty \tag{4.16}$$

for some α and for both J^+ and J^-, then

$$\sum_j (|E_j(J)| - 2)^\alpha < \infty \tag{4.17}$$

Proof. Define

$$b_0 = -\sum_{j=1}^{\infty} b_j$$

so if f is extended to $\mathbb{Z}$ by setting $f_k = 0$ for $k \leq 0$, we have $b = \tilde{f} - f$. Thus, by (4.8) as operator on $\ell^2(\mathbb{Z})$,

$$-b \geq \tfrac{1}{2}\left[-(2 - H_0) - 2(f^2 + \tilde{f}^2)\right]$$

so

$$2 - H_0 - b \geq \tfrac{1}{2}\left[(2 - H_0) - 2(f^2 + \tilde{f}^2)\right]$$

Now restrict to functions supported on $\mathbb{Z}_+$ to get

$$2 - J \geq \tfrac{1}{2}\left[2 - J^+\right] \tag{4.18}$$

This yields (4.14) and (4.1) then yields (4.15). The two together imply (4.17). ∎

Example. Let $b_n = \frac{\beta(-1)^n}{n}$. Then $f_n \sim -\frac{1}{2}\frac{\beta(-1)^n}{n} + O(\frac{1}{n^2})$ and the leading term in $2n^2(f^2 + \tilde{f}^2) = 4(\frac{1}{2}\beta)^2 = \beta^2$. By Theorem A.6, if $\beta^2 < \frac{1}{4}$, $J(a = 1, b)$ has finitely many eigenvalues, that is, $|\beta| < \frac{1}{2}$ produces finitely many eigenvalues.

On the other hand, if $|\beta| > 1$, it is known [11] that H has an infinite number of bound states. It would be interesting to determine the exact value of the coupling constant, where the shift from finitely many to infinitely many bound states takes place.

Proof of the First Assertion in Theorem 3. By (4.18), if b_n has the form (1.4) and $J^\pm$ is formed with $b_n^\pm = 2e_n^\pm \pm 2(f_n^2 + f_{n+1}^2)$, then

$$2 \mp J \geq \tfrac{1}{2}\,[2 \mp J^\pm]$$

If (1.9) holds, then

$$\limsup_n \, n^2[|\tilde{b}_n^\pm|] < \tfrac{1}{4}$$

so $J^\pm$ have finitely many bound states by Theorem A.6. $\square$

5. Oscillatory Jacobi Matrices

In this section, we will prove Theorems 1, 2, and 3 by accommodating general values of a_n within the bounds of the last section. Recall that R acts as $Ru(n) = u(n+1)$ and we defined $\delta_+ = R - 1$. It will be convenient to write the Jacobi matrix $J = aR + R^*a + b$ in "divergence form," that is, write it as $J = -\delta_+^* g \delta_+ + q$. Let us first consider the whole-line case: Given sequences a, b on $\mathbb{Z}_+$, we extend them to sequences on $\mathbb{Z}$ by setting $a_n = 1$ and $b_n = 0$ for $n \leq 0$. We denote the corresponding operator on $\ell^2(\mathbb{Z})$ by K. With $a^\sharp = R^*aR$, that is, $a_n^\sharp = a_{n-1}$, and $-\delta_+^* g \delta_+ = -R^*gR + gR + R^*g - g$, we see that $g = a$ and $q = b + a + a^\sharp$. Thus, recalling $\delta_+^*\delta_+ = 2 - H_0$,

$$\begin{aligned} K &= -\delta_+^* a \delta_+ + b + a + a^\sharp \\ &= H_0 + b + a + a^\sharp - 2 - \delta_+^*(a-1)\delta_+ \end{aligned}$$

which shows

$$\langle u, Ku\rangle = \langle u, H_0 u\rangle + \langle u, (b + a + a^\sharp - 2)u\rangle - \langle \delta_+ u, (a-1)\delta_+ u\rangle$$

By restriction to u's supported on $\mathbb{Z}_+$, we get

$$\langle u, Ju\rangle = \langle u, J_0 u\rangle + \langle u, (b + a + a^\sharp - 2)u\rangle - \langle \delta_+ u, (a-1)\delta_+ u\rangle \tag{5.1}$$

where one should keep in mind that $a_1^\sharp = 1$.

We first estimate the third term in (5.1). Writing a as in (1.4), that is, $a = 1 + c + \delta_+ d$, it reads

$$\langle \delta_+ u, (a-1)\delta_+ u\rangle = \langle \delta_+ u, c\delta_+ u\rangle + \langle \delta_+ u, (\delta_+ d)\, \delta_+ u\rangle \tag{5.2}$$

With $(x)_- = \max(-x, 0)$, the negative part, we have

$$\langle \delta_+ u, c\delta_+ u\rangle \geq -\langle \delta_+ u, c_- \delta_+ u\rangle$$

$$
\begin{aligned}
&= -\sum_n (c_n)_- \big(|u(n+1)|^2 + |u(n)|^2 - 2\operatorname{Re}(\overline{u(n+1)}u(n))\big) \\
&\geq -\sum_n 2(c_n)_- \big(|u(n+1)|^2 + |u(n)|^2\big) \\
&= -2\langle u, (c_- + c_-^\sharp)u\rangle
\end{aligned} \tag{5.3}
$$

where one should keep in mind that $c_1^\sharp = 0$. For the last term in (5.2), we note that by (4.11),

$$\langle \delta_+ u, (\delta_+ d)\delta_+ u\rangle \geq -\langle \delta_+ u, A\delta_+ u\rangle$$

where

$$A = A_\varepsilon = \varepsilon(2 - J_0) + \frac{1}{2\varepsilon}\,(d^2 + \tilde{d}^2)$$

Now since $A \geq 0$,

$$
\begin{aligned}
\langle \delta^+ u, A\delta^+ u\rangle &= \|A^{1/2}(R-1)u\|^2 \\
&\leq 2\|A^{1/2}Ru\|^2 + 2\|A^{1/2}u\|^2 \\
&= \langle u, [2(R^*AR) + 2A]u\rangle
\end{aligned}
$$

We have $R^*J_0R = J_0$, $R^*\tilde{f}R = f$, and $R^*fR = f^\sharp$. Thus we arrive at

$$\langle \delta_+ u, (\delta_+ d)\delta_+ u\rangle \geq -\langle u, Bu\rangle \tag{5.4}$$

with

$$B = 4\varepsilon(2 - J_0) + \varepsilon^{-1}[(d^\sharp)^2 + 2d^2 + \tilde{d}^2]$$

Writing $b = e + \delta_+ f$, as in (1.5), and putting (5.1), (5.2), (5.3), and (5.4) together, we have

$$
\begin{aligned}
\langle u, (2-J)u\rangle \geq& (1-4\varepsilon)\langle u, (2-J_0)u\rangle \\
&- \langle u, (e + |c| + |c^\sharp| + \varepsilon^{-1}((d^\sharp)^2 + 2d^2 + \tilde{d}^2))u\rangle \\
&- \langle u, (\delta_+ f + (\delta_+ d)^\sharp + \delta_+ d)u\rangle
\end{aligned} \tag{5.5}
$$

Estimating the last term in (5.5) again with the help of (4.11) yields

$$
\begin{aligned}
\langle u, (2-J)u\rangle \geq& (1-(\mu+\nu+4\varepsilon))\langle u, (2-J_0)u\rangle \\
&- \langle u, (e + |c| + |c^\sharp|)u\rangle - \frac{1}{2\nu}\langle u, (f^2 + \tilde{f}^2)u\rangle \\
&- \left(\frac{1}{\mu} + \frac{1}{\varepsilon}\right)\langle u, ((d^\sharp)^2 + 2d^2 + \tilde{d}^2)u\rangle
\end{aligned} \tag{5.6}
$$

Choosing $\mu = \nu = \varepsilon = \frac{1}{12}$, we get

$$2 - J \geq \tfrac{1}{2}\,[2 - J_0 - W] \tag{5.7}$$

where

$$W = 2e + 2|c| + 2|c^\sharp| + 12[(f)^2 + (\tilde{f})^2] + 48[(d^\sharp)^2 + 2(d)^2 + (\tilde{d})^2] \quad (5.8)$$

Proof of Theorem 1. (1.6) and (5.8) imply

$$\sum |W_n| < \infty$$

Thus, by Hundertmark-Simon [14], (1.7) holds for the eigenvalues of $J_0 \pm W_0$. By (5.7), (4.1), and the min-max principle, (1.7) holds for J.

Moreover, by (1.3) and (1.6), $\sum(a_n - 1)$ is conditionally convergent and, by (1.6), $\sum(a_n - 1)^2 < \infty$. It follows that $\sum \log(a_n)$ is conditionally convergent. Thus, by Theorem 1 of Simon-Zlatoš [31], $Z(J) < \infty$. □

Proof of Theorem 3. By (5.8) and (1.10), for large n,

$$|W_n| \le \frac{1-\varepsilon}{4n^2}$$

for some $\varepsilon > 0$. It follows, by Theorem A.6, that $J_0 \pm W$ has only finitely many bound states. Hence, by (4.1), (5.7), and the min-max principle, J has finitely many bound states. □

Appendix A. Finiteness of the Eigenvalue Spectrum for Potentials of a Definite Sign

We need information on finiteness results for nonoscillatory potentials. For Schrödinger operators, these results are well known, but we include some discussion here for two reasons: Optimal constants for Jacobi matrices are not known. The weak $L^{\nu/2}$ results we discuss are new. We begin with a version of Hardy's inequality with optimal constant:

Theorem A.1. *Let $H_0 = -\frac{d^2}{dx^2}$ on $L^2(0,\infty)$ with $u(0) = 0$ boundary conditions. Let V be a bounded function on $[0,\infty)$ with $V(x) \to 0$ at infinity. Then*

(i) *If $V(x) \ge (4x^2)^{-1}$ for all (resp. all large) x, then $H_0 + V$ has no (resp. finitely many) bound states. In particular, for any $\varphi \in Q(H_0)$, the form domain of H_0,*

$$\int \frac{|\varphi(x)|^2}{4x^2}\,dx \le \int |\nabla\varphi(x)|^2\,dx \quad \text{(A.1)}$$

This is known as Hardy's inequality.

(ii) *If $V(x) \le -(1+\varepsilon)(4x^2)^{-1}$ for $x > R_0$ and some $R_0, \varepsilon > 0$, then $H_0 + V$ has an infinite number of bound states.*

Remark. We only assume V bounded to avoid technicalities. In fact, one can use (A.1) to discuss V's with $V(x) \geq -\frac{1}{4}x^{-2} - c$.

Proof. Sturm's theory (see [27, pp. 90–94]) says that the number of negative eigenvalues of $H_0 + V$ is precisely the number of zeros of $-u''(x) + V(x)u(x) = 0$, $u(0) = 0$, and that any other solution of $-w'' + Vw = 0$ has a zero between any two zeros of u, and vice-versa. Thus, if some solution, w, of $-w'' + Vw = 0$, is positive, $H_0 + V$ has no eigenvalues, and if it has an infinity of zeros, $H_0 + V$ has an infinity of eigenvalues.

$u(x) = x^{1/2}$ solves $-u'' - \frac{1}{4}x^{-2}u = 0$, showing (i). On the other hand, $u(x) = x^\alpha$ with $\alpha(\alpha - 1) = \frac{1}{4}(1 + \varepsilon)$ solves $-u'' - \frac{1}{4}(1 + \varepsilon)x^{-2}u = 0$. If $\varepsilon > 0$, α has an imaginary part and $\mathrm{Re}(x^\alpha)$ has an infinity of zeros. This plus a comparison theorem implies the results. $\square$

Remark. There are two other ways to prove Hardy's inequality: Let $a = \frac{d}{dx} - \frac{1}{2x}$. Then $a^*a = H_0 - (4x^2)^{-1}$; a careful version of this proof requires consideration of boundary conditions at $x = 0$. Second (Herbst [12]), (A.1) is equivalent to $x^{-1}p^{-2}x^{-1} \leq 4$. Changing variables from x to $e^u = x$, using the explicit form $p^{-2}(x, y) = \max(x, y)$ for the kernel of p^{-2} with Dirichlet boundary conditions at zero, one gets that $x^{-1}p^{-2}x^{-1}$ is unitarily equivalent to convolution with $e^{-\frac{1}{2}|u|}$ on $L^2(\mathbb{R})$. This operator has norm $\int_{-\infty}^{\infty} e^{-\frac{1}{2}|u|}\, du = 4$. This argument also shows that the operator has continuous spectrum, so if its norm is larger than 1, a Birman-Schwinger-type argument provides an alternate proof of an infinity of bound states.

For reasons that will become clear when we discuss the two-dimensional case, we need more on the borderline $-\frac{1}{4}x^{-2}$ case.

Theorem A.2. *Let*

$$X_\gamma(x) = -\frac{1}{4x^2} - \gamma\chi_{(2,\infty)}(x)\,\frac{1}{x^2(\log x)^2} \tag{A.2}$$

Let V be a bounded function on $[0, \infty)$ with $V(x) \to 0$ as $x \to \infty$. Then

(i) *If $V(x) \geq X_{\gamma=1/4}(x)$ for large x, then $H_0 + V$ has finitely many bound states.*

(ii) *If $V(x) \leq X_\gamma(x)$ for some $\gamma > \frac{1}{4}$ and all large x, then $H_0 + V$ has infinitely many bound states.*

Proof. Let $u_\alpha = x^{1/2}(\log x)^\alpha$ in the region $x > 2$. Then

$$-u'' + X_{\gamma=-\alpha(\alpha-1)}u = 0$$

by a direct calculation. The proof is now identical to that of Theorem A.1. $\square$

Theorem A.3. *Let $H_0 = -\Delta$ on $L^2(\mathbb{R}^\nu)$. Let V be a bounded function on $\mathbb{R}^\nu$ with $V(x) \to 0$ as $|x| \to \infty$. Then*

(i) *If $\nu \geq 3$ or on $L^2([0,\infty))$ with Dirichlet boundary conditions at 0 and $V(x) \geq -\frac{(\nu-2)^2}{4}|x|^{-2}$, then $-\Delta + V$ has no negative spectrum. If this holds for all $|x| > R_0$, then $-\Delta + V$ has finite negative spectrum. In any event, one has Hardy's inequality,*
$$\frac{(\nu-2)^2}{4} \int \frac{|\varphi(x)|^2}{|x|^2}\, d^\nu x \leq \int |\nabla\varphi(x)|^2\, d^\nu x \tag{A.3}$$

(ii) *If $\nu = 2$ and $V(x) \geq -\frac{1}{4}(|x|\log|x|)^{-2}$ for all $|x| \geq R_0$, then $-\Delta+V$ has finite negative spectrum.*

(iii) *If $\nu \neq 2$ and $V(x) \leq -(1+\varepsilon)\frac{(\nu-2)^2}{4}|x|^{-2}$ for $|x| \geq R_0$, then $-\Delta+V$ has infinite negative spectrum.*

(iv) *If $\nu = 2$ and $V(x) \leq -(1+\varepsilon)\frac{1}{4}(|x|\log|x|)^{-2}$ for all $|x| > R_0$, then $-\Delta + V$ has an infinite negative spectrum.*

Proof. By the min-max principle, it suffices to consider the case where V is spherically symmetric. In that case, $-\Delta+V$ is unitarily equivalent (see [26, pp. 160–161]) to a discrete sum $\oplus H_{\ell,m}$ on $\oplus L^2([0,\infty), dr)$ where
$$H_{\ell,m} = -\frac{d^2}{dr^2} + \frac{(\nu-1)(\nu-3)}{4}\,\frac{1}{r^2} + \frac{\kappa_\ell}{r^2} + V$$
where $\kappa_{\ell=0} = 0$ and all other κ's have $\kappa_\ell > 0$ and $\kappa_\ell \to \infty$. Since $\frac{(\nu-1)(\nu-3)}{4} - \frac{1}{4} = \frac{(\nu-2)^2}{4}$, this result follows from the previous two theorems. $\square$

The following result seems to be new:

Theorem A.4. *Let $\nu \geq 3$. Let $V(x)$ be a function on $\mathbb{R}^\nu$ so that for any α, $m(\alpha) \equiv |\{x \mid |V(x)| > \alpha\}|$ is finite. Suppose*
$$\lim_{\alpha\downarrow 0} \alpha^{\nu/2} m(\alpha) < \tau_\nu \left(\frac{\nu-2}{2}\right)^\nu \tag{A.4}$$
where τ_ν is the volume of the unit ball in $\mathbb{R}^\nu$. Then $-\Delta+V$ has a finite number of bound states.

Proof. Let $V_0 = \frac{(\nu-2)^2}{4}\frac{1}{|x|^2}$. $m_0(\alpha) \equiv |\{x \mid V_0(x) > \alpha\}| = \tau_\nu(\frac{\nu-2}{2})^\nu \alpha^{-\nu/2}$ so (A.4) implies $V = V_1 + V_2$ where $V_1 \in L^{\nu/2}$ and V_2 has a spherical rearrangement, V_2^* (see [20]) with
$$V_2^* \leq (1-\varepsilon)V_0^*$$
for some $\varepsilon > 0$. Now $-\Delta+V = \varepsilon(-\Delta)+V_1+(1-\varepsilon)[-\Delta+(1-\varepsilon)^{-1}V_2]$. By the Cwikel-Lieb-Rozenblum [10, 19, 28] bound, $\varepsilon(-\Delta)+V_1$ has finite

negative spectrum. By (A.3),

$$\|V_0^{1/2}(-\Delta)^{-1}V_0^{1/2}\| \leq 1$$

The Brascamp-Lieb-Luttinger inequality [2] shows

$$\||V_2|^{1/2}(-\Delta)^{-1}|V_2|^{1/2}\| \leq \|V_2^{*1/2}(-\Delta)V_2^{*1/2}\|$$

It follows that $-\Delta + (1-\varepsilon)^{-1}V_2$ has no negative spectrum. $\square$

We note that the main part of this paper has results that extend some of these results to Schrödinger operators with oscillatory potentials; see Theorem 2.2. We now turn to the discrete Jacobi case, beginning with

Theorem A.5. *Let J_0 be the free Jacobi matrix. Then*

$$\sum_{n=1}^{\infty} \frac{1}{4n^2}\,|u(n)|^2 \leq (u,(2-J_0)u) \tag{A.5}$$

Remarks. 1. We will see below that $\frac{1}{4}$ is the optimal constant in this inequality, that is, it is false if $\frac{1}{4}$ is replaced by a larger constant.

2. However, $\frac{1}{4n^2}$ can be replaced by $\frac{1}{4n^2} + \frac{5}{32n^4}$ or, more generally, $[(1+\frac{1}{n})^{1/2} + (1-\frac{1}{n})^{1/2} - 2]$.

Proof. There is a Sturm theory in the discrete case [25, 33]. One needs to look at zeros of the linear interpolation of u. In particular, if b_n is such that there is a positive solution u_0 of

$$(J_0 + b)u_0 = 2u_0 \tag{A.6}$$

then $(2 - J_0 - b) \geq 0$. Let $u_0(n) = n^{1/2}$ for $n \geq 0$. Define for $n \geq 1$,

$$b_n = \frac{u_0(n+1) + u_0(n-1)}{u_0(n)} - 2 = \left(1+\frac{1}{n}\right)^{1/2} + \left(1-\frac{1}{n}\right)^{1/2} - 2$$

Thus (A.6) is obeyed so $(2-J_0-b) \geq 0$ or $\sum b_n|u(n)|^2 \leq (u,(2-J_0)u)$ for any u. Since $(1-x)^{1/2} = 1 - \sum_{n=1}^{\infty} c_n x^n$ with $c_n \geq 0$ and $c_2 = \frac{1}{8}$, $b_n \geq \frac{1}{4n^2}$. $\square$

Theorem A.6. *Let J be a Jacobi matrix with*

$$\limsup n^2|a_n - 1| = \gamma_a \tag{A.7}$$

$$\limsup n^2|b_n| = \gamma_b \tag{A.8}$$

both finite with

$$2\gamma_a + \gamma_b < \tfrac{1}{4} \tag{A.9}$$

Then J has finitely many bound states outside $[-2,2]$.

Remarks. 1. As we will see, the $\frac{1}{4}$ in (A.9) cannot be improved.

2. In [7], Chihara proves J has finitely many eigenvalues if

$$\limsup\bigl(n^2[(a_n^2-1)\pm\tfrac{1}{2}(b_n+b_{n-1})]\bigr)<\tfrac{1}{16} \tag{A.10}$$

(We take his Jacobi matrix and multiply by 2 to get the $[-2,2]$ rather than $[-1,1]$ normalization; then his c_n and λ_n are related to ours by $a_n=\sqrt{4\lambda_{n+1}}$, $b_n=2c_n$.) This leads to $2\gamma_a+\gamma_b<\frac{1}{16}$, so our result, which is best possible, is better by a factor of 4.

3. Because having no eigenvalues remains true if the a_n's are decreased, (A.7) can be replaced by $n^2(a_n-1)_+$, although it still must be true that $a_n\to 1$ as $n\to\infty$.

Proof. By (4.1), it suffices to prove the spectrum above 2 is finite. Pick ε so that

$$\frac{2\gamma_a+\gamma_b+3\varepsilon}{1-\varepsilon}\leq\frac{1}{4} \tag{A.11}$$

By changing a_n and b_n on a finite set (which, because it is a finite rank perturbation of J, cannot change the finiteness of the number of eigenvalues), we can assume for all n,

$$|a_{n-1}-1|+|a_n-1|\leq\frac{2(\gamma_a+\varepsilon)}{n^2}\qquad a_n-1\geq-\varepsilon\qquad |b_n|\leq\frac{\gamma_b+\varepsilon}{n^2} \tag{A.12}$$

By (5.1), we then have

$$\begin{aligned}(u,(2-J)u)&\geq(1-\varepsilon)(u,(2-J_0)u)-\sum_{n=1}^\infty\frac{2\gamma_a+\gamma_b+3\varepsilon}{n^2}\,u_n^2\\&\geq(1-\varepsilon)\biggl[(u,(2-J_0)u)-\tfrac{1}{4}\sum_{n=1}^\infty\frac{u_n^2}{n^2}\biggr]\geq 0\end{aligned}$$

where we first use (A.11) and then (A.5). ☐

In the other direction, we have

Theorem A.7. *Let J be a Jacobi matrix with*

$$\liminf n^2(a_n-1)=\gamma_a \tag{A.13}$$

$$\liminf n^2 b_n=\gamma_b \tag{A.14}$$

with $\gamma_a\geq 0$, $\gamma_b\geq 0$, and

$$2\gamma_a+\gamma_b>\tfrac{1}{4} \tag{A.15}$$

Suppose also that

$$\lim_{n\to\infty}|a_n-1|+|b_n|=0$$

Then J has an infinity of eigenvalues in $[2,\infty)$.

Remark. The existence of $O(\frac{1}{n^2})$ potentials with an infinity of eigenvalues evoked some interest because Case [4] claimed that if $\sup n^2[|a_n - 1| + |b_n|] < \infty$, there were only finitely many eigenvalues. Chihara [7] produced a counterexample with $b_n \sim \frac{1}{2n^2}$, $a_n - 1 \sim \frac{3}{8n^2}$ (after changing to our normalization), so $2\gamma_a + \gamma_b = \frac{5}{4}$, larger than the needed $\frac{1}{4}$ our theorem allows.

Proof. If there are only finitely many eigenvalues, the solution of $(J - 2)u = 0$ with $u(0) = 0$ has only finitely many zeros so, by restricting to the region beyond the zeros and using Sturm theory, we see there is an N_0 so

$$u(n) = 0,\ n \le N_0 \Rightarrow \langle u, (2-J)u\rangle \ge 0 \tag{A.16}$$

Define $\tilde{a}_n = \min(a_n, 1 + \frac{\gamma_a}{n^2})$, $\tilde{b}_n = \min(b_n, \frac{\gamma_b}{n^2})$ so

$$\lim[n^2(\tilde{a}_n - 1) + n^2(\tilde{a}_{n-1} - 1) + n^2\tilde{b}_n] = 2\gamma_a + \gamma_b > \tfrac{1}{4} \tag{A.17}$$

By (5.1) and (A.16) if $u(n) = 0$ for $n \le N_0$,

$$\begin{aligned}
0 &\le (u, (2-J)u) \\
&= \sum_{n=1}^{\infty} a_n(u(n) - u(n+1))^2 + \sum_{n=1}^{\infty}(-b_n - (a_n - 1) - (a_{n-1} - 1))u(n)^2 \\
&\le \sum_{n=1}^{\infty} a_n(u(n) - u(n+1))^2 + \sum_{n=1}^{\infty}(-\tilde{b}_n - (\tilde{a}_n - 1) - (\tilde{a}_{n-1} - 1)]u(n)^2
\end{aligned} \tag{A.18}$$

since $b_n \ge \tilde{b}_n$, etc.

Given $\varphi \in C_0^\infty(0,\infty)$ and $\ell = 1, 2, \dots$, pick

$$u_n^\ell = \sqrt{\ell}\,\varphi\left(\frac{n}{\ell}\right) \tag{A.19}$$

Since $\operatorname{supp}(\varphi)$ is a compact subset of $(0,\infty)$, $u_n^{(\ell)} = 0$ if $n \le \varepsilon\ell$ for some $\varepsilon > 0$, so (A.18) holds for ℓ large. Since $a_n \to 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n(u_n^{(\ell)} - u_{n+1}^{(\ell)})^2 &= \sum_{n=1}^{\infty} a_n \left(\frac{\varphi(\frac{n}{\ell}) - \varphi(\frac{n+1}{\ell})}{\frac{1}{\ell}}\right)^2 \frac{1}{\ell} \\
&\to \int \varphi'(x)^2\,dx
\end{aligned}$$

Similarly, by (A.17),

$$\sum_{n=1}^{\infty}[-\tilde{b}_n - (\tilde{a}_n - 1) - (\tilde{a}_{n-1} - 1)]u_n^2$$

$$= \sum_{n=1}^{\infty} n^2[-\tilde{b}_n - (\tilde{a}_n - 1) - (\tilde{a}_{n-1} - 1)] \, \frac{\varphi(\frac{n}{\ell})^2}{(\frac{n}{\ell})^2} \, \frac{1}{\ell}$$

$$\to - \int \biggl(\frac{2\gamma_a + \gamma_b}{x^2} \biggr) \varphi(x)^2 \, dx$$

we thus have that

$$-\frac{d^2}{dx^2} - \frac{2\gamma_a + \gamma_b}{x^2} \geq 0$$

violating Theorem A.1(ii). This contradiction proves that J must have infinitely many eigenvalues. □

References

[1] M.L. Baeteman and K. Chadan, *Scattering theory with highly singular oscillating potentials*, Ann. Inst. H. Poincaré Sect. A (N.S.) **24** (1976), 1–16.

[2] H.J. Brascamp, E.H. Lieb, and J.M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. **17** (1974), 227–237.

[3] F. Calogero, *Upper and lower limits for the number of bound states in a given central position*, Comm. Math. Phys. **1** (1965), 80–88.

[4] K.M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **15** (1974), 2166–2174.

[5] K. Chadan, *The number of bound states of singular oscillating potentials*, Lett. Math. Phys. **1** (1975/1977), 281–287.

[6] K. Chadan and A. Martin, *Inequalities on the number of bound states in oscillating potentials*, Comm. Math. Phys. **53** (1977), 221–231.

[7] T.S. Chihara, *Orthogonal polynomials whose distribution functions have finite point spectra*, SIAM J. Math. Anal. **11** (1980), 358–364.

[8] M. Combescure, *Spectral and scattering theory for a class of strongly oscillating potentials*, Comm. Math. Phys. **73** (1980), 43–62.

[9] M. Combescure and J. Ginibre, *Spectral and scattering theory for the Schrödinger operator with strongly oscillating potentials*, Ann. Inst. H. Poincaré Sect. A (N.S.) **24** (1976), 17–30.

[10] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. **106** (1977), 93–100.

[11] D. Damanik, D. Hundertmark, R. Killip, and B. Simon, in preparation.

[12] I.W. Herbst, *Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$*, Comm. Math. Phys. **53** (1977), 285–294.

[13] D. Hundertmark, E.H. Lieb, and L.E. Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator*, Adv. Theor. Math. Phys. **2** (1998), 719–731.

[14] D. Hundertmark and B. Simon, *Lieb-Thirring inequalities for Jacobi matrices*, J. Approx. Theory, to appear.

[15] R.S. Ismagilov, *The spectrum of the Sturm-Liouville equation with oscillating potential*, Math. Notes **37** (1985), 476–482; Russian original in Mat. Zametki **37** (1985), 869–879, 942.

[16] A.R. Its and V.B. Matveev, *Coordinatewise asymptotic behavior for Schrödinger's equation with a rapidly oscillating potential*, in "Mathematical Questions in the Theory of Wave Propagation" Vol. 7, Zap. Naučn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) **51** (1975), 119–122, 218. [Russian]

[17] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math., to appear.

[18] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Comm. Math. Phys. **194** (1998), 1–45.

[19] E.H. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Amer. Math. Soc. **82** (1976), 751–753.

[20] E.H. Lieb and M. Loss, *Analysis*, 2nd edition. Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, R.I., 2001.

[21] E.H. Lieb and W. Thirring, *Bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett. **35** (1975), 687–689. Errata **35** (1975), 1116.

[22] E.H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, in "Studies in Mathematical Physics. Essays in Honor of Valentine Bargmann," pp. 269–303 (E.H. Lieb, B. Simon, and A.S. Wightman, eds), Princeton University Press, Princeton, NJ, 1976.

[23] V.B. Matveev and M.M. Skriganov, *Wave operators for a Schrödinger equation with rapidly oscillating potential*, Dokl. Akad. Nauk SSSR **202** (1972), 755-757. [Russian]

[24] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **18** (1979), no. 213, 185 pp.

[25] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Springer, Berlin, 1992.

[26] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[27] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.

[28] G.V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. AN SSSR **202** (1972), 1012–1015; Izv. VUZov, Matematika **1** (1976), 75–86.

[29] A. Sarkar, *Spectrum of a Schrödinger operator with a class of damped oscillating potentials*, J. Indian Inst. Sci. **60** (1978), 65–71.

[30] M. Schechter, *Wave operators for oscillating potentials*, Lett. Math. Phys. **2** (1977/1978), 127–132.

[31] B. Simon and A. Zlatoš, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*, preprint

[32] M.M. Skriganov, *The spectrum of a Schrödinger operator with rapidly oscillating potential*, in "Boundary Value Problems of Mathematical Physics" Vol. 8, Trudy Mat. Inst. Steklov. **125** (1973), 187–195, 235. [Russian]

[33] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs Vol. 72, American Mathematical Society, Providence, R.I., 2000.

[34] T. Weidl, *On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$*, Comm. Math. Phys. **178** (1996), 135–146.