STABILITY OF SINGULAR SPECTRAL TYPES UNDER DECAYING PERTURBATIONS

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Dedicated to Jean Michel Combes on his 60th birthday

ABSTRACT. We look at invariance of a.e. boundary condition spectral behavior under perturbations, W, of half-line, continuum or discrete Schrödinger operators. We extend the results of del Rio, Simon, Stolz from compactly supported W's to suitable short-range W. We also discuss invariance of the local Hausdorff dimension of spectral measures under such perturbations.

1. INTRODUCTION

We want to discuss aspects of the spectral theory of Schrödinger operators on a half-line, both continuous

$$(H\psi)(x) = -\psi''(x) + V(x)\psi(x) \tag{1.1}$$

on $L^2(0, \infty; dx)$ and discrete

$$(h\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n) \tag{1.2}$$

on $\ell^2(\{1, 2, \dots\})$ with $\psi(0)$ determined by the boundary condition. These operators have a boundary condition determined by a parameter θ in $[0, \pi)$:

$$\psi(0)\cos(\theta) + \psi'(0)\sin(\theta) = 0 \tag{1.3}$$

in the continuum case and

$$\psi(0)\cos\theta + \psi(1)\sin(\theta) = 0 \tag{1.4}$$

in the discrete case. Thus (1.4) is equivalent to defining

$$(h_\theta\psi)(1) = \psi(2) + [V(1) - \tan(\theta)]\psi(1).$$

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In the continuum case, we will suppose that $V(x)$ is locally integrable and sometimes that it is bounded from below for reasons that will become clear. We will need to impose a mild restriction on the growth of the potential in Theorem 1.9 on Lyapunov behavior (both in the continuum and discrete cases).

We will use H_θ and h_θ to indicate the operators with boundary condition. It is well known (see, e.g., Simon [20]) that there are spectral measures $d\rho_\theta(\lambda)$ for H_θ and h_θ (so that H_θ or h_θ is unitarily equivalent to multiplication by λ on $L^2(\mathbb{R}, d\rho_\theta(\lambda))$) normalized so that

$$\int_{\theta=0}^{\pi} d\rho_\theta(\lambda)\frac{d\theta}{\pi} = d\lambda. \tag{1.5}$$

A major theme in this paper (as in many recent papers) is the relation of spectral properties with solutions of the differential/difference equation. Given V and θ, for each $\lambda \in \mathbb{C}$, we will define $\varphi_{1,\theta}(\lambda, x)$ (or $\varphi_{1,\theta}(\lambda, n)$) to be the solution of

$$H\varphi = \lambda\varphi \qquad (\text{or } h\varphi = \lambda\varphi) \tag{1.6}$$

(intended as a differential/difference equation with no L^2 condition at ∞) obeying the boundary condition (1.3)/(1.4) and normalized by

$$\varphi_{1,\theta}(\lambda, 0) = \sin(\theta) \qquad \varphi'_{1,\theta}(\lambda, 0) = \cos(\theta) \tag{1.7}$$

(or $\varphi_{1,\theta}(\lambda, 1) = \cos(\theta)$ in the discrete case). We will also define

$$\varphi_{2,\theta} \equiv \varphi_{1,(\theta-\pi/2)}.$$

While we consider $\theta \in [0, \pi)$ in the basic definition of $\varphi_{1,\theta}$, it makes sense for all θ with $\varphi_{1,\theta+n\pi} = (-1)^n\varphi_{1,\theta}$. In particular, in the last equation $\theta - \pi/2$ lies in $[-\pi/2, \pi/2)$. With this definition, the Wronskian obeys

$$W(\varphi_{1,\theta}, \varphi_{2,\theta}) = 1 \tag{1.8}$$

with $W(f, g) = fg' - f'g$ in the continuum case and $W(f, g)(n) = f(n)g(n+1) - f(n+1)g(n)$ in the discrete case.

Following Jitomirskaya-Last [9], for $L > 0$, we define

$$\|f\|_L^2 = \int_0^L |f(x)|^2\, dx$$

in the continuum case and

$$\|f\|_L^2 = \sum_{n=1}^{[L]} |f(n)|^2 + (L - [L])|f([L]+1)|^2$$

in the discrete case (so $\|f\|_L^2$ is the obvious analog at integer L, with linear interpolation in between).

When one looks at the decomposition of $d\rho_\theta$ into spectral types, for example, into a.c., s.c., and pure point pieces (see Reed-Simon [14]), a basic pair of facts says that the a.c. spectrum is stable and the singular spectrum is unstable — explicitly (see Simon [20] for references), the essential support of $d\rho_\theta^{\text{ac}}$ is θ independent, while for any pair $\theta \neq \theta'$, $d\rho_\theta^{\text{sing}}$ and $d\rho_{\theta'}^{\text{sing}}$ are mutually singular. These facts seem to be at variance with the notion that spectral properties should depend on the behavior of V at infinity since they suggest that $d\rho^{\text{sing}}$ will be unstable under perturbations of compact support. The resolution of this conundrum is the idea of del Rio, Simon, and Stolz [5] that one should look at the union over θ of spectral supports. Explicitly, we proceed as follows:

Definition (Gilbert-Pearson [7]**).** We say there is a subordinate solution at energy $\lambda \in \mathbb{R}$ if and only if there is some $\theta \in [0, \pi)$ so $\lim_{L\to\infty} \|\varphi_{1,\theta}\|_L / \|\varphi_{2,\theta}\|_L = 0$. θ is necessarily unique and we call it $\theta(\lambda)$.

Definition.

$$P = \{\lambda \mid \varphi_{1,\theta(\lambda)} \in L^2\}$$

$$S = \{\lambda \mid \text{there is a subordinate solution but } \varphi_{1,\theta(\lambda)} \notin L^2\}$$

$$L = \{\lambda \mid \text{ there is no subordinate solution}\}$$

When we need to discuss the V-dependence of these sets, we will write $P(V)$, etc.

Then:

Theorem 1.1. (i) $P = \cup_\theta \, \sigma_{\text{pp}}(H_\theta)$.

(ii) $L =$ *essential support of* $\sigma_{\text{ac}}(H_\theta)$ *for all* θ.

(iii) *For any* θ, $d\rho_\theta^{\text{sc}} = d\rho_\theta(S \cap \cdot)$ *and if* $\tilde{S}$ *is any other set with that property, then* $|S \triangle \tilde{S}| = 0$ *where* $|\cdot|$ *is Lebesgue measure.*

Remarks. 1. This is close to a theorem in [5], although S and L are defined differently there.

2. σ_{pp} in (i) means the set of eigenvalues, not their closure.

3. (i) is obvious since $\lambda \in \sigma_{\text{pp}}(H_{\theta(\lambda)})$ if and only if $\varphi_{1,\theta(\lambda)}(\lambda, \cdot) \in L^2$.

4. (ii) is the main result of Gilbert-Pearson [7].

5. That $d\rho_\theta^{\text{sc}}(P) = 0$ is obvious since $d\rho_\theta^{\text{sing}}$ is mutually singular to each $d\rho_{\theta'}^{\text{pp}}$ for $\theta' \neq \theta'$ and $d\rho_\theta^{\text{sc}}$ is obviously mutually singular to $d\rho_\theta^{\text{pp}}$.

6. That $d\rho_\theta^{\text{sc}}(L) = 0$ is a result of Gilbert-Pearson showing that $d\rho_\theta^{\text{sc}} = d\rho_\theta(S \cap \cdot)$.

7. The $|S \triangle \tilde{S}| = 0$ result follows from (1.5).

Since P, L, S are defined purely in terms of the behavior of solutions at infinity, the following result of del Rio et al. [5] is immediate:

Theorem 1.2. *Let $V = V_0 + W$ where W has compact support. Then $P(V) = P(V_0)$, $L(V) = L(V_0)$, $S(V) = S(V_0)$.*

A major theme of this paper will be to examine when this result still holds for W's not of compact support. Before discussing our theorems, we will further refine the set S in connection with the breakdown of singular spectrum according to Hausdorff measures and dimensions.

As usual for $\alpha \in (0,1)$, α-dimensional Hausdorff measure is defined on Borel sets, T, by

$$h^\alpha(T) \equiv \lim_{\delta\to 0} \inf_{\delta-\text{covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha,$$

where a δ-cover is a countable collection of intervals each of length at most δ so $T \subset \cup_{\nu=1}^\infty b_\nu$. h^1 is Lebesgue measure and h^0 is counting measure.

Given $\alpha \in [0,1]$ (following Rogers and Taylor [16, 17]; see also Last [12]), we define a measure μ to be α-continuous (αc) if $\mu(S) = 0$ for any set S with $h^\alpha(S) = 0$ and α-singular (αs) if it is supported on a set of S with $h^\alpha(S) = 0$. For every such α and any measure μ, one can uniquely decompose $\mu = \mu^{\alpha\mathrm{c}} + \mu^{\alpha\mathrm{s}}$ with $\mu^{\alpha\mathrm{c}}$ α-continuous and $\mu^{\alpha\mathrm{s}}$, α-singular.

We call a measure zero-dimensional if it is supported on a set S with $h^\alpha(S) = 0$ for all $\alpha > 0$. We call it one-dimensional if it is α-continuous for all $\alpha < 1$.

It will be useful, following Jitomirskaya-Last, to have a pair of inverse functions $A, B : [0,1]$ to $[0,1]$ by

$$\begin{aligned} B(\alpha) &= \alpha/(2-\alpha) \\ A(\beta) &= 2\beta/(1+\beta). \end{aligned}$$

Definition. Let $\lambda \in S$, the set of energies for which there is a non-L^2 subordinate solution. Define

$$\beta(\lambda) = \liminf_{L\to\infty} \big[\ln \|\varphi_{1,\theta(\lambda)}\|_L \big/ \ln \|\varphi_{2,\theta(\lambda)}\|_L \big].$$

Notice that since $\varphi_{1,\theta} \notin L^2$, $\|\varphi_{1,\theta(\lambda)}\|_L \to \infty$ as $L \to \infty$ and since φ_1 is subordinate, eventually $\|\varphi_{2,\theta}\|_L \geq \|\varphi_{1,\theta}\|_L$, and thus

$$\lim_{L\to\infty} \|\varphi_{2,\theta}\|_L = \infty \tag{1.9}$$

and

$$0 \leq \beta(\lambda) \leq 1.$$

When we want to indicate the V-dependence of β, we will write $\beta(\lambda; V)$. We note the following elementary:

Proposition 1.3. *If $\beta > \beta(\lambda)$, then*

$$\underline{\lim}\, \|\varphi_1\|_L / \|\varphi_2\|_L^\beta = 0 \tag{1.10}$$

and if $\beta < \beta(\lambda)$,

$$\lim \|\varphi_1\|_L / \|\varphi_2\|_L^\beta = \infty. \tag{1.11}$$

Proof. Write

$$\|\varphi_1\|_L / \|\varphi_2\|_L^\beta = \exp\left[\ln \|\varphi_2\|_L \left\{ \frac{\ln \|\varphi_1\|_L}{\ln \|\varphi_2\|_L} - \beta \right\}\right].$$

By (1.9), $\ln \|\varphi_2\|_L \to \infty$. If $\beta > \beta(\lambda)$, then there is a subsequence where the expression in $\{\ \}$ goes to $\beta(\lambda) - \beta < 0$, so a subsequence where the expression in $[\ ]$ goes to $-\infty$ and (1.10) holds. If $\beta < \beta(\lambda)$, then eventually the expression in $\{\ \}$ is larger than $\frac{1}{2}(\beta(\lambda) - \beta)$, and so (1.11) holds. □

For each β_0, decompose S into four sets:

$$\begin{aligned}
S_{\beta_0}^{++} &= \{\lambda \mid \beta_0 > \beta(\lambda)\} \\
S_{\beta_0}^{--} &= \{\lambda \mid \beta_0 < \beta(\lambda)\} \\
S_{\beta_0}^{+} &= \{\lambda \mid \beta_0 = \beta(\lambda) \text{ and (1.10) holds for } \beta_0 = \beta(\lambda)\} \\
S_{\beta_0}^{-} &= \{\lambda \mid \beta_0 = \beta(\lambda) \text{ and } \underline{\lim}\, \|\varphi_1\|_L / \|\varphi_2\|^{\beta_0} > 0\}.
\end{aligned}$$

Thus (1.10) holds for $\beta = \beta_0$ if and only if $\lambda \in S_{\beta_0}^{++} \cup S_{\beta_0}^{+}$.

It follows from Theorem 1.1 and the discussion following equation (2.2) of Jitomirskaya-Last [9] that

Theorem 1.4. *Let $\beta_0 = B(\alpha_0)$.*

(i) $d\rho_\theta^{\alpha_0 c} = d\rho_\theta((S_{\beta_0}^{-} \cup S_{\beta_0}^{--} \cup L) \cap \cdot)$

(ii) $d\rho_\theta^{\alpha_0 s} = d\rho_\theta((S_{\beta_0}^{+} \cup S_{\beta_0}^{++} \cup P) \cap \cdot)$

(iii) *$d\rho_\theta^{\mathrm{sc}}$ is one-dimensional for a.e. θ if and only if $\beta = 1$ a.e. on S.*

(iv) *$d\rho_\theta^{\mathrm{sc}}$ is zero-dimensional for a.e. θ if and only if $\beta = 0$ a.e. on S.*

Remark. More generally, $d\rho_\theta^{\mathrm{sc}}$ has exact dimension α_0 for a.e. θ if $\beta = B(\alpha_0)$ for a.e. $\lambda \in S$.

Clearly, β only depends on V near infinity, so we extend the result of del Rio et al. [5] to handle dimensional decomposition of $d\rho$ via

Theorem 1.5. *Let $V = V_0 + W$ where W has compact support. Then $\beta(\lambda; V) = \beta(\lambda; V_0)$.*

The purpose of this paper is to study when invariance results of the genre of Theorems 1.2 and 1.5 extend to cases where W does not have compact support but has "suitable" decay; that is, we want to determine what suitable decay is. For the a.c. spectrum, the standard rate of decay is $W \in L^1$:

Theorem 1.6. *In the continuum case, suppose V_0 and $V \equiv V_0 + W$ are such that $H_0 + V_0$ and $H_0 + V$ are bounded below by $\varepsilon H_0 - c$. In the discrete case, no hypothesis is needed on V_0. Suppose that $W \in L^1$ (or ℓ^1). Then*

$$|L(V) \triangle L(V_0)| = 0. \tag{1.12}$$

Proof. In the discrete case, W is trace class, and in the continuum case, $(H_0+1)^{-1/2}W(H_0+1)^{-1/2}$ is trace class. So $(H_0+V+c-1)^{-1} - (H_0+V_0+c+1)^{-1}$ is trace class. The trace class theory of scattering [15] implies that H_0 on $\mathcal{H}_{\rm ac}(H_0)$ is unitarily equivalent to H_0+V_0 on $\mathcal{H}_{\rm ac}(H_0+V_0)$ from which (1.12) follows by Theorem 1.1. $\square$

Remark. We conjecture that (1.12) holds if W is merely assumed in L^2. In [11], we made this conjecture when $V_0 = 0$ and it was proven by Deift-Killip [4]. Killip [10] proved the result when V_0 is periodic. We conjecture the result for all V_0.

We now turn to the substantially new results in this paper. As spectrum moves from the most smooth (a.c.) to the least smooth (point), we need to successively strengthen the conditions on the perturbation W.

We begin with several results we prove in Section 3 concerning point spectrum that all hold in the discrete and continuum case.

Theorem 1.7. *For each $\lambda \in P(V_0)$, define*

$$f_+(\lambda, x) = (1+|x|) \sup_{|y|\leq x} |\varphi_{2,\theta(\lambda)}(y)|. \tag{1.13}$$

Suppose that for all $\lambda \in Q \subseteq P(V_0)$, we have that

$$\int |W(x)| f_+(\lambda, x)\, dx < \infty$$

and that the L^2 solution is bounded. Then $Q \subseteq P(V_0 + W)$.

Remarks. 1. In (1.13), one can replace $(1+|x|)$ by $(1+|x|)^\gamma$ for any $\gamma > \frac{1}{2}$.

2. By a Sobolev estimate if $\varphi, \varphi' \in L^2$, then $\varphi \in L^\infty$, so, for example, if V_0 is bounded from below, L^2 solutions will be bounded.

When V_0 is bounded, f_+ does not grow faster than exponentially for any λ.

Corollary 1.8. *Let V_0 be bounded from below and suppose that*

$$\int |W(x)|\, e^{A|x|}\, dx < \infty$$

for all $A > 0$. Then

$$P(V_0) = P(V_0 + W).$$

Finally, we have a result on preservation of Lyapunov behavior. Recall that we say there is Lyapunov behavior at energy λ if the transfer matrix

$$T_\lambda(0, x) = \begin{pmatrix} \varphi'_{1,\theta}(x) & \varphi'_{2,\theta}(x) \\ \varphi_{1,\theta}(x) & \varphi_{2,\theta}(x) \end{pmatrix}$$

obeys

$$\lim_{x\to\infty} \frac{1}{|x|} \ln \|T_\lambda(0,x)\| \equiv \gamma(\lambda). \tag{1.14}$$

Assume in addition that

$$\lim_{x\to\infty} \frac{1}{|x|} \ln(1 + |V(x)|) = 0. \tag{1.15}$$

Theorem 1.9. *Suppose V_0 satisfies* (1.15)*, has Lyapunov behavior at energy λ and that for some $\varepsilon > 0$,*

$$\int |W(x)|\, e^{\varepsilon|x|}\, dx < \infty.$$

Then $V_0 + W$ has Lyapunov behavior at λ with the same value of γ.

Remarks. 1. If $\gamma > 0$, we have much more than merely the same Lyapunov behavior.

2. Theorem 1.9 isn't new. It is essentially a special case of Theorem 4.I of [18].

3. There is an obvious discrete analog of this theorem.

In Section 4, we will discuss stability of singular spectrum and its components. Our results will hold only for energies with an extra condition.

Definition. An energy λ is called regular if and only if for some θ ($= \theta(\lambda)$ if there is a subordinate solution) we have for all $\varepsilon > 0$,

$$\|\varphi_{1,\theta}\|_L \leq C_\varepsilon L^{1/2+\varepsilon}. \tag{1.16}$$

By the general theory of eigenfunction expansions [1, 19], a.e. λ is regular both with respect to each $d\rho_\theta$, and so by (1.5) for a.e. λ with respect to Lebesgue measure $d\lambda$. Indeed, we could replace $L^{1/2+\varepsilon}$ by $L^{1/2}(\ln L)^\kappa$ for any $\kappa > \frac{1}{2}$.

Remark. If $V(x) = -\frac{3}{16}\,x^{-2}$ for large x, then the subordinate solution at $\lambda = 0$ is $\sim x^{1/4}$ at infinity. So $\|\varphi_1\|_L \sim L^{3/4}$ and $\lambda = 0$ is not a regular energy, so not all energies need to be regular.

In the discrete case, constancy of the Wronskian implies

$$\|\varphi_{1,\theta}\|_L \, \|\varphi_{2,\theta}\|_L \geq \tfrac{1}{2}(L-1), \tag{1.17}$$

but in the continuum case, this is not automatic since the Wronskian involves φ'. But, by a Sobolev estimate, if V is bounded from below (uniform locally L^1 will do!), then

$$\|\varphi_{1,\theta}\|_L \, \|\varphi_{2,\theta}\|_L \geq c(L-1) \tag{1.18}$$

for some c, dependent on V and λ (see e.g. [19]), and so we will need to suppose that V is bounded from below in the continuum case.

Remark. The case $V(x) = -x$ where $\|\varphi_{1,\theta}\|_L \sim \|\varphi_{2,\theta}\|_L \sim L^{1/4}$ shows (1.18) really can fail if V is unbounded from below.

Here are the theorems we will prove in Section 4.

Theorem 1.10. *In the continuum case, suppose V_0 is bounded from below. Let $\lambda \in S(V_0)$ be a regular energy with $\beta(\lambda, V_0) = 1$. Suppose that*

$$|W(x)| \leq C(1+|x|)^{-1-\varepsilon} \tag{1.19}$$

for some $\varepsilon > 0$. Then $\lambda \in S(V_0 + W)$ with $\beta(\lambda, V_0 + W) = 1$. In particular, if, for V_0, H_θ has one-dimensional spectrum for a.e. θ, the same is true for $V_0 + W$.

Theorem 1.11. *In the continuum case, suppose V_0 is bounded from below. Let $\lambda \in S(V_0)$ be a regular energy. Suppose that for all $\eta > 0$,*

$$|W(x)| \leq C_\eta (1+|x|)^{-\eta}. \tag{1.20}$$

Suppose that $\beta(\lambda, V_0) \neq 0$. Then $\lambda \in S(V_0 + W)$ and $\beta(\lambda, V_0 + W) = \beta(\lambda, V_0)$. Suppose $\beta(\lambda, V_0) = 0$. Then either $\lambda \in S(V_0 + W)$ with $\beta(\lambda, V_0 + W) = 0$ or $\lambda \in P(V_0 + W)$.

Remarks. 1. The latter shows that having zero-dimensional spectrum is preserved under perturbations obeying (1.20), although to preserve point spectrum, we need a stronger exponential bound.

2. In fact, our proof shows that for a given $\beta(\lambda, V_0) = \beta_0$, we only need (1.20) for some

$$\eta > \frac{1}{\beta_0}\,.$$

In terms of the case of Hausdorff dimension α, one needs

$$\eta > \frac{2}{\alpha} - 1. \tag{1.21}$$

We will prove our new results, Theorem 1.7, Corollary 1.8, and Theorems 1.9–1.11, by proving stability of the asymptotics of solutions of the Schrödinger differential/difference equation. We use φ_- for $\varphi_{1,\theta(\lambda)}$, the subordinate solution with potential V_0, and φ_+ for $\varphi_{2,\theta(\lambda)}$. The basic construction we will use is variation of parameters. That is, we will write (in the continuum case):

$$\psi(x) = u_1(x)\varphi_-(x) + u_2(x)\varphi_+(x) \tag{1.22}$$

$$\psi'(x) = u_1(x)\varphi_-'(x) + u_2(x)\varphi_+'(x). \tag{1.23}$$

With $u(x) = \binom{u_1(x)}{u_2(x)}$, the differential equation for ψ is equivalent, given the normalization (1.7), to

$$u'(x) = A(x)u(x) \tag{1.24}$$

with

$$A(x) = -W(x)\begin{pmatrix} \varphi_+(x)\varphi_-(x) & \varphi_+(x)^2 \\ -\varphi_-(x)^2 & -\varphi_+(x)\varphi_-(x) \end{pmatrix}. \tag{1.25}$$

(1.25) is sometimes written (e.g., in [9]) in the integral form:

$$\begin{aligned}\psi(x) &= u_1(x_0)\varphi_-(x) + u_2(x_0)\varphi_+(x) \\ &\quad + \int_{x_0}^{x} W(y)[\varphi_+(x)\varphi_-(y) - \varphi_-(x)\varphi_+(y)]\psi(y)\,dy.\end{aligned} \tag{1.26}$$

In the discrete case, the result is similar. One writes

$$\psi(n) = u_1(n)\varphi_-(n) + u_2(n)\varphi_+(n) \tag{1.27}$$

$$\psi(n-1) = u_1(n)\varphi_-(n-1) + u_2(n)\varphi_+(n-1). \tag{1.28}$$

(1.24) becomes

$$u(n+1) - u(n) = A(n)u(n), \tag{1.29}$$

where

$$A(n) = -W(n)\begin{pmatrix} \varphi_+(n)\varphi_-(n) & \varphi_+(n)^2 \\ -\varphi_-(n)^2 & -\varphi_+(n)\varphi_-(n) \end{pmatrix} \tag{1.30}$$

or its integral form

$$\begin{aligned}\psi(n) &= u_1(n_0)\varphi_-(n) + u_2(n_0)\varphi_+(n) \\ &\quad + \sum_{j=n_0}^{n} W(j)[\varphi_+(n)\varphi_-(j) - \varphi_-(n)\varphi_+(j)]\psi(j).\end{aligned} \tag{1.31}$$

The standard control for perturbing solutions at infinity is to require $\int_{x_0}^\infty \|A(x)\|\,dx < \infty$. For the diagonal matrix elements of A, that cannot be improved without detailed oscillation estimates, but it is well known that one can try to trade off the growth of one off-diagonal matrix element by the decay of the other. In Section 2, we present a version of this fact made for our applications. These ideas are not new; for example, our method of proof is patterned after problem XI.97 of Reed-Simon [15]. In Section 3, we present the results of stability of a solution L^2 at ∞ and in Section 4, the results on stability of polynomially bounded solutions. The Appendix discusses some results concerning the preservation of WKB asymptotic behavior of solutions.

2. A Perturbation Lemma

In this preliminary section, we will be interested in solutions of

$$u'(x) = A(x)u(x), \tag{2.1}$$

where

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \tag{2.2}$$

is in $L^1_{\mathrm{loc}}[0,\infty)$ and

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \tag{2.3}$$

is a two-component vector. By a solution of (2.1), we mean an absolutely continuous function so that (2.1) holds for a.e. x. As usual, given any x_0 and $\omega \in \mathbb{C}^2$, there is a unique solution of (2.1) with $u(x_0) = \omega$.

We will use a pair of non-negative functions $f_\pm(x)$ with

$$f_+(x)f_-(x) \geq 1 \tag{2.4}$$

and f_+ monotone increasing and f_- monotone decreasing (in some applications, we will take $f_\pm = e^{2(\pm\gamma+\varepsilon)|x|}$ so you can have this example in mind). Define

$$G(x) = \max(|a_{11}(x)| + |a_{12}(x)|f_-(x), |a_{21}(x)|f_+(x) + |a_{22}(x)|). \tag{2.5}$$

Lemma 2.1. *Define* $\|\cdot\|_x^\pm$ *as norms on* $\mathbb{C}^2$ *by*

$$\begin{aligned} \|\omega\|_x^+ &= \max(|\omega_1|, |\omega_2|f_+(x)) \\ \|\omega\|_x^- &= \max(f_-(x)|\omega_1|, |\omega_2|). \end{aligned}$$

Then

$$\|A(x)\omega\|_x^\pm \leq G(x)\|\omega\|_x^\pm. \tag{2.6}$$

Proof. We will prove the $\|\cdot\|^+$ result. The $\|\cdot\|^-$ is similar. Note that

$$\begin{aligned} |(A(x)\omega)_1| &\leq |a_{11}(x)|\,|\omega_1| + |a_{12}|f_+(x)^{-1}f_+(x)|\omega_2| \\ &\leq [|a_{11}(x)| + |a_{12}|f_+^{-1}(x)]\,\|\omega\|_x^+ \leq G(x)\|\omega\|_x^+ \end{aligned}$$

since $f_+^{-1} \leq f_-$ by (2.4) and

$$f_+(x)|(A(x)\omega)_2| \leq [|a_{21}(x)f_+(x)|\omega_1| + |a_{22}(x)|f_+(x)|\omega_2|]$$

$$\begin{aligned} &\leq (|a_{21}(x)| f_+(x) + |a_{22}(x)|) \|\omega\|_x^+ \\ &\leq G(x) \|\omega\|_x^+. \end{aligned}$$

□

Theorem 2.2. *Suppose f_+ is monotone increasing, f_- is monotone decreasing,* (2.4) *holds, and*

$$\int_x^\infty G(y)\, dy < \infty.$$

Then there exist solutions $u^\pm$ of (2.1) *so that as $x \to \infty$*

(i) $u_1^-(x) \to 1, \quad f_+(x) u_2^-(x) \to 0$
(ii) $u_1^+(x) f_-(x) \to 0, \quad u_2^+(x) \to 1.$

Proof. Define $u^{-(n)}$ by

$$\begin{aligned} u^{-(0)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u^{-(n+1)}(x) &= -\int_x^\infty A(y) u^{-(n)}(y)\, dy, \end{aligned}$$

where we will deal with the convergence of the integral below. Since f_+ is increasing, if $y > x$, then $\|\omega\|_x^+ \leq \|\omega\|_y^+$. Thus

$$\begin{aligned} \|u^{-(n+1)}(x)\|_x^+ &\leq \int_x^\infty \|A(y) u^{-(n)}(y)\|_x^+\, dy \\ &\leq \int_x^\infty \|A(y) u^{-(n)}(y)\|_y^+\, dy \\ &\leq \int_x^\infty G(y)\ \|u^{-(n)}(y)\|_y^+\, dy \end{aligned}$$

by (2.6). Thus

$$\sup_{y \geq x} \|u^{-(n+1)}(y)\|_y^+ \leq \sup_{y \geq x} \|u^{-(n)}(y)\|_y^+ \int_x^\infty G(y)\, dy$$

proving convergence of the integral and

$$\sup_{y \geq x} \|u^{-(n)}(y)\|_y^+ \leq \left[\int_x^\infty G(y)\, dy \right]^n$$

inductively.

It follows that

$$u^-(y) \equiv \sum_{n=0}^\infty u^{-(n)}(y)$$

converges for $y \geq x_0$ where $\int_{x_0}^{\infty} G(y)\,dy < 1$ and that for such y, $\int_y^{\infty} A(w)u^-(w)\,dw$ converges and

$$u^-(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_y^{\infty} A(w)u^-(w)\,dw$$

so u^- solves (2.1). Since $\|u^-(y) - \binom{1}{0}\|_y^+ \to 0$ as $y \to \infty$, we obtain (i).

Define $\tilde{u}^{+(n)}$ by

$$\tilde{u}^{+(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\tilde{u}^{+(n+1)}(x) = \int_{x_0}^{x} A(y)\tilde{u}^{+(n)}(y)\,dy$$

for x_0 chosen so that

$$\int_{x_0}^{\infty} G(y)\,dy \leq \tfrac{1}{3}. \tag{2.7}$$

As above, using the fact that if $y < x$, then $\|\omega\|_x^- \leq \|\omega\|_y^-$ since f_- is decreasing, we have

$$\sup_{x \geq x_0} \|\tilde{u}^{+(n)}(x)\|_x^- \leq \left(\int_x^{\infty} G(y)\,dy \right)^n \leq \left(\frac{1}{3} \right)^n.$$

As in the $\|\cdot\|^+$ case, we see that $\sum_{n=0}^{\infty} \tilde{u}^{+(n)} = \tilde{u}^+$ converges for $y > x_0$ and $\tilde{u}^+$ solves (2.1) and obeys

$$\tilde{u}^+(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{x_0}^{x} A(y)\tilde{u}^+(y)\,dy.$$

In particular,

$$\tilde{u}_2^+(\infty) = 1 + \int_{x_0}^{\infty} A(y)\tilde{u}^+(y)\,dy$$

exists and $|\tilde{u}_2^+(\infty) - 1| \leq \frac{1}{2}$ so $\tilde{u}_2^+(\infty) \equiv \alpha > 0$. Define

$$u^+ = \alpha^{-1}\tilde{u}^+$$

and so obtain a vector-valued function u^+ with $u_2^+ \to 1$ and $|u_1^+ f_-|$ bounded. We will show that if $f_- \to 0$, then $u_1^+ f_- \to 0$. When f_- does not go to zero, we will provide an alternative construction of u^+.

To prove that $u_1^+ f_- \to 0$ if $f_- \to 0$, write for $x_0 < y < x$:

$$\begin{aligned} u_1^+(x)f_-(x) = f_-(x) &\int_{x_0}^{y} (A(w)u^+(w))_1\,dw \\ &+ \int_y^{x} f_-(x)f_-(w)^{-1}f_-(w)(A(w)u^+(w))_1\,dw \end{aligned}$$

so, since f_- is monotone decreasing,

$$|u_1^+(x)f_-(x)| \leq f_-(x) \int_{x_0}^y |(A(w)u^+(w))_1|\, dw + \int_y^\infty G(w)\|u^+(w)\|_w^-\, dw. \tag{2.8}$$

Given ε, pick y so the second integral in (2.8) is less than $\varepsilon/2$ and then, since $f_- \to 0$, x so that the first term is less than $\varepsilon/2$. Thus $u_1^+ f_- \to 0$.

If $f_-(x)$ has a non-zero limit as $x \to \infty$, then since f_- is monotone, $f_-(x) \geq c$. Thus, $A(x) \in L^1$, and by the same construction as used for u_- (i.e., integrating from infinity), one can construct $u^+(x) \to \binom{1}{0}$ as $x \to \infty$. □

The situation in the discrete case is similar. (2.1) becomes

$$u(n+1) - u(n) = A(n)u(n). \tag{2.9}$$

$f_\pm$ obey (2.4), although they are only defined (and monotone) on $n = 1, 2, \dots$. $G(n)$ is defined as in (2.5). The analog of Theorem 2.2 holds with $\int_x^\infty G(y)\, dy < \infty$ replaced by

$$\sum_{n_0}^\infty G(n) < \infty$$

and x going through discrete values. The proof is identical with obvious changes — for example, the formula for $u^{-(n+1)}$ becomes

$$u^{-(n+1)}(j) = -\sum_{k=j}^\infty A(k)u^{-(n)}(k).$$

We owe to F. Gesztesy an illuminating remark about our result, Theorem 2.2, namely the special case $f_+ f_- = 1$ (which is true in some of the applications we will make) follows quickly from Levinson's theorem [6, 13]. One variant of Levinson's theorem says:

Proposition 2.3. *Let A be a 2×2 matrix of the form $A_1 + A_2$ where $\int_{t_0}^\infty \|A_1(s)\|\, ds < \infty$, A_2 is diagonal with*

$$A_2(s) = \begin{pmatrix} \alpha_1(s) & 0 \\ 0 & \alpha_2(s) \end{pmatrix}$$

so that

$$\int_{t_0}^t \mathrm{Re}[\alpha_1(s) - \alpha_2(s)] \tag{2.10}$$

is either bounded below or bounded above. Then, there exist solutions $\varphi_{1,2}$ of

$$\dot{\varphi} = A\varphi$$

so that

$$\varphi_1(t)\, e^{-\int_{t_0}^t \alpha_1(s)\,ds} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\varphi_2\, e^{-\int_{t_0}^t \alpha_2(s)\,ds} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $t \to \infty$.

Remark. This is essentially equivalent to the general 2×2 case.

To apply this to the situation of Theorem 2.2, given a solution, u, of (2.2), let φ be defined by $\varphi_1 = u_1$, $\varphi_2 = u_2 f_+$. Then

$$\varphi' = (A_1 + A_2)\varphi,$$

where

$$A_1 = \begin{pmatrix} a_{11}(x) & a_{12}(x) f_+^{-1}(x) \\ a_{21}(x) f_+(x) & a_{22}(x) \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & f_+'/f_+(x) \end{pmatrix}.$$

By hypothesis ($\int_{x_0}^\infty G(x) < \infty$), $A_1 \in L^1$ and the function (2.10) is $\log(f_+(t_0)/f_+(t))$ which is bounded by 0 (since f_+ is monotone). The two Levinson's theorem solutions obey (i) and (ii) of Theorem 2.2.

3. Stability of Point Spectra

In this section, we will prove Theorems 1.7 and 1.9. We will only consider the continuum case; the discrete case is similar.

Proof of Theorem 1.7. Since $\lambda \in P(V_0)$, $\varphi_- \equiv \varphi_{1,\theta(\lambda)}$ is in L^2 and so by hypothesis, also in L^∞ [19]. Pick $\varphi_+ \equiv \varphi_{2,\theta(\lambda)}$ and use variation of parameters (1.22)/(1.23). A has the form (1.25). Let f_+ be given by (1.13). Since φ_- is bounded,

$$|W(x)\varphi_+(x)\varphi_-(x)| \le C f_+(x)|W(x)|$$
$$|W(x)|\,|\varphi_+(x)|^2 f_+(x)^{-1} \le f_+(x)|W(x)|$$
$$|W(x)|\,|\varphi_-(x)|^2 f_+(x) \le C f_+(x)|W(x)|.$$

So if $f_-(x) \equiv f_+(x)^{-1}$, we have that G given by (2.5) obeys

$$|G(x)| \le C f_+(x) W(x).$$

Thus, Theorem 2.2 is applicable, so there is a solution, ψ, of the perturbed Schrödinger equation of the form:

$$\psi(x) = u_1^-(x)\varphi_-(x) + u_2^-(x)\varphi_+(x) \tag{3.1}$$

with u_1^- bounded and with $u_2^-(x)f_+(x)$ bounded. Since $\varphi_- \in L^2$, $u_1^-\varphi_- \in L^2$. Moreover, since $u_2^- f_+$ is bounded, (1.13) says that

$$|u_2^-(x)\varphi_+(x)| \leq C(1+|x|)^{-1}$$

which is also in L^2. Thus $\psi \in L^2$. □

Corollary 1.8 follows immediately since f_+ is exponentially bounded. Results of this genre are well known; see, for example, Hartman [8]. We proceed to prove Theorem 1.9:

Proof of Theorem 1.9. If $\gamma = 0$, any solution, φ, of the unperturbed equation has

$$|\varphi(x)| + |\varphi'(x)| \leq C_\varepsilon \, e^{\varepsilon|x|} \tag{3.2}$$

so, by hypothesis, the A of (1.25) is in L^1 for any choice of $\varphi_\pm$. Thus, by standard theory (or Theorem 2.2 with $f_+ = f_- = 1$), any solution ψ also obeys (3.2) which implies that $\gamma = 0$.

Now suppose that $\gamma > 0$. By the Ruelle-Osceledec Theorem [18], if V satisfies (1.15), there is a solution $\varphi_-(x)$ ($\equiv \varphi_{1,\theta(\lambda)}$) for the V_0 equation with

$$\lim_{x\to\infty} \left[\frac{1}{2|x|} \ln[|\varphi_-(x)|^2 + |\varphi_-'(x)|^2]\right] = -\gamma.$$

Any linearly independent solution and, in particular, $\varphi_+ = \varphi_{2,\theta(\lambda)}$ obeys

$$\lim_{x\to\infty} \left[\frac{1}{2|x|} \ln[|\varphi_+(x)|^2 + |\varphi_+'(x)|^2]\right] = \gamma.$$

In particular, for any $\varepsilon_1 > 0$,

$$|\varphi_+(x)| \leq C_{\varepsilon_1} e^{(\gamma+\varepsilon_1)|x|}, \qquad |\varphi_-(x)| \leq C_{\varepsilon_1} \, e^{-(\gamma-\varepsilon_1)|x|}. \tag{3.3}$$

Pick $f_\pm(x) = e^{(\pm 2\gamma + 2\varepsilon_1)|x|}$ where ε_1 is chosen so that $\varepsilon_1 < \gamma$ (so f_- is decreasing) and $\varepsilon_1 < \frac{1}{4}\varepsilon$ where ε is given in the hypothesis of the theorem. By the estimates of (3.3),

$$|G(x)| \leq e^{4\varepsilon_1|x|}|W(x)|$$

so $G \in L^1$ since $4\varepsilon_1 < \varepsilon$. Theorem 2.2 applies and we get solutions $\psi_\pm$ of the perturbed equation with

$$|\psi_+ - u_2^+\varphi_+| \leq |u_1^+||\varphi_-| \leq C_{\varepsilon_1}|u_1^+|f_- \, e^{(\gamma-\varepsilon_1)|x|}$$

and a similar estimate for ψ_+'. It follows that

$$\frac{\|\psi_+\|_x}{\|\varphi_+\|_x} - 1 \to 0$$

as $x \to \infty$ where $\|g\|_x = (|g(x)|^2 + |g'(x)|^2)^{1/2}$. Similarly $\|\psi_-\|_x / \|\varphi_-\|_x \to 1$. Thus not only is the Lyapunov exponent the same, but even the subexponential corrections are unchanged. □

4. Power Law Theorems

In this section we will prove the following result that essentially includes Theorems 1.10 and 1.11 as corollaries. (We will need to make an additional argument for $\beta = 0$.)

Theorem 4.1. *In the continuum case, suppose V_0 is bounded from below. Let $\lambda \in S_0(V_0)$ be a regular energy with $\beta(\lambda, V_0) > 0$. Suppose that*

$$|W(x)| \leq C(1 + |x|)^{-\eta}$$

for some $\eta > \beta(\lambda, V_0)^{-1}$. Then $\lambda \in S(V_0 + W)$ and

$$\beta(\lambda, V_0 + W) = \beta(\lambda, V_0).$$

Interestingly enough, we will apply Theorem 2.2 in a situation where $f_+ f_- \neq 1$ but is strictly bigger. Essentially, we will not want to take f_- as small as f_+^{-1} because we will need the error estimate $u_1^+ f_- \to 0$ to be stronger than $u_1^+ f_+^{-1} \to 0$.

To employ the ideas of Jitomirskaya-Last, we need to relate estimates involving an integral of a product of φ_+, φ_-, W and $1, f_+$ or f_- to $\|\cdot\|_L$. The following is useful:

Lemma 4.2. *If*

$$|Q(x)| \leq C_1 (1 + |x|)^{-a} \tag{4.1}$$

and

$$\|\varphi_+\|_L \|\varphi_-\|_L \leq C_2 (1 + L)^b \tag{4.2}$$

and $a > b$, then

$$\int_0^\infty |Q(x) \varphi_+(x) \varphi_-(x)| \, dx < \infty.$$

Proof. Let $g(x) = \int_0^x |\varphi_+(y) \varphi_-(y)| \, dy$. By the Schwarz inequality and (4.2),

$$|g(x)| \leq C_2 (1 + |x|)^b \tag{4.3}$$

and, of course,

$$g(0) = 0. \tag{4.4}$$

Use (4.1) to write

$$\int_0^L |Q(x) \varphi_+(x) \varphi_-(x)| \, dx$$

$$\begin{aligned}&\leq C_1 \int_0^L (1+|x|)^{-a}\, \frac{dg}{dx}\, dx \\ &= C_1 a \int_0^L (1+|x|)^{-a-1} g(x)\, dx + C_1(1+|L|)^{-a} g(L).\end{aligned}$$

There is no boundary term at $x=0$ by (4.4).

Now use (4.3) and $b<a$ to see

$$\begin{aligned}\lim_{L\to\infty} \int_0^L |Q(x)\varphi_+(x)\varphi_-(x)|\, dx &\leq C_1 C_2\, a \int_0^\infty (1+|x|)^{b-a-1}\, dx \\ &< \infty.\end{aligned}$$

□

The next step is obtaining power-law upper and lower bounds on $\|\varphi_\pm\|_L$. In principle, the upper and lower powers could be different with oscillation between the two powers of growth.

Lemma 4.3. *Let λ be a regular energy with $\lambda \in S$ and $\beta(\lambda)>0$. Let $\varphi_- = \varphi_{1,\theta(\lambda)}$ and $\varphi_+ = \varphi_{2,\theta(\lambda)}$. Then for any $\varepsilon>0$, there are constants C_1, C_2, C_3, C_4 (ε-dependent) so that for L large,*

$$C_2 L^{1-1/2\beta-\varepsilon} \leq \|\varphi_-\|_L \leq C_1 L^{1/2+\varepsilon} \tag{4.5}$$

$$C_4 L^{1/2-\varepsilon} \leq \|\varphi_+\|_L \leq C_3 L^{1/2\beta+\varepsilon}. \tag{4.6}$$

Proof. The definition of regularity says (1.16) which is the C_1 estimate in (4.5). (1.17) then implies the C_4 estimate in (4.6).

By (1.11), if $\tilde\beta<\beta$, then $\|\varphi_-\|_L \geq \|\varphi_+\|_L^{\tilde\beta}$ for L large which, given the C_1 estimate, implies the C_3 estimate in (4.6). Using (1.17) again, we get the C_2 estimate in (4.5). □

At first sight, it might appear that all one needs on $\|\varphi_\pm\|_L$ are upper bounds because they are all that enter in proving the applicability of Theorem 2.2. But one wants to apply Theorem 2.2 to show that

$$\frac{\|\psi_+\|_L}{\|\varphi_+\|_L} \to 1, \qquad \frac{\|\psi_-\|_L}{\|\varphi_-\|_L} \to 1 \tag{4.7}$$

as $L\to\infty$. Consider the second part of (4.7). We have

$$\psi_- = u_1^- \varphi_- + u_2^- \varphi_+.$$

Since $u_1^- \to 1$, we have that

$$\frac{|\|\psi_-\|_L - \|\varphi_-\|_L|}{\|\varphi_-\|_L} \leq \frac{\|\psi_- - \varphi_-\|_L}{\|\varphi_-\|_L} \leq \frac{\|\psi_- - u_1^-\varphi_-\|_L}{\|\varphi_-\|_L} + o(1)$$

and so it is natural to prove the desired relation by showing

$$\frac{\|u_2^- \varphi_+\|_L}{\|\varphi_-\|_L} \to 0.$$

All we basically know about u_2^- is $f_+ u_2^- \to 0$. Thus

Lemma 4.4. *Suppose $G(x) \in L^1$. In order for (4.7) to hold, it suffices that for large L,*

$$\frac{\|\varphi_+\|_L}{\|\varphi_-\|_L} \leq C f_+(L) \tag{4.8}$$

and

$$\frac{\|\varphi_-\|_L}{\|\varphi_+\|_L} \leq C f_-(L). \tag{4.9}$$

By (4.5)/(4.6), we have (4.8) if $f_+(L) = L^{\mu_+}$ with

$$\frac{1}{2\beta} - \left(1 - \frac{1}{2\beta}\right) < \mu_+.$$

If we only apply the similar bound for (4.9), we see that we need $f_-(L) \geq L^{2\varepsilon}$ which is incompatible with f_- decreasing. We therefore do not gain from (4.5)/(4.6) and instead define $f_-(x) \equiv 1$ so that (4.9) holds since $\|\varphi_-\|_L$ is subordinate. Thus we will take

$$f_+(x) = x^{\mu_+}, \qquad f_-(x) = 1 \tag{4.10}$$

with

$$\mu_+ > \frac{1}{\beta} - 1. \tag{4.11}$$

Proof of Theorem 4.1. By the above analysis, if we take f_+, f_- to obey (4.10)/(4.11), we have (4.7) so long as Theorem 2.2 is applicable. But (4.7) implies that $[\|\psi_-\|_L / \|\psi_+\|_L^{\tilde\beta}] / [\|\varphi_-\|_L / \|\varphi_+\|_L^{\tilde\beta}] \to 1$ and thus by Proposition 1.3, $\beta(\lambda, V_0 + W) = \beta(\lambda, V_0)$.

To apply Theorem 2.2, we need G to be in L^1. By Lemma 4.2 and the upper bounds in (4.5)/(4.6), this is true if the following three inequalities hold

$$\frac{1}{2} + \frac{1}{2\beta} < \eta \quad \longleftarrow (a_{11}, a_{22} \text{ terms}) \tag{4.12}$$

$$1 + \mu_+ < \eta \quad \longleftarrow (a_{21} \text{ terms}) \tag{4.13}$$

$$\frac{1}{\beta} < \eta \quad \longleftarrow (a_{12} \text{ terms}). \tag{4.14}$$

By the basic hypothesis of the theorem, $\eta > \beta^{-1}$ and, of course, $\beta^{-1} \geq 1$. Thus (4.12) and (4.14) hold, and to get (4.13) and (4.11), we need only choose $\mu_+ > 0$ so that

$$\frac{1}{\beta} < 1 + \mu_+ < \eta.$$

This can be done since $\beta^{-1} \geq 1$. $\square$

Theorem 1.10 is an immediate corollary of Theorem 4.1 as is Theorem 1.11 if $\beta(\lambda, V_0) \neq 0$. In case $\beta(\lambda, V_0) = 0$, then we claim $\beta(\lambda, V_0 + W) = 0$ for if not, we can turn this argument around (think of $V_0 = (V_0 + W) - W$) and find that $\beta(\lambda, V_0) = \beta(\lambda, V_0 + W) \neq 0$. That means $\beta(\lambda, V_0 + W) = 0$ which implies $\lambda \in P(V_0 + W)$ or $\lambda \in S(V_0 + W)$ with $\beta = 0$.

The condition $\eta > \beta^{-1}$ of Theorem 4.1 is needed because we assume no extra information about the behavior of $\|u_1\|_L$ and $\|u_2\|_L$ other than the value of β. If one has additional information, one can often do better. Here is an extreme example, but one that holds in some explicit examples.

Definition. We say there is power Lyapunov-Osceledec behavior with exponent $\gamma > 0$ at energy λ if and only if there exist solutions $\varphi_{1,\theta(\lambda)}$ and $\varphi_{2,\theta(\lambda)}$ with

$$\lim_{x\to\infty} \frac{\ln[|\varphi_{1,\theta(\lambda)}(x)|^2 + |\varphi'_{1,\theta(\lambda)}(x)|^2]}{\ln|x|} = -\gamma$$
$$\lim_{x\to\infty} \frac{\ln[|\varphi_{2,\theta(\lambda)}(x)|^2 + |\varphi'_{2,\theta(\lambda)}(x)|^2]}{\ln|x|} = \gamma.$$

Notes. 1. In the discrete case, replace $\varphi'(x)$ by $\varphi(n+1)$.

2. Under these circumstances, if $\gamma < 1/2$ and V is bounded, we have $\|\varphi_1\|_L \sim L^{-\gamma+1/2}$, $\|\varphi_2\|_L \sim L^{\gamma+1/2}$ (where $\sim$ means up to factors of L^ε) so $\lambda \in S(V_0)$ and $\beta(\lambda) = (1/2 - \gamma)/(1/2 + \gamma)$ and $\alpha(\lambda) = 1 - 2\gamma$.

3. One example where it is known [11] there is power Lyapunov-Osceledec behavior is the discrete $n^{-1/2}$ decaying Anderson model where $V_\omega(n) = \lambda n^{-1/2} X_\omega(n)$ where the X_ω are bounded i.i.d.'s with $E(X_\omega) = 0$, $E(X_\omega^2) = 1$. Then there is power Lyapunov-Osceledec behavior with $\gamma = \lambda^2/(8 - 2E^2)$ in the region $|E| < 2$.

Theorem 4.5. *Suppose V_0 has power Lyapunov-Osceledec behavior with $\gamma > 0$ at energy λ and that*

$$\int (1 + |x|)^\varepsilon |W(x)|\, dx < \infty$$

for some $\varepsilon > 0$. *Then* $V_0 + W$ *has power Lyapunov-Osceledec behavior at energy* λ *with the same value of* γ.

The proof is essentially identical to the proof of Theorem 1.9 (in Section 3) with $f_\pm(x) = (1+|x|)^{2(\pm\gamma+\varepsilon)}$.

This shows the improvement over the power in Theorem 4.1. Instead of $\eta > \beta^{-1}$, we only need $\eta > 1$.

5. Appendix: WKB Asymptotic Behavior

In this appendix, we illustrate with an example how Theorem 2.2 can be used to obtain precise asymptotic behavior of solutions in some concrete situations (where the perturbation does not even have to be decaying). Namely, we show how to use Theorem 2.2 to prove the existence of WKB solutions at $+\infty$ for

$$-\psi'' + V\psi = \lambda\psi, \tag{5.1}$$

when $V = V_1 + V_2$ with

$$V_1 \in L^1, \quad V_2' \in L^1, \quad V_2(x) \to 0 \qquad \text{as } x \to \infty, \tag{5.2}$$

when $\lambda \neq 0$. For $\lambda > 0$, it is well known that such solutions exist (see, e.g., [21]). For $\lambda < 0$, one can also apply Levinson's theorem (see [13] or [3], Theorem 8.1) to prove this result, but it is nice to get it from Theorem 2.2. As a preliminary, we note one can try an Ansatz, (1.22)/(1.21) for solving (5.1) even if $\varphi_\pm$ do not solve a related Schrödinger equation. The result is that u still obeys (1.24) but A is now given by

$$A(x) = w(x)^{-1} \begin{pmatrix} -\varphi_+(x)(H_\lambda\varphi_-)(x) & -\varphi_+(x)(H_\lambda\varphi_+)(x) \\ \varphi_-(x)(H_\lambda\varphi_-)(x) & \varphi_-(x)(H_\lambda\varphi_+)(x) \end{pmatrix}, \tag{5.3}$$

where

$$w(x) = \varphi_-(x)\varphi_+'(x) - \varphi_+(x)\varphi_-'(x) \tag{5.4}$$

and H_λ is the differential expression

$$H_\lambda = -\frac{d^2}{dx^2} + V - \lambda. \tag{5.5}$$

We can now prove

Theorem 5.1. *Let* V *obey* (5.2) *and* $\lambda \neq 0$. *If* $\lambda < 0$, *let*

$$\varphi_\pm(x) = \exp(\pm\eta(x)),$$

where

$$\eta(x) = \int_{s_0}^x \sqrt{-\lambda + V_2(s)}\, ds,$$

and s_0 is chosen so that $|V_2(s)| \leq |\lambda|$ for $s > s_0$. If $\lambda > 0$, let

$$\varphi_\pm = \exp(\pm\eta(x)),$$

where

$$\eta = i \int_{s_0}^{x} \sqrt{\lambda - V_2(s)}\, ds.$$

Then there exist solutions $\psi_\pm$ of (5.1) so

$$\psi_\pm(x) = \varphi_\pm(x)\,(1 + o(1))$$
$$\psi'_\pm(x) = \varphi'_\pm(x)\,(1 + o(1))$$

as $x \to \infty$.

Proof. Consider first the case $\lambda < 0$. Then $\varphi'_\pm = \pm\eta' e^{\pm\eta}$ and $\varphi''_\pm = (\pm\eta'' + (\eta')^2)\varphi_\pm$ and thus, since $(\eta')^2 = -\lambda + V_2$,

$$\begin{aligned} H_\lambda \varphi_\pm &= [\pm\eta'' + V_1]\varphi_\pm \\ &= \left[\frac{\pm V_2'}{2\eta'} + V_1\right]\varphi_\pm \end{aligned}$$

so we define

$$Q_\pm = \frac{\pm V_2'}{2\eta'} + V_1.$$

Since $\eta' \to \sqrt{-\lambda}$ as $x \to \infty$, we see that $Q_\pm \in L^1$. Moreover,

$$w(x) = 2\eta' \to 2\sqrt{-\lambda} \qquad \text{as } x \to \infty. \tag{5.6}$$

It follows with $f_\pm = \varphi_\pm^2$ (so $f_+ = f_-^{-1}$) and A given by (5.3) that $G(x) \in L^1$ since $Q_\pm(x) \in L^1$.

Applying Theorem 2.2, there are solutions $\psi_\pm$ with

$$\begin{aligned} \psi_\pm &= \varphi_\pm(1 + o(1)) + \varphi_\mp(f_\mp^{-1})o(1) \\ &= \varphi_\pm(1 + o(1)) \end{aligned}$$

and similarly for $\psi'_\pm$.

The calculation for $\lambda > 0$ is similar, except we use $|\varphi_+| = |\varphi_-| = 1$ in that case to pick $f_+ = f_- = 1$. □

In this paper, we considered only perturbations which are absolutely integrable. It is reasonable to ask what one can expect for stronger perturbations, for example, in situations where there is Lyapunov behavior. While in general the picture is not complete, we provide a sample result which gives L^2 stability under additional assumptions on the behavior of solutions of the unperturbed equation. As a bonus, we also obtain a stronger version of Theorem 5.1 in the case $\lambda < 0$.

Proposition 5.2. *Assume that there exist functions $\varphi_\pm(\lambda, x)$ such that $(-\frac{d^2}{dx^2} + V - \lambda)\varphi_\pm = U_\pm \varphi_\pm$, with $U_\pm \in L^2$, and that the inverse of the Wronskian $W[\varphi_-, \varphi_+]^{-1}$ is bounded. Define functions*

$$\eta_\pm(\lambda, x) = \varphi_\pm e^{\pm \int_0^x \frac{U_\pm \varphi_+ \varphi_-}{W[\varphi_-, \varphi_+]}\, dt} \tag{5.7}$$

and the kernel

$$K(x,y) = \varphi_+^2(x)\varphi_-^2(y) e^{-\int_x^y \frac{(U_- + U_+)\varphi_- \varphi_+}{W[\varphi_-, \varphi_+]}\, dt}. \tag{5.8}$$

Assume in addition that

$$\inf_x |\varphi_+(x)\varphi_-(x)| \geq c > 0 \tag{5.9}$$

and

$$\int\limits_0^\infty \sup_x |K(x, x+y)|\, dy < \infty, \ \ \sup_{y \geq x} |K(x,y)| \leq C. \tag{5.10}$$

Then there exist solutions $\psi_\pm$ of the equation $(H + V - \lambda)\psi_\pm = 0$ with the asymptotic behavior

$$\psi_\pm(\lambda, x) = \eta_\pm(\lambda, x)(1 + o(1)). \tag{5.11}$$

Remarks. 1. In order for (5.10) to hold, one needs, roughly speaking, Lyapunov behavior at λ and moreover $\phi_+(x,\lambda)\phi_-(x,\lambda) \sim$ const (or grows very slowly) for large x.

2. In the case where $V_2 = 0$, the result follows from the Hartman-Wintner theorem (see, e.g., [6]).

3. Notice that the asymptotic behavior of solutions of the perturbed equation differs from $\varphi_\pm$ by an additional factor.

Before sketching the proof, let us illustrate the result with the following generalization of Theorem 5.1 for $\lambda < 0$.

Corollary 5.3. *Assume that $V = V_1 + V_2$, $V_1 \in L^2$, $V_2' \in L^2$, $V_2(x) \to 0$ as $x \to \infty$. Then for $\lambda < 0$ there exist solutions $\psi_\pm$ of the equation $-\psi'' + V\psi = \lambda\psi$ such that*

$$\psi_\pm(x, \lambda) = \eta_\pm(x, \lambda)(1 + o(1)).$$

Here $\eta(x, \lambda)$ is given by (5.7) with

$$\varphi_\pm(x, \lambda) = \exp\left(\int_{s_0}^x \sqrt{-\lambda + V_2(s)}\, ds \right),$$

and s_0 is such that $|V_2(s)| < |\lambda|$ for $s > s_0$.

Remark. For $\lambda > 0$, the result is generally not true. It holds for a.e. $\lambda > 0$ for $V_1 \in L_p$, $V_2' \in L^p$ with $p < 2$ [2]. It is not known if the result remains true for $p = 2$ and a.e. $\lambda > 0$.

Proof. Choosing $\varphi_\pm$ as in the statement of the corollary, one directly verifies that all conditions of Proposition 5.2 hold. Notice that the Lyapunov behavior is preserved, since

$$W[\phi_-, \phi_+] = -2\sqrt{-\lambda + V_2},$$
$$U_\pm = V_1 \mp \frac{V_2'}{2\sqrt{-\lambda + V_2}}$$

and therefore the additional factor in (5.7) is bounded by $e^{Cx^{1/2}}$. □

We now sketch the proof of Proposition 5.2. Seeking solution $\psi(x)$ of the equation $-\psi'' + V\psi = \lambda\psi$, apply variation of parameters-type transformation

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} \varphi_- & \varphi_+ \\ \varphi_-' & \varphi_+' \end{pmatrix} u(x),$$

obtaining a system

$$u'(x) = \frac{1}{W[\varphi_-, \varphi_+]} \begin{pmatrix} -U_-\varphi_-\varphi_+ & -U_+\varphi_+^2 \\ U_-\varphi_-^2 & U_+\varphi_-\varphi_+ \end{pmatrix} u(x).$$

Do one more transformation to bring this system to a simpler form:

$$u(x) = \begin{pmatrix} e^{-\int_0^x \frac{U_-\varphi_-\varphi_+}{W[\varphi_-,\varphi_+]}\,dt} & 0 \\ 0 & e^{\int_0^x \frac{U_+\varphi_-\varphi_+}{W[\varphi_-,\varphi_+]}\,dt} \end{pmatrix} z(x),$$

then

$$z'(x) = \begin{pmatrix} 0 & -\frac{U_+\varphi_+^2}{W[\varphi_-,\varphi_+]} e^{\int_0^x \frac{(U_- + U_+)\varphi_-\varphi_+}{W[\varphi_-,\varphi_+]}\,dt} \\ -\frac{U_-\varphi_-^2}{W[\varphi_-,\varphi_+]} e^{-\int_0^x \frac{(U_- + U_+)\varphi_-\varphi_+}{W[\varphi_-,\varphi_+]}\,dt} & 0 \end{pmatrix} z(x). \tag{5.12}$$

One can obtain the formal series for solutions of (5.12) by iteration; starting with the vector $(1, 0)^T$ will lead to the solution $\psi_-(x)$. Properties (5.9) and (5.10) allow one to prove the convergence of this series and (5.11) using elementary estimates. We leave the details to the interested reader.

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