RESONANCES IN ONE DIMENSION AND FREDHOLM DETERMINANTS

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Abstract. We discuss resonances for Schrödinger operators in whole- and half-line problems. One of our goals is to connect the Fredholm determinant approach of Froese to the Fourier transform approach of Zworski. Another is to prove a result on the number of antibound states — namely, in a half-line problem there are an odd number of antibound states between any two bound states.

1. Introduction

In this paper, we want to discuss resonances and antibound states in one dimension for Schrödinger operators $H = H_0 + V$, where H_0 is one of the following:

Case 1: $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$

Case 2: $-\frac{d^2}{dx^2}$ with $u(0) = 0$ boundary conditions on $L^2(0,\infty)$

Case 3: $-\frac{d^2}{dx^2}$ with $u'(0) + hu(0) = 0$ boundary condition on $L^2(0,\infty)$

Case 4: $-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2}$ on $L^2(0,\infty)$; $\ell = 1, 2, \dots$.

We will often consider Case 2 as the $\ell = 0$ case of Case 4. We will normally suppose V has compact support, although many of our results only require

$$\int e^{a|x|} |V(x)| \, dx < \infty \tag{1.1}$$

for all $a > 0$ (and some only that (1.1) hold for suitable $a > 0$).

The operator H_0 has spectrum $[0,\infty)$ except for Case 3 with $h < 0$ which has a single eigenvalue at energy $-h^2$. That means that $(H_0 + \kappa^2)^{-1}$ is a well-defined operator in the region $\operatorname{Re}\kappa > 0$ (except for a pole at $\kappa = -h$ in Case 3 with $h < 0$). Its integral kernel

$$(H_0 + \kappa^2)^{-1}(x,y) \equiv G_0(x,y;\kappa)$$

has an explicit formula in terms of exponential functions in Cases 1–3 and Bessel functions in Case 4. For example

$$G_0(x,y;\kappa) = e^{-\kappa|x-y|}/2\kappa \qquad \text{(Case 1)} \tag{1.2}$$

$$G_0(x,y;\kappa) = e^{-\kappa x_>} \sinh(\kappa x_<)/\kappa \qquad \text{(Case 2)} \tag{1.3}$$

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where $x_< = \min(x,y)$ and $x_> = \max(x,y)$.

From these explicit formulae, $G_0(x,y;\kappa)$ as a function of κ has an analytic continuation to the entire κ plane except for a simple pole at $\kappa = 0$ in Case 1 and at $\kappa = -h$, in Case 3 with $h \geq 0$.

By the explicit formula for G_0, it can be seen that when (1.1) holds for all $a > 0$:

$$K(\kappa)(x,y) = V(x)^{1/2} G_0(x,y;\kappa) |V(y)|^{1/2} \tag{1.4}$$

is in $L^2(\mathbb{R}\times\mathbb{R}, dx\,dy)$ for all κ (except for the simple pole in Cases 1 and 3) and so defines an analytic Hilbert-Schmidt operator-valued function on $\mathbb{C}$ (or $\mathbb{C}\backslash\{0\}$, or $\mathbb{C}\backslash\{-h\}$). In (1.4), $V(x)^{1/2}$ is short for

$$\begin{aligned} V(x)^{1/2} &= V(x)/|V(x)|^{1/2} && \text{if } V(x) \neq 0 \\ &= 0 && \text{if } V(x) = 0. \end{aligned}$$

$K(\kappa)$ is called the Birman-Schwinger kernel.

It is well known (see, e.g., [7, 23] that

Birman-Schwinger Principle. $H_0 + V$ has $-\kappa^2$ as an eigenvalue (with $\operatorname{Re}\kappa > 0$) if and only if $K(\kappa)$ has eigenvalue -1.

With this in mind, one defines

Definition. κ is called a resonance energy if $\operatorname{Re}\kappa < 0$ and $K(\kappa)$ has eigenvalue -1. The multiplicity of the resonance is the algebraic multiplicity of the eigenvalue. If $\operatorname{Im}\kappa = 0$ as well, we call κ an antibound state.

Of course, $-\kappa^2$ is really the energy, but since κ is the natural parameter, we will abuse terminology. This definition is the one of Froese [8, 9]. Melrose and Zworski and their school in numerous papers (e.g., [29, 31]) have defined resonance in terms of poles of suitable elements of the analytically contained S-matrix. It follows from our results below (Proposition 2.9 and 2.10) that the definitions agree and are equivalent to the solution of $-u'' + Vu = -\kappa^2 u$ with $u(x) = e^{-\kappa x}$ (a suitable Hankel function in Case 4) near $+\infty$ obeys the boundary condition at zero in Cases 2 and 3, is $e^{+\kappa x}$ near $-\infty$ in Case 1 and decreases as $x \downarrow 0$ in Case 4.

While one has obviously that $K(\kappa)$ is Hilbert-Schmidt, the following is true:

Theorem 1. *$K(\kappa)$ is trace class for all κ (except for the poles in Cases 1 and 3).*

This is a result of Froese [8] for Cases 1 and 2, but it seems worthwhile to give an alternate proof that also works in Case 4 (Case 3 is an easy consequence of Case 2). We do this in Section 2. Once one has Theorem 1, it is natural to define

$$d(\kappa) = \det(1 + K(\kappa)) \tag{1.5}$$

in which case the resonances or bound states are precisely the zeros of $d(\kappa)$ (or $\kappa d(\kappa)$ in Case 1 or $(\kappa + h)\, d(\kappa)$ in Case 3). Determinants have also been used by Melrose [20] and Zworski [30] in their work on resonances.

Our main goals in this paper are the following:

(1) Zworski [29] proved his theorem on counting resonances by realizing the inverse of a suitable S-matrix element as a Laplace transform. Froese's alternate proof directly analyzes the asymptotics of $d(\kappa)$. In this paper, we link the two methods by showing the Fredholm determinant can be realized as a Laplace transform. This provides an alternate to using Melin's theory [19].
(2) We want to present a new result on counting antibound states. During the final preparation of this manuscript, I received a preprint of Kargaev-Korotyaev [13] who independently found this result.

To be more specific, in Section 2, we will prove expansion formulae for the Fredholm determinants $d(\kappa)$ in Cases 1 and 2 (this part also works for Cases 3 and 4, but we do not state these results explicitly since we will only handle Cases 1 and 2 in Section 3).

Theorem 2. *In Case 2 (half-line with $u(0) = 0$ boundary conditions):*

$$d(\kappa) = 1 + \sum_{n=1}^{\infty} d_n(\kappa) \tag{1.6}$$

with

$$d_n(\kappa) = \int_{x_0 \equiv 0 < x_1 < \cdots < x_n} V(x_1) \dots V(x_n) \prod_{j=1}^{n} \frac{\sinh(\kappa(x_j - x_{j-1}))}{\kappa} e^{-\kappa x_n} \, dx_1 \dots dx_n. \tag{1.7}$$

Theorem 3. *In Case 1 ($H_0 \equiv -\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$):*

$$d(\kappa) = 1 + \sum_{n=1}^{\infty} d_n(\kappa) \tag{1.8}$$

with

$$\begin{aligned} d_n(\kappa) = \int_{x_1 < \cdots < x_n} & V(x_1) \dots V(x_n) \\ & \left[\prod_{j=2}^{n} \frac{\sinh(\kappa(x_j - x_{j-1}))}{\kappa} \right] \frac{e^{-\kappa(x_n - x_1)}}{2\kappa} \, dx_1 \dots dx_n. \end{aligned} \tag{1.9}$$

Notes. 1. The product in (1.9) has $n-1$ terms and is empty in case $n = 1$ so that

$$d_1(k) = \frac{1}{2\kappa} \int V(x) \, dx. \tag{1.10}$$

2. (1.9) implies that if we look at the problem of λV with a coupling constant λ added

$$\kappa d(\kappa) = \kappa + \lambda \int V(x) \, dx + \lambda^2 F(\lambda, \kappa)$$

with F analytic near $\lambda = \kappa = 0$. It follows that if $\int V(x)\,dx < 0$, there is a bound state for small λ with energy $(-\kappa^2)$ given by $-\lambda^2(\int V(x)\,dx)^2 + O(\lambda^3)$ and an antibound state at that energy if $\int V(x)\,dx > 0$. This recovers old results of Landau-Lifshitz [16, Section 45] (see also Simon [25]).

Once we have these expansions, we can use

$$\frac{\sinh \kappa y}{\kappa} = \int_0^y e^{-2\kappa\alpha}\, e^{\kappa y}\, d\alpha$$

to obtain the following Laplace transform representations:

Theorem 4. *In Case 2, let $a = \sup(\text{supp}(V))$. Then*

$$d(\kappa) = 1 + \int_0^a t(\alpha)\, e^{-2\alpha\kappa}\, d\alpha$$

for a suitable L^1-function t on $[0,a]$ with $a \in \text{supp}(t)$.

Theorem 5. *In Case 1, let $[a,b]$ be the convex hull of the support of V. Then*

$$\kappa d(k) = \kappa + \tfrac{1}{2} \int_a^b V(x)\, dx + \tfrac{1}{2} \int_0^{b-a} t(\alpha)\, e^{-2\alpha\kappa}\, d\alpha,$$

where $b - a \in \text{supp}(t)$.

Following Zworski, these Laplace transform formulae allow one to use Titchmarsh's theorem [28] to obtain the result of Regge [24] and Zworski [29] on the density of resonances.

While we will directly write down these Laplace transforms, we could use a less direct result. As we will show in Section 2, the expansion in Theorem 2 lets us identify $d(\kappa)$ with a Jost function, the value of the Jost solution at $x = 0$ (this is not a new result; it is due to Jost-Pais [12]). But Levin [17] has a Laplace transform formula for the Jost solution as used extensively by Marchenko [18] and that allows one to prove Theorem 4 from Theorem 2 (the estimate we use to show $a \in \text{supp}(t)$ is in Marchenko's book).

Our final results are on a different subject

Theorem 6. *In a half-line problem (Cases 2, 3 or 4), suppose h has n bound states $0 < \kappa_1 < \cdots < \kappa_n$. Then each interval $(-\kappa_{j+1}, -\kappa_j)$ has an odd number of antibound states and, in particular, at least one antibound state. In particular, there are at least $(n-1)$ antibound states.*

We will prove Theorems 1, 2, and 3 in Section 2, Theorems 4 and 5 in Section 3 and Theorem 6 in Section 4. An appendix has a result on finite determinants that we need in Section 2.

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2. Expansion of the Fredholm Determinant

In this section, we want to first prove that $K(\kappa)$ is trace class for all $\kappa \in \mathbb{C}$ (or $\mathbb{C}\backslash\{0\}$), that is, prove Theorem 1. Then we present an expansion of the determinant, $\det(1+K(\kappa))$, in the $\ell = 0$ case (Theorems 2 and 3).

We begin with the whole-line case, so

$$K(\kappa)(x,y) \equiv V(x)^{1/2}\, \frac{e^{-\kappa|x-y|}}{2\kappa}\, |V(y)|^{1/2}.$$

Since V has compact support, $K(\kappa)(x,y)$ is L^2 in $\mathbb{R}\times\mathbb{R}$ for all $\kappa \in \mathbb{C}\backslash\{0\}$ and the kernel is analytic. Thus,

Proposition 2.1. *$K(\kappa)$ is Hilbert-Schmidt for all $\kappa \in \mathbb{C}\backslash\{0\}$ and analytic in κ.*

For $\operatorname{Re}\kappa > 0$, we can write

$$K(\kappa) = V^{1/2}(x)(p^2+\kappa^2)^{-1}|V(x)|^{1/2}.$$

It is a basic fact [26, Theorem 4.1] that if $f,g \in L^2(\mathbb{R})$, then $f(p)g(x)$ is Hilbert-Schmidt with

$$\|f(p)g(x)\|_2 \le (2\pi)^{-1/2}\|f\|_2\|g\|_2. \tag{2.1}$$

It follows immediately that

Proposition 2.2. *If $g,h \in L^2(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, then $g(x)f(p)h(x)$ is trace class and*

$$\|g(x)f(p)h(x)\|_1 \le (2\pi)^{-1}\|g\|_2\|h\|_2\|f\|_1.$$

Note next that if $|\operatorname{Arg}\kappa| \in (\frac{\pi}{4},\frac{\pi}{2})$,

$$\int \frac{d\rho}{|p^2+\kappa^2|} \le \frac{c}{|\operatorname{Im}\kappa|}\left[1+\log\big(|\operatorname{Im}\kappa|/\operatorname{Re}\kappa\big)\right],$$

so we have

Proposition 2.3. *For $|\operatorname{Arg}\kappa| \in (\frac{\pi}{4},\frac{\pi}{2})$, $K(\kappa)$ is trace class with*

$$\|K(\kappa)\|_1 \le c\|V\|_1\left[\frac{1}{|\operatorname{Im}\kappa|}\left[1+\log\big(|\operatorname{Im}\kappa|/\operatorname{Re}\kappa\big)\right]\right]. \tag{2.2}$$

Next, we note that

$$\begin{aligned}(K(\kappa)-K(-\kappa))(x,y) &= V^{1/2}(x)\,\frac{2\cosh(\kappa|x-y|)}{\kappa}\,V(y)^{1/2}\\ &= V^{1/2}(x)\,\frac{2\cosh(\kappa(x-y))}{\kappa}\,|V(y)|^{1/2}\end{aligned}$$

is a rank 2 operator. It follows that

Proposition 2.4. *If $\operatorname{Re}(\kappa) > 0$, $K(\kappa)-K(-\kappa)$ is trace class with*

$$\|K(\kappa)-K(-\kappa)\|_1 \le 2\int |V(x)|\, e^{2(\operatorname{Re}\kappa)|x|}\,dx.$$

As a result

Proposition 2.5. *If* $\mathrm{Re}(\kappa) < 0$, $K(\kappa)$ *is trace class with*
$$\|K(\kappa)\|_1 \leq C(1 + \log(|\operatorname{Im}\kappa|)), \tag{2.3}$$
where C *is a constant bounded on each half-annulus* $\{\kappa \mid \operatorname{Re}\kappa < 0,\ 0 < A < |\kappa| < B\}$.

Now given κ_0 with $\kappa_0 = i\alpha$, $\alpha \in \mathbb{R}\backslash\{0\}$, write
$$K(\kappa_0) = \int_{-\pi}^{\pi} K\left(\kappa_0 + \frac{\alpha}{2}\, e^{i\theta}\right) \frac{d\theta}{2\pi}\,.$$
Since $\|K(\kappa_0 + \frac{\alpha}{2}\, e^{i\theta}\|_1$ has only logarithmic singularities at $\theta = \pm\frac{\pi}{2}$, we conclude

Proposition 2.6. $K(\kappa)$ *is trace class on* $\mathbb{C}\backslash\{0\}$.

This is just Theorem 1 in this case.

The proof of Theorem 1 in the $\ell(\ell{+}1)/r^2$ case depends on an eigenfunction expansion for $h_\ell^{(0)}$. Let
$$u_\ell(k,x) = kx\, j_\ell(kr),$$
where j_ℓ is a spherical Bessel function. Then [1],
$$(F_\ell\varphi)(k) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty u_\ell(k,x)\varphi(x)\, dx \tag{2.4}$$
is a unitary map of $L^2(\mathbb{R},dx)$ to $L^2(\mathbb{R},dk)$, that is,
$$\int_0^\infty |(F_\ell\varphi)(k)|^2\, dk = \int_0^\infty |\varphi(x)|^2\, dx \tag{2.5}$$
and
$$(F_\ell h_\ell^{(0)}\varphi)(k) = k^2(F_\ell\varphi)(k). \tag{2.6}$$

While the usual proof [26] that $\|f(p)g(x)\|_2 = (2\pi)^{-1/2}\|f\|_2\|g\|_2$ comes from writing $f(p)$ as a convolution operator, here is another proof. Let V be the Fourier transform and let M_f be multiplications by $f(x)$. Then
$$f(p)g(x) = V^{-1}M_f V M_g$$
so since V^{-1} is unitary $\|f(p)g(x)\|_2 = \|M_f V M_g\|$ but $M_f V M_g$ has integral kernel:
$$(M_f V M_g)(p,x) = f(p)\, \frac{e^{-ipx}}{\sqrt{2\pi}}\, g(x)$$
from which the Hilbert-Schmidt norm formula is immediate. Similarly,

Proposition 2.7. *If* $f,g \in L^2(0,\infty)$, *then* $f(\sqrt{h_\ell^{(0)}}\,)g(x)$ *is Hilbert-Schmidt and*
$$\left\| f\left(\sqrt{h_\ell^{(0)}}\right) g(x)\right\|_2 \leq C\|f\|_2\|g\|_2. \tag{2.7}$$

Proof. As above,

$$f\left(\sqrt{h_\ell^0}\right)g(x) = F_\ell^{-1} M_f F_\ell M_g,$$

so since F_ℓ is unitary (2.5), we have

$$\left\| f\left(\sqrt{h_\ell^{(0)}}\right)g(x)\right\|_2 = \|M_f F_\ell M_g\|$$

and the latter has integral kernel

$$f(k)\sqrt{\tfrac{2}{\pi}}\, u_\ell(k,x) g(x)$$

so (2.7) follows from $|u(k,x)| \le C_1$ (see [1]). □

With (2.7) in hand, the proof of Theorem 1 is essentially identical to the proof in the whole-line case. The $\ell = 0$ situation is identical to the $u(0) = 0$ boundary conditions for the half-line $-\frac{d^2}{dx^2}$ case and then the general h bound condition situation follows from the fact that the difference of the resolvents is rank 1 when h changes. That completes the proof of Theorem 1.

Once we know $K(\kappa)$ is trace class, we can form $d(\kappa) = \det(1 + K(\kappa))$. We turn to the expansion of the Fredholm determinant. We will do this for a general trace class operator, A, with an integral kernel $A(x, y)$ on $\mathbb{R}$ (or $[0,\infty)$) of the form

$$A(x,y) = V(x)^{1/2} G_0(x,y) |V(y)|^{1/2}, \tag{2.8}$$

where G_0 has the form

$$G_0(x,y) = f_-(x_<) f_+(x_>), \tag{2.9}$$

with $x_< = \min(x,y)$, $x_> = \max(x,y)$. We will show that

Proposition 2.8. *Let A be a trace class operator of the form* (2.8)/(2.9). *Then*

$$\begin{aligned}\det(1+A) =& 1 + \sum_{n=1}^\infty \int_{x_1<\cdots<x_n} V(x_1)\dots V(x_n) f_-(x_1) f_+(x_n) \\ & \prod_{j=1}^{n-1} [f_+(x_{j+1})f_-(x_j) - f_+(x_j) f_-(x_{j+1})]\, dx_1 \dots dx_n.\end{aligned} \tag{2.10}$$

Remarks. 1. We are vague about convergence issues since in the applications we make, they are trivial. In general, it certainly suffices that $\int |V(x)|\, [|f_+(x)| + |f_-(x)| + 1]^2\, dx < \infty$.

2. Theorems 2 and 3 are immediate corollaries of Proposition 2.8 given that

$$e^{-\kappa x}(2\kappa)^{-1}\, e^{+\kappa y} - e^{-\kappa y}(2\kappa)^{-1}\, e^{+\kappa x} = \kappa^{-1}\sinh(\kappa(y-x))$$

and

$$e^{-\kappa x}\kappa^{-1}\sinh(\kappa y)-e^{-\kappa y}\kappa^{-1}\sinh(\kappa x)=\kappa^{-1}\sinh(\kappa(y-x)).$$

3. While we are interested in the cases given by Theorems 2 and 3 because of the application in the next section, the proposition applies directly to the $\ell(\ell+1)/x^2$ example; f_- is a Bessel function (of imaginary argument) and f_+ is a suitable Hankel function.

4. Jost-Pais [12] have a related result.

Proof. As shown in [26] (and essentially due to Fredholm)

$$\det(1+A)=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int A\begin{pmatrix}x_1\dots x_n\\ x_1\dots x_n\end{pmatrix}dx_1\dots dx_n, \tag{2.11}$$

where

$$A\begin{pmatrix}x_1\dots x_n\\ y_1\dots y_n\end{pmatrix}=\det([A(x_i,y_j)]_{i,j=1,\dots,n}).$$

Since $A\binom{x_1\dots x_n}{x_1\dots x_n}$ is symmetric in the x's, we can remove the $n!$ and integrate over $x_1<\dots<x_n$. In that case,

$$A\begin{pmatrix}x_1\dots x_n\\ x_1\dots x_n\end{pmatrix}=V(x_1)\dots V(x_n)\det(f_-(x_{\min(i,j)})f_+(x_{\max(i,j)})).$$

Determinants of this form are discussed in the appendix where it is shown that

$$\begin{aligned}&\det(f_-(x_{\min(i,j)})f_+(x_{\max(i,j)}))\\ &\qquad=f_+(x_n)f_-(x_j)\prod_{j=1}^{n-1}[f_+(x_{j+1})f_-(x_j)-f_+(x_j)f_-(x_{j+1})],\end{aligned}$$

which proves the proposition. ☐

We close this section by using the expansions in Theorem 2 and 3 to identify $d(\kappa)$ with quantities related to the Jost function. Consider first the half-line case. Let $d(\kappa,x_0)$ be the $d(\kappa)$-function for the potential $\{V(x+x_0)\}_{x\geq 0}$. Then, with

$$F(x,y;\kappa)=\frac{\sinh(\kappa(y-x))}{\kappa}, \tag{2.12}$$

we have that

$$\begin{aligned}&d(\kappa,x_o)\\ &\quad=1+\sum_{n=1}^{\infty}\int_{0<x_1<\dots<x_n}F(0,x_1;\kappa)V(x_0+x_1)F(x_1,x_i;\kappa)\dots V(x_0+x_n)\\ &\qquad\qquad e^{-\kappa x_n}\,dx_1\dots dx_n\end{aligned}$$

$$= 1 + e^{\kappa x_0} \sum_{n=1}^{\infty} \int_{x_0<x_1<\cdots<x_n} F(x_0,x_1;\kappa)V(x_1)F(x_1,x_n)\dots V(x_n)$$
$$e^{-\kappa x_n}\, dx_1 \dots dx_n$$

by a change of variables. That means if we define

$$f(x,\kappa) = e^{-\kappa x}\, d(\kappa, x), \tag{2.13}$$

then f obeys

$$f(x,\kappa) = e^{-\kappa x} + \sum_{n=1}^{\infty} \int_{x<x_1<\cdots<x_n} F(x,x_1;\kappa)V(x_1)F(x_1,x_2;\kappa)\dots V(x_n)$$
$$e^{-\kappa x_n}\, dx_1 \dots dx_n, \tag{2.14}$$

so that f obeys the integral equation

$$f(x,\kappa) = e^{-\kappa x} + \int_x^\infty F(x,y;\kappa)V(y)f(y,\kappa)\, dy, \tag{2.15}$$

which implies that f obeys

$$-f'' + Vf = -\kappa^2 f \tag{2.16}$$

with the boundary condition

$$f(x,\kappa) = e^{-\kappa x} \qquad \text{if } x \ge a = \sup[\operatorname{supp} V]. \tag{2.17}$$

Thus,

Proposition 2.9. *$f(x,\kappa)$ is the Jost solution, that is, solution of* (2.16) *which obeys* (2.17). *In particular, $d(\kappa)$ is the Jost function, that is, $f(x = 0,\kappa)$.*

Remark. It is a known result that the Jost function is a Fredholm determinant. See Jost-Pais [12] and Simon [26]. Our proof is related to that of Jost-Pais. Similarly, Proposition 2.10 below is known; it follows by combining formulae in Newton [21].

There is a related expression for the whole-line case. First, we need some notation. Let $[a,b]$ be the convex hull of the support of V. The Jost solutions $f_\pm(x,\kappa)$ are the solutions of (2.16) that obey the boundary conditions

$$f_\pm(x,\kappa) = e^{\mp\kappa x} \qquad \text{for } \pm x \ge |a| + |b|. \tag{2.18}$$

Given two C^1 functions f, g, we define their Wronskian by

$$W(f,g)(x) = f'(x)g(x) - f(x)g'(x). \tag{2.19}$$

As usual, if f, g obey the same second-order differential equation, W is constant and we denote its value as $W(f,g)$. Finally, we define the free Jost solutions

$$f_\pm^{(0)}(x,\kappa) = e^{\mp\kappa x}.$$

Proposition 2.10. *In the whole-line case*

$$\kappa d(\kappa) = \tfrac{1}{2}\, W(f_-(\,\cdot\,,\kappa), f_+(\,\cdot\,,\kappa)). \tag{2.20}$$

Proof. By looking at the expansion (2.14) of the Jost solution $f_+(x,\kappa)$, and the expansion of $d(\lambda)$ given by Theorem 3, we see that

$$d(\kappa) = 1 + \frac{1}{2\kappa}\int_a^b e^{\kappa x} V(x) f_+(x,\kappa)\, dx \tag{2.21}$$

so

$$\kappa d(\kappa) = \kappa + \tfrac{1}{2}\int_a^b f_-^{(0)}(x,\kappa) V(x) f_+(x,\kappa)\, dx. \tag{2.22}$$

But since $f_+ = f_+^{(0)}$ for $x > b$,

$$\kappa = \tfrac{1}{2}\, W(f_-^{(0)}, f_+)(b)$$

and

$$f_-^{(0)}(x,\kappa) V(x) f_+(x,\kappa) = -\frac{d}{dx}\, W(f_-^{(0)}, f_+),$$

so (2.22) shows that

$$\kappa d(\kappa) = \tfrac{1}{2}\, W(f_-^{(0)}, f_+)(a).$$

But for $x \le a$, $f_-^{(0)} = f_-$, so

$$\kappa d(\kappa) = \tfrac{1}{2}\, W(f_-, f_+)(a) = \tfrac{1}{2}\, W(f_-, f_+)(x)$$

for all x. $\square$

Remarks. 1. Proposition 2.9 can be rephrased in a form close to (2.20). Namely, if $u(x,\kappa)$ is the solution of (2.16) obeying $u(0,\kappa) = 0$, $u'(0,\kappa) = 1$, then $d(\kappa) = W(u, f)$.

2. Similarly, in the $\ell(\ell+1)/x^2$ case, if $f_\ell(x,\kappa)$ is the solution of the equation

$$-u'' + \frac{\ell(\ell+1)}{x^2}\, u + Vu = \kappa^2 u,$$

which is given by $kxh^{(1)}(\kappa x)$ if $x \ge a = \sup(\operatorname{supp}(V))$ and u_ℓ is the solution that obeys $\lim_{x\downarrow 0} \frac{u(x,r)}{kxj_\ell(kx)} = 1$, then

$$d(\kappa) = W(u, f).$$

In the next section, we will need an additional function and a relation between d and this new function. Let $c(\kappa)$ be defined by

$$\kappa c(\kappa) \equiv \tfrac{1}{2}\, W(f_-(\,\cdot\,,\kappa), f_+(\,\cdot\,,-\kappa)). \tag{2.23}$$

Proposition 2.11.

$$\text{(i)} \qquad f_-(\,\cdot\,,\kappa) = -c(\kappa) f_+(\,\cdot\,,\kappa) + d(\kappa) f_+(\,\cdot\,,-\kappa) \tag{2.24}$$

$$\text{(ii)} \qquad d(\kappa) d(-\kappa) = 1 + c(\kappa) c(-\kappa) \tag{2.25}$$

Proof. (2.23) is a direct consequence of (2.20), (2.23) and

$$W(f_+(\,\cdot\,,\kappa), f_+(\,\cdot\,,-\kappa)) = -2\kappa$$

since $f_+(x,\kappa) = e^{-\kappa x}$ for x near $+\infty$.

Writing (2.24) for κ and $-\kappa$ and

$$W(f_-(\,\cdot\,,\kappa), f_-(\,\cdot\,,-\kappa) = 2\kappa$$

(since $f_-(x,\kappa) = e^{\kappa x}$ for x near $-\infty$), we have that

$$2\kappa = -2\kappa c(\kappa)c(-\kappa) + 2\kappa d(\kappa)d(-\kappa).$$

which implies (2.25). □

3. Determinants as Laplace Transforms

Our goal in this section is to prove Theorems 4 and 5. Given the expansion of Theorems 2 and 3, the argument is similar to part of the construction of the A-function in [27]. We will use

$$\frac{\sinh \kappa x}{\kappa} = \int_{-x/2}^{x/2} e^{-2\ell\kappa}\, d\ell \tag{3.1}$$

$$= \int_0^x e^{x\kappa} e^{-2\ell\kappa}\, d\ell. \tag{3.2}$$

We start with the half-line case. Using (1.7) and (3.2), we have that

$$d(\kappa) = 1 + \sum_{n=1}^\infty d_n(\kappa) \tag{3.3}$$

$$d_n(\kappa) = \int_{0<x_1<\dots<x_n;\, 0<\ell_1<x_1,\, 0<\ell_2<x_2-x_1,\dots,0<\ell_n<x_n-x_{n-1}} V(x_1)\dots V(x_n)$$

$$e^{-2\sum_{j=1}^n \ell_j\kappa}\, dx_1\dots dx_n\, d\ell_1\dots d\ell_n \tag{3.4}$$

$$\equiv \int t_n(\alpha)\, e^{-2\alpha\kappa}\, d\alpha, \tag{3.5}$$

where

$$t_n(\alpha) = \int_{0<x_1<\dots<x_n} V(x_1)\dots V(x_n) R_n(x_1,\dots,x_n;\alpha)\, dx_1\dots dx_n \tag{3.6}$$

with

$$R_n(x_1,\dots,x_n;\alpha)$$

$$= \int_{0<\ell_1<x_1,\, 0<\ell_2<x_2-x_1,\dots,0<\ell_n<(x_n-x_{n-1})} \delta\biggl(\sum_{j=1}^n \ell_j - \alpha\biggr)\, d\ell_1\dots d\ell_n. \tag{3.7}$$

Lemma 3.1. (i) *$R_n = 0$ if $\alpha > x_n$*

(ii) $|R_n| \le \frac{\alpha^{n-1}}{(n-1)!}$

(iii) *For $n \ge 2$, R_n is C^1 and* $|\frac{\partial R_n}{\partial\alpha}| \le 2\,\frac{\alpha^{n-2}}{(n-2)!}$.

Proof. (i) By the inequalitites on ℓ_j, $\sum \ell_j \leq (x_1 + (x_2 - x_1) + \cdots + (x_n - x_{n-1})) = x_n$.

(ii) Clearly, $|R_n| \leq \int_{0<\ell_1,\, 0<\ell_2, \dots, 0<\ell_n} \delta(\sum_{j=1}^n \ell_j - \alpha)\, d\ell_1 \dots d\ell_n = \frac{\alpha^{n-1}}{(n-1)!}$ by induction.

(iii)

$$
\begin{aligned}
\frac{\partial R_n}{\partial \alpha} &= \int_{\dots} \delta' \biggl(\sum_{j=1}^n \ell_j - \alpha \biggr)\, d\ell_1 \dots d\ell_n \\
&= \int_{\dots} \delta \biggl(\sum_{j=1}^{n-1} \ell_j - \alpha \biggr)\, d\ell_1 \dots d\ell_{n-1} \\
&\qquad - \int_{\dots} \delta \biggl(\sum_{j=1}^{n-1} \ell_j - \alpha - (x_n - x_{n-1}) \biggr)\, d\ell_1 \dots d\ell_{n-1}.
\end{aligned}
$$

Now use the estimation method of (ii). □

Proposition 3.2. *Let* $a = \sup(\operatorname{supp}(V))$*.*

(i)

$$
d(\kappa) = 1 + \int_0^a t(\alpha)\, e^{-2\alpha\kappa}\, d\alpha \tag{3.8}
$$

with $t(\alpha)$ *a continuous function.*

(ii) t *is an absolutely continuous function with*

$$
|t'(\alpha) + V(\alpha)| \leq C \int_\alpha^a |V(y)|\, dy. \tag{3.9}
$$

(iii) *The convex hull of the support of* $\delta(\alpha) + t(\alpha)$ *is* $[0, a]$*.*

Proof. By (3.6) and (ii) of Lemma 3.1, we have that

$$
|t_n(\alpha)| \leq \frac{\alpha^{n-1}}{(n-1)!} \biggl[\int_\alpha^a |V(y)|\, dy \biggr] \frac{1}{(n-1)!} \biggl(\int_0^a |V(y)|\, dy \biggr)^{n-1}
$$

so

$$
t(\alpha) = \sum_{n=1}^\infty t_n(\alpha)
$$

converges and we can justify the interchange of sum and integral to obtain (3.8).

Similarly, by (iii) of Lemma 3.1, $\sum_{n=2}^\infty t_n(\alpha)$ is C^1 with derivative $g(\alpha)$ that obeys

$$
\begin{aligned}
|g(\alpha)| &\leq \sum_{n=2}^\infty (2\alpha^{n-2})/(n-2)! \biggl[\int_\alpha^a V(y)\, dy \biggr] \frac{1}{(n-1)!} \biggl(\int_0^a |V(y)|\, dy \biggr) \\
&\leq C \int_\alpha^a |V(y)|\, dy.
\end{aligned}
$$

By a direct calculation.

$$t_1(\alpha) = \int_\alpha^a V(y)\,dy$$

so that (3.9) is proven.

Clearly, $t(\alpha)$ is supported on $[0,a]$ and $\delta + t$ has 0 in its support. That means we need only show t is non-zero on each interval $[a-\varepsilon, a]$. Suppose t vanishes identically on such an interval. Then by (3.9),

$$|V(\alpha)| \le C \int_\alpha^a |V(y)|\,dy$$

for $\alpha > a - \varepsilon$. So for $x \in [a-\varepsilon, a]$,

$$\int_x^a |V(y)|\,dy \le |a-x| C \int_x^a |V(y)|\,dy$$

so V vanishes on $[a-\varepsilon', a)$ with $\varepsilon' = \min(\varepsilon, \frac{1}{C})$. Since $a = \sup(\mathrm{supp}(V))$ by hypothesis, this is a contradiction which shows that $a \in \mathrm{supp}(t)$. $\square$

This implies Theorem 4. Proposition 3.2 also makes it easy to construct potentials with an infinity of antibound states. The basic idea is close to those of Titchmarsh [28] on zeros of Laplace transforms and a similar analysis (but using Melin's theory and on the whole line) has been made by Zworski [29]. Define intervals $I_1, I_2, \dots \subset [0,1]$ by $I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, \frac{3}{4}], \dots, I_n = [1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$. Let $a_n = 2^n\, e^{-2^{n+1}}$ and

$$V(x) = (-1)^n a_n \qquad \text{if } x \in I_n.$$

It is easy to modify V to be C^∞. The infinity of oscillations of V is critical since if V has a definite sign near $x = a$, it is easy to see that $t(\alpha)$ has a definite sign, and so $f(-\kappa)$ does for κ near infinity and thus there are only finitely many antibound states.

Since a_n is decreasing, $|t'(\alpha) + V(\alpha)| \le C 2^{-n} a_n$ by (ii) of Proposition 3.2. Thus for n large, $t'(\alpha)$ is very close to $(-1)^{n+1} a_n$ on I_n. Thus

$$\begin{aligned}\int_0^1 t'(\alpha)\, e^{2\alpha\kappa}\, d\alpha &\sim \sum_{n=1}^\infty (-1)^{n+1} a_n 2^{-n}\, e^{2(1-2^{-n})\kappa} \\ &\sim e^{2\kappa} \sum_{n=1}^\infty (-1)^{n+1} \exp(-2^{-n-1}\kappa - 2^{n+1}).\end{aligned}$$

For $\kappa_m = 2^{2(m+1)}$, it is easy to see this sum is dominated by the term with $n = m$. Since $\int_0^1 t'(\alpha)\, e^{2\alpha\kappa}\, d\alpha \sim -2\kappa \int_0^1 t(\alpha)\, e^{2\alpha\kappa}\, d\alpha$, we concluded that

$$(-1)^m f(-2^{2(m+1)}) > 0$$

for m large, and so $f(-\kappa)$ has infinitely many zeros as $\kappa \to \infty$.

Once one has a half-line potential with an infinity of antibound states, the same is true of whole-line problems with suitable even potentials.

We now turn to the whole-line case and Theorem 5. We will use (2.21)

$$2\kappa d(\kappa) = 2\kappa + \int_a^b e^{\kappa x} V(x) f_+(x,\kappa)\, dx, \tag{3.10}$$

as well as the following formula proven in a similar way:

$$\begin{align} 2\kappa c(\kappa) &= \int_a^b e^{\kappa x} V(x) f_+(x,-\kappa)\, d\lambda \tag{3.11} \\ &= \int_a^b f_-(x,\kappa) V(x)\, e^{\kappa x}\, dx. \tag{3.12} \end{align}$$

Proposition 3.3. *In the whole-line case:*

(i)

$$\kappa d(\kappa) = \kappa + \tfrac{1}{2}\int_a^b V(x)\, dx + \tfrac{1}{2}\int_0^{b-a} t(\alpha) e^{-2\alpha\kappa}\, d\alpha \tag{3.13}$$

for an L^1-function t on $[0, b-a]$.

(ii)

$$\kappa c(\kappa) = \tfrac{1}{2}\int_a^b e^{2\kappa\alpha} s(\alpha)\, d\alpha, \tag{3.14}$$

where $s(\alpha)$ is a function in $L^1(a,b)$.

(iii)

$$\begin{align*} |s(\alpha) - V(\alpha)| &\leq C \int_a^\alpha |V(y)|\, dy \\ &\leq C \int_\alpha^b |V(y)|\, dy. \end{align*}$$

(iv) $a, b \in \operatorname{supp}(s)$.

Proof. (i) We begin with the extension of (3.8) which implies that

$$f_+(x,\kappa) = e^{-\kappa x} + \int_x^b t_+(\alpha, x)\, e^{-2\kappa\alpha}\, e^{+\kappa x}. \tag{3.15}$$

Plugging this into (3.10),

$$\begin{align*} \kappa d(\kappa) &= \kappa + \tfrac{1}{2}\int_a^b V(x)\, dx + \tfrac{1}{2}\int_a^b dx \int_x^b t_+(\alpha,x)\, e^{-2\kappa(\alpha - x)}\, d\alpha \\ &= \kappa + \tfrac{1}{2}\int_a^b V(x)\, dx + \tfrac{1}{2}\int_0^{b-a} \biggl(\int_a^{b-\beta} t_+(\beta + x, x)\, dx\biggr) e^{-2\beta\kappa}\, d\beta, \end{align*}$$

which is (3.13) if $t(\alpha) = \int_a^{b-\alpha} t_+(\alpha + x, x)\, dx$.

(ii) Plugging (3.15) into (3.11) yields

$$\kappa c(\kappa) = \tfrac{1}{2}\int_a^b e^{2\kappa x} V(x)\, dx + \tfrac{1}{2}\int_a^b V(x)\, dx \biggl(\int_x^b t_+(\alpha, x)\, e^{2\kappa\alpha}\, d\alpha\biggr)$$

which is (3.14) if

$$s(\alpha) = V(\alpha) + \int_a^\alpha t_+(\alpha,x)V(x)\,dx. \tag{3.16}$$

Using (3.12) instead of (3.11) yields

$$s(\alpha) = V(\alpha) + \int_\alpha^b t_-(\alpha,x)V(x)\,dx. \tag{3.17}$$

(iii) This follows from (3.16)/(3.17) with

$$c = \sup_{\alpha,x} |t_\pm(\alpha,x)|.$$

(iv) This follows as in the proof of (iii) of Proposition 3.2. □

(3.13) is Theorem 5 if we prove that $b-a \in \operatorname{supp}(t)$. We give a proof of this fact that is part of Zworski's proof [29] translated to our language; Froese [8] has a different proof.

Proposition 3.4. *$b-a \in \operatorname{supp}(t)$.*

Proof. Write $\kappa d(\kappa) = \int m(\alpha)\,e^{-2\alpha\kappa}\,d\alpha$ and $\kappa c(\kappa) = \int n(\alpha)\,e^{-2\alpha\kappa}\,d\alpha$ in distributional sense where

$$m(\alpha) = \tfrac{1}{2}\delta'(\alpha) + \biggl(\tfrac{1}{2}\int_a^b V(x)\,dx\biggr)\delta(\alpha) + \tfrac{1}{2}\,t(\alpha)$$

and

$$n(\alpha) = \tfrac{1}{2}\,s(-\alpha).$$

By (2.25) and the uniqueness of inverse Laplace transforms

$$(m * \tilde{m})(\alpha) = \tfrac{1}{4}\,\delta''(\alpha) + (n * \tilde{n})(\alpha), \tag{3.18}$$

where $\tilde{m}(\alpha) = m(-\alpha)$ and $*$ is convolution. Since $a, b \in \operatorname{supp}(s)$, we have that $a-b$ and $b-a$ lie in $\operatorname{supp}[n*\tilde{n}]$. If $\operatorname{supp}[m] \subset [0,c]$, then $\operatorname{supp}[m*\tilde{m}] \subset [-c,c]$. Hence by (3.18), $b-a \in \operatorname{supp}[m]$. □

4. Antibound States

Our goal in this section is to prove Theorem 6. We will provide three rather different proofs, two that I found and one supplied to me by G.M. Graf [11]. We will present them first in the case when $u(0)=0$ boundary conditions with V of compact support, and then make some remarks about the other cases.

First Proof of Theorem 6. We begin by noting some properties of the Fredholm determinant $d(\kappa)$:

(i) If $\kappa \neq 0$, either $d(\kappa) \neq 0$ or $d(-\kappa) \neq 0$ (or both). This is true because if $d(\pm\kappa) = 0$, then the Jost solution $f(x, \pm\kappa)$ vanishes at $x = 0$, so the function solving $-u'' + qu = -\kappa^2 u$ with $u(0) = 0$, $u'(0) = 1$ is $\alpha e^{\mp\kappa x}$ for $x > a \equiv \max \operatorname{supp}(V)$. This cannot happen for both κ and $-\kappa$ since $u \neq 0$ for x large.

(2) If $d(\kappa = 0) = 0$, then $d'(\kappa = 0) \neq 0$. This well-known result holds because $\frac{\partial}{\partial\kappa} f(x,\kappa)\big|_{\kappa=0}$ and $f(x,\kappa)\big|_{\kappa=0}$ both solve $-u''+qu = 0$ and are equal to x and 1 for $x > a$. As with the proof of (i), it cannot happen that both $d(\kappa = 0) = 0$ (in which case the solution with $u(0) = 0$, $u'(0) = 1$ is a constant for $x > a$) and $\frac{\partial d}{\partial\kappa}(\kappa = 0) = 0$ (in which case u is equal to αx for $x > a$).

(iii) If $\kappa_n(\lambda)$ are the bound state "energies" of $-\frac{d^2}{dx^2} + \lambda q$, then $\kappa_n(\lambda)$ is increasing in λ for λ in $(L_n; \infty)$ where L_n is the infimum over those λ with n or more bound states. This follows from first-order perturbation theory (Feynman-Hellman theorem), which implies that $\frac{\partial(-\kappa_n^2)}{\partial\lambda} = \langle\varphi, q\varphi\rangle \leq -\kappa_n^2/\lambda < 0$.

(iv) There are no zeros of f in the quadrants $\operatorname{Re}\kappa > 0$, $\operatorname{Im}\kappa \neq 0$ since they would correspond to imaginary eigenvalues.

(v) Resonances (zeros with $\operatorname{Re}\kappa < 0$, $\operatorname{Im}\kappa \neq 0$) occur in complex conjugate pairs.

This means as λ increase and we look at eigenvalues of $-\frac{d^2}{dx^2} + \lambda q$, the only way bound states can change is by an antibound state turning into a bound state. However, antibound states can change due to a complex pair of resonances turning into a pair of antibound states or vice-versa.

Now imagine λ increasing past L_1. A single antibound state turns into a bound state $\kappa_1(\lambda)$. For λ near L_1, there is exactly one bound state/antibound state near 0 since $d'(L_1) \neq 0$ by (ii). In particular, there are no antibound states in $(-\kappa_1(\lambda), 0)$. Since $-\kappa_1(\lambda)$ cannot be an antibound state, no antibound states can pass along the real axis into or out of $(-\kappa_1(\lambda), 0)$ for $\lambda \in (L_1, L_2)$. Only pairs of resonances can produce pairs of antibound states. Thus, for $\lambda \in (L_1, L_2)$, there are an even number of antibound states in $(-\kappa_1(\lambda), 0)$. At $\lambda = L_2$, a single antibound state turns into a bound state leaving an odd number in $(-\kappa_1(\lambda), -\kappa_2(\lambda))$. As λ increases, antibound states only get added or subtracted in pairs, so the number stays odd. The argument is simple for each similar interval $(-\kappa_n(\lambda), -\kappa_{n+1}(\lambda))$ for $\lambda > L_{n+1}$. □

Second Proof of Theorem 6. This proof has some connection with an old paper of Ciafaloni-Menotti [6], who discuss a related result concerning alternation as coupling constant is changed rather as κ is varied. We define

$$\frac{d(-\kappa)}{d(\kappa)} = e(\kappa). \tag{4.1}$$

If κ_j is the jth bound state $0 < \kappa_n < \kappa_{n-1} < \cdots < \kappa_1$, then $d(\kappa_j) = 0$ and $d(-\kappa_j) \neq 0$ (as in (i) in the last proof). We will show that

$$\lim_{\kappa \to \kappa_j} (\kappa - \kappa_j) e(\kappa) < 0. \tag{4.2}$$

It follows that $e(\kappa_j + \varepsilon) < 0$ and $e(\kappa_{j-1} - \varepsilon) > 0$ for ε small and so, by continuity, $e(\kappa)$ has an odd number of zeros (counting multiplicity) in (κ_j, κ_{j-1}). But zeros of $d(-\kappa)$ are the same as zeros of $e(\kappa)$.

We will prove (4.2) by using a representation for $e(\kappa)$ due to Froese [8, 9]. In this case

$$d(\kappa) = \det(1 + K(\kappa))$$

$$K(\kappa)(x,y) = V(x)^{1/2} \frac{\sinh(\kappa x_<)}{\kappa} e^{-\kappa x_>} |V(y)|^{1/2}$$

so

$$[K(\kappa) - K(-\kappa)](x,y) = 2\kappa^{-1} V(x)^{1/2} \sinh(\kappa x) \sinh(\kappa y) |V(y)|^{1/2}$$

is rank 1; call it $A(\kappa)$. Then

$$\begin{aligned} d(-\kappa) &= d(\kappa) \det(1 - (1 + K(\kappa))^{-1} A(\kappa)) \\ &= d(\kappa)[1 - \mathrm{Tr}((1 + K(\kappa))^{-1} A(\kappa))] \end{aligned}$$

so [8, 9]:

$$e(\kappa) = 1 - 2\kappa^{-1} \langle |V|^{1/2} \varphi_0, (1 + K(\kappa))^{-1} V^{1/2} \varphi_0 \rangle, \tag{4.3}$$

where $\varphi_0(x) = \sinh(\kappa x)$.

Let $L(\kappa) = V^{1/2}(h_0 + V + \kappa^2)^{-1} |V|^{1/2}$. Then, by the resolvent formula:

$$L(\kappa) = K(\kappa) - K(\kappa) L(\kappa)$$

or

$$(1 + K(\kappa))^{-1} = 1 - L(\kappa).$$

Then by (4.3)

$$e(\kappa) = 1 - 2\kappa^{-1} \langle \varphi_0, V\varphi_0 \rangle + 2\kappa^{-1} \langle V\varphi_0, (h_0 + V + \kappa^2)^{-1} V\varphi_0 \rangle.$$

It follows from this that

$$\lim_{\kappa \to \kappa_j} (\kappa - \kappa_j) e(\kappa_j) = -2\kappa_j^{-1} (2\kappa_j)^{-1} |\langle V\varphi_0, \eta_j \rangle|^2,$$

where η_j is the normalized eigenvector for $H_0 + V$ at energy $-\kappa_j^2$. Thus, the limit is non-positive. Since e has a pole, the limit is non-zero, and so (4.2) is proven. □

Third Proof of Theorem 6 (Graf [11]). Let $u(x,\kappa)$ solve the equation $-u'' + qu = -\kappa^2 u$ with $u(0) = 0$, $u'(0) = 1$. As is well known, $u(x, \kappa_j)$ has $j - 1$ zeros, so since $u(x,\kappa) > 0$ for x small, $(-1)^{j-1} u(x, \kappa_j) > 0$ for x near infinity. Since q has compact support, say $[0, a]$,

$$u(x,\kappa) = \alpha(\kappa)\, e^{-\kappa x} + \beta(\kappa)\, e^{\kappa x}, \quad x \geq a$$

and α and β are continuous since $u(x,\kappa)$, $u'(x,\kappa)$ are, and α,β can be expressed in terms of suitable u,u' data. By the sign condition on u near infinity

$$(-1)^{j-1}\alpha(\kappa_j) > 0, \tag{4.4}$$

where one has strict positivity since u cannot be identically zero on (a,∞).

By the association of f and the Jost function, $f(-\kappa)=0$ if and only if the solution which is $e^{\kappa x}$ near infinity vanishes at $x=0$, that is, if and only if $\alpha(\kappa)=0$. By (4.4), α has an odd number of zeros in (κ_j,κ_{j-1}). $\square$

These proofs were stated for the case where H has $u(0)=0$ boundary conditions, but each proof can be modified to handle the other half-line cases. For example, the third proof accommodates $u'(0)+hu(0)=0$ boundary conditions by looking at the solution obeying that boundary condition and $u(0)=1$ normalization. And it handles an $\frac{\ell(\ell+1)}{x^2}$ term by replacing exponentials by suitable Bessel functions.

The proofs also accommodate potentials with superexponential decay with minor modification. For example, the second proof applies verbatim to such potentials.

An illuminating example is the Bargmann potential [5] with Jost function (suggested by Newton [22])

$$f(\kappa) = \frac{(\kappa-k_1)(\kappa-k_2)}{(\kappa+k_3+ik_4)(\kappa+k_3-ik_4)}\,,$$

where $k_j>0$ for $j=1,2,3,4$. Such a potential has no antibound states and seemingly violates Theorem 6. The point, of course, is that q is not superexponential in this case, but only decays as $e^{-\alpha x}$ with $\alpha=\min(k_1,k_2,k_3)$. In general, Theorem 6 does hold for potentials with exponential decay, say, bounded by $e^{-\alpha x}$, but only for intervals $(-\kappa_j,-\kappa_{j-1})$ with $\kappa_j<\alpha$.

Appendix A. The Determinant of a Green's Matrix

Let $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^N$ be two sequences and let C be the $N\times N$ matrix

$$c_{ij} = a_{\min(i,j)}b_{\max(i,j)}; \qquad 1\le i,j\le N. \tag{A.1}$$

Our main goal in this appendix is to give a simple proof of the following theorem of Barrett-Feinsilver [3], which we used in Section 2:

Theorem A.1. *Let C be a matrix given by* (A.1). *Then*

$$\det(C) = b_N(a_Nb_{N-1}-a_{N-1}b_N)(a_{N-1}b_{N_2}-a_{N-2}b_{N-1})\dots(a_2b_1-a_1b_2)a_1. \tag{A.2}$$

Remarks. 1. Barrett-Feinsilver [3] actually state the theorem for matrices closely related to ones given by A.1 and express the result in terms of the

c_{ij}, viz:

$$\det(C) = \prod_{i=1}^{N-1} (c_{i,i}c_{i+1,i-1} - c_{i,i+1}c_{i+1,i}) \Big/ \prod_{i+1}^{N} c_{ii}.$$

2. The proof in [3] is combinatoric but has the advantage of generalizations in a variety of directions [2, 15, 4].

3. An idea similar to our proof in a related context appears in Jost-Pais [12].

4. Matrices of type (A.1) were dubbed Green's matrices by Karlin [14] since they are a discrete analog of one-dimensional Green's functions. Indeed, they occur in Section 2 in connection with free Green's functions. And they arise as the inverses of symmetric tridiagonal matrices

$$J = \begin{pmatrix} x_1 & y_1 \dots & 0 \\ y_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots y_{N-1} & x_N \end{pmatrix} \tag{A.3}$$

in that there is a one-one correspondence realized by the inverse between invertible matrices of the form (A.1) and invertible matrices of the form (A.3). This is a theorem of Gantmacher-Krein [10] (see also Barrett [2]). Indeed, given an invertible (A.3), a can be determined by

$$a_1 = 1; \quad x_1 a_1 + y_1 a_2 = 0; \quad y_{n-1}a_{n-1} + x_n a_n + y_n a_{n+1} = 0, \quad n = 2, \dots N-1 \tag{A.4}$$

$$y_{N-1}b_{N-1} + x_N b_n = 0; \quad y_{n-1}b_{n-1} + x_n b_n + y_n b_{n+1} = 0, \quad n = 2, \dots, N-1, \tag{A.5}$$

where b is normalized by

$$y_n(a_{n+1}b_1 - b_{n+1}a_n) = 1, \qquad n = 1, \dots, N-1. \tag{A.6}$$

[*Note:* (A.4)/(A.5) imply the left side of (A.6) is independent of n and it is non-zero if J is invertible.] Conversely, given a matrix (A.1), one defines y_n by (A.6). [*Note:* $\det(C) \neq 0$ means $(a_{n-1}b_n - b_{n-1}a_n) \neq 0$ by Theorem A.1.] And then one defines x_n by (A.4) or (A.5).

Proof of Theorem A.1. We prove the result by induction in N. The result for $N = 1$ is obvious. If we can prove the result when $b_{N-1} \neq 0$, it follows by continuity for all b_{N-1}, so suppose $b_{N-1} \neq 0$. Consider the last row and column of C. It has $a_N b_N$ in the corner and otherwise every element is $a_j b_N$ for some $j \in \{1, \dots, N-1\}$. It follows that

$$\det(C) = a_N b_N \det(C_{N-1}) + b_N^2 F(a_1, \dots, a_{n-1}; b_1, \dots b_{N-1})$$

where C_{N-1} is the $(N-1) \times (N-1)$ Green's matrix with the last row and column removed and F is some function of $\{a_i\}_{i=1}^{N-1}$ and $\{b_i\}_{i=1}^{N-1}$.

Fix these values of a and b and think of a_N and b_N as variable and $\det(C)$ as a function of them. When $a_N = a_{N-1}$ and $b_N = b_{N-1}$, then C has the identical rows, so $\det(C) = 0$, that is,

$$a_{N-1}b_{N-1}\det(C_{N-1}) + b_{N-1}^2 F = 0$$

so, since $b_{N-1} \neq 0$, $F = -a_{N-1}\det(C_{N-1})/b_{N-1}$ and thus

$$\det(C) = b_N(a_N b_{N-1} - a_{N-1}b_N)\det(C_{N-1})/b_{N-1}. \tag{A.7}$$

But by induction, we can assume (A.2) for $\det(C_{N-1})$ and thus (A.7) implies (A.2) for $\det(C)$. $\square$

References

[1] W.O. Amrein, J.M. Jauch, and K.B. Sinha, *Scattering Theory in Quantum Mechanics*, W.A. Benjamin, 1977.

[2] W. Barrett, *A theorem on inverses of tridiagonal matrices*, Linear Alg. Appl. **27** (1979), 211–217.

[3] W. Barrett and P. Feinsilver, *Inverses of bounded matrices*, Linear Alg. Appl. **41** (1981), 111–130.

[4] W. Barrett and C. Johnson, *Determinantal formulae for matrices with sparse inverses*, Linear Alg. Appl. **56** (1984), 73–88.

[5] K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Springer, New York, 1989.

[6] M. Ciafaloni and P. Menotti, *Analysis of S-matrix singularities by means of operator technique*, Nuovo Cimento (10) **35** (1965), 160–193.

[7] H. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators With Application to Quantum Mechanics and Global Geometry*, Springer, Berlin, 1987.

[8] R. Froese, *Asymptotic distribution of resonances in one dimension*, J. Diff. Eqns. **137** (1997), 251–272.

[9] R. Froese, *Upper bounds for the resonance counting function in odd dimension*, Canadian J. Math. **50** (1998), 538–546.

[10] F.R. Gantmacher and M.G. Krein, *Oscillation Matrices, Oscillation Kernels and Mechanical Systems* (German), Akademie-Verlag, Berlin, 1960.

[11] G.M. Graf, private communication.

[12] R. Jost and A. Pais, *On the scattering of a particle by a static potential*, Phys. Rev. **82** (1951), 840–851.

[13] P. Kargaev and E. Korotyaev, *Inverse resonance scattering and conformal mappings for Schrödinger operators*, preprint.

[14] S. Karlin, *Total Positivity, Vol. 1*, Stanford University Press, Stanford, 1968.

[15] D.J. Klein, *Treediagonal matrices and their inverses*, Linear Alg. Appl. **42** (1982), 109–117.

[16] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, Pergamon Press, London, 1958.

[17] B. Ya. Levin, *Fourier- and Laplace-type transformations by means of solutions of a second-order differential equation* (Russian), Dokl. Akad. Nauk SSSR **106** (1956), 187–190.

[18] V.A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[19] A. Melin, *Operator methods for inverse scattering on the real line*, Commun. Partial Diff. Eq. **10** (1985), 677–766.

[20] R. Melrose, *Polynomial bound on the number of scattering poles*, J. Funct. Anal. **53** (1983), 287–303.

[21] R. Newton, *Inverse scattering, I. One dimension*, J. Math. Phys. **21** (1980), 493–505.
[22] R. Newton, private communication.
[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.
[24] T. Regge, *Analytic properties of the scattering matrix*, Nuovo Cimento (10) **8** (1958), 671–679.
[25] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Phys. **97** (1976), 279–288.
[26] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.
[27] B. Simon, *A new approach to inverse spectral theory, I. Fundamental formalism*, Ann. of Math. **150** (1999), 1–29.
[28] E.C. Titchmarsh, *The zeros of certain integral functions*, Proc. London Math. Soc. **25** (1926), 283–302.
[29] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (1987), 277–296.
[30] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311–323.
[31] M. Zworski, *Resonances in physics and geometry*, Notices Amer. Math. Soc. **46** (1999), 319–328.

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