# MODIFIED PRÜFER AND EFGP TRANSFORMS AND DETERMINISTIC MODELS WITH DENSE POINT SPECTRUM

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Abstract. We provide a new proof of the theorem of Simon and Zhu that in the region $|E| < \lambda$ for a.e. energies, $-\frac{d^2}{dx^2} + \lambda\cos(x^\alpha)$, $0 < \alpha < 1$ has Lyapunov behavior with a quasi-classical formula for the Lyapunov exponent. We also prove Lyapunov behavior for a.e. $E \in [-2,2]$ for the discrete model with $V(j^2) = e^j$, $V(n) = 0$ if $n \notin \{1,4,9,\dots\}$. The arguments depend on a direct analysis of the equations for the norm of a solution.

## §1. Introduction

In this paper, we will consider half-line Schrödinger operators

$$H_\theta = -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

on $L^2(0,\infty)$ with $u(0)\cos(\theta) + u'(0)\sin(\theta) = 0$ boundary conditions and the discrete analog on $\ell^2(\mathbb{Z}^+)$, $\mathbb{Z}^+ = \{1,2,3,\dots\}$,

$$(h_\alpha u)(n) = \begin{cases} u(n+1) + u(n-1) + V(n)u(n) & n \leq 2 \\ u(2) + (V(1)+\alpha)u(1) & n = 1 \end{cases} \tag{1.2}$$

where $\alpha$ plays the role of a boundary condition.

We are interested in models where $H_\theta$ has dense point spectrum in some interval $[a,b]$. By general instability results [3,7], this cannot happen for all $\theta$ but can and does for a.e. $\theta$ if $[a,b] \subset \mathrm{spec}(H_\theta)$, and if for a.e. $E \in [a,b]$, there is a solution $-u'' + Vu = Eu$ which is

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$L^2$ at infinity [4,15,16]. The first examples of such operators involved random $V$'s where one proves dense point spectrum for a.e. $V$.

Examples which are deterministic were first found by Gordon [6] (also see [9]), who showed it for problems with very high and sparse but not too sparse barriers. Simon-Zhu [17] proved a similar result for slowly oscillating potentials like $V(x) = \lambda \cos(x^\alpha)$; $0 < \alpha < 1$. Attention on the first class was focused by work of Simon-Spencer [15], and on the second by work of Behncke [1] and Stolz [18] — these authors showed the absence of a.c. spectrum.

Our goal here is to obtain dense pure point spectrum by direct control of the asymptotics of the transfer matrix $T(0,x)$, defined by $T(0,x)\binom{u'(0)}{u(0)} = \binom{u'(x)}{u(x)}$ for solutions of

$$-u'' + Vu = Eu \tag{1.3}$$

in the continuum case, and $T(0,n)\binom{u(1)}{u(0)} = \binom{u(n+1)}{u(n)}$ for solutions of

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{1.4}$$

in the discrete case. It follows from results of Ruelle [13] that if $\lim_{n\to\infty} \frac{1}{n} \ln \|T(0,n)\| > 0$, then there is an $L^2$ solution (here we include existence and finiteness of the limit). The same idea works for other situations, for example, if $\lim_{n\to\infty} \frac{1}{n^\gamma} \ln \|T(0,n)\| > 0$ for any $\gamma > 0$; see [11].

For the case $V(x) = \lambda \cos(x^\alpha)$, that $\lim_{n\to\infty} \frac{1}{n} \ln \|T(0,n)\|$ exists for a.e. $E$ and an explicit formula for the limit was found by Simon and Zhu [17]. In Section 6, we will prove

**Theorem 1.1.** *Let $V(x) = 1 + \cos(x^\alpha)$; $0 < \alpha < 1$. Let $x_n = (2\pi n)^{1/2}$; let $a(n) = n^{(1-\alpha)/2\alpha}$, and let $\{E_j^{(n)}\}_{j=1}^\infty$ be the eigenvalues of*

$$-\frac{d^2}{dx^2} + V(x); \qquad u(x_{n-1}) = u(x_n) = 0$$

*on $L^2(x_{n-1}, x_n; dx)$ and let*

$$\bar{A} = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \bigcup_{j=1}^{\infty} (E_j^{(n)} - e^{-a(n)}, E_j^{(n)} + e^{-a(n)})$$

*(so that $\bar{A}$ is a $G_\delta$ dense in $[0,\infty)$ of Lebesgue measure zero).*

*Let $E \in (0,2)\backslash\bar{A}$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|T_E(0,n)\| = \frac{1}{2\pi} \int_{\{y \mid 1+\cos(y) \geq E\}} (1 + \cos(y) - E)^{1/2}\, dy.$$

*Remarks.* 1. The forbidden set $\bar{A}$ in [17] is larger; they conjecture that our $\bar{A}$ is the "right" one. One should be able to use WKB methods to describe $\bar{A}$ more completely.

2. It is known [1,18] that for $E > 2$, the limit exists and is zero.

3. $V(x)$ can be replaced by any $f(x^\alpha)$ where $f$ is any $C^2$ periodic function with a finite number of critical points in each period.

Unlike Simon-Zhu [17], we will directly attack the transfer matrix by using a transformation idea. In the continuum case, we transform from $u'(x), u(x)$ to $R(x), \theta(x)$ defined by ($k = \sqrt{E}$):

$$u'(x) = kR(x)\cos(\theta(x)) \tag{1.5a}$$

$$u(x) = R(x)\sin(\theta(x)). \tag{1.5b}$$

Then (1.3) is equivalent to

$$\frac{d(\theta(x))}{dx} = k - \frac{V(x)}{k}\sin^2(\theta(x)) \tag{1.6}$$

$$\frac{d\ln R}{dx}(x) = \frac{1}{2k}V(x)\sin(2\theta(x)). \tag{1.7}$$

In [10], together with A. Kiselev, we have shown how to exploit these formulas in a variety of spectral situations, and our main goal here is to show that they are useful in many tunnelling calculations.

In the classically forbidden region where $V(x) > k^2$, (1.6) tends to drive $\theta$ toward values where the left side vanishes, that is,

$$\sin(\theta) = \pm\sqrt{\frac{k^2}{V(x)}}\,. \tag{1.8}$$

At such points,

$$\frac{1}{2k}V(x)\sin(2\theta(x)) = \pm\sqrt{V(x)-k^2}\,. \tag{1.9}$$

The solutions of (1.8) where (1.9) has the plus sign are attracting, which is why $R$ grows like $\exp\big(+\int\sqrt{V(x)-k^2}\,\big)$.

In the classically allowed region where $V(x) < k^2$, it is useful to define $R, \theta$ in a slightly different way. Define

$$k(x) = \sqrt{k^2 - V(x)} \tag{1.10a}$$

$$u'(x) = k(x)R_w(x)\cos(\theta_w(x)) \tag{1.10b}$$

$$u(x) = R_w(x)\sin(\theta_w(x)) \tag{1.10c}$$

which we will call WKB-Prüfer variables. Then, $R_w, \theta_w$ obey

$$\frac{d\theta_w(x)}{dx} = k(x) - \frac{k'(x)}{2k(x)}\sin(2\theta_w(x)) \tag{1.11}$$

$$\frac{d\ln R_w(x)}{dx} = -\frac{k'}{k}\cos^2(\theta_w(x)). \tag{1.12}$$

For later purposes, note that $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$ suggests we define

$$\tilde{R}_w(x) = k(x)^{1/2} R_w(x) \tag{1.13}$$

in which case,

$$\frac{d \ln \tilde{R}_w(x)}{dx} = -\frac{k'}{2k} \cos(2\theta_w(x)). \tag{1.14}$$

Equations (1.11)/(1.14) can be found, for example, in [2].

To the extent that purely oscillatory terms are unimportant because they average to zero, we have $\tilde{R}_w(x) = \text{constant}$, $\theta_w(x) = \int^x k(y)\,dy$ so that $u$ has the the WKB form,

$$k(x)^{-1/2} \sin\biggl(\theta_0 + \int_{x_0}^{x} k(y)\,dy\biggr).$$

As we will see in the appendix, this makes (1.11/1.14) ideal tools for WKB approximations in the classically allowed region.

For the discrete case, we need an analog of modified Prüfer variables, and these are provided by what we have called EFGP variables after contributions in [5,8,12]. Define $R(n), \theta(n)$ by

$$R(n)\cos(\theta(n)) = u(n) - \cos(k)u(n-1) \tag{1.15a}$$
$$R(n)\sin(\theta(n)) = (\sin k)u(n-1), \tag{1.15b}$$

where $E \in (-2,2)$ and $k$ are related by $E = 2\cos(k)$. Then

$$\bar{\theta}(n) \equiv \theta(n) + k; \qquad \nu_k(n) \equiv -\frac{V(n)}{\sin(k)} \tag{1.16}$$

$$\cot(\theta(n+1)) = \cot(\bar{\theta}(n)) + \nu_k(n) \tag{1.17}$$

$$\frac{R(n+1)^2}{R(n)^2} = 1 + \nu_k(n)\sin(2\bar{\theta}(n)) + \nu_k(n)^2 \sin^2(\bar{\theta}(n)). \tag{1.18}$$

We will also need the following relation between $\theta$ and $u$:

$$\frac{u(n)}{u(n-1)} = \frac{\sin(\bar{\theta}(n))}{\sin(\theta(n))}\,. \tag{1.19}$$

For any $\theta_0 \in [0,\pi)$, define $R(x,\theta_0)$ (resp. $R(n,\theta_0)$) to solve (1.6/1.7) (resp. (1.17/1.18)) with $\theta(0) = \theta_0$ (resp. $\theta(1) = \theta_0$) and $R(0) = 1$ (resp. $R(1) = 1$). Then the behavior of any two $R$'s determines the growth of $T$ in the sense that for any fixed $k$ with $k > 0$ (resp. $k \in (0,\pi)$),

$$C_1(k)R(n,\theta_1) \leq \|T(0,n)\| \leq C_2(k,\theta_1,\theta_2)\max(R(n,\theta_1), R(n,\theta_2)),$$

where $C_1$ and $C_2$ are constants independent of $n$ and $V$. In particular,

**Proposition 1.2.** *If for $\theta_1 \neq \theta_2$ (both in $[0,\pi)$) we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln R(n,\theta_1) = \lim_{n\to\infty} \frac{1}{n} \ln R(n,\theta_2) = \gamma,$$

*then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|T(n,0)\| = \gamma.$$

As already mentioned, in the classically forbidden region, the basic equations push $R$ to want to grow as $\exp\bigl(+\int \sqrt{V(y)-E}\,dy\bigr)$ or else to decay as $\exp\bigl(-\int \sqrt{V(y)-E}\,dy\bigr)$. In examples like $\cos(x^\alpha)$, forbidden and allowed regions alternate. Our strategy will be to prove one of three possibilities occurs:

(i) All forbidden regions are decay regions for $x$ sufficiently large. In that case, $u$ will be in $L^2$.
(ii) All forbidden regions are growth regions for $x$ sufficiently large. In that case, $R$ grows in the expected WKB manner.
(iii) Arbitrarily far out, there will be a growing region followed by a decaying region. In that case, we can cut off $u$ at the centers of those forbidden regions and get a very good approximate eigenfunction, and so see that $E \in \bar{A}$.

So if $E \notin \bar{A}$, either $R$ grows in the expected way or $u$ is $L^2$. Since at most one $\theta_0$ can lead to an $L^2$ solution, we can always find two $\theta$'s with the expected growth and so use Proposition 1.2.

In Section 2, we discuss a discrete model with sparse growing barriers for which $\lim_{n\to\infty}$
$\frac{1}{n} \lim \ln \|T(0,n)\| > 0$. This shows the use of EFGP variables. In Section 3, we discuss a model like $\cos(x^\alpha)$ but where cos is replaced by a periodized step function. Sections 4–6 present our proof of Theorem 1.1. An appendix discusses bounded transfer matrices in the region $E > 2$ in the $1+\cos(x^\alpha)$ model if $\alpha < \frac{1}{2}$.

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## §2. A Model of Simon-Spencer Type

In this section, we will study the following model on $\ell^2([1,\infty))$,

$$\begin{aligned} (H_\alpha u)(n) &= u(n+1) + u(n-1) + V(n)u(n) \qquad && n \geq 2 \\ &= u(n+1) + \alpha u(n) && n = 1 \end{aligned}$$

where

$$\begin{aligned} V(j^2) &= e^{\beta j} \qquad && j \geq 2 \\ V(n) &= 0 && n \neq 4, 9, 16, \dots \end{aligned}$$

$\alpha$ plays the role of a boundary condition. $\beta$ is a parameter, $\beta > 0$.

Define

$$A_m = \bigcup_{j=1}^{m-1} (2\cos(\pi j/m) - e^{-\sqrt{m}}, 2\cos(\pi j/m) + e^{-\sqrt{m}})$$

and let

$$\bar{A} = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_{2m+1}$$

so $\bar{A}$ is a dense $G_\delta$ in $[-2,2]$ of Lebesgue measure zero.

We will the prove the following theorem:

**Theorem 2.1.** *Suppose $E \notin \bar{A}$ is in $[-2,2]$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|T(n,0)\| = \frac{\beta}{2}. \tag{2.1}$$

*For a.e. $\alpha$, $H_\alpha$ has dense point spectrum in $[-2,2]$ with eigenfunctions decaying as $e^{-\beta n/2}$.*

*Remarks.* 1. By "eigenfunctions decaying as $e^{-\beta n/2}$," we mean $\lim_{n\to\infty} \ln(|u(n)|^2 + |u(n+1)|^2)^{1/2}/n = -\frac{\beta}{2}$.

2. Since $\overline{\lim}\, |V(n)| = \infty$, the results of Simon-Spencer [15] imply $\sigma_{\mathrm{ac}}(H_\alpha) = \emptyset$. Gordon [6] and Kirsch-Molchanov-Pastur [9] proved that for some potentials of Simon-Spencer type (where the distances between the bumps are not too large), $H$ has dense point spectrum for a.e. boundary condition. Their methods apply to the problem discussed here. Our method is different and identifies the set $\bar{A}$ and the Lyapunov exponent $\gamma(E) = \frac{\beta}{2}$.

3. A similar result holds if $V([j^\beta]) = e^{\nu j^\mu}$ for any $\mu > 0$ and $\beta > 1$ (here $[j^\beta]$ is the greatest integer less than $j^\beta$). Then, $\lim_{n\to\infty} \frac{\ln \|T(n,0)\|}{n^\zeta} = \frac{\nu}{\mu+1}$ where $\zeta = (\mu+1)/\beta$. Where we use Ruelle's result [13] in the argument below, one instead uses its extension in [11].

*Proof.* It obviously suffices to prove that for $E \notin \bar{A}$, $\lim \frac{1}{n} \ln \|T(n,0) u_{\theta_0}\| = \frac{\beta}{2}$ for $u_{\theta_0} = (\cos(\theta_0), \sin(\theta_0))$ and at least two out of three values of $\theta_0$, say, $\theta_0 = 0$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. Pick $\theta_0 = 0$ and let $k$ be defined by $E = 2\cos(k)$ and let $\theta(n), R(n)$ be the EFGP variables for this value of $\theta_0$ and $E$.

Assume that for $j \geq j_0$,

$$|\bar{\theta}(j^2)| \geq \exp(-j^{2/3}). \tag{2.2}$$

We then have that by (1.18), $R(n)^2$ is constant for $n = j^2+1, \dots, (j+1)^2$ and jumps from $n = j^2$ to $n = j^2+1$. By (1.16) and (1.18),

$$\begin{aligned}\frac{R(j^2+1)^2}{R(j^2)^2} &\leq 1 + \frac{e^{\beta j}}{|\sin k|} + \frac{e^{2\beta j}}{\sin^2(k)} \\ &\geq 1 + \frac{\sin^2 \exp(-j^{2/3})}{\sin^2(k)}\, e^{2\beta j} - \frac{e^{\beta j}}{\sin(k)}\end{aligned}$$

for $j \geq j_0$. From these inequalities and $\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$, one easily sees that $\lim_{n\to\infty} \frac{\ln R(j^2+1)}{j^2} = \frac{\beta}{2}$ and then

$$\lim_{n\to\infty} \frac{\ln R(n)}{n} = \frac{\beta}{2}\,. \tag{2.3}$$

We need to examine (2.2). We will prove that at least one of the following holds for $E, \theta_0$ fixed:

(i) (2.2) holds; or
(ii) $E \in \bar{A}$; or
(iii)

$$\sum_n \|T(n,0)u_{\theta_0}\|^2 < \infty. \tag{2.4}$$

If we prove this and $E \notin \bar{A}$, then for each of $\theta_0 = 0$, $\theta_0 = \frac{\pi}{4}$, and $\theta_0 = \frac{\pi}{2}$, one of (i) or (iii) must hold. Since (2.4) can hold for at most one $\theta_0$ (by constancy of the Wronskian), (2.2) must hold for at least two $\theta_0$'s and so (2.3) holds for two $\theta_0$'s, and thus (2.1) holds.

Once (2.1) holds, application of Ruelle's theorem [13] implies that for $E \notin \bar{A}$, there exists an initial $u_E$ so that $\lim \ln \|T(0,n)u_E\|/n = -\frac{\beta}{2}$, and then the Simon-Wolff [16] method proves point spectrum for a.e. $\alpha$ (see, e.g., [4,14]).

Thus, we need only prove that one of the three alternatives (i)–(iii) above holds. Suppose neither (i) nor (ii) holds. We will prove that (iii) holds.

Since (ii) is assumed false, there exists $j_0$ large so that Lemma 2.2 holds and so that $E \notin \bar{A}_{2j-1}$ for $j \geq j_0$. In particular, alternative (a) of Lemma 2.2 does not hold. Suppose $j_1 \geq j_0$ and $|\bar{\theta}(j_1^2)| \leq \exp(-j_1^{2/3})$. Since alternative (b) of Lemma 2.2 holds, we can iterate and see that (2.5) holds for $j = j_0+1, \dots, j_1$.

If alternative (i) fails, there are $j_1$'s going to infinity with $|\bar{\theta}(j_1^2)| \leq \exp(-j_1^{2/3})$, so (2.7) holds for all $j \geq j_0$, and thus $|R(n)| \leq Ce^{-(\alpha-\epsilon)n/4}$ so (2.4) holds. $\square$

**Lemma 2.2.** *There is a $j_0$ (depending only on $k$ and $\beta$) so that if $j \geq j_0$ and $|\bar{\theta}(j^2)| \leq \exp(-j^{2/3})$, then either*

(a) *For some $\ell \in \{1, \dots, 2j-2\}$, $|E - 2\cos(\frac{\pi\ell}{2j-1})| \leq e^{-\sqrt{2j-1}}$, or*
(b) *$|\bar{\theta}(j-1)^2)| \leq \exp(-(j-1)^{2/3})$ and*

$$R(j^2) \leq e^{-\beta j/2} R((j-1)^2). \tag{2.5}$$

*Proof.* By (1.19),

$$|u(j^2)| = \left|\frac{R(j^2)}{\sin k}\,\sin(\bar{\theta}(j^2))\right| \leq CR(j^2)\exp(-j^{2/3}), \tag{2.6}$$

where $C$ will be used to indicate a constant depending only on $k$. $C$ can vary from formula to formula!

Let $q = (j-1)^2$. If (recall $R(n) = R(j^2)$ if $q < n \leq j^2$)

$$|u(q)| \leq R(j^2)\exp(-j^{2/3}), \tag{2.7}$$

then $\tilde{u} \equiv u \restriction (j = q+1, \dots, j^2-1)$ is an extremely accurate trial function for $\widetilde{H}_0$, the Jacobi matrix on $(j = q+1, \dots, j^2-1)$, for by (2.6), (2.7)

$$\|(\widetilde{H}_0 - E)\tilde{u}\|^2 \leq CR(j)^2 \exp(-2j^{2/3})$$

while by an elementary estimate,

$$\|\tilde{u}\|^2 \geq CjR(j)^2.$$

Thus, by taking $j_0$ large enough, we can be certain that

$$\frac{\|(\widetilde{H}_0 - E)\tilde{u}\|}{\|u\|} \leq \exp(-\sqrt{2j-1}\,)$$

if $j \geq j_0$. Since the eigenvalues of $\widetilde{H}_0$ are $\{2\cos(\frac{\pi\ell}{2j-1})\}_{\ell=1,\dots,2j-2}$, we conclude if $j \geq j_0$ and (2.7) holds, then alternative (a) holds.

So suppose that (2.7) fails. Since $u(q)^2 + u(q+1)^2 \geq CR(q+1)^2 = CR(j^2)$, we conclude that

$$\left|\frac{u(q)}{u(q+1)}\right| \geq C\exp(-j^{2/3}). \tag{2.8}$$

Thus, using the eigenfunction equation,

$$\left|\frac{u(q-1)}{u(q)}\right| \geq e^{\beta j} - |E| - C\exp(j^{2/3}) \geq \frac{1}{2}\, e^{\beta(j-1)} \tag{2.9}$$

if $j \geq j_0$ and $j_0$ is sufficiently large.

By (1.19), (2.9) implies

$$\left|\frac{\sin(\bar{\theta}(q))}{\sin(\theta(q))}\right| \leq 2e^{-\beta(j-1)}$$

so for $j \geq j_0$ with $j_0$ large, we have $|\bar{\theta}(q)| \leq Ce^{-\beta(j-1)} \leq \exp(-(j-1)^{2/3})$ verifying one of the conclusions.

Moreover, by (2.8) and (2.9) ($C$ is a constant whose value changes!),

$$\begin{aligned} R(j)^2 = R(q+1)^2 &\leq C(u(q)^2 + u(q-1)^2) \\ &\leq Ce^{-2\beta j}[1 + \exp(2j^{2/3})]u(j-1)^2 \\ &\leq Ce^{-2\beta j}[1 + \exp(2j^{2/3})]R(q)^2 \\ &\leq e^{-\beta j} R(q)^2 \end{aligned}$$

if $j \geq j_0$ with $j_0$ large. This proves (2.5). $\square$

## §3. A Warm-up Problem

In this section, we treat an elementary tunnelling problem that, because $V$ is piecewise constant, avoids some of the technicalities we will need in the $\cos(x^\alpha)$ case. Throughout this section, let $[x] \equiv$ greatest integer less than $x$, and define

$$\begin{aligned} f(x) &= \tfrac{1}{2}(1+(-1)^{[x]}) \\ V(x) &= f(x^\alpha) \end{aligned} \tag{3.1}$$

with $0 < \alpha < 1$. Thus, $f$ is 1 (resp. 0) for $x$ in $[0,1)\cup[2,3)\cup[4,5)\cdots$ (resp. $[1,2)\cup[3,4)\cup\cdots$). $V$ consists of regions $1,2,\dots$, where $V$ is first 1, then 0, then $1,\dots$ and region $n$ runs from $(n-1)^{1/\alpha}$ to $n^{1/\alpha}$ and has approximate width $\alpha^{-1}n^{(1/\alpha)-1}$ going to infinity. We will use $Q_1, L_1, Q_2, L_2, \dots$ for the regions and $|Q_1|, |L_1|, \dots$ for their widths. $L_n$ is the $n^{\text{th}}$ region where $V$ is 0.

For each $L$, consider the potential $W_L$ which is 0 on $[0,L]$ and 1 for other $x$. Let $e_1(L) < \cdots < e_{m_L}(L)$ denote the eigenvalues of $-\frac{d^2}{dx^2}+W_L$ of energy less than 1. A Sturm oscillation argument shows that $m_L = \frac{L}{2\pi}+O(1)$ as $L\to\infty$.

Define

$$A_j = \bigcup_{k=1}^{m_{L_j}} \big(e_k(L_j) - e^{-\sqrt{|Q_j|}}, e_k(L_j) + e^{-\sqrt{|Q_j|}}\big)$$

and let

$$\bar{A} = \bigcap_{\ell=1}^{\infty} \bigcup_{j=\ell}^{\infty} A_j$$

which is a dense $G_\delta$ of $[0,1]$ of Lebesgue measure zero.

**Theorem 3.1.** *Let $H_\theta = -\frac{d^2}{dx^2}+V(x)$ with $V$ given by (3.1) with $0<\alpha<1$ and $\theta$ boundary conditions at 0. Suppose $E \in (0,1)\backslash\bar{A}$. Then*

$$\lim_{x\to\infty} \frac{1}{|x|} \ln\|T(0,x)\| = \frac{1}{2}\sqrt{1-E}\,. \tag{3.2}$$

*Proof.* Let $Q_i = (x_i, y_i)$, $L_i = (y_i, x_{i+1})$. We will look at three values of $\theta_0^{(k)}$, say, $0, \frac{\pi}{4}, \frac{\pi}{2}$ for $k=1,2,3$ and let $R_k(x), \theta_k(x)$ be the solution of the usual modified Prüfer equations with $\theta(0) = \theta_0^{(i)}$ and $R_k(0)=1$. Our goal is to prove that if $E\in(0,1)\backslash\bar{A}$, then for each $k$, either $\|T(0,x)u_{\theta_k}\|\in L^2$ or $\lim_{n\to\infty}\frac{1}{|x|}\ln R_k(x) = \frac{1}{2}\sqrt{1-E}$. Since at most one $k$ can have $\|T(0,x)u_{\theta_k}\|\in L^2$, we conclude (3.2).

In each region $L_j$, $T(y_j, x)$ is just the free transfer matrix at energy $E$ and so $\|T(y_j, x)\| \leq C$ if $x \in L_j$. Thus, $|\ln R_i(x_{j+1}) - \ln R_i(y_j)| \leq C$ and thus

$$\lim_{n\to\infty} \frac{1}{x_{n+1}} \sum_{j=1}^{n} |\ln R(x_{j+1}) - \ln R(y_j)| \to 0$$

since $x_{n+1} \sim (2n)^{1/\alpha}$ and thus $\frac{n}{x_{n+1}} \to 0$. That means we only have to control the change of $R$ in the tunnelling regions $Q_i$.

Define an angle $\eta$ in $(0, \frac{\pi}{2})$ by $\sin(\eta) = k$ so $\cos(\eta) = \sqrt{1-E}$. Then in the regions $Q_i$, (1.6) can be rewritten as

$$\frac{d\theta}{dx} = \frac{1}{k}\,(\sin^2(\eta) - \sin^2(\theta)). \tag{3.3}$$

The equation (3.3) with $\theta$ thought of as running mod $2\pi$ has a simple structure. There are four fixed points where $\sin^2(\theta) = \sin^2(\eta)$, viz. $\theta = \pm\eta$, $\pi \pm \eta$. The fixed points at $\eta$ and $\pi + \eta$ are attracting, and the ones at $-\eta$ and $\pi - \eta$ are repelling. As $x$ increases, $\theta$ moves away from the neighboring repeller and toward the neighboring attractor. For definiteness, we will talk about $\theta$ in the interval $(-\eta, \eta)$ and suppose $\eta < \frac{\pi}{2}$, but a similar argument works for any other interval.

For $x \in Q_i$,

$$\frac{1}{2k}\,V(x)\sin(2\eta) = \frac{1}{k}\,\sin(\eta)\cos(\eta) = \sqrt{1-E}$$

and one sees similarly that at the two attracting fixed points $(2k)^{-1}V(x)\sin(2\theta)$ is $\sqrt{1-E}$, and at the two repelling fixed points, it is $-\sqrt{1-E}$, and for regions near $-\eta$, $\ln R$ decreases by $(\delta x)\sqrt{1-E}$.

Fix $\epsilon$ small and define $\eta_1$ by $(2k)^{-1}\sin(2\eta_1) = \sqrt{1-E} - \epsilon$. Let $\ell_0$ be the length of $x$ it takes a solution of (3.3) to run from $-\eta_1$ to $\eta_1$. Consider the region $(x_k, x_k + |Q_k|^{2/3})$ at the start of $Q_k$. Suppose $\theta(x_k) \in (-\eta, \eta)$ in accordance with our simplifying assumption. If $\theta(x_k + |Q_k|^{2/3}) \leq -\eta_1$, then

$$\ln R(x_k + |Q_k|^{2/3}) \leq \ln R(x_k) - (\sqrt{1-E} - \epsilon)\,|Q_k|^{2/3}. \tag{3.4}$$

If $\theta(x_k + |Q_k|^{2/3}) \geq -\eta_1$, then once $x \geq x_k + |Q_k|^{2/3} + \ell_0$, we have $\frac{d\ln R}{dx} \geq \sqrt{1-E} - \epsilon$, and so

$$\ln R(y_k) \geq \ln R(x_k) + (\sqrt{1-E} - \epsilon)(|Q_k| - |Q_k|^{2/3} - \ell_0) - (|Q_k|^{2/3} + \ell_0). \tag{3.5}$$

Thus for such intervals, either (3.4) or (3.5) holds. For $(\eta, -\eta + \pi)$ intervals (or if $\eta > \frac{\pi}{2}$), we need to deal with $\sqrt{1-E} + \epsilon$ instead of $\sqrt{1-E} - \epsilon$. The net result is that

$$\left|\ln R(y_k) - \ln R(x_k) - \sqrt{1-E}\,|Q_k|\,\right| \leq \epsilon\,|Q_k| + C\ell_0 + C\,|Q_k|^{2/3}. \tag{3.6}$$

If we can show that (3.4) fails for large $k$, then for $y$ large,

$$\left|\ln R(y) - \tfrac{1}{2}\sqrt{1-E}\,|y|\,\right| \leq \tfrac{1}{2}\epsilon|y| + o(y). \tag{3.7}$$

So, since $\epsilon$ is arbitrary, we obtain the desired result.

Suppose (3.4) holds. Go back to $Q_{k-1} = (x_{k-1}, y_{k-1})$. Again, for simplicity, suppose $\theta(y_{k-1}) \in (-\eta, \eta)$. If $\theta(y_{k-1} - |Q_k|^{2/3}) \geq \eta_1$, then

$$\ln R(y_{k-1} - |Q_k|^{2/3}) \leq \ln R(y_{k-1}) - (\sqrt{1-E} - \epsilon)|Q_k|^{2/3}. \tag{3.8}$$

If $\theta(y_{k-1} - |Q_k|^{2/3}) \leq \eta_1$, then $\theta(x) \leq -\eta_1$ for $x_{k-1} \leq x \leq y_{k-1} - |Q_k|^{2/3} - \ell_0$, and (assuming $k$ is so large that $|Q_k| \geq 2|Q_k|^{2/3} + \ell_0$) we conclude that (3.4) holds for $k-1$ replacing $k$. Moreover, $R(y_k) \leq R(x_k)\exp(-\frac{1}{2}\sqrt{1-E}\,|Q_k|)$ (again for $k$ large).

If (3.4) and (3.8) hold, we can smoothly cut off $u$ at $y_{k-1} - \sqrt{Q_k}$ and $x_k + \sqrt{Q_k}$ and get a trial function for $-\frac{d^2}{dx^2} + W_{L_k}$, and so we see that $|E - e_j(|L_k|)| \leq e^{-\sqrt{Q_k}}$. As in the last section, we see that one of the following holds:

(1) $E \in \bar{A}$
(2) (3.5) holds for all large $k$ (and so (3.7) holds)
(3) $u \in L^2$.

As explained at the start of the proof, this suffices. $\square$

## §4. The Classically Allowed Region

In proving Theorem 1.1, we will break up $[0, \infty)$ into three regions where $V(x) \leq E - \epsilon_0$, where $V(x) \geq E - \epsilon_0$, and where $|V(x) - E| \leq \epsilon_0$. Here $\epsilon_0$ is a parameter we will take to zero eventually, using the fact that we can show the contribution of the $|V(x) - E| \leq \epsilon_0$ region to $\lim \frac{\ln R(x)}{x}$ is bounded by $C\epsilon_0$. In this section, we will control the contribution of the classically allowed region where $V(x) \leq E - \epsilon_0$. The goal will be to show that each oscillation of $V$ contributes at most a constant $C$ to $\ln R(x)$, so that, since $x^{-1}\#$ of oscillations $\to 0$, the classically allowed region makes no contribution to $\gamma$ (as it makes no contribution to the integral in Theorem 1.1).

**Theorem 4.1.** *Fix $0 \leq A \leq B < E$. Suppose that $V$ is a $C^1$ function on $(a, b)$ so that there is a $c \in (a, b)$ with*

(i) *$V(x)$ is monotone decreasing on $(a, c)$ and monotone increasing on $(c, b)$.*
(ii) *$A \leq V(x) \leq B$ on $(a, b)$.*

*Fix $\theta_0$ and let any $R(x)$ solve (1.7) on $[a, b]$ with $\theta(a) = \theta_0$. Then there is a constant $C$ (depending only on $A, B, E$ but not on $V, a, b$ or $\theta_0$ ) so that*

$$|\ln R(a) - \ln R(b)| \leq C.$$

*Remarks.* 1. The proof shows that one can take

$$C = \ln(E/(E-B)) + \ln((E-A)/(E-B)).$$

2. That $V$ be piecewise monotone is convenient but not critical. In general, one gets

$$C = \ln(E/(E-B)) + \frac{1}{2}\int_a^b \left| \frac{d}{dx} \ln(E - V(x)) \right| dx.$$

*Proof.* We use what we called WKB-modified Prüfer variables, that is, we let $k(x) = \sqrt{E - V(x)}$ and $R_w(x)^2 = u(x)^2 + (u'(x))^2/k(x)^2$. Then by (ii) and $A \geq 0$,

$$R(x)^2 \leq R_w(x)^2 \leq \frac{E}{E-B} R(x)^2$$

so

$$|\ln R(x) - \ln R_w(x)| \leq \tfrac{1}{2} \ln(E/(E-B)). \tag{4.1}$$

By (1.12),

$$\left| \frac{d \ln R_w(x)}{dx} \right| \leq \left| \frac{k'(x)}{k(x)} \right|$$

so

$$\begin{aligned} |\ln R_w(a) - \ln R_w(b)| &\leq \int_a^b \left| \frac{d}{dx} \ln k(x) \right| dx \\ &= \int_a^c \frac{d}{dx} \ln k(x) - \int_c^b \frac{d}{dx} \ln k(x) \\ &= \ln\left(\frac{k(c)}{k(a)}\right) + \ln\left(\frac{k(c)}{k(b)}\right) \end{aligned}$$

since $\frac{d}{dx}k \geq 0$ on $(a,c)$ and $\frac{d}{dx}k \leq 0$ on $(c,b)$. So

$$|\ln R_w(a) - \ln R_w(b)| \leq \frac{1}{2} \times 2 \ln\left(\frac{E-A}{E-B}\right). \tag{4.2}$$

(4.1) and (4.2) prove the theorem. □

## §5. The Classically Forbidden Region

Our goal in this section is to prove the following:

**Theorem 5.1.** *Let $0 < E < 2$. Let $V$ be the potential of Theorem 1.1. Let $R(x), \theta(x)$ be the solution of (1.6/1.7) for some $\theta_0$ with $\|T(x,0)u_{\theta_0}\| \notin L^2$. Suppose $E \notin \bar{A}$. Then*

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{x} \int_{\{y \mid V(y) \geq E+\epsilon_0; 0 \leq y \leq x\}} \frac{d}{dy} \ln R(y) \\ &\qquad = \frac{1}{2\pi} \int_{\{y \mid 1+\cos(y) \geq E+\epsilon_0; 0\leq y\leq 2\pi\}} (1+\cos(y) - E)^{1/2}\, dy. \end{aligned} \tag{5.1}$$

*Proof.* The argument is very similar to that in Section 3, so we will focus on the new elements. In $\{y \mid V(y) \geq E + \epsilon_0\}$, let $k(x) \equiv \sqrt{V(x) - E}$ and $\eta(x) = \text{Arcsin}(\sqrt{E/V(x)}\,)$ so that (because of the $\epsilon_0$ cutoff) $|k'(x)| \leq Cx^{1-\alpha}$ and $|\eta'(x)| \leq Cx^{1-\alpha}$. Notice that

$$\frac{1}{2k}\, V(x) \sin(2\eta(x)) = k(x). \tag{5.2}$$

Pick $\epsilon \leq \frac{1}{2} \min(\eta(x), \frac{\pi}{2} - \eta(x)) \equiv \epsilon_1$. $\epsilon_1 > 0$ because of the $\epsilon_0$ cutoff and $E > 0$. We claim that

(a) If $|\theta - \eta(x)| < \epsilon$, then $|\frac{1}{2k} V(x) \sin(2\theta) - k(x)| \leq D\epsilon$
(b) $k - \frac{V(x)}{k} \sin^2(\eta(x) - \epsilon) \geq Y > 0$ uniformly in $x$
(c) $k - \frac{V(x)}{k} \sin^2(\eta(x) + \epsilon) \leq -Y < 0$ uniformly in $x$.

Here $D$ and $Y$ are fixed $\epsilon$ independent (but they are $\epsilon_0$ dependent) non-zero constants. (a) holds by (5.2). (b), (c) follow from the monotonicity of $\sin^2$ in $(0, \frac{\pi}{2})$ and the condition $\epsilon \leq \epsilon_1$.

We claim in any interval where $V(x) \geq E+\epsilon_0$ and $|x|$ is sufficiently large, as $x$ increases, once $x \in (\eta(x) - \epsilon, \eta(x) + \epsilon) \equiv I_1$, it remains in that interval. For $\frac{d}{dx}[\theta(x) - \eta(x)] \geq E - Cx^{1-\alpha}$ at $\theta = \eta - \epsilon$ and $\leq -E + Cx^{1-\alpha}$ at $\theta = \eta + \epsilon$. Similarly, once $\theta$ leaves $(-\eta - \epsilon, \eta + \epsilon) \equiv I_2$, it stays outside it; and in a finite distance $\ell_0$, it moves from anywhere outside $I_2$ into $I_1$ (or the interval $(\pi + \eta - \epsilon, \pi + \eta + \epsilon)$).

By mimicking the arguments in Section 3, we see that either $E \in \bar{A}$ or $\|T(x,0)u_{\theta_0}\| \in L^2$ or else

$$\lim_{x\to\infty} \frac{1}{x} \int_{\{y \mid V(y) \geq E+\epsilon_0; 0 \leq y \leq x\}} \left| \frac{d}{dy} \ln R(y) - k(y) \right| \leq D\epsilon.$$

Since we can take $\epsilon$ to zero and

$$\frac{1}{x} \int_{\{y \mid V(y) \geq E+\epsilon_0; 0 \leq y \leq x\}} k(y)\, dy = \frac{1}{2\pi} \int_{\{1+\cos(y) \geq E+\epsilon_0; 0 \leq y \leq 2\pi\}} (1 + \cos(y) - E)^{1/2}\, dy$$

the theorem is proven. $\square$

## §6. Putting It Together

Here we will prove Theorem 1.1. Suppose $E \notin \bar{A}$ and $\theta_0$ is such that $\|T(x,0)u_{\theta_0}\| \notin L^2$. Let $R(x)$ solve (1.7) with $\theta(x) = \theta_0$. Fix $\epsilon_0 < 0$ and consider the three regions:

$$\begin{aligned} Z(1) &: \{x \mid V(x) \leq E - \epsilon_0\} \\ Z(2) &: \{x \mid |V(x) - E| \leq \epsilon_0\} \\ Z(3) &: \{x \mid V(x) \geq E + \epsilon_0\}. \end{aligned}$$

In Section 4, we proved that

$$\frac{1}{x} \int_{Z(1) \cap \{y \leq x\}} \left( \frac{d}{dy} \ln R(y) \right) dy \to 0.$$

In Section 5, we proved that

$$\overline{\lim}\left|\frac{1}{x}\int_{Z(3)\cap\{y\leq x\}}\left(\frac{d}{dy}\ln R(y)\right)dy-\frac{1}{2\pi}\int_{\{y|1+\cos(y)\geq E;0\leq y\leq 2\pi\}}(1-\cos(y)-E)^{1/2}\,dy\right| \leq D_0\epsilon_0$$

for a constant $D_0$.

By (1.7), $|\frac{d}{dy}\ln R(y)| \leq \frac{2}{2k}$. Moreover, it is clear that $\overline{\lim}\,\frac{1}{x}|Z(2)\cap\{y\leq x\}|\leq D_1\epsilon_0$ for some constant $D_1$.

Thus, we have

$$\overline{\lim}\left|\frac{1}{x}\left[\ln R(x)-\ln R(0)\right]-\frac{1}{2\pi}\int_{\{y|1+\cos(y)\leq E;0\leq y\leq 2\pi\}}(1+\cos(y)-E)^{1/2}\,dy\right|\leq D_2\epsilon_0$$

with $D_2 = D_0 + \frac{D_1}{2k}$. Taking $\epsilon_0$ to zero, we prove that

$$\frac{1}{x}\ln R(x)\to\frac{1}{2\pi}\int_{\{y|1+\cos(y)\geq E;0\leq y\leq 2\pi\}}(1+\cos(y)-E)^{1/2}\,dy.$$

Since at most one $\theta_0$ has $\|T(x,0)u_{\theta_0}\|\in L^2$, we see that $\frac{1}{x}\ln\|T(x,0)\|$ has the required limit. ☐

## Appendix: WKB Prüfer Variables and Bounded Transfer Matrices

In this appendix, we will show how to use WKB-Prüfer variables to show for $E > 1$, the transfer matrix for $\cos(x^\alpha)$ potentials is bounded. This is a result of Behncke [1] and Stolz [18] whose proof is not unrelated. Their method is basically a variation of parameters, and this appendix reiterates the idea of [10] that modified Prüfer variables are often a suitable replacement for variation of parameters.

Recall the definition (1.13) for $\tilde{R}_w(x)$ and $\tilde{\theta}_w(x)$. They obey

$$\frac{d\theta_w}{dx} = k(x)+\frac{1}{2}\,\frac{k'}{kx}\,\sin(2\theta_2(x)) \tag{A.1}$$

$$\frac{d\ln\tilde{R}_w}{dx} = -\frac{k'}{2k}\,\cos(2\theta_w(x)). \tag{A.2}$$

Let $V(x) = \cos(x^\alpha)$, with $\frac{1}{2} < \alpha < 1$ and $E > 1$. Then $k(x) = \sqrt{E-V(x)} > \sqrt{|E-1|}$ is bounded away from zero. Moreover, we have for $j = 0, 1, 2, \dots$ and $|x| \geq 1$:

$$\left|\frac{d^j k(x)}{dx^j}\right| \leq C_j(1+|x|)^{-j(1-\alpha)} \tag{A.3}$$

$$\left|\frac{d^j}{dx^j}\left(\frac{k'}{k}\right)\right| \leq D_j(1+|x|)^{-(j+1)(1-\alpha)}. \tag{A.4}$$

In particular, for $x$ large, $\frac{d\theta}{dx} \geq \sqrt{|E-1|} - D_0(1+|x|)^{-(1-\alpha)} > 0$. By (A.1) and (A.3/A.4), we see

$$\left| \frac{d^2}{dx^2}\, \theta_w(x) \right| \leq x^{-(1-\alpha)}. \tag{A.5}$$

Integrate (A.2) to get (where $x_0$ is picked so large that $k(x) > \delta > 0$ for $x > x_0$)

$$\ln \tilde{R}_w(x) - \ln \tilde{R}_w(x_0) = \int_x^{x_0} -\frac{k'}{4k}\, \frac{1}{\frac{d\theta_w}{dx}}\, \frac{d}{dx}\, (\sin(2\theta_w(x)))\, dx$$

and integrate by parts. By (A.4) and (A.5), the integrand bounded by $(1+|x|)^{-2(1-\alpha)}$ is integrable, so $\tilde{R}_w(x)$ is bounded.

*Remarks.* 1. One doesn't gain anything by iterating the integration by parts because $\frac{d^j}{dx^j}\theta_w(x)$ only falls as $(1+|x|)^{-2(1-\alpha)}$.

2. One also sees by integrating by parts that $\theta_w(x) - \int_{x_0}^x k(y)\, dy$ has a limit, and so one can prove there are WKB-type solutions.

3. By using higher-order modifications, it should be possible to accommodate $0 < \alpha \leq \frac{1}{2}$.

4. All this proof requires, if one keeps track of the derivatives, is that $V$ is $C^2$ and

(i) $\sup_x V(x) = V_+ < \infty$, $\inf_x V(x) > -\infty$
(ii) $V'(x) \to 0$ as $x \to \infty$
(iii) $V' \in L^2$, $V'' \in L^1$.

One obtains a bounded transfer matrix if $E > V_+$.

5. The point of this is that bounded transfer matrices imply purely a.c. spectrum [1,11,18].

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