Set theory and dynamical systems

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ABSTRACT. We give an introduction to a recent direction of research in set theory, developed primarily over the last 15–20 years, and discuss its connections with aspects of dynamical systems and in particular rigidity phenomena in the context of ergodic theory.

1. Introduction

The general context of this work is the development of a theory of complexity of classification problems in mathematics. From another point of view it can be thought of as the study of "definable" or Borel cardinality theory of quotient spaces (vs. the "classical" or Cantor cardinality theory).

Classification Problems. A classification problem is given by:

- A collection of objects X.
- An equivalence relation E on X.

A *complete classification* of X up to E consists of:

- A set of invariants I.
- A map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$.

For this to be of any interest both I, c must be as explicit and concrete as possible. Here are some examples of classification problems and their complete invariants:

EXAMPLE 1. *Classification of finitely generated (f.g.) abelian groups up to isomorphism.*
INVARIANTS: *(Essentially) finite sequences of integers.*

EXAMPLE 2. *Classification of Bernoulli automorphisms up to conjugacy by entropy* (Ornstein).
INVARIANTS: *Reals.*

EXAMPLE 3. *Classification of increasing homeomorphisms of* $[0, 1]$ *up to conjugacy.*
INVARIANTS: *(Essentially) countable linear orderings up to isomorphism.*

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Most often the collection of objects we try to classify can be viewed as forming a "nice" space, namely a standard Borel space, i.e., a Polish (complete separable metric) space with its associated Borel structure and the equivalence relation E turns out to be *Borel* or *analytic* (as a subset of X^2). We will concentrate primarily below on Borel equivalence relations.

The theory of Borel equivalence relations studies the set-theoretic nature of possible (complete) invariants and develops a mathematical framework for measuring the complexity of classification problems.

The following simple concept is basic in organizing this study.

Definition 1. *Let $(X, E), (Y, F)$ be Borel equivalence relations. Then E is (Borel)* reducible *to F, in symbols*

$$E \leq_B F,$$

if there is Borel map $f : X \to Y$ such that

$$x \; E \; y \Leftrightarrow f(x) \; F \; f(y).$$

The intuitive meaning of this concept can be expressed in two ways:

- The classification problem represented by E is at most as complicated as that of F.
- F-classes are complete invariants for E.

Definition 2. *E is* bi-reducible *to F if E is reducible to F and vice versa. Let*

$$E \sim_B F \Leftrightarrow E \leq_B F \text{ and } F \leq_B E.$$

We also put:

Definition 3.

$$E <_B F \Leftrightarrow E \leq_B F \text{ and } F \not\leq_B E.$$

Let us now discuss the previous and some further examples in the light of this concept.

Example 4. *(Isomorphism of f.g. abelian groups) $\sim_B$ $(=_{\mathbb{N}})$.*

Example 5. *(Conjugacy of Bernoulli automorphisms) $\sim_B$ $(=_{\mathbb{R}})$.*

Example 6. *(Isomorphism of torsion-free abelian groups of rank 1) $\sim_B$ E_0* (Baer)*, where E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by*

$$x \; E_0 \; y \Leftrightarrow \exists n \forall m \geq n (x_m = y_m).$$

Example 7. *(Conjugacy of discrete spectrum measure preserving transformations) $\sim_B$ E_c* (Halmos-von Neumann)*, where E_c is the equivalence relation on $\mathbb{T}^{\mathbb{N}}$ given by*

$$(x_n) \; E_c \; (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$

Example 8. *(Conjugacy of increasing homeomorphisms of $[0, 1]$) $\sim_B$ (Isomorphism of countable linear orderings).*

Borel cardinality theory. The preceding concepts can be also interpreted as the basis of a "definable" or Borel cardinality theory for quotient spaces.

- $E \le_B F$ means that there is a Borel injection of X/E into Y/F, i.e., an injection that has a Borel lifting to X, Y. This can be understood as saying that X/E has Borel cardinality less than or equal to that of Y/F, in symbols
$$|X/E|_B \le |Y/F|_B.$$
- $E \sim_B F$ means that X/E and Y/F have the same Borel cardinality, in symbols
$$|X/E|_B = |Y/F|_B.$$
- $E <_B F$ means that X/E has strictly smaller Borel cardinality than Y/F, in symbols
$$|X/E|_B < |Y/F|_B.$$

2. The Borel reducibility hierarchy

Below X stands for the equality relation on X, $=_X$. We clearly have:

$$1 <_B 2 <_B 3 \cdots <_B \mathbb{N} <_B E$$

and this is an initial segment of the Borel reducibility hierarchy. The first non-trivial result is now the following:

THEOREM 1 (Silver [**18**]). *For every Borel E, either $E \le_B \mathbb{N}$ or $\mathbb{R} \le_B E$.*

Thus we have the following continuation of the hierarchy:

$$1 <_B 2 <_B 3 \cdots <_B \mathbb{N} <_B \mathbb{R} <_B E$$

Note that $E \le_B \mathbb{R}$ means that there is a standard Borel space Y and a Borel map $f : X \to Y$ such that $x \; E \; y \Leftrightarrow f(x) = f(y)$. Such E are called *concretely classifiable* or *smooth*. A canonical example of a non-smooth E is the equivalence relation E_0 defined above. So $\mathbb{R} <_B E_0$.

The next step in the hierarchy is given by the following result:

THEOREM 2 (Harrington-Kechris-Louveau [**7**]). *For any Borel E, either $E \le_B \mathbb{R}$ or $E_0 \le_B E$.*

This is called the *General Glimm-Effros Dichotomy* because its first special instances where discovered by Glimm [**6**] and Effros [**3**] in connection with work in operator algebras.

Thus we have:

$$1 <_B 2 <_B 3 \cdots <_B \mathbb{N} <_B \mathbb{R} <_B E_0 <_B E$$

and this is an initial segment of the reducibility hierarchy.

The proofs of these two dichotomies, which are about very simple classical concepts of descriptive set theory, i.e., Borel sets and functions, use methods of *effective descriptive set theory*, which are based on computability theory, i.e., the theory of algorithms, Turing machines, etc. No "classical" type proofs are known.

This hierarchy of Borel cardinalities looks so far like the wellordered hierarchy of Cantor cardinalities. However the linearity of $\leq_B$ breaks down after E_0. Various examples have been discovered rather early in this theory. Here are some relatively more recent ones:

EXAMPLE 9 (Kechris-Louveau [**15**]). *The following equivalence relations on $\mathbb{R}^{\mathbb{N}}$ are incomparable:*

$$(x_n)\ E_1\ (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m)$$
$$(x_n)\ E_2\ (y_n) \Leftrightarrow \lim_{n\to\infty} (x_n - y_n) = 0$$

So the picture is as follows:

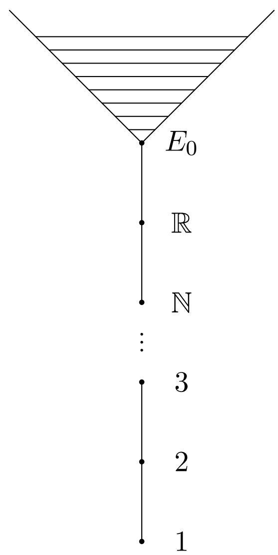

FIGURE 1

The Borel equivalence relations above E_0 that have been analyzed so far fall into exactly 4 types and it may be that they all do. This is partially supported by a series of results of Hjorth, Kechris, Louveau and others, see, e.g., Hjorth-Kechris [**11**] for more detailed explanations. Below we use the following definitions.

DEFINITION 4. *For a Polish group G, Polish space X, and a continuous or Borel action of G on X, we denote by E_G^X the induced (orbit) equivalence relation. (Equivalence relations of the form E_G^X are analytic but not necessarily Borel.)*

DEFINITION 5. *S_∞ is the infinite symmetric group.*

Definition 6. Γ *denotes an arbitrary countable (discrete) group.*

Definition 7.

$$(x_n)\ E_0\ (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m),\ \textit{on } 2^{\mathbb{N}}$$

$$(x_n)\ E_1\ (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m),\ \textit{on } \mathbb{R}^{\mathbb{N}}$$

$$(x_n)\ E_2\ (y_n) \Leftrightarrow \lim_{n\to\infty} (x_n - y_n) = 0,\ \textit{on } \mathbb{R}^{\mathbb{N}}$$

$$E_3 = (E_0)^{\mathbb{N}},\ \textit{on } (2^{\mathbb{N}})^{\mathbb{N}}$$

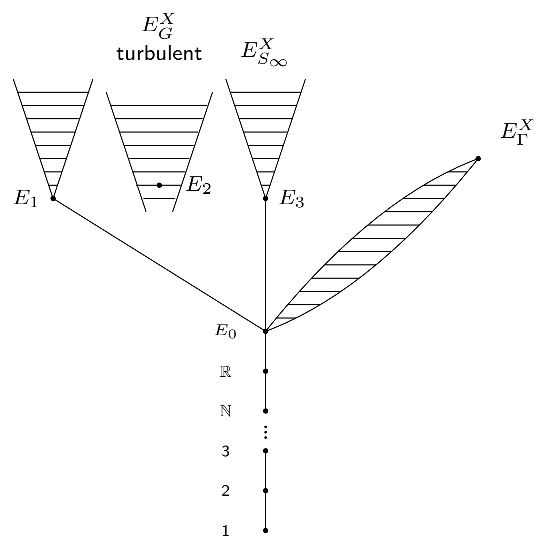

Figure 2

This figure shows the division of Borel equivalence relations (above E_0) into four types which we label (1)–(4), starting from the left. Class (1) consists of all $E \geq E_1$. All the other classes (2)–(4) consist of E that satisfy $E \leq E_G^X$, for some G, X as in Definition 4. Class (2) consists of all such E that are above, in the sense of $\leq_B$, some particularly complex E_G^X, i.e., those induced by the so-called *turbulent actions*, see Hjorth [**8**]. Class (3) consists of the equivalence relations E that are $\leq_B E_{S_\infty}^X$ for some action of S_∞ and are also $\geq_B E_3$. Finally, class (4) consists of all E that are $\leq_B E_\Gamma^X$ for some countable group Γ.

We emphasize that this is only a conjectural picture. Although counterexamples may be found that do not belong to any of these classes, the picture helps organize the study of Borel equivalence relations and it may very well be the case that most natural examples fall into one of these four classes.

3. Countable Borel equivalence relations

We will now concentrate on the Borel equivalence relations in the fourth class of Figure 2, i.e., those that are Borel reducible to equivalence relations of the form E_Γ^X, for some countable group Γ. The latter are also called *countable* in view of what follows.

Definition 8. *E is* countable *if every E-class is countable.*

Example 10. *Any equivalence relation, E_Γ^X, induced by a Borel action of a countable group Γ on X.*

We actually have:

Theorem 3 (Feldman-Moore [**4**]). *Every countable Borel equivalence relation E is of the form E_Γ^X.*

A totally different example of a countable Borel equivalence relation familiar to logicians is the following:

Example 11. *Turing equivalence.*

There are also many Borel equivalence relations, which although not literally countable, fall in this domain, since they are Borel bireducible to countable ones. Here are some representative examples.

Example 12 (Kechris [**14**]). *E_G^X for G a second countable locally compact group (e.g., a Lie group).*

Example 13 (Hjorth-Kechris [**9**]). *Isomorphism of countable structures that are of "finite type", e.g., finitely generated groups, locally finite trees, finite rank torsion-free abelian groups, finite transcendence degree fields, etc.*

Example 14 (Hjorth-Kechris [**10**]). *Conformal equivalence of Riemann surfaces*

We will now consider the structure of $\leq_B$ on the countable Borel equivalence relations. We refer the reader to Dougherty–Jackson–Kechris [**2**] and Jackson–Kechris–Louveau [**13**] for more information.

The simplest countable equivalence relations are the smooth ones, which have a trivial structure. The next more complicated ones are the so-called hyperfinite.

Definition 9. *E is* hyperfinite *if $E = \bigcup_n E_n$, with E_n Borel, increasing and finite (i.e., having equivalence classes that are finite).*

Theorem 4 (Slaman-Steel [**19**], Weiss [**22**]). *E is hyperfinite iff it is of the form $E_{\mathbb{Z}}^X$.*

Which groups always give hyperfinite equivalence relations? A necessary condition is that they have to be amenable (i.e., admit a left-invariant finitely additive probability measure). The following asks whether the converse is true.

PROBLEM 1 (Weiss [**22**]). *If* Γ *is amenable, is* E_Γ^X *hyperfinite?*

For finitely generated groups, the next result is essentially all that is known so far.

THEOREM 5 (Jackson–Kechris–Louveau [**13**]). *If* Γ *is finitely generated of polynomial growth, then* E_Γ^X *is hyperfinite.*

Very recently, Gao-Jackson [**5**] proved that this is also true for any abelian group Γ.

The hyperfinite equivalence relations have been classified both under bireducibilty and isomorphism.

THEOREM 6 (Dougherty–Jackson–Kechris [**2**]). *i) Up to Borel bireducibility, there is only one non-smooth, hyperfinite equivalence relation, namely* E_0*.*

ii) Up to Borel isomorphism, there are exactly countably many non-smooth, aperiodic (i.e., having infinite classes), hyperfinite equivalence relations, namely

$$E_t, E_0, 2E_0, 3E_0, \dots, nE_0, \aleph_0 E_0, E_s.$$

Here nE_0 is the direct sum of n copies of E_0 for $1 \leq n \leq \aleph_0$, E_t is the *tail equivalence relation* on $2^{\mathbb{N}}$, i.e., $xE_t y \Leftrightarrow \exists n \exists m \forall k (x_{n+k} = y_{m+k})$ and E_s is the aperiodic part of the *shift equivalence relation* on $2^{\mathbb{Z}}$.

The hyperfinite equivalence relations are the simplest non-trivial countable equivalence relations. At the other end are the most complex ones, the so-called *universal* ones.

THEOREM 7 (see [**13**]). *There is a universal countable Borel equivalence relation,* E_∞*, i.e., one that satisfies* $E \leq_B E_\infty$*, for all countable* E*.*

EXAMPLE 15. $E_\infty \sim_B$ *(the shift equivalence relation on* 2^{F_2}*).*

Here F_n is the free group with n generators. The following shows that E_∞ is not hyperfinite.

THEOREM 8 (see [**13**]).

$$E_0 <_B E_\infty.$$

There are countable equivalence relations that are neither hyperfinite nor universal.

THEOREM 9 (see [**13**]). *There exist intermediate countable Borel equivalence relations* E*, i.e.,*

$$E_0 <_B E <_B E_\infty.$$

EXAMPLE 16. $E =$ *(the free part of the shift equivalence relation on* 2^{F_2}*).*

This is a typical example of a *treeable* equivalence relation. These were first studied by S. Adams in ergodic theory.

Since the early 1990's only a small finite number of intermediate equivalence relations were known and they were linearly ordered under $\leq_B$. This lead to the following basic problems:

- Are there infinitely many?
- Does non-linearity occur here?

These were answered by the following result.

THEOREM 10 (Adams-Kechris [**1**]). *Every Borel partial order embeds into $\leq_B$ on the countable equivalence relations.*

In Figure 3 we give a schematic picture of the structure of countable Borel equivalence relations. On the left side of the figure we have listed some representative examples of classification problems whose complexity is measured by equivalence relations in this domain. For example, the isomorphism problem for finitely generated groups is bireducible to E_∞ (Thomas–Velickovic [**21**]).

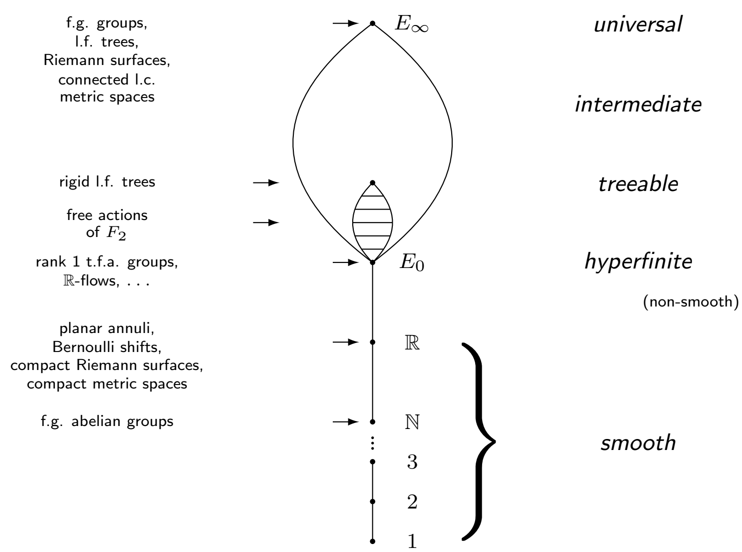

FIGURE 3

4. Set theoretic rigidity

The proof of the preceding theorem of Adams-Kechris used *Zimmer's cocycle superrigidity theory* for ergodic actions of linear algebraic groups and their lattices, see Zimmer [**23**].

The key point is that there is a phenomenon of *set theoretic rigidity* analogous to the *measure theoretic rigidity* phenomenon discovered by Zimmer.

- **(Measure theoretic rigidity)** Under certain circumstances, when a countable group acts preserving a probability measure, the equivalence relation associated with the action together with the measure "encode" or "remember" a lot about the group (and the action).
- **(Set theoretic rigidity)** Such information is simply encoded in the Borel cardinality of the (quotient) orbit space.

To illustrate this, let us mention some set theoretic rigidity results.

THEOREM 11 (Adams-Kechris [**1**]).

$$|\mathbb{T}^m/\mathrm{GL}_m(\mathbb{Z})|_B = |\mathbb{T}^n/\mathrm{GL}_n(\mathbb{Z})|_B \Leftrightarrow m = n.$$

Here $\mathrm{GL}_m(\mathbb{Z})$ is the group of $n \times n$ matrices in $\mathbb{Z}$ with determinant ± 1. It acts in the obvious way on $\mathbb{T}^n$. This result shows that the dimension is coded in the Borel cardinality of the quotient space. It also implies the existence of infinitely many distinct up to $\sim_B$ countable Borel equivalence relations.

Below $\Gamma_p = \mathrm{SO}_7(\mathbb{Z}[1/p])$, p prime. Also E_p is the free part of the shift equivalence relation on 2^{Γ_p}.

THEOREM 12 (Adams-Kechris [**1**]).

$$E_p \leq_B E_q \Leftrightarrow p = q.$$

In particular this shows that there are infinitely many incomparable under $\leq_B$ countable Borel equivalence relations.

Below let $\cong_n$ be isomorphism of torsion-free abelian groups of rank at most n, i.e., subgroups of $(\mathbb{Q}^n, +)$. This can be seen to be (up to $\sim_B$) a countable Borel equivalence relation.

THEOREM 13 (S. Thomas, [**20**]).

$$(\cong_m) \sim_B (\cong_n) \Leftrightarrow m = n.$$

Thus the rank is encoded in the Borel cardinality of the isomorphism types. This result has important implications for the classical classification problem for finite rank torsion-free abelian groups.

Recently Hjorth-Kechris [**12**] developed a set theoretic rigidity theory for product groups that has several applications in the study of countable Borel equivalence relations – but also in ergodic theory. They also use ergodic theoretic methods, like cocycle reduction techniques, actions on boundaries, etc. (Also, independently, Monod-Shalom [**16**] and Popa [**17**] have recently proved important rigidity results for product groups in the context of ergodic theory – it is yet unclear what is the relationship between these theories.)

Here are a few results from the work of Hjorth-Kechris. Below, for any group Γ, we let E_Γ be the free part of the shift equivalence relation on 2^Γ.

THEOREM 14 (Hjorth–Kechris [**12**]).

$$E_{(\mathbb{Z}_p \star \mathbb{Z}_p)\times\mathbb{Z}} \leq_B E_{(\mathbb{Z}_q \star \mathbb{Z}_q)\times\mathbb{Z}} \Leftrightarrow p = q.$$

Note however that:

$$E_{(\mathbb{Z}_p \star \mathbb{Z}_p)} \sim_B E_{(\mathbb{Z}_q \star \mathbb{Z}_q)}.$$

The next result concerns the distinction between the equivalence relation $E_{F_2^n}$ induced by the shift action of the product of n copies of F_2 (*shift of the product*) and the product equivalence relation of n copies of the shift action of F_2, i.e., $(E_{F_2})^n$ (*product of the shift*). It can be best summarized in a picture.

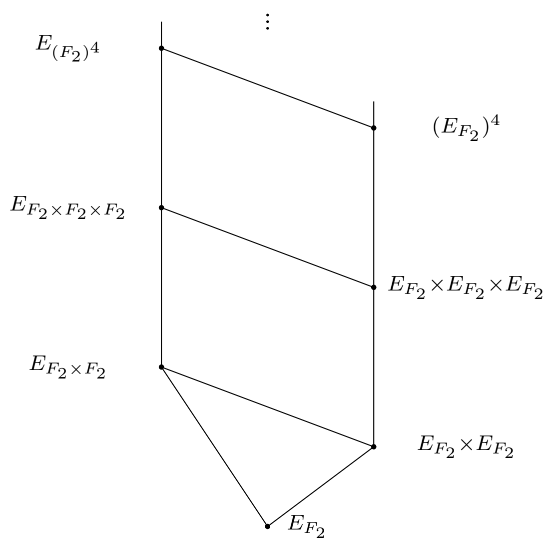

FIGURE 4

Finally an application to ergodic theory.

THEOREM 15 (Hjorth–Kechris [**12**]). *Suppose H_0, H_1 are non-amenable, torsion-free, hyperbolic groups and Δ_0, Δ_1 are infinite amenable groups. Let each $H_i \times \Delta_i$ act freely on X_i with invariant, probability measure, so that the action is ergodic on Δ_i, $i = 1, 2$. If the action of $H_0 \times \Delta_0$ is (stably) orbit equivalent to the action of $H_1 \times \Delta_1$, then $H_0 \cong H_1$.*

We conclude with the following remark about methodology. The theory of countable Borel equivalence relations points to an interesting phenomenon. Although one is dealing here with very simple set theoretic notions (countable Borel equivalence relations and Borel reducibility) most basic questions about them (like existence of intermediate or incomparable ones) have been answered by using rather sophisticated ergodic theory methods, and this certainly represents an interesting application of ergodic theory to set theory. At this time no other methods to study these problems are known.

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