THE COMPLEXITY OF CLASSIFICATION PROBLEMS IN ERGODIC THEORY

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The last two decades have seen the emergence of a theory of set theoretic complexity of classification problems in mathematics. In these lectures we will discuss recent developments concerning the application of this theory to classification problems in ergodic theory.

The first lecture will be devoted to a general introduction to this area. The next two lectures will give the basics of Hjorth's theory of turbulence, a mixture of topological dynamics and descriptive set theory, which is a basic tool for proving strong non-classification theorems in various areas of mathematics.

In the last three lectures, we will show how these ideas can be applied in proving a strong non-classification theorem for orbit equivalence. Given a countable group Γ, two free, measure-preserving, ergodic actions of Γ on standard probability spaces are called *orbit equivalent* if, roughly speaking, they have the same orbit spaces. More precisely this means that there is an isomorphism of the underlying measure spaces that takes the orbits of one action to the orbits of the other. A remarkable result of Dye and Ornstein-Weiss asserts that any two such actions of amenable groups are orbit equivalent. Our goal will be to outline a proof of a dichotomy theorem which states that for any non-amenable group, we have the opposite situation: The structure of its actions up to orbit equivalence is so complex that it is impossible, in a vey strong sense, to classify them (Epstein-Ioana-Kechris-Tsankov).

Beyond the method of turbulence, an interesting aspect of this proof is the use of many diverse of tools from ergodic theory. These include: unitary representations and their associated Gaussian actions; rigidity properties of the action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus and separability arguments (Popa, Ioana), Epstein's co-inducing construction for generating actions of a group from actions of another, quantitative aspects of inclusions of equivalence relations (Ioana-Kechris-Tsankov) and the use of percolation on Cayley graphs of groups and the theory of costs in proving a measure theoretic analog of the von Neumann Conjecture, concerning the "inclusion" of free groups in non-amenable ones (Gaboriau-Lyons). Most of these tools will be introduced as needed along the way and no prior knowledge of them is required.

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1. Lecture I. A Survey

A. Classification problems in ergodic theory.

Definition 1.1. A *standard measure space* is a measure space (X, μ), where X is a standard Borel space and μ a non-atomic Borel probability measure on X.

All such spaces are isomorphic to the unit interval with Lebesgue measure.

Definition 1.2. A *measure-preserving transformation* (mpt) on (X, μ) is a measurable bijection T such that $\mu(T(A)) = \mu(A)$, for any Borel set A.

Examples 1.3.

- $X = \mathbb{T}$ with the usual measure; $T(z) = az$, where $a \in \mathbb{T}$, i.e., T is a *rotation*.
- $X = 2^{\mathbb{Z}}, T(x)(n) = x(n-1)$, i.e., the *shift transformation*.

Definition 1.4. A mpt T is *ergodic* if every T-invariant measurable set has measure 0 or 1.

Any irrational, modulo π, rotation and the shift are ergodic. The *ergodic decomposition theorem* shows that every mpt can be canonically decomposed into a (generally continuous) direct sum of ergodic mpts.

In ergodic theory one is interested in classifying ergodic mpts up to various notions of equivalence. We will consider below two such standard notions.

- *Isomorphism or conjugacy*: A mpt S on (X, μ) is isomorphic to a mpt T on (Y, ν), in symbols $S \cong T$, if there is an isomorphism φ of (X, μ) to (Y, ν) that sends S to T, i.e., $S = \varphi^{-1} T \varphi$.
- *Unitary isomorphism*: To each mpt T on (X, μ) we can assign the unitary (Koopman) operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$ given by $U_T(f)(x) = f(T^{-1}(x))$. Then S, T are unitarily isomorphic, in symbols $S \cong^u T$, if U_S, U_T are isomorphic.

Clearly $\cong$ implies $\cong^u$ but the converse fails.

We state two classical classification theorems:

- (Halmos-von Neumann [HvN42]) An ergodic mpt has *discrete spectrum* if U_T has discrete spectrum, i.e., there is a basis consisting of eigenvectors. In this case the eigenvalues are simple and form a (countable) subgroup of $\mathbb{T}$. It turns out that up to isomorphism these are exactly the ergodic rotations in compact metric groups G : $T(g) = ag$, where $a \in G$ is such that $\{a^n : n \in \mathbb{Z}\}$ is dense in G. For such T, let $\Gamma_T \leq \mathbb{T}$ be its group of eigenvalues. Then we have:

$$S \cong T \Leftrightarrow S \cong^u T \Leftrightarrow \Gamma_S = \Gamma_T.$$

- (Ornstein [Orn70]) Let $Y = \{1, \ldots, n\}$, $\bar{p} = (p_1, \cdots, p_n)$ a probability distribution on Y and form the product space $X = Y^{\mathbb{Z}}$ with the product measure μ. Consider the *Bernoulli shift* $T_{\bar{p}}$ on X. Its *entropy* is the real number $H(\bar{p}) = -\sum_i p_i \log p_i$. Then we have:

$$T_{\bar{p}} \cong T_{\bar{q}} \Leftrightarrow H(\bar{p}) = H(\bar{q})$$

(but all the shifts are unitarily isomorphic).

We will now consider the following question: Is it possible to classify, in any reasonable way, general ergodic mpts?

We will see how ideas from descriptive set theory can throw some light on this question.

B. Complexity of classification.

We will next give an introduction to recent work in set theory, developed primarily over the last two decades, concerning a theory of complexity of classification problems in mathematics, and then discuss its implications to the above problems.

A classification problem is given by:

- A collection of objects X.
- An equivalence relation E on X.

A *complete classification* of X up to E consists of:

- A set of invariants I.
- A map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$.

For this to be of any interest both I, c must be as explicit and concrete as possible.

Example 1.5. Classification of Bernoulli shifts up to isomorphism (Ornstein).

Invariants: Reals.

Example 1.6. Classification of ergodic measure-preserving transformations with discrete spectrum up to isomorphism (Halmos-von Neumann).

Invariants: Countable subsets of $\mathbb{T}$.

Example 1.7. Classification of unitary operators on a separable Hilbert space up to isomorphism (Spectral Theorem).

INVARIANTS: Measure classes, i.e., probability Borel measures on a Polish space up to measure equivalence.

Most often the collection of objects we try to classify can be viewed as forming a "nice" space, namely a standard Borel space, and the equivalence relation E turns out to be *Borel* or *analytic* (as a subset of X^2).

For example, in studying mpts the appropriate space is the Polish group $\mathrm{Aut}(X, \mu)$ of mpts of a fixed (X, μ), with the so-called weak topology. (As usual, we identify two mpts if they agree a.e.) Isomorphism then corresponds to conjugacy in that group, which is an analytic equivalence relation. Similarly unitary isomorphism is an analytic equivalence relation (in fact, it is Borel, using the Spectral Theorem). The ergodic mpts form a G_δ set in $\mathrm{Aut}(X, \mu)$.

The theory of equivalence relations studies the set-theoretic nature of possible (complete) invariants and develops a mathematical framework for measuring the complexity of classification problems.

The following simple concept is basic in organizing this study.

Definition 1.8. Let (X, E), (Y, F) be equivalence relations. E is *(Borel) reducible* to F, in symbols

$$E \leq_B F,$$

if there is Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

Intuitively this means:

- The classification problem represented by E is at most as complicated as that of F.
- F-classes are complete invariants for E.

Definition 1.9. E is *(Borel) bi-reducible* to F if E is reducible to F and vice versa:

$$E \sim_B F \Leftrightarrow E \leq_B F \text{ and } F \leq_B E.$$

We also put:

Definition 1.10.

$$E <_B F \Leftrightarrow E \leq_B F \text{ and } F \not\leq_B E.$$

Example 1.11. (Isomorphism of Bernoulli shifts) $\sim_B$ $(=_\mathbb{R})$

Example 1.12. (Isomorphism of ergodic discrete spectrum mpts)$\sim_B$ E_c, where E_c is the equivalence relation on $\mathbb{T}^{\mathbb{N}}$ given by

$$(x_n)\ E_c\ (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$

Example 1.13. (Isomorphism of unitary operators)$\sim_B$ ME, where ME is the equivalence relation on the Polish space of probability Borel measures on $\mathbb{T}$ given by

$$\mu \text{ ME } \nu \Leftrightarrow \mu \ll \nu \text{ and } \mu \ll \nu.$$

The preceding concepts can be also interpreted as the basis of a "definable" or Borel cardinality theory for quotient spaces.

- $E \leq_B F$ means that there is a Borel injection of X/E into Y/F, i.e., X/E has Borel cardinality less than or equal to that of Y/F, in symbols
$$|X/E|_B \leq |Y/F|_B$$
(A map $f : X/E \to Y/F$ is called Borel if it has a Borel *lifting* $f^* : X \to Y$, i.e., $f([x]_E) = [f^*(x)]_F$.)
- $E \sim_B F$ means that X/E and Y/F have the same Borel cardinality, in symbols
$$|X/E|_B = |Y/F|_B$$
- $E <_B F$ means that X/E has strictly smaller Borel cardinality than Y/F, in symbols
$$|X/E|_B < |Y/F|_B$$

C. Non-classification results for isomorphism and unitary isomorphism.

Definition 1.14. An equivalence relation E on X is called *concretely classifiable* if $E \leq_B (=_Y)$, for some Polish space Y, i.e., there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x) = f(y)$.

Thus isomorphism of Bernoulli shifts is concretely classifiable. However in the 1970's Feldman showed that this fails for arbitrary mpts (in fact even for the so-called K-automorphisms, a more general class of mpts than Bernoulli shifts).

Theorem 1.15 (Feldman [Fel74]). *Isomorphism of ergodic mpts is not concretely classifiable.*

One can also see that isomorphism of ergodic discrete spectrum mpts is not concretely classifiable.

An equivalence relation is called *classifiable by countable structures* if it can be Borel reduced to isomorphism of countable structures (of some given type, e.g., groups, graphs, linear orderings, etc.). More precisely, given a countable language L, denote by X_L the space of L-structures with universe $\mathbb{N}$. This is a Polish space. Denote by $\cong$ the equivalence relation of isomorphism in X_L. We say that an equivalence relation is classifiable by countable structures if it is Borel reducible to isomorphism on X_L, for some L.

Such types of classification occur often, for example, in operator algebras, topological dynamics, etc.

It follows from the Halmos-von Neumann theorem that isomorphism (and unitary isomorphism) of ergodic discrete spectrum mpts is classifiable by countable structures. On the other hand we have:

Theorem 1.16 (Kechris-Sofronidis [KS01]). ME *is not classifiable by countable structures and thus isomorphism of unitary operators is not classifiable by countable structures.*

Theorem 1.17 (Hjorth [Hjo01]). *Isomorphism and unitary isomorphism of ergodic mpts cannot be classified by countable structures.*

This has more recently been strengthened as follows:

Theorem 1.18 (Foreman-Weiss [FW04]). *Isomorphism and unitary isomorphism of ergodic mpts cannot be classified by countable structures on any generic class of ergodic mpts.*

One can now in fact calculate the exact complexity of unitary isomorphism.

Theorem 1.19 (Kechris [Kec10]).

i) *Unitary isomorphism of ergodic mpts is Borel bireducible to measure equivalence.*
ii) *Measure equivalence is Borel reducible to isomorphism of ergodic mpts.*

While isomorphism of ergodic mpts is clearly analytic, Foreman-Rudolph-Weiss also showed the following:

Theorem 1.20 (Foreman-Rudolph-Weiss [FRW06]). *Isomorphism of ergodic mpts is not Borel.*

However recall that unitary isomorphism of mpts is Borel.

It follows from the last two theorems that

$$(\cong^u) \ <_B \ (\cong),$$

i.e., isomorphism of ergodic mpts is strictly more complicated than unitary isomorphism.

We have now seen that the complexity of unitary isomorphism of ergodic mpts can be calculated exactly and there are very strong lower bounds for isomorphism but its exact complexity is unknown. An obvious upper bound is the universal equivalence relation induced by a Borel action of the automorphism group of the measure space (see [BK96] for this concept).

Problem 1.21. Is isomorphism of ergodic mpts Borel bireducible to the universal equivalence relation induced by a Borel action of the automorphism group of the measure space?

More generally one also considers in ergodic theory the problem of classifying measure-preserving actions of countable (discrete) groups Γ on standard measure spaces. The case $\Gamma = \mathbb{Z}$ corresponds to the case of single transformations. We will now look at this problem from the point of view of the preceding theory.

We will consider again isomorphism (also called conjugacy) and unitary isomorphism of actions. Two actions of the group Γ are isomorphic if there is a measure-preserving isomorphism of the underlying spaces that conjugates the actions. They are unitarily isomorphic if the corresponding unitary representations (the Koopman representations) are isomorphic.

We can form again in a canonical way a Polish space $A(\Gamma, X, \mu)$ of all measure-preserving actions of Γ on (X, μ), in which the ergodic actions form again a G_δ subset of $A(\Gamma, X, \mu)$, and then isomorphism and unitary isomorphism become analytic equivalence relations on this space. We can therefore study their complexity using the concepts introduced earlier.

Theorem 1.22 (Foreman-Weiss [FW04], Hjorth [Hjo97])**.** *For any infinite countable group Γ, isomorphism of free, ergodic, measure-preserving actions of Γ is not classifiable by countable structures.*

Theorem 1.23 (Kechris [Kec10])**.** *For any infinite countable group Γ, unitary isomorphism of free, ergodic, measure-preserving actions of Γ is not classifiable by countable structures.*

Recall that an action $(\gamma, x) \mapsto \gamma \cdot x$ is *free* if for any $\gamma \in \Gamma \setminus \{1\}$, $\gamma \cdot x \neq x$, a.e.

Except for abelian Γ, where we have the same picture as for $\mathbb{Z}$, it is unknown however how isomorphism and unitary isomorphism relations compare with ME. However Hjorth and Tornquist have recently shown that unitary isomorphism is a Borel equivalence relation. Finally, it is again not known what is the precise complexity of these two equivalence relations. Is isomorphism Borel bireducible to the universal equivalence relation induced by a Borel action of the automorphism group of the measure space?

D. Non-classification of orbit equivalence.

There is an additional important concept of equivalence between actions, called orbit equivalence. The study of orbit equivalence is a very active area today that has its origins in the connections between ergodic theory and operator algebras and the pioneering work of Dye.

Definition 1.24. Given an action of the group Γ on X we associate to it the orbit equivalence relation E^X_Γ, whose classes are the orbits of the action. Given measure-preserving actions of two groups Γ and Δ on spaces (X, μ) and (Y, ν), resp., we say that they are *orbit equivalent* if there is an isomorphism of the underlying measure spaces that sends E^X_Γ to E^Y_Δ (neglecting null sets as usual).

Thus isomorphism clearly implies orbit equivalence but not vice versa.

Here we have the following classical result.

Theorem 1.25 (Dye [Dye59, Dye63], Ornstein-Weiss [OW80])**.** *Every two free, ergodic, measure-preserving actions of amenable groups are orbit equivalent.*

Thus there is a single orbit equivalence class in the space of free, ergodic, measure-preserving actions of an amenable group Γ.

The situation for non-amenable groups has taken much longer to untangle. For simplicity, below "action" will mean "free, ergodic, measure-preserving action." Schmidt [Sch81], showed that every non-amenable group which does not have Kazhdan's property (T) admits at least two non-orbit equivalent actions and Hjorth [Hjo05] showed that every non-amenable group with property (T) has continuum many non-orbit equivalent actions. So every non-amenable group has at least two non-orbit equivalent actions.

For general non-amenable groups though very little was known about the question of how many non-orbit equivalent actions they might have. For example, until recently only finitely many distinct examples of non-orbit equivalent actions of the free, non-abelian groups were known. Gaboriau-Popa [GP05] finally showed that the free non-abelian groups have continuum many non-orbit equivalent actions (for an alternative treatment see Tornquist [Tor06] and the exposition in Hjorth [Hjo09]).

In an important extension, Ioana [Ioa07] showed that every group that contains a free, non-abelian subgroup has continuum many such actions. However there are examples of non-amenable groups that contain no free, non-abelian subgroups (Ol'shanski [Ol'80]).

Finally, the question was completely resolved by Epstein.

Theorem 1.26 (Epstein [Eps08])**.** *Every non-amenable group admits continuum many non-orbit equivalent free, ergodic, measure-preserving actions.*

This still leaves open however the possibility that there may be a concrete classification of actions of some non-amenable groups up to orbit equivalence. However the following has been now proved by combining very recent work of Ioana-Kechris-Tsankov and the work of Epstein.

Theorem 1.27 (Epstein-Ioana-Kechris-Tsankov [IKT09])**.** *Orbit equivalence of free, ergodic, measure-preserving actions of any non-amenable group is not classifiable by countable structures.*

Thus we have a very strong dichotomy:

- If a group is amenable, it has exactly one action up to orbit equivalence.
- If it non-amenable, then orbit equivalence of its actions is unclassifiable in a strong sense.

In the rest of these lectures, we will give an outline of the proof of Theorem 1.27.

2. Lecture II. Turbulence and classification by countable structures

A. The space of countable structures.

Definition 2.1. A countable *signature* is a countable family $L = \{f_i\}_{i\in I} \cup \{R_j\}_{j\in J}$ of *function symbols* f_i, with f_i of arity $n_i \geq 0$, and *relation symbols* R_j, with R_j of arity $m_j \geq 1$. A *structure* for L or L-*structure* has the form

$$\mathcal{A} = \langle A, \{f_i^{\mathcal{A}}\}_{i\in I}, \{R_j^{\mathcal{A}}\}_{j\in J}\rangle,$$

where A is a nonempty set, $f_i^{\mathcal{A}} : A^{n_i} \to A$, and $R_j^{\mathcal{A}} \subseteq A^{m_j}$.

Example 2.2. If $L = \{\cdot, 1\}$, where $\cdot$ and 1 are binary and nullary function symbols respectively, then a group is any L-structure $\mathcal{G} = \langle G, \cdot^{\mathcal{G}}, 1^{\mathcal{G}}\rangle$ that satisfies the group axioms. Similarly, using various signatures, we can study structures that correspond to fields, graphs, etc.

We are interested in countably infinite structures here, so we can always take (up to isomorphism) $A = \mathbb{N}$.

Definition 2.3. Denote by X_L the *space of (countable) L-structures*, i.e.,

$$X_L = \prod_{i\in I} \mathbb{N}^{(\mathbb{N}^{n_i})} \times \prod_{j\in J} 2^{(\mathbb{N}^{m_j})}.$$

With the product topology ($\mathbb{N}$ and 2 being discrete) this is a Polish space.

Definition 2.4. Let S_∞ be the *infinite symmetric group* of all permutations of $\mathbb{N}$. It is a Polish group with the pointwise convergence topology. It acts continuously on X_L: If $\mathcal{A} \in X_L$, $g \in S_\infty$, then $g\cdot\mathcal{A}$ is the isomorphic copy of $\mathcal{A}$ obtained by applying g. For example, if $I = \emptyset$, $\{R_j\}_{j\in J}$ consists of a single binary relation symbol R, and $\mathcal{A} = \langle \mathbb{N}, R^{\mathcal{A}}\rangle$, then $g\cdot\mathcal{A} = \mathcal{B}$, where $(x,y) \in R^{\mathcal{B}} \Leftrightarrow (g^{-1}(x), g^{-1}(y)) \in R^{\mathcal{A}}$. This is called the *logic action* of S_∞ on X_L.

Clearly, $\exists g(g\cdot\mathcal{A} = \mathcal{B}) \Leftrightarrow \mathcal{A} \cong \mathcal{B}$, i.e., the equivalence relation induced by this action is isomorphism.

Logic actions are universal among S_∞-actions.

Theorem 2.5 (Becker-Kechris [BK96, 2.7.3])**.** *There is a countable signature L such that for every Borel action of S_∞ on a standard Borel* *space X there is a Borel equivariant injection $\pi : X \to X_L$ (i.e. $\pi(g \cdot x) = g \cdot \pi(x)$). Thus every Borel S_∞-space is Borel isomorphic to the logic action on an isomorphism-invariant Borel class of L-structures.*

B. Classification by countable structures.

Definition 2.6. Let E be an equivalence relation on a standard Borel space X. We say the E *admits classification by countable structures* if there is a countable signature L and a Borel map $f : X \to X_L$ such that $xEy \Leftrightarrow f(x) \cong f(y)$, i.e., $E \leq_B \cong_L \big(= \cong |X_L \big)$.

By Theorem 2.5 this is equivalent to the following: There is a Borel S_∞-space Y such that $E \leq_B E^Y_{S_\infty}$, where $x E^Y_{S_\infty} y \Leftrightarrow \exists g \in S_\infty (g \cdot x = y)$ is the equivalence relation induced by the S_∞-action on Y.

Examples 2.7.

- If E is concretely classifiable, then E admits classification by countable structures.
- Let X be an uncountable Polish space. Define E_c on $X^{\mathbb{N}}$ by
$$(x_n) E_c (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$
Then E_c is classifiable by countable structures. (E_c is, up to Borel isomorphism, independent of X.)
- (Giordano-Putnam-Skau [GPS95]) Topological orbit equivalence of minimal homeomorphisms of the Cantor set is classifiable by countable structures.
- (Kechris [Kec92]) If G is Polish locally compact and X is a Borel G-space, then E^X_G admits classification by countable structures.

Hjorth developed in [Hjo00] a theory called *turbulence* that provides the basic method for showing that equivalence relations *do not* admit classification by countable structures. (Beyond [Hjo00] extensive expositions of this theory can be found in [Kec02], [Kan08] and [Gao09].)

Theorem 2.8 (Hjorth [Hjo00, 3.19])**.** *Let G be a Polish group acting continuously on a Polish space X. If the action is* turbulent, *then E^X_G cannot be classified by countable structures. In particular if E is an equivalence relation and $E^X_G \leq_B E$ for some turbulent action of a Polish group G on X, then E cannot be classified by countable structures.*

The rest of this lecture will be devoted to explaining the concept of turbulence and sketching some ideas in the proof the Theorem 2.8.

C. Turbulence.

Let G be a Polish group acting continuously on a Polish space X. Below U (with various embellishments) is a typical nonempty open set in X and V (with various embellishments) is a typical open symmetric nbhd of $1 \in G$.

Definition 2.9. The (U, V)-*local graph* is given by

$$xR_{U,V}y \Leftrightarrow x, y \in U \,\&\, \exists g \in V\ (g \cdot x = y).$$

The (U, V)-*local orbit* of $x \in U$, denoted $\mathcal{O}(x, U, V)$, is the connected component of x in this graph.

Remark 2.10. If $U = X$ and $V = G$, then $\mathcal{O}(x, U, V) = G \cdot x =$ the orbit of x.

Definition 2.11. A point $x \in X$ is *turbulent* if $\forall U \ni x\, \forall V \big(\overline{\mathcal{O}(x, U, V)}$ has nonempty interior$\big)$.

It is easy to check that this property depends only on the orbit of x, so we can talk about *turbulent orbits.*

The action is called *(generically) turbulent* if

(i) every orbit is meager,
(ii) there is a dense, turbulent orbit.

Remark 2.12. This implies that the set of dense, turbulent orbits is comeager (see [Kec02, 8.5]).

Proposition 2.13. *Let G be a Polish group acting continuously on a Polish space X and let $x \in X$. Suppose there is a neighborhood basis $\mathcal{B}(x)$ for x such that for all $U \in \mathcal{B}(x)$ and any open nonempty set $W \subseteq U$, there is a continuous path $(g_t)_{0 \le t \le 1}$ in G with $g_0 = 1$, $g_1 \cdot x \in W$ and $g_t \cdot x \in U$ for each t. Then x is turbulent.*

Proof. It is enough to show that $\mathcal{O}(x, U, V)$ is dense in U for all $U \in \mathcal{B}(x)$ and open symmetric nbhds V of $1 \in G$. Fix such a U, V and take any nonempty open $W \subseteq U$ and let (g_t) be as above. Using uniform continuity, we can find $t_0 = 0 < t_1 < \cdots < t_k = 1$ such that $g_{t_{i+1}} g_{t_i}^{-1} \in V$ for all $i < k$. Let $h_1 = g_{t_1} g_{t_0}^{-1} = g_{t_1}$, $h_2 = g_{t_2} g_{t_1}^{-1}, \ldots,$ $h_k = g_{t_k} g_{t_{k-1}}^{-1}$. Then $h_i \in V$ and

$$h_i \cdot h_{i-1} \cdots h_1 \cdot x = g_{t_i} \cdot x \in U, \quad \forall 1 \le i \le k$$

and $g_1 \cdot x = h_k \cdot h_{k-1} \cdots h_1 \cdot x \in W$, so $g_1 \cdot x \in \mathcal{O}(x, U, V) \cap W \neq \emptyset$. $\square$

Example 2.14. The action of the Polish group $(c_0, +)$ (with the sup-norm topology) on $(\mathbb{R}^{\mathbb{N}}, +)$ (with the product topology) by translation is turbulent. The orbits are the cosets of c_0 in $(\mathbb{R}^{\mathbb{N}}, +)$, so they are dense and meager. Also, any $x \in \mathbb{R}^{\mathbb{N}}$ is turbulent. Clearly the sets of the form $x + C$, with $C \subseteq \mathbb{R}^{\mathbb{N}}$ a convex open nbhd of 0, form a nbhd basis for $x \in \mathbb{R}^{\mathbb{N}}$. Now c_0 is dense in any such C, so $x+(c_0 \cap C)$ is dense in $x+C$. Let $g \in c_0 \cap C$. We will find a continuous path $(g_t)_{0 \leq t \leq 1}$ from 1 to g in c_0 such that $x+g_t \in x+C$ for each t. Clearly $g_t = tg \in c_0 \cap C$ works.

Remark 2.15. There are groups that admit no turbulent action. For example Polish locally compact groups and S_∞.

D. Generic ergodicity.

Definition 2.16. Let E be an equivalence relation on a Polish space X and F an equivalence relation on a Polish space Y. A *homomorphism* from E to F is a map $f : X \to Y$ such that $xEy \Rightarrow f(x)Ff(y)$.

We say that E is *generically F-ergodic* if for every Baire measurable homomorphism f there is a comeager set $A \subseteq X$ which f maps into a single F-class.

Example 2.17. Assume $E = E_G^X$ is induced by a continuous G-action of a Polish group G on a Polish space X with a dense orbit. Then E is generically $=_Y$-ergodic, for any Polish space Y. (*Proof:* Let $f : X \to Y$ be a Baire measurable homomorphism from E_G^X to $=_Y$ and let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of nonempty open subsets of Y. For each n the set $f^{-1}(U_n) \subseteq X$ is a G-invariant set and has the property of Baire. Thus, it is enough to show that if $A \subseteq X$ is G-invariant and has the property of Baire, then it is either meager or comeager. Otherwise there are open nonempty $U, U' \subseteq X$ with A comeager in U and $X \setminus A$ comeager in U'. Since there is a dense orbit, there is a $g \in G$ such that $W = g \cdot U \cap U' \neq \emptyset$ and since $g \cdot A = A$ we have that both A, $X \setminus A$ are comeager in W, a contradiction.)

In particular, if every G-orbit is also meager, then E is not concretely classifiable.

Let as before E_c be the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by

$$(x_n)E_c(y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$

(Note that this is Borel isomorphic to the one defined in 1.12.)

Theorem 2.18 (Hjorth [Hjo00, 3.21])**.** *The following are equivalent:*

(i) *E is generically E_c-ergodic.*

(ii) *E is generically $E^Y_{S_\infty}$-ergodic for any Borel S_∞-space Y.*

For another reference for the proof, see also [Kec02, 12.3].

It follows that if E satisfies these properties and if every E-class is meager, then E cannot be classified by countable structures.

Theorem 2.19 (Hjorth [Hjo00]). *If a Polish group G acts continuously on a Polish space X and the action is turbulent, then E^X_G is generically E_c-ergodic, so cannot be classified by countable structures.*

Proof. (Following the presentation in [Kec02, 12.5]) Assume $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ is a Baire measurable homomorphism from E^X_G to E_c. Let $A(x) = \{f(x)_n : n \in \mathbb{N}\}$ so that $xE^X_G y \Rightarrow A(x) = A(y)$.

Step 1. Let $A = \{a \in 2^{\mathbb{N}} : \forall^* x\,(a \in A(x))\}$, where $\forall^* x$ means "on a comeager set of x." Then A is countable.

$\big($*Proof of Step 1:* The function f is continuous on a dense G_δ set $C \subseteq X$. We have then $a \in A$ if and only if $\forall^* x \in C\,(a \in A(x))$. The set $B = \bigcup_n \{(x,a) \in C \times 2^{\mathbb{N}} : a = f(x)_n\}$ is Borel, and thus so is $A = \{a : \forall^* x \in C\,(x,a) \in B\}$. So if A is uncountable, it contains a Cantor set D. Then $\forall^* a \in D\,\forall^* x (a \in A(x))$, so, by Kuratowski-Ulam, $\forall^* x \forall^* a \in D\,(a \in A(x))$, thus for some x, $A(x)$ is uncountable, which is obviously absurd. $\square$[*Step 1*]$\big)$

Step 2. $\forall^* x\,(A(x) = A)$, which completes the proof.

One proceeds by assuming that this fails, which implies that $\forall^* x (A \subsetneqq A(x))$ and deriving a contradiction. Let C be a dense G_δ set such that $f|C$ is continuous. Then one finds appropriate $a \notin A$, $l \in \mathbb{N}$ and (using genericity arguments) $z_i \in C$, $z \in C$ with $z_i \to z$ and $f(z_i)_l = a$ but $a \notin A(z)$. By continuity, $a = f(z_i)_l \to f(z)_l$, so $f(z)_l = a$, i.e., $a \in A(z)$, a contradiction. The point z is found using the fact that $a \notin A$, so $\forall^* y (a \notin A(y))$ and the sequence z_i is obtained from the turbulence condition. The detailed proof follows.

$\big($*Proof of Step 2.* Otherwise, since there is a dense orbit, the invariant set $\{x : A = A(x)\}$ (which has the Baire property) must be meager. It follows that $C_1 = \{x : A \subsetneqq A(x)\}$ is comeager. Let

$$C_2 = \{x : \forall^* y \forall U \forall V\,(x \in U \Rightarrow G \cdot y \cap \overline{\mathcal{O}(x,U,V)} \neq \emptyset)\},$$

so that C_2 is comeager as well, since it contains all turbulent points.

Next fix a comeager set $C_0 \subseteq X$ with $f|C_0$ continuous. For $B \subseteq X$ let

$$C_B = \{x : x \in B \Leftrightarrow \exists \text{ open nbhd } U \text{ of } x \text{ with } \forall^* y \in U (y \in B)\}.$$

Then, if B has the Baire property, C_B is comeager (see [Kec95, 8G]). Finally fix a countable dense subgroup $G_0 \subseteq G$ and find a countable collection $\mathcal{C}$ of comeager sets in X with the following properties:

(i) $C_0, C_1, C_2 \in \mathcal{C}$;

(ii) $C \in \mathcal{C}, g \in G_0 \Rightarrow g \cdot C \in \mathcal{C}$;

(iii) $C \in \mathcal{C} \Rightarrow C^* = \{x : \forall^* g(g \cdot x \in C)\} \in \mathcal{C}$.

(iv) If $\{V_n\}$ enumerates a local basis of open symmetric nbhds of 1 in G, then, letting
$$A_{l,n} = \{x : \forall^* g \in V_n \, (f(x)_l = f(g \cdot x)_l)\},$$
we have that $C_{A_{l,n}} \in \mathcal{C}$.

(v) If $\{U_n\}$ enumerates a basis for X, and
$$C_{m,n,l} = \{x : x \notin U_m \text{ or } \forall^* g \in V_n \, (f(x)_l = f(g \cdot x)_l)\},$$
then $\mathcal{C}$ contains all $C_{m,n,l}$ which are comeager.

For simplicity, if $x \in \bigcap \mathcal{C}$ (and there are comeager many such x), we call x "generic".

So fix a generic x. Then there is $a \notin A$ so that $a \in A(x) = A(g \cdot x)$ for all g. So $\forall g \exists l (a = f(g \cdot x)_l)$, thus there is an $l \in \mathbb{N}$ and open nonempty $W \subseteq G$ so that $\forall^* g \in W \, (f(g \cdot x)_l = a)$. Fix $p_0 \in G_0 \cap W$ and V a basic symmetric nbhd of 1 so that $V p_0 \subseteq W$. Let $p_0 \cdot x = x_0$, so that x_0 is generic too, and $\forall^* g \in V \, (f(g \cdot x_0)_l = a)$. Now $\forall^* g \in V \, (g \cdot x_0 \in C_0)$, so we can find $g_i \in V$, $g_i \to 1$ with $g_i \cdot x_0 \in C_0$ and $f(g_i \cdot x_0)_l = a$, so as $g_i \cdot x_0 \to x_0 \in C_0$, by continuity we have $f(x_0)_l = a$. Also since $\forall^* g \in V \, (f(x_0)_l = f(g \cdot x_0)_l)$ and x_0 is generic, using (iv) we see that there is a basic open U with $x_0 \in U$, such that
$$\forall^* z \in U \forall^* g \in V \, (f(z)_l = f(g \cdot z)_l),$$
i.e., if $U = U_m$, $V = V_n$, then $C_{m,n,l}$ is comeager, so by (v) it is in $\mathcal{C}$. Since $a \notin A$, $\{y : a \notin A(y)\}$ is not meager, so choose y generic with $a \notin A(y)$ and also
$$\forall \tilde{U}, \tilde{V} (x_0 \in \tilde{U} \Rightarrow G \cdot y \cap \overline{\mathcal{O}(x_0, \tilde{U}, \tilde{V})} \neq \emptyset).$$
Thus we have $G \cdot y \cap \overline{\mathcal{O}(x_0, U, V)} \neq \emptyset$. So choose $g_0, g_1, \dots \in V$ so that if $g_i \cdot x_i = x_{i+1}$, then $x_i \in U$ and some subsequence of (x_i) converges to some $y_1 \in G \cdot y$. Fix a compatible metric d for X.

Since $\forall^* h(h \cdot x_0$ is generic) and $\forall^* g \in V \, (f(g \cdot x_0)_l = a)$, we can find h_1 so that $h_1 g_0 \in V$, $g_1 h_1^{-1} \in V$, $\overline{x}_1 = h_1 \cdot x_1 \in U$, $d(x_1, \overline{x}_1) < \frac{1}{2}$, $\overline{x}_1 = h_1 \cdot g_0 \cdot x_0$ is generic. Then $\forall^* g \in V (f(\overline{x}_1)_l = f(g \cdot \overline{x}_1)_l)$ (as $\overline{x}_1 \in C_{m,n.l}$), and $f(\overline{x}_1)_l = a$, so also $\forall^* g \in V \, (f(g \cdot \overline{x}_1)_l = a)$. Note that $g_1 h_1^{-1} \cdot \overline{x}_1 = x_2$ and $g_1 h_1^{-1} \in V$, so since $\forall^* h(h \cdot \overline{x}_1$ is generic) and $\forall^* g \in V \, (f(g \cdot \overline{x}_1)_l = a)$, we can find h_2 so that $h_2 g_1 h_1^{-1} \in V$, $g_2 h_2^{-1} \in V$,

$\overline{x}_2 = h_2 \cdot x_2 \in U$, $d(x_2, \overline{x}_2) < \frac{1}{4}$, $\overline{x}_2 = h_2 g_1 h_1^{-1} \cdot \overline{x}_1$ is generic. Then $\forall^* g \in V\ (f(\overline{x}_2)_l = f(g \cdot \overline{x}_2)_l)$, and $f(\overline{x}_2)_l = a$, so $\forall^* g \in V(f(g \cdot \overline{x}_2) = a)$, etc.

Repeating this process, we get $x_0, \overline{x}_1, \overline{x}_2, \ldots$ generic and belonging to the (U, V)-local orbit of x_0, so that some subsequence of $\{\overline{x}_i\}$ converges to y_1 and $\forall^* g \in V\ (f(g \cdot \overline{x}_i)_l = a)$. Now

$$\forall^* g(g \cdot \overline{x}_i \in C_0), \quad \forall^* g(g \cdot y_1 \in C_0), \quad \forall^* g \in V(f(g \cdot \overline{x}_i)_l = a),$$

so fix g satisfying all these conditions. Then for some subsequence $\{n_i\}$ we have $\overline{x}_{n_i} \to y_1$, so $g \cdot \overline{x}_{n_i} \to g \cdot y_1$ and $g \cdot \overline{x}_{n_i}, g.y_1 \in C_0$, so by continuity, $a = f(g \cdot \overline{x}_{n_i})_l \to f(g \cdot y_1)_l$, so $f(g \cdot y_1)_l = a$, i.e., $a \in A(g \cdot y_1) = A(y_1) = A(y)$, a contradiction. $\square$[*Step 2*])

$\square$

3. Lecture III. Turbulence in the irreducible representations

Let H be a separable complex Hilbert space. We denote by $U(H)$ the *unitary group* of H, i.e., the group of Hilbert space automorphisms of H. The strong topology on $U(H)$ is generated by the maps $T \in U(H) \mapsto T(x) \in H$ $(x \in H)$, and it is the same as the *weak topology* generated by the maps $T \in U(H) \mapsto \langle T(x), y\rangle \in \mathbb{C}$ $(x, y \in H)$. With this topology $U(H)$ is a Polish group.

If now Γ is a countable (discrete) group, $\mathrm{Rep}(\Gamma, H)$ is the space of unitary representations of Γ on H, i.e., homomorphisms of Γ into $U(H)$ or equivalently actions of Γ on H by unitary transformations. It is a closed subspace of $U(H)^\Gamma$, equipped with the product topology, so it is a Polish space. The group $U(H)$ acts continuously on $\mathrm{Rep}(\Gamma, H)$ via conjugacy $T \cdot \pi = T\pi T^{-1}$ (where $T\pi T^{-1}(\gamma) = T \circ \pi(\gamma) \circ T^{-1}$) and the equivalence relation induced by this action is *isomorphism* of unitary representations: $\pi \cong \rho$.

A representation $\pi \in \mathrm{Rep}(\Gamma, H)$ is *irreducible* if it has no non-trivial invariant closed subspaces. Let $\mathrm{Irr}(\Gamma, H) \subseteq \mathrm{Rep}(\Gamma, H)$ be the space of irreducible representations. Then it can be shown that $\mathrm{Irr}(\Gamma, H)$ is a G_δ subset of $\mathrm{Rep}(\Gamma, H)$, and thus also a Polish space in the relative topology (see [Kec10, H.5]).

From now on we assume that H is ∞-dimensional.

Thoma [Tho64] has shown that if Γ is abelian-by-finite then we have $\mathrm{Irr}(\Gamma, H) = \emptyset$, but if Γ is not abelian-by-finite then $\cong |\mathrm{Irr}(\Gamma, H)$ is not concretely classifiable. Hjorth [Hjo97] extended this by showing that it is not even classifiable by countable structures. This is proved by showing that the conjugacy action is turbulent on an appropriate conjugacy invariant closed subspace of $\mathrm{Irr}(\Gamma, H)$ (see [Kec10, H.9]). We will need below this result for the case $\Gamma = \mathbb{F}_2$ = the free group with two generators, so we will state and sketch the proof of the following stronger result for this case.

Theorem 3.1 (Hjorth [Hjo00])**.** *The conjugacy action of $U(H)$ on $\mathrm{Irr}(\mathbb{F}_2, H)$ is turbulent.*

Proof. Note that we can identify $\mathrm{Rep}(\mathbb{F}_2, H)$ with $U(H)^2$ and the action of $U(H)$ on $U(H)^2$ becomes

$$T \cdot (U_1, U_2) = (TU_1T^{-1}, TU_2T^{-1}).$$

Step 1: $\mathrm{Irr}(\mathbb{F}_2, H)$ is dense G_δ in $\mathrm{Rep}(\mathbb{F}_2, H) = U(H)^2$.

$\big($*Proof of Step 1.* Note that if $U(n) = U(\mathbb{C}^n)$, then, with some canonical identifications, $U(1) \subseteq U(2) \subseteq \cdots \subseteq U(H)$ and $\overline{\bigcup_n U(n)} = U(H)$.

Now $U(n)$ is compact, connected, so, by a result of Schreier-Ulam [SU35], the set of $(g, h) \in U(n)^2$ such that $\overline{\langle g, h\rangle} = U(n)$ is a dense G_δ in $U(n)^2$. By Baire Category this shows that the set of $(g, h) \in U(H)^2$ with $\overline{\langle g, h\rangle} = U(H)$ is dense G_δ, since for each nonempty open set N in $U(H)$, the set of $(g, h) \in U(H)^2$ that generate a subgroup intersecting N is open dense. Thus the generic pair $(g, h) \in U(H)^2$ generates a dense subgroup of $U(H)$ so, viewing (g, h) as a representation, any (g, h)-invariant closed subspace A is invariant for all of $\overline{\langle g, h\rangle} = U(H)$, hence $A = \{0\}$ or $A = H$ and the generic pair is irreducible.

$\square$[*Step 1*]$\big)$

Step 2: Every orbit is meager.

$\big($*Proof of Step 2.* It is enough to show that every conjugacy class in $U(H)$ is meager. This is a classical result but here is a simple proof recently found by Rosendal. (This proof is general enough so it works in other Polish groups.)

For each infinite $I \subseteq \mathbb{N}$, let $A(I) = \{T \in U(H) : \exists i \in I\, (T^i = 1)\}$. It is easy to check that $A(I)$ is dense in $U(H)$. (Use the fact that $\bigcup_n U(n)$ is dense in $U(H)$. Then it is enough to approximate elements of each $U(n)$ by $A(I)$ and this can be easily done using the fact that elements of $U(n)$ are conjugate in $U(n)$ to diagonal unitaries.) Let now $V_0 \supseteq V_1 \supseteq \cdots$ be a basis of open nbhds of 1 in $U(H)$ and put $B(I, k) = \{T \in U(H) : \exists i \in I(i > k \text{ and } T^i \in V_k)\}$. This contains $A(I \setminus \{0, \ldots, k\})$, so is open dense. Thus

$$C(I) = \bigcap_k B(I, k) = \{T : \exists (i_n) \in I^{\mathbb{N}}\, (T^{i_n} \to 1)\}$$

is comeager and conjugacy invariant. If a conjugacy class C is nonmeager, it will thus be contained in all $C(I)$, $I \subseteq \mathbb{N}$ infinite. Thus $T \in C \Rightarrow T^n \to 1$, so letting d be a left invariant metric for $U(H)$ we have for $T \in C$, $d(T, 1) = d(T^{n+1}, T^n) \to 0$, whence $T = 1$, a contradiction.

$\square$[*Step 2*]$\big)$

Step 3: There is a dense conjugacy class in $\mathrm{Irr}(\mathbb{F}_2, H)$ (so the set of all $\pi \in \mathrm{Irr}(\mathbb{F}_2, H)$ with dense conjugacy class is dense G_δ in $\mathrm{Irr}(\mathbb{F}_2, H)$).

$\big($*Proof of Step 3.* As $\mathrm{Irr}(\mathbb{F}_2, H)$ is dense G_δ in $\mathrm{Rep}(\mathbb{F}_2, H)$ it is enough to find $\pi \in \mathrm{Rep}(\mathbb{F}_2, H)$ with dense conjugacy class in $\mathrm{Rep}(\mathbb{F}_2, H)$ – then the set of all such π's is dense G_δ so intersects $\mathrm{Irr}(\mathbb{F}_2, H)$. Let (π_n) be dense in $\mathrm{Rep}(\mathbb{F}_2, H)$ and let $\pi \cong \bigoplus_n \pi_n$, $\pi \in \mathrm{Rep}(\mathbb{F}_2, H)$. This π easily works. $\square$[*Step 3*]$\big)$

Remark 3.2. The also gives an easy proof of a result of Yoshizawa: There exists an irreducible representation of $\mathbb{F}_2$ which weakly contains any representation of $\mathbb{F}_2$.

Step 4: Let $\pi \in \mathrm{Irr}(\mathbb{F}_2, H)$ have dense conjugacy class. Then π is turbulent.

Thus by Steps 2,3,4, the conjugacy action of $U(H)$ on $\mathrm{Irr}(\mathbb{F}_2, H)$ is turbulent.

$\big($*Proof of Step 4.*

Lemma 3.3. *For $\rho, \sigma \in \mathrm{Irr}(\mathbb{F}_2, H)$, the following are equivalent:*

(i) $\rho \cong \sigma$,

(ii) $\exists (T_n) \in U(H)^{\mathbb{N}}$ *such that* $T_n \cdot \rho = T_n \rho T_n^{-1} \to \sigma$ *and no subsequence of* (T_n) *converges in the weak topology of* $B_1(H) = \{T \in B(H) : ||T|| \leq 1\}$ *(a compact metrizable space) to* 0.

(Here $B(H)$ is the set of bounded linear operators on H.)

$\big($*Proof of Lemma 3.3.* Use the compactness of the unit ball $B_1(H)$ and Schur's Lemma (see, e.g., [Fol95, 3.5 (b)]): if $\pi_1, \pi_2 \in \mathrm{Irr}(\mathbb{F}_2, H)$, then $\pi_1 \cong \pi_2 \Leftrightarrow (\exists S \in B(H) \setminus \{0\}$ such that $\forall \gamma \in \mathbb{F}_2\, (S\pi_1(\gamma) = \pi_2(\gamma)S))$. $\square$[*Lemma 3.3*]$\big)$

Lemma 3.4. *Given any nonempty open $W \subseteq \mathrm{Irr}(\mathbb{F}_2, H)$ (open in the relative topology) and orthonormal $e_1, \ldots, e_p \in H$, there are orthonormal $e_1, \ldots, e_p, e_{p+1}, \ldots, e_q$ and $T \in U(H)$ such that*

(i) $T(e_i) \perp e_j$, $\forall i, j \leq q$;

(ii) $T^2(e_i) = -e_i$, $\forall i \leq q$;

(iii) $T = \mathrm{id}$ *on* $(H_0 \oplus T(H_0))^\perp$, *where* $H_0 = \langle e_1, \ldots, e_q \rangle$;

(iv) $T \cdot \pi \in W$.

$\big($*Proof of Lemma 3.4.* We can assume that

$$W = \{\rho \in \mathrm{Irr}(\mathbb{F}_2, H) : \forall \gamma \in F\ \forall i, j \leq q \\ |\langle \rho(\gamma)(e_i), e_j \rangle - \langle \sigma(\gamma)(e_i), (e_j) \rangle| < \epsilon\}$$

for some $e_1, \ldots, e_p, e_{p+1}, \ldots, e_q$, $F \subseteq \mathbb{F}_2$ finite, $\epsilon > 0$, and $\sigma \in \mathrm{Irr}(\mathbb{F}_2, H) \setminus U(H) \cdot \pi$ (since this set is comeager, hence dense). So, by Lemma 3.3, there is a sequence $(T_n) \in U(H)^{\mathbb{N}}$ with $T_n \cdot \pi \to \sigma$, $T_n \xrightarrow{w} 0$, so also $T_n^{-1} \xrightarrow{w} 0$. Thus for all large enough n, $e_1, \ldots, e_q, T_n^{-1}(-e_1), \ldots, T_n^{-1}(-e_q)$ are linearly independent. So apply Gram-Schmidt to get an orthonormal set $e_1, \ldots, e_q, f_1^{(n)}, \ldots, f_q^{(n)}$ with the same span. Then for all $i \leq q$, $||T_n^{-1}(-e_i) - f_i^{(n)}|| \to 0$ (as $\langle T_n^{-1}(-e_i), e_j \rangle \to 0$, $\forall i, j \leq q$). Define $S_n \in U(H)$ by $S_n(e_i) = f_i^{(n)}$, $S_n(f_i^{(n)}) = -e_i$, $\forall i \leq q$, and $S_n = \mathrm{id}$ on $\langle e_1, \dots, e_q, f_1^{(n)}, \dots, f_q^{(n)} \rangle^\perp$. Then if n is large enough, $T = S_n$ works. $\square$[*Lemma 3.4*])

We now show that π is turbulent. We will apply Proposition 2.13 of Lecture II. Fix a basic nbhd of π of the form

$$U = \{\rho \in \mathrm{Irr}(\mathbb{F}_2, H) : \forall \gamma \in F \ \forall i, j \leq k$$
$$|\langle \rho(\gamma)(e_i), e_j \rangle - \langle \pi(\gamma)(e_i), (e_j) \rangle| < \epsilon\},$$

$\epsilon > 0, F \subseteq \mathbb{F}_2$ finite, $e_1, \dots, e_k$ orthonormal. Let $e_1, \dots, e_k, e_{k+1}, \dots, e_p$ be an orthonormal basis for the span of $\{e_1, \dots, e_k\} \cup \{\pi(\gamma)(e_i) : \gamma \in F, 1 \leq i \leq k\}$, and let $W \subseteq U$ be an arbitrary nonempty open set. Then let $e_1, \dots, e_p, e_{p+1}, \dots, e_q$ and T be as in Lemma 3.4 (so that $T \cdot \pi \in W$). It is enough to find a continuous path $(T_\theta)_{0 \leq \theta \leq \pi/2}$ in $U(H)$ with $T_0 = 1$, $T_{\pi/2} = T$, and $T_\theta \cdot \pi \in U$ for all θ. Take

$$T_\theta(e_i) = (\cos\theta) e_i + (\sin\theta) T(e_i)$$
$$T_\theta(T(e_i)) = (-\sin\theta) e_i + (\cos\theta) T(e_i),$$

for $i = 1, \dots, q$ and let $T_\theta = \mathrm{id}$ on $(H_0 \oplus T(H_0))^\perp$, where $H_0 = \langle e_1, \dots, e_q \rangle$. Then one can easily see that $T_\theta \cdot \pi \in U$, for all θ.

$\square$[*Step 4*])

$\square$

4. Lecture IV. Non-classification of orbit equivalence by countable structures, Part A: Outline of the proof and Gaussian actions.

Our goal in the remaining three lectures is to prove the following result.

Theorem 4.1 (Epstein-Ioana-Kechris-Tsankov [IKT09, 3.12])**.** *Let Γ be a countable non-amenable group. Then orbit equivalence for measure-preserving, free, ergodic (in fact mixing) actions of Γ is not classifiable by countable structures.*

We will start be giving a *very rough* idea of the proof and then discussing the (rather extensive) set of results needed to implement it.

A. Definitions.

A *standard measure space* (X, μ) is a standard Borel space X with a non-atomic probability Borel measure μ. All such spaces are isomorphic to $[0, 1]$ with Lebesgue measure on the Borel sets.

The measure algebra MALG_μ of μ is the algebra of Borel sets of X, modulo null sets, with the topology induced by the metric $d(A, B) = \mu(A\Delta B)$. Let $\mathrm{Aut}(X, \mu)$ be the group of measure-preserving automorphisms of (X, μ) (again modulo null sets) with the *weak topology*, i.e., the one generated by the maps $T \mapsto T(A)$ ($A \in \mathrm{MALG}_\mu$). It is a Polish group.

If Γ is a countable group, denote by $A(\Gamma, X, \mu)$ the space of measure-preserving actions of Γ on (X, μ) or equivalently, homomorphisms of Γ into $\mathrm{Aut}(X, \mu)$. It is a closed subspace of $\mathrm{Aut}(X, \mu)^\Gamma$ with the product topology, so also a Polish space. We say that $a \in A(\Gamma, X, \mu)$ is *free* if $\forall \gamma \neq 1\, (\gamma \cdot x \neq x, \text{a.e.})$, and it is *ergodic* if every invariant Borel set $A \subseteq X$ is either null or conull. We say that $a \in A(\Gamma, X, \mu)$, $b \in A(\Gamma, Y, \nu)$ are *orbit equivalent*, $a \,\text{Œ}\, b$, if, denoting by E_a, E_b the equivalence relations induced by a, b respectively, E_a is isomorphic to E_b, in the sense that there is a measure-preserving isomorphism of (X, μ) to (Y, ν) that sends E_a to E_b (modulo null sets). Thus Œ is an equivalence relation on $A(\Gamma, X, \mu)$ and Theorem 4.1 asserts that it cannot be classified by countable structures if Γ is not amenable.

B. Idea of the proof.

We start with the following fact.

Theorem 4.2. *To each $\pi \in \mathrm{Rep}(\Gamma, H)$, we can assign in a Borel way an action $a_\pi \in A(\Gamma, X, \mu)$ (on some standard measure space (X, μ)), called the* Gaussian action *associated to π, such that*

(i) $\pi \cong \rho \Rightarrow a_\pi \cong a_\rho$,
(ii) *If $\kappa_0^{a_\pi}$ is the Koopman representation on $L_0^2(X, \mu)$ associated to a_π, then $\pi \leq \kappa_0^{a_\pi}$.*

$\big($*Explanations:*

(i) $a, b \in A(\Gamma, X, \mu)$ are *isomorphic*, $a \cong b$ if there is a $T \in \mathrm{Aut}(X, \mu)$ taking a to b,
$$T\gamma^a T^{-1} = \gamma^b, \quad \forall \gamma \in \Gamma.$$
(Here we let $\gamma^a = a(\gamma)$.)
(ii) Let $a \in A(\Gamma, X, \mu)$. Let $L_0^2(X, \mu) = \{f \in L^2(X, \mu) : \int f = 0\} = \mathbb{C}^\perp$. The *Koopman representation* κ_0^a of Γ on $L_0^2(X, \mu)$ is given by $(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x)$.
(iii) If $\pi \in \mathrm{Rep}(\Gamma, H)$, $\rho \in \mathrm{Rep}(\Gamma, H')$, then $\pi \leq \rho$ iff π is isomorphic to a subrepresentation of ρ, i.e., the restriction of ρ to an invariant, closed subspace of H'.$\big)$

So let $\pi \in \mathrm{Irr}(\mathbb{F}_2, H)$ and look at $a_\pi \in A(\mathbb{F}_2, X, \mu)$. We will then modify a_π, in a Borel and isomorphism preserving way, to another action $a(\pi) \in A(\mathbb{F}_2, Y, \nu)$ (for reasons to be explained later) and finally apply a construction of Epstein to "co-induce" appropriately $a(\pi)$, in a Borel and isomorphism preserving way, to an action $b(\pi) \in A(\Gamma, Z, \rho)$, which will turn out to be free and ergodic (in fact *mixing* i.e., $\mu(\gamma \cdot A \cap B) \to \mu(A)\mu(B)$ as $\gamma \to \infty$, for every Borel A, B). So we finally have a Borel function $\pi \in \mathrm{Irr}(\mathbb{F}_2, H) \mapsto b(\pi) \in A(\Gamma, Z, \rho)$.

Put
$$\pi R \rho \Leftrightarrow b(\pi) \, \text{Œ} \, b(\rho).$$
Then R is an equivalence relation on $\mathrm{Irr}(\mathbb{F}_2, H)$, and $\pi \cong \rho \Rightarrow \pi R \rho$ (since $\pi \cong \rho \Rightarrow a_\pi \cong a_\rho \Rightarrow a(\pi) \cong a(\rho) \Rightarrow b(\pi) \cong b(\rho)$).

Fact: R has countable index over $\cong$ (i.e., every R-class contains only countably many $\cong$-classes).

If now Œ on $A(\Gamma, Z, \rho)$ admitted classification by countable structures, so would R on $\mathrm{Irr}(\mathbb{F}_2, H)$. So let $F : \mathrm{Irr}(\mathbb{F}_2, H) \to X_L$ be Borel, where X_L is the standard Borel space of countable structures for a signature L, with
$$\pi R \rho \Leftrightarrow F(\pi) \cong F(\rho).$$
Therefore $\pi \cong \rho \Rightarrow F(\pi) \cong F(\rho)$. By Theorem 3.1 and Theorems 2.18, 2.19, there is a comeager set $A \subseteq \mathrm{Irr}(\mathbb{F}_2, H)$ and $\mathcal{A}_0 \in X_L$ such that $F(\pi) \cong \mathcal{A}_0$, $\forall \pi \in A$. But every $\cong$-class in $\mathrm{Irr}(\mathbb{F}_2, H)$ is meager, so

by the previous fact every R-class in $\mathrm{Irr}(\mathbb{F}_2, H)$ is meager, so there are R-inequivalent $\pi, \rho \in A$, and thus $F(\pi) \ncong F(\rho)$, a contradiction.

$\big($*Sketch of proof of Theorem 4.2.* (See [Kec10, Appendix E]) For simplicity we will discuss the case of real Hilbert spaces, the complex case being handled by appropriate complexifications.

Let H be an infinite-dimensional, separable, real Hilbert space. Consider the product space $(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$, where μ is the normalized, centered Gaussian measure on $\mathbb{R}$ with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Let $p_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $i \in \mathbb{N}$, be the projection functions. The closed linear space $\langle p_i \rangle \subseteq L^2_0(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$ (real valued) has countable infinite dimension, so we can assume that $H = \langle p_i \rangle \subseteq L^2_0(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$.

Lemma 4.3. *If $S \in O(H) =$ the orthogonal group of H (i.e., the group of Hilbert space automorphisms of H), then we can extend uniquely S to $\overline{S} \in \mathrm{Aut}(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$ in the sense that the Koopman operator $O_{\overline{S}} : L^2_0(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}}) \to L^2_0(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$ defined by $O_{\overline{S}}(f) = f \circ (\overline{S})^{-1}$ extends S, i.e., $O_{\overline{S}}|H = S$.*

Thus if $\pi \in \mathrm{Rep}(\Gamma, H)$, we can extend each $\pi(\gamma) \in O(H)$ to $\overline{\pi(\gamma)} \in \mathrm{Aut}(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$. Let $a_\pi \in A(\Gamma, \mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$ be defined by $a_\pi(\gamma) = \overline{\pi(\gamma)}$. This clearly works.

$\big($*Proof of Lemma 4.3:* The p_i's form an orthonormal basis for H. Let $S(p_i) = q_i \in H$. Then let $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by $\theta(x) = (q_0(x), q_1(x), \dots)$. Then θ is 1-1, since the σ-algebra generated by (q_i) is the Borel σ-algebra, modulo null sets, so (q_i) separates points modulo null sets. Moreover θ preserves $\mu^{\mathbb{N}}$. This follows from the fact that every $f \in \langle p_i \rangle$ (including q_i) has centered Gaussian distribution and the q_i are independent, since $\mathbb{E}(q_i q_j) = \langle q_i, q_j \rangle = \langle p_i, p_j \rangle = \delta_{ij}$. Thus $\theta \in \mathrm{Aut}(\mathbb{R}^{\mathbb{N}}, \mu^{\mathbb{N}})$. Now put $\overline{S} = \theta^{-1}$. $\square$[*Lemma 4.3*]$\big)$

$\square$[*Theorem 4.2*]$\big)$

5. Lecture V. Non-classification of orbit equivalence by countable structures, Part B: An action of $\mathbb{F}_2$ on $\mathbb{T}^2$ and a separability argument.

Recall that our plan consists of the three steps

$$\pi \xrightarrow{(1)} a_\pi \xrightarrow{(2)} a(\pi) \xrightarrow{(3)} b(\pi),$$

where π is an irreducible unitary representation of $\mathbb{F}_2$, a_π is the corresponding Gaussian action of $\mathbb{F}_2$, (2) is the "perturbation" to a new action of $\mathbb{F}_2$ and (3) is the co-inducing construction, which from the $\mathbb{F}_2$-action $a(\pi)$ produces a Γ-action $b(\pi)$.

We already discussed step (1).

A. Properties of the co-induced action.

Let's summarize next the key properties of the co-inducing construction (3) that we will need and discuss this construction in Lecture VI. Below we write $\gamma^a \cdot x$ for $\gamma^a(x)$.

Theorem 5.1. *Let Γ be non-amenable. Given $a \in A(\mathbb{F}_2, Y, \nu)$ we can construct $b \in A(\Gamma, Z, \rho)$ and $a' \in A(\mathbb{F}_2, Z, \rho)$ (Z, ρ independent of a) with the following properties:*

(i) $E_{a'} \subseteq E_b$,
(ii) *b is free and ergodic (in fact mixing),*
(iii) *a' is free,*
(iv) *a is a factor of a' via a map $f : Z \to Y$ (f independent of a), i.e., $f(\delta^{a'} \cdot z) = \delta^a \cdot f(z)$ ($\delta \in \mathbb{F}_2$), and $f_*\rho = \nu$, and in fact for every a'-invariant Borel set $A \subseteq Z$ of positive measure, if $\rho_A = \frac{\rho|A}{\rho(A)}$, then a is a factor of $(a'|A, \rho_A)$ via f.*
(v) *If a free action $\overline{a} \in A(\mathbb{F}_2, \overline{Y}, \overline{\nu})$ is a factor of the action a via $g : Y \to \overline{Y}$*

$$\begin{tikzcd} Y \arrow[r, "g"] & \overline{Y} \\ Z \arrow[u, "f"] & \end{tikzcd}$$

then for $\gamma \in \Gamma \setminus \{1\}$, $gf(\gamma^b \cdot z) \neq gf(z)$, ρ-a.e.

Moreover the map $a \mapsto b$ is Borel and preserves isomorphism.

B. A separability argument.

We now deal with construction (2). The key here is a particular action of $\mathbb{F}_2$ on $\mathbb{T}^2$ utilized first to a great effect by Popa and then Ioana [Ioa07]. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $(\mathbb{T}^2, \lambda)$ (λ is Lebesgue measure) in the usual way by matrix multiplication

$$A \cdot (z_1, z_2) = (A^{-1})^t \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

This is free, measure-preserving, and ergodic.

Fix also a copy of $\mathbb{F}_2$ with finite index in $\mathrm{SL}_2(\mathbb{Z})$ (see, e.g., [New72, VIII.2]) and denote the restriction of this action to $\mathbb{F}_2$ by α_0. It is also free, measure-preserving and ergodic. For any $c \in A(\mathbb{F}_2, X, \mu)$, we let $a(c) \in A(\mathbb{F}_2, \mathbb{T}^2 \times X, \lambda \times \mu)$ be the product action

$$a(c) = \alpha_0 \times c$$

(i.e. $\gamma^{a(c)} \cdot (z, x) = (\gamma^{\alpha_0} \cdot z, \gamma^c \cdot x)$). Then in our case we take

$$a(\pi) = a(a_\pi) = \alpha_0 \times a_\pi.$$

The key property of the passage from c to $a(c)$ is the following separability result established by Ioana (in a somewhat different context – but his proof works as well here).

Below, for each $c \in A(\mathbb{F}_2, X, \mu)$, with $a(c) = \alpha_0 \times c \in A(\mathbb{F}_2, Y, \nu)$, where $Y = \mathbb{T}^2 \times X$, $\nu = \lambda \times \mu$, we let $b(c) \in A(\Gamma, Z, \rho)$, $a'(c) \in A(\mathbb{F}_2, Z, \rho)$ come from $a(c)$ via Theorem 5.1.

Theorem 5.2 (Ioana [Ioa07])**.** *If $(c_i)_{i\in I}$ is an uncountable family of actions in $A(\mathbb{F}_2, X, \mu)$ and $(b(c_i))_{i\in I}$ are mutually orbit equivalent, then there is uncountable $J \subseteq I$ such that if $i, j \in J$, we can find Borel sets $A_i, A_j \subseteq Z$ of positive measure which are respectively $a'(c_i), a'(c_j)$-invariant and $a'(c_i)|A_i \cong a'(c_j)|A_j$ with respect to the normalized measures ρ_{A_i}, ρ_{A_j}.*

Proof. We have the following situation, letting $b_i = b(c_i)$: $(b_i)_{i\in I}$ is an uncountable family of pairwise orbit equivalent free, ergodic actions in $A(\Gamma, Z, \rho)$, $a'_i = a'(c_i)$ are free in $A(\mathbb{F}_2, Z, \rho)$, with $E_{a'_i} \subseteq E_{b_i}$, and α_0 is a factor of a'_i via a map $p : Z \to \mathbb{T}^2$ such that

$$(*) \qquad \text{for } \gamma \in \Gamma \setminus \{1\},\ i \in I, \quad p(\gamma^{b_i} \cdot z) \neq p(z), \quad \rho\text{-a.e.}$$

Here $p = \mathrm{proj} \circ f$, where f is as in (iv) of Theorem 5.1 and $\mathrm{proj} : Y = \mathbb{T}^2 \times X \to \mathbb{T}^2$ is the projection. This follows from (v) of Theorem 5.1

with $\overline{a} = \alpha_0 \in A(\mathbb{F}_2, \mathbb{T}^2, \lambda)$ and $g = \text{proj}$.

$$\begin{array}{ccc} Z & & a'_i \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ \mathbb{T}^2 & & \alpha_0 \end{array}$$

By applying to each b_i, a'_i a measure preserving transformation $T_i \in \mathrm{Aut}(Z, \rho)$, i.e., replacing b_i by Tb_iT^{-1} and a'_i by Ta'_iT^{-1}, which we just call again b_i and a'_i by abuse of notation, we can clearly assume that there is E such that $E_{b_i} = E$ for each $i \in I$. Then α_0 is a factor of a'_i via $p_i = p \circ T_i^{-1}$ and $(*)$ holds as well for p_i instead of p.

Consider now the σ-finite measure space (E, P) where for each Borel set $A \subseteq E$, $P(A) = \int |A_z| \, \mathrm{d}\rho(z)$.

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ by matrix multiplication gives a semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, and similarly $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ (as we view $\mathbb{F}_2$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$). The key point is that $(\mathbb{F}_2 \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has the so-called *relative property* (T):

$\exists$ finite $Q \subseteq \mathbb{F}_2 \ltimes \mathbb{Z}^2$, $\epsilon > 0$, such that for any unitary representation $\pi \in \mathrm{Rep}(\mathbb{F}_2 \ltimes \mathbb{Z}^2, H)$, if v is a (Q, ϵ)-invariant unit vector (i.e., $||\pi(q)(v) - v|| < \epsilon$, $\forall q \in Q$), then there is a $\mathbb{Z}^2$-invariant vector w with $||v - w|| < 1$

(see [BHV08, 4.2] and also [Hjo09, 2.3]).

Given now $i, j \in I$, we will define a representation $\pi_{i,j} \in \mathrm{Rep}(\mathbb{F}_2 \ltimes \mathbb{Z}^2, L^2(E, P))$. Identify $\mathbb{Z}^2$ with $\widehat{\mathbb{T}^2}$ = the group of characters of $\mathbb{T}^2$ so that $\tilde{m} = (m_1, m_2) \in \mathbb{Z}^2$ is identified with $\chi_{\tilde{m}}(z_1, z_2) = z_1^{m_1} z_2^{m_2}$. Via this identification, the action of $\mathbb{F}_2$ on $\mathbb{Z}^2$ by matrix multiplication is identified with the shift action of $\mathbb{F}_2$ on $\widehat{\mathbb{T}^2}$: $\delta \cdot \chi(t) = \chi(\delta^{-1} \cdot t)$, $\delta \in \mathbb{F}_2$, $\chi \in \widehat{\mathbb{T}^2}$, $t \in \mathbb{T}^2$. Then the semidirect product $\mathbb{F}_2 \ltimes \mathbb{Z}^2$ is identified with $\mathbb{F}_2 \ltimes \widehat{\mathbb{T}^2}$ and multiplication is given by

$$(\delta_1, \chi_1)(\delta_2, \chi_2) = (\delta_1\delta_2, \chi_1(\delta_1 \cdot \chi_2)).$$

If $\chi \in \widehat{\mathbb{T}^2}$, let $\eta^i_\chi = \chi \circ p_i : Z \to \mathbb{T}$. Then define $\pi_{i,j}$ as follows

$$\pi_{i,j}(\delta, \chi)(f)(x, y) = \eta^i_\chi(x)\overline{\eta^j_\chi(y)} f((\delta^{-1})^{a'_i} \cdot x, (\delta^{-1})^{a'_j} \cdot y)$$

for $\delta \in \mathbb{F}_2$, $\chi \in \widehat{\mathbb{T}^2}$. To check that this is a representation note that

$$(**) \qquad \begin{aligned} \eta^i_{\delta\cdot\chi}(x) &= (\delta \cdot \chi)(p_i(x)) \\ &= \chi((\delta^{-1})^{\alpha_0} \cdot p_i(x)) = \chi(p_i((\delta^{-1})^{a'_i} \cdot x)) \\ &= \eta^i_\chi((\delta^{-1})^{a'_i} \cdot x) \end{aligned}$$

and similarly for j.

Then if $v = 1_\Delta$, where $\Delta = \{(z,z) : z \in Z\}$, using the separability of $L^2(Z,\rho)$ and $L^2(E,P)$ one can find $J \subseteq I$ uncountable such that if $i,j \in J$, then

$$||\pi_{i,j}(q)(v) - v||_{L^2(E,P)} < \epsilon,$$

$\forall q \in Q$. Indeed, note that

$$||\pi_{i,j}(\delta,\chi)(v) - v||^2_{L^2(E,P)} = 2\int \Re(1 - fg)\mathrm{d}\rho,$$

where $f = \eta^i_\chi \overline{\eta^j_\chi}$ and $g = 1_{\{z:(\delta^{-1})^{a'_i}\cdot z = (\delta^{-1})^{a'_j}\cdot z\}}$. Since $1 - fg = (1-f) + (1-g) - (1-f)(1-g)$, this is bounded by

$$2(||1-f||_{L^2(Z,\rho)} + ||1-g||_{L^2(Z,\rho)} + ||1-f||_{L^2(Z,\rho)}||1-g||_{L^2(Z,\rho)}).$$

Now

$$||1-f||_{L^2(Z,\rho)} = ||\eta^i_\chi - \eta^j_\chi||_{L^2(Z,\rho)}$$

and $|1-g| = 1_{\{z:(\delta^{-1})^{a'_i}\cdot z \neq (\delta^{-1})^{a'_j}\cdot z\}}$, so denoting by f^δ_i the characteristic function of the graph of $(\delta^{-1})^{a'_i}$ and similarly for f^δ_j,

$$||1-g||_{L^2(Z,\rho)} = \frac{1}{2}||f^\delta_i - f^\delta_j||_{L^2(E,P)}$$

and so by the separability of $L^2(Z,\rho)$ and $L^2(E,P)$, there is $J \subseteq I$ uncountable, so that if $i,j \in J$, then

$$||\pi_{i,j}(q)(v) - v||_{L^2(E,P)} < \epsilon, \quad \forall q \in Q.$$

So by relative property (T), there is $f \in L^2(E,P)$ with $||f - 1_\Delta|| < 1$ and f is $\mathbb{Z}^2$-invariant (for $\pi_{i,j}$), i.e.,

$$f(x,y) = \eta^i_\chi(x)\overline{\eta^j_\chi(y)}f(x,y), \quad \forall \chi \in \widehat{\mathbb{T}^2}.$$

Since $f \neq 0$ (as $||1_\Delta|| = 1$ and $||f - 1_\Delta|| < 1$) the set

$$S = \{(x,y) \in E : \eta^i_\chi(x) = \eta^j_\chi(y), \forall \chi \in \widehat{\mathbb{T}^2}\}$$

has positive P-measure.

Using the fact that E is generated by b_j, and the fact that characters separate points, it follows that for almost all $x \in X$, there is at most one y such that $(x,y) \in S$. Indeed, fix x such that $(x,y_1) \in S$, $(x,y_2) \in S$ with $y_1 \neq y_2$. Then $y_1 = \gamma_1^{b_j}\cdot x$, $y_2 = \gamma_2^{b_j}\cdot x$, $\gamma_1 \neq \gamma_2$. But also

$$\forall \chi\big(\eta^i_\chi(x) = \eta^j_\chi(\gamma_1^{b_j}\cdot x) = \eta^j_\chi(\gamma_2^{b_j}\cdot x)\big).$$

But $\widehat{\mathbb{T}^2}$ separates points, so

$$p_j(\gamma_1^{b_j}\cdot x) = p_j(\gamma_2^{b_j}\cdot x).$$

Let $\gamma_1\gamma_2^{-1} = \gamma \neq 1$. Then

$$p_j(\gamma^{b_j} \cdot \gamma_2^{b_j} \cdot x) = p_j(\gamma_2^{b_j} \cdot x)$$

which happens only on a null set by $(*)$ for p_j.

Let

$$A_i = \{x : \exists \text{ unique } y\ (x, y) \in S\}, \quad A_j = \{y : \exists x \in A_i\ (x, y) \in S\}.$$

Then $\rho(A_i) > 0$ (as $P(S) > 0$). Now if $(x, y) \in S$ then by $(**)$

$$(\delta^{a'_i} \cdot x, \delta^{a'_j} \cdot y) \in S, \quad \forall \delta \in \mathbb{F}_2,$$

so A_i and A_j are respectively a'_i-invariant and a'_j-invariant sets of positive measure. Let $\varphi : A_i \to A_j$ be defined by $\varphi(x) = y \Leftrightarrow (x, y) \in S$. Then

$$\begin{aligned}\varphi(x) = y &\Leftrightarrow (x, y) \in S \\ &\Leftrightarrow (\delta^{a'_i} \cdot x, \delta^{a'_j} \cdot y) \in S \\ &\Leftrightarrow \varphi(\delta^{a'_i} \cdot x) = \delta^{a'_j} \cdot y,\end{aligned}$$

i.e., φ shows that $a'_i|A_i \cong a'_j|A_j$. $\square$

C. Completion of the proof.

We now complete the proof of Theorem 4.1.

We have $a'(\pi), b(\pi)$ as in Theorem 5.1 coming from $a(\pi)$ in step (3). Recall that we defined

$$\pi R \rho \Leftrightarrow b(\pi) \mathrel{\text{Œ}} b(\rho).$$

To complete the proof of Theorem 4.1, we only had to show that R has countable index over $\cong$. We now prove this:

Assume, toward a contradiction, that $(\pi_i)_{i\in I}$ is an uncountable family of pairwise non-isomorphic representations in $\mathrm{Irr}(\mathbb{F}_2, H)$ such that if $b_i = b(\pi_i)$, then (b_i) are pairwise orbit equivalent. Recall the chain:

$$\pi_i \to a_{\pi_i} = c_i \to a(\pi_i) = a(c_i) = \alpha_0 \times c_i \to b(c_i) = b_i,\ a'(c_i) = a'_i.$$

By Theorem 5.2, we can find uncountable $J \subseteq I$ such that if $i, j \in J$, there are a'_i, a'_j-resp. invariant Borel sets A_i, A_j of positive measure, so that $a'_i|A_i \cong a'_j|A_j$. Moreover by property (iv) of Theorem 5.1, $a(c_i)$ is a factor of $a'_i|A_i$ and similarly for $a(c_j)$, $a'_j|A_j$.

Fix $i_0 \in J$. Then for any $j \in J$, fix A_{i_0}, A_j as in Theorem 5.2. We have

$$\pi_j \le \kappa_0^{a_{\pi_j}}\ (= \kappa_0^{c_j}) \le \kappa_0^{\alpha_0 \times c_j}\ (= \kappa_0^{a(c_j)}) \le \kappa_0^{a'_j|A_j} \cong \kappa_0^{a'_{i_0}|A_{i_0}} \le \kappa_0^{a'_{i_0}}.$$

Thus (π_j) is (up to isomorphism) an uncountable family of pairwise non-isomorphic *irreducible* subrepresentations of ${\kappa_0}^{a'_{i_0}}$, a contradiction.

6. Lecture VI. Non-classification of orbit equivalence by countable structures, Part C: Co-induced actions

It only remains to prove Theorem 5.1 from Lecture V. This is based on a co-inducing construction due to Epstein [Eps08].

A. Co-induced actions.

We have two countable groups Δ and Γ (in our case Δ will be $\mathbb{F}_2$) and are given a *free*, *ergodic*, measure-preserving action a_0 of Δ on (Ω, ω) and a measure-preserving action b_0 of Γ on (Ω, ω) with $E_{a_0} \subseteq E_{b_0}$ (note that b_0 is also ergodic). Let $N = [E_{b_0} : E_{a_0}] =$ (the number of E_{a_0}-classes in each E_{b_0}-class) $\in \{1, 2, 3, \dots, \aleph_0\}$. Work below with $N = \aleph_0$ the other cases being similar.

Given these data we will describe Epstein's *co-inducing* construction that, given any $a \in A(\Delta, Y, \nu)$, will produce $b \in A(\Gamma, Z, \rho)$, where $Z = \Omega \times Y^{\mathbb{N}}$, $\rho = \omega \times \nu^{\mathbb{N}}$, called the *co-induced action of a, modulo* (a_0, b_0),

$$b = \mathrm{CInd}(a_0, b_0)^{\Gamma}_{\Delta}(a),$$

which will satisfy Theorem 5.1 of Lecture V. See [IKT09, §3] for more details.

Put $E = E_{a_0}$, $F = E_{b_0}$. We can then find a sequence $(C_n) \in \mathrm{Aut}(\Omega, \omega)^{\mathbb{N}}$ of *choice functions*, i.e., $C_0 = \mathrm{id}$ and $\{C_n(w)\}$ is a transversal for the E-classes contained in $[w]_F$.

To prove this, define first a sequence (D_n) of Borel choice functions as follows: define the equivalence relation on Γ

$$\gamma \sim_w \delta \Leftrightarrow (\gamma^{b_0} \cdot w) E (\delta^{b_0} \cdot w).$$

Let $\{\gamma_{n,w}\}$ be a transversal for $\sim_w$ with $\gamma_{0,w} = 1$ and put $D_n(w) = \gamma^{b_0}_{n,w} \cdot w$.

We can then use the ergodicity of E to modify (D_n) to a sequence (C_n) of 1-1 choice functions (which are then in $\mathrm{Aut}(\Omega, \omega)$) as follows; see [IKT09, 2.1]. Fix $n \in \mathbb{N}$ and consider D_n. As it is countable-to-1, let $\Omega = \bigsqcup_{k=1}^{\infty} Y_k$ be a Borel partition such that $D_n|Y_k$ is 1-1. Let then $Z_k = D_n(Y_k)$, so that $\mu(Z_k) = \mu(Y_k)$. Since E is ergodic, there is $T_k \in \mathrm{Aut}(\Omega, \omega)$ with $T_k(w)Ew$, a.e., such that $T_k(Z_k) = Y_k$ (see, e.g., [KM04, 7.10]). Let then $C_n(w) = T_k(D_n(w))$, if $w \in Y_k$. We have $C_n(w) E D_n(w)$ and C_n is 1-1. So $\{C_n\}$ are choice functions and each C_n is 1-1.

Using the $\{C_n\}$ we can define the *index cocycle*

$$\varphi_{E,F} = \varphi : F \to S_\infty = \text{the symmetric group of } \mathbb{N}$$

given by

$$\varphi(w_1, w_2)(k) = n \Leftrightarrow [C_k(w_1)]_E = [C_n(w_2)]_E$$

(cocycle means: $\varphi(w_2, w_3)\varphi(w_1, w_2) = \varphi(w_1, w_3)$ whenever $w_1 F w_2 F w_3$). Define also for each $(w_1, w_2) \in F$, $\vec{\delta}(w_1, w_2) \in \Delta^{\mathbb{N}}$ by

$$(\vec{\delta}(w_1, w_2)_n)^{a_0} \cdot C_{\varphi(w_1,w_2)^{-1}(n)}(w_1) = C_n(w_2).$$

Now S_∞ acts on the product group $\Delta^{\mathbb{N}}$ by shift: $(\sigma \cdot \vec{\delta})_n = \vec{\delta}_{\sigma^{-1}(n)}$, where $\vec{\delta} = (\vec{\delta}_n) \in \Delta^{\mathbb{N}}$. So we can form the semi-direct product $S_\infty \ltimes \Delta^{\mathbb{N}}$, with multiplication

$$(\sigma_1, \vec{\delta}_1)(\sigma_2, \vec{\delta}_2) = (\sigma_1\sigma_2, \vec{\delta}_1(\sigma_1 \cdot \vec{\delta}_2)).$$

Given $a \in A(\Delta, Y, \nu)$, we then have a measure-preserving action of $S_\infty \ltimes \Delta^{\mathbb{N}}$ on $(Y^{\mathbb{N}}, \nu^{\mathbb{N}})$ by

$$((\sigma, \vec{\delta}) \cdot \vec{y})_n = (\vec{\delta_n})^a \cdot \vec{y}_{\sigma^{-1}(n)}.$$

Finally we have a cocycle for the action b_0, $\psi : \Gamma \times \Omega \to S_\infty \ltimes \Delta^{\mathbb{N}}$ given by

$$(*) \qquad \psi(\gamma, w) = \big(\varphi(w, \gamma^{b_0} \cdot w), \vec{\delta}(w, \gamma^{b_0} \cdot w)\big)$$

(cocycle means: $\psi(\gamma_1\gamma_2, w) = \psi(\gamma_1, \gamma_2 \cdot w)\psi(\gamma_2, w)$). Finally let

$$b = \mathrm{CInd}(a_0, b_0)^{\Gamma}_{\Delta}(a)$$

be the skew product $b = b_0 \ltimes_\psi (Y^{\mathbb{N}}, \mu^{\mathbb{N}})$, i.e., for $\gamma \in \Gamma$

$$\begin{aligned}\gamma^b \cdot (w, \vec{y}) &= (\gamma^{b_0} \cdot w, \psi(\gamma, w) \cdot \vec{y}) \\ &= \big(\gamma^{b_0} \cdot w, (n \mapsto (\vec{\delta}(w, \gamma^{b_0} \cdot w)_n)^a \cdot \vec{y}_{\varphi(w,\gamma^{b_0}\cdot w)^{-1}(n)})\big).\end{aligned}$$

We also let $a' = a_0 \ltimes_{\psi'} (Y^{\mathbb{N}}, \mu^{\mathbb{N}})$, where ψ' is the cocycle for the action a_0 given by replacing b_0 by a_0 in $(*)$. Thus for $\delta \in \Delta$

$$\delta^{a'} \cdot (w, \vec{y}) = \big(\delta^{a_0} \cdot w, (n \mapsto (\vec{\delta}(w, \delta^{a_0} \cdot w)_n)^a \cdot \vec{y}_{\varphi(w,\delta^{a_0}\cdot w)^{-1}(n)}\big).$$

We verify some properties of a', b needed in Theorem 5.1:

(i) $E_{a'} \subseteq E_b$: trivial as $E_{a_0} \subseteq E_{b_0}$.

(ii) b is free: trivial as b_0 is free.

(iii) a' is free: trivial as a_0 is free.

(iv) Let $f : Z = \Omega \times Y^{\mathbb{N}} \to Y$ be given by $f(w, \vec{y}) = \vec{y}_0$. Then a is a factor of a' via f (this follows from $C_0(w) = w$).

Next we show that if $A \subseteq \Omega \times Y^{\mathbb{N}}$ has positive measure and is a'-invariant, then $f_*\rho_A = \nu$. Let $B \subseteq Y$, $\nu(B) = 1$ be a-invariant such that $\nu|B$ is the unique a-invariant probability measure on B. Then $\rho(f^{-1}(B)) = 1$, so $f_*\rho_A$ lives on B and then $f_*\rho_A = \nu$.

(v) If $\overline{a} \in A(\Delta, \overline{Y}, \overline{\nu})$ is a free action which is a factor of a via $g : Y \to \overline{Y}$, then for $\gamma \in \Gamma \setminus \{1\}$, $gf(\gamma^b \cdot z) \neq gf(z)$, ρ-a.e: Fix $\gamma \in \Gamma \setminus \{1\}$. We need to show that

$$(**) \qquad g\big((\vec{\delta}(w, \gamma^{b_0} \cdot w)_0)^a \cdot \vec{y}_{\varphi(w,\gamma^{b_0}\cdot w)^{-1}(0)}\big) \neq g(\vec{y}_0)$$

for almost all $w, \vec{y}$. Assume not, i.e. for positively many $w, \vec{y}$ $(**)$ fails. We can also assume that $\varphi(w, \gamma^{b_0} \cdot w)^{-1}(0) = k$ is fixed for the $w, \vec{y}$ and $\vec{\delta}(w, \gamma^b \cdot w)_0 = \delta$ is also fixed. Thus $g(\delta^a \cdot \vec{y}_k) = \delta^{\overline{a}} \cdot g(\vec{y}_k) = g(\vec{y}_0)$ on a set of positive measure of $w, \vec{y}$. If $k \neq 0$ this is false using Fubini. If $k = 0$, then $\delta = 1$ by the freeness of $\overline{a}$, so $w = \gamma^{b_0} \cdot w$ for a positive set of w, contradicting the freeness of b_0.

B. Small subequivalence relations.

In general it is not clear that the second part of (ii) in 5.1, i.e., "b is ergodic" is true. However if $E = E_{a_0}$ is "small" in $F = E_{b_0}$ in the sense to be described below, then this will be the case and in fact b will be mixing.

For $\gamma \in \Gamma$, let

$$|\gamma|_E = \omega(\{w : (w, \gamma \cdot w) \in E\}) \in [0, 1].$$

We say that $E = E_{a_0}$ is *small* in $F = E_{b_0}$, if $|\gamma|_E \to 0$ as $\gamma \to \infty$.

Theorem 6.1 (Ioana-Kechris-Tsankov [IKT09, 3.3])**.** *In the above notation and assuming also that b_0 is mixing, if E is small in F, then* for any $a \in A(\Delta, Y, \nu)$, $b = \mathrm{CInd}(a_0, b_0)^{\Gamma}_{\Delta}(a)$ *is mixing.*

We will omit the somewhat technical proof.

C. A measure theoretic version of the von Neumann Conjecture.

Thus to complete the proof of Theorem 5.1 we will need to show that for $\Delta = \mathbb{F}_2$, Γ non-amenable, there are free, ergodic $a_0 \in A(\Delta, \Omega, \omega)$, and $b_0 \in A(\Gamma, \Omega, \omega)$ mixing with $E_{a_0} \subseteq E_{b_0}$ and E_{a_0} small in E_{b_0}.

This is based on a construction of Gaboriau-Lyons [GL09], using ideas from probability theory as well as the theory of costs, who proved that there are such a_0, b_0 without considering the smallness condition, which was later established by Ioana-Kechris-Tsankov. The Gaboriau-Lyons result provided an affirmative answer to a measure theoretic version of von Neumann's Conjecture.

Since a full account of the background theory needed in this construction will take us too far afield, we will only give a very rough sketch of the ideas involved.

By some simple manipulations this result can be reduced to the case where Γ is non-amenable and finitely generated (note that every non-amenable countable group contains a non-amenable finitely generated one).

For a fixed finite set of generators $S \subseteq \Gamma$, we denote by $\mathrm{Cay}(\Gamma, S)$ its Cayley graph with (oriented) edge set $\mathcal{E}$ (and of course vertex set Γ): $(\gamma, \delta) \in \mathcal{E}$ iff $\exists s \in S\, (\delta = \gamma s)$. Γ acts freely on this graph by left multiplication and thus acts on $\Omega = \{0,1\}^{\mathcal{E}}$ by shift. We can view $w \in \{0,1\}^{\mathcal{E}}$ as the subgraph with vertex set Γ and edges e being those $e \in \mathcal{E}$ with $w(e) = 1$. The connected components of this graph are called the *clusters* of w.

On $\Omega = \{0,1\}^{\mathcal{E}}$ we put the product measure $\mu_p = \nu_p^{\mathcal{E}}$, where $0 < p < 1$, and $\nu_p(\{1\}) = p$. It is invariant under the action of Γ and is called the *Bernoulli bond percolation*. This action is also mixing and free. For an appropriate choice of p, we will take $\omega = \mu_p$ and $b_0 =$ this Bernoulli action.

We now define a subequivalence relation $E^{cl} \subseteq E_{b_0} = F$ (called the *cluster equivalence relation*) by

$$(w_1, w_2) \in E^{cl} \Leftrightarrow \exists \gamma (\gamma^{-1} \cdot w_1 = w_2 \,\&\, \gamma \text{ is in the cluster of 1 in } w_1).$$

Each E^{cl}-class $[w]_{E^{cl}}$ carries in a natural way a graph structure isomorphic to the cluster of 1 in w.

Now Pak and Smirnova-Nagnibeda [PSN00] show that one can choose S and p so that μ_p-a.e. the subgraph given by w has infinitely many infinite clusters each with infinitely many ends. (A connected, locally finite graph has *infinitely many ends* if for every k there is a finite set of vertices which upon removal leave at least k infinite connected components in the remaining graph.) It follows that the set $U^\infty \subseteq \Omega$ given by

$$w \in U^\infty \Leftrightarrow [w]_{E^{cl}} \text{ is infinite}$$

has positive $\omega\,(= \mu_p)$-measure and by a result of Gaboriau [Gab00, IV.24(2)] $E^{cl}|U^\infty$ has normalized cost that is finite but > 1. (For the definition and the theory of cost see Gaboriau [Gab00], [Gab10] and also Kechris-Miller [KM04], [Hjo09].) Also it turns out that $E^{cl}|U^\infty$ is ergodic. By a standard extension process this gives a subequivalence relation $E' \subseteq F$ such that E' is ergodic and has finite cost > 1. Using the theory of cost, by a result of Kechris-Miller and independently Pichot (see, e.g., [KM04, 28.11]), E' can be assumed to be treeable and then by a result of Hjorth [Hjo06] (see also [KM04, 28.2]), this gives a free, ergodic action $a_0 \in A(\mathbb{F}_2, \Omega, \omega)$ with $E_{a_0} \subseteq E' \subseteq F = E_{b_0}$.

To make sure now that E_{a_0} is small in E_{b_0} one can either choose above p with more care or else one starts with any a_0, b_0 as above and coinduces by (a_0, b_0) an appropriate Bernoulli percolation $\overline{a}$ of $\mathbb{F}_2$ to get $\overline{b}_0$ and then shows that one can find a small subequivalence relation $E_{\overline{a}_0} \subseteq E_{\overline{b}_0}$ generated by a free, ergodic action $\overline{a}_0$ of $\mathbb{F}_2$.

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