A Systematic Method for Besicovitch's $\frac{1}{2}$-Problem I:

A new fundamental perspective on the geometry of sets, and almost everywhere removal of the flatness hypothesis

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Abstract

Recall that Besicovitch's $\frac{1}{2}$-Conjecture (1938) is the assertion that the lower spherical density of purely unrectifiable 1-sets (at least in the plane) is bounded above by $\frac{1}{2}$ a.e. In recent work we verified this conjecture under a measure-theoretic flatness hypothesis imposed a.e. for all sufficiently small scales. As a consequence of what we call the small scale construction of loops which we present in this paper (first discovered in the author's 1997 Yale dissertation but hitherto unpublished), we can now remove this hypothesis from almost every point, and all but one scale (for each such point). This paper however succeeds in accomplishing a much more fundamental goal. Indeed, we present the two fundamental ideas which were required for our program. The second is the small scale construction mentioned above, whereas the first is the (long sought after) completely general construction of (large scale) loops which can be done for *any* closed set, and, in fact, (quite surprisingly) turns out to be completely independent of the problem (though immediately and fundamentally applicable to it). This was a bit of a shock to the author as our only true motivation for pursuing it so deligintly was that it was exhibited in every attempt by the author on a counterexample for this conjecture. The complete generality of this construction immediately allows us to present a new fundamental perspective on the geometry of sets, replacing a rather natural, familiar, and (much) older one which is invariably taken as self-evident in geometrical analysis. As far as the $\frac{1}{2}$-Problem is concerned, the two constructions basically perform a "hand-shake" in the sense that, when the geometry of the set (and hence its associated loop) arising from the first construction does not allow for the required estimates, the small scale construction

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becomes the relevant one and starts to provide a higher resolution information to do the job. We have thus arrived very close to our original goal, as we expect that a perfect merger of the two constructions (which is the subject of work in progress by the author) will completely solve the problem (at least in its original form for 1-sets in the plane). But, in fact, we have actually achieved a perhaps higher goal, in the sense that the method is essentially geometrically optimal, that is the construction directly associates the estimates with the geometry with basically no hidden information. The exact same procedure can be started immediately for 1-sets in R^n ($n \geq 3$) which was expected. More surprisingly however, the most fundamental construction (of the general large scale loops) is completely independent of dimension or codimension, and we now expect that our approach will generalize to m-sets, though we remark that some technical issues (unrelated to the method itself) need to be resolved first. Finally, we remark that although we stay focused in this paper on the $\frac{1}{2}$-Conjecture (in the plane), we expect this new perspective on the geometry of sets to have an impact on other important problems, in fact we can even say that the $\frac{1}{2}$-problem is basically a *symptom* of the failure to recognize the proper perspective on the geometry of sets in general. Furthermore, the older, less complete, perspective lies at the foundation of various theories especially in (but not limited to) geometric measure theory and related areas. We therefore expect more "symptoms" to be exposed in the near future with hopefully accompanying solutions. For the time being we have to consider our belief philosophical, but we give (brief) mention in some specific important problems of particular interest to the author (in connection with singular integrals and capacities) where we highly expect a strong link. It is our hope that this new perspective will be taken very seriously by the experts. We suspect it will prove quite valuable especially in problems where very little information is a priori given to determine or restrict the geometry of a set while strong control on the geometry is required in the conclusion.

Contents

1 Introduction

1.1 Historical Background

In his two classic papers [Be1],[Be 2] of 1928, 1938, A.S. Besicovitch succeeded in describing the geometry of 1-sets in the plane to a great extent, giving several characterizations of rectifiability and pure unrectifiability. Most of his results were impressively optimal. We now give a brief account of one fundamental problem which Besicovitch took up in both of these papers using intriguing methods yet with limited success, thereby leaving us with a long-standing conjecture which became known as the $\frac{1}{2}$-problem. First we recall that $E \subset R^n$ is called an m-set if $0 < H^m(E) < \infty$, where H^m the Hausdorff measure of dimension m, and E is said to be purely unrectifiable if $H^m(E \cap \Gamma) = 0$ for all Γ a Lipschitz image of R^m. Recall also that the spherical density $\Theta^m(E,x) \equiv \lim_{r\to 0^+} \frac{H^m(E\cap B(x,r))}{(2r)^m}$, and the lower spherical density $\Theta^m_*(E,x) \equiv \lim_{r\to 0^+} \frac{H^m(E\cap B(x,r))}{(2r)^m}$. We remark that throughout this paper (and this area of research) Hausdorff measure is defined by allowing all sets of a given diameter in a cover (this observation is especially important when making statements regarding purely unrectifiable sets) so that

$$H^m(E) = \lim_{\delta\to 0^+}\inf \left\{ \sum (diamE_i)^m : E \subset \bigcup_i E_i, diamE_i < \delta \right\}.$$

It is well-known that this definition implies that $\lim \sup_{\substack{diamU \leq \delta \\ x \in U \\ \delta \to 0^+}} \frac{H^m(E \cap U)}{(diamU)^m} \leq 1$ for H^m a.e. $x \in E$. From differentiability of Lipschitz maps we know that for rectifiable sets in R^n, the spherical density exists a.e. and equals 1 (see e.g. [Fal], or [Ma1]). In metric spaces this is recent [Ki]. Results of Besicovitch for 1-sets in the plane, and the celebrated result of Preiss [Pr] for m-sets in R^n give that the existence of density implies rectifiability. Now, let $\sigma_m(R^n)$ be the smallest number such that, if E is a purely unrectifiable m-set in R^n, then $\Theta_*^m(E,x) \leq \sigma_m(R^n)\ H^m$ a.e.. The first result on $\sigma_1(R^2)$ is in [Be1], that $\sigma_1(R^2) \leq 1 - 10^{-2576}$, and the second is in [Be 2], that $\sigma_1(R^2) \leq \frac{3}{4}$. This was generalized to $\sigma_1(R^n)$ by Moore [Mo]. Preiss and Tišer [PT] improved this to $\frac{2+\sqrt{46}}{12} \simeq 0.7312$, where they actually allowed any metric space instead of R^n. We will remark on the metric space case in the concluding remarks of the paper (we basically conjecture against it). In higher dimensions we only have $\sigma_2(R^3) \leq 1$ [Mar], and $\sigma_m(R^n) < 1$ [Ma2], $\sup_m \sigma_m(R^n) < 1$ [Ch]. There is mounting evidence that our method will work in higher dimensions just the same, and should produce the right bounds, though we do not necessarily wish to believe in any specific number and, in fact, the whole point is that our method turns the problem into an effective and concrete (albeit tedious) computation (basically a discrete variational/packing problem). We must remark however that we do not know exactly at this point how pure unrectifiability should be encoded for m-sets as for 1-sets in the setup of the problem, but we try to offer some suggestions in this regard and, while this is certainly necessary to go further in higher dimensions, we do not expect this to be the most delicate piece or the reason for any specific number. It is our hope that some experts will take up these issues and thus open the way for (most likely) important new results for m-sets.

It is our vision that, in the near future, even the higher dimensional theory can be brought to optimal form using the same principles of this paper (the most fundamental construction in our paper is independent of dimension and/or codimension), and seems to be just as natural to compute density bounds in any dimension (provided we first establish a reasonable way to encode rectifiability).

1.2 Recent Works

In recent works we proved a slightly stronger result than the following (which is even valid in a Hilbert space or a Banach space with smooth norm):

Let $B(x,r)$ be a ball centered at x and of radius r. Now let $\beta_E^1(x,r) = \frac{1}{r}\left[\text{width of thinnest strip } S \text{ such that } H^1((E \cap B(x,r)) \backslash S) = 0\right]$. See [Jo], [DS] for extensive rectifiability studies using quadratic summs involving such numbers. We remark however that our usage of these numbers is completely different both technically and fundamentally. We proved [Far 2]:

Theorem 1 *If $E \subset R^2$ is a purely unrectifiable 1-set, and* $\lim \sup_{r \to 0^+} \beta_E^1(x,r) \leq \frac{1}{4}$ *for a.e. $x \in E$, then $\Theta_*^1(E,x) \leq \frac{1}{2}$ a.e.*

Chronologically (completely opposite to publication dates) this was first proved with $\limsup \beta \leq \frac{1}{10}$ a.e. in [Far 3] using a method heavily dependent on the hypothesis and basically by contradiction; first constructing a finite set resembling sharp saw-teeth (vertices of a piece-wise linear graph with sharp slopes) and then showing that such set cannot lie in a quasi-arc with small constant in contradiction to the flatness hypothesis (small beta-numbers). This method evolved into the deeper more specialized method of [Far 2]. The crucial observation (which we included in each paper) is that we always observed that a certain collection of points always resembled a loop (will be defined properly later in the paper). This observation was taken quite seriously by the author for one specific reason, and that is that every attempt on a counterexample by the author produced a similar collection of points, but a general construction eluded us for sometime. Finally in [Far 1] which became the author's official dissertation, we succeeded in finding what seemed to be the most natural construction, which removed our flatness hypothesis from all but a countable set of balls. It was clear to the author that this was somehow the right approach to the problem since it also succeeded in building the required loops under significantly reduced restrictions. Unfortunately it took the author some time to realize the impact of this on partial results which we here publish for the first time, and which remove the flatness hypothesis from almost every point and scale (leaving it only on a countable set). Still, our program at that point had to produce these loops with an "external" condition which may have seemed artificial. This is because a general construction of these loops for any closed set was needed to make that condition simply a special case, preventing the general construction from completely collapsing for certain geometries. The whole point of the present paper is that these two fundamental ideas are presented, with some emphasis on the general construction because it turns out to be completely independent of the problem and lies at the foundation of our understanding of the geometry of sets, and we wish to emphasize its importance. While we cannot blame the reader for assuming it may only be important for density problems, we wish to strongly suggest otherwise. On the one hand, it replaces a general principle ubiquitous in geometric analysis which says that an arbitrary closed set is basically a topological object with holes (if any exists). This principle lies at the roots of (at least) geometric measure theory where arbitrary sets are often the start of problems. As of this writing we cannot conclusively give other important examples, but there is growing suspicion by the author that it will have an impact (at least) on problems involving analytic/harmonic Lip_1 capacities and singular integrals. We give more (still sketchy) details in the concluding remarks. We truly hope that the experts will seriously consider this principle in the context of either classical or modern problems or theories. It represents neither fancy machinery nor a useful theorem with specific applications. Instead, its simplicity and generality seem to change one's view of a problem or even its "setup"). Instead of being lost in a "sea of points", one is able to start the problem with a global representation of the set. In our opinion, this simple and basic improvement on our understanding of the geometry of sets is completely redundant if not useless in problems where the features of the geometry of the set are a priori known. It becomes quite valuable however if one starts the problem with little geometrical information on the set, which is then the object of the required results. This is certainly the case for density problems as is now evident to us (and hopefully will be the case for the reader), but we suspect the same will be true in (at least) problems with singular integrals.

It is my honor (though with great sadness) that I dedicate this paper to the memory of my good friend and mentor, the great analyst Tom Wolff who, with little words, said so much.

Acknowledgement 2 *I would like to thank Guy David for careful reading of, and comments on, the small-scale construction of loops originating in [Far 1], Peter Jones for inspiring this project in the first place and for suggesting the earlier flatness hypothesis as a testing ground for our program which goes back to 1993 when the author (as part of his Yale dissertation) became convinced, based on evidence from attempts on counterexamples, that these loops were every bit real and this paper is the first time we can construct them in both large and small scales. Finally, I am overwhelmed with gratitude to David Preiss for continuous support and encouragement over the past three years since I first met him in the summer of 1998, and for updating my knowledge of this area of research at that time.*

2 The General Setup Of The Problem In Simplified Terms

I would like to thank David Preiss and the referee of [Far 2] for teaching me this elegant setup of the problem. the approach is based on blow-up techniques and weak limits (via tangent measures). We will be sketchy as the reader can find details in [PT].

2.1 Basic Density Properties

¿From elementary geometric measure theory (see e.g. [Fal], or [Ma1]), we have the following classical theorems:

Theorem 3 *If E is an m-set, then $\Theta^m(E,x) = 0$ H^m -a.e. $x \notin E$;*

Theorem 4 *Suppose E is an m-set, then*

$$\lim \sup_{\substack{x\in U \\ diam(U)\to 0}} \frac{H^m(E\cap U)}{(diam(U))^m} = 1,$$

for H^m-a.e. $x \in E$, where the sets U are allowed to be rather general (e.g. convex, or closed etc.);

Theorem 5 *Suppose E is an m-set, $\Theta_*^m(E,x) > \sigma > 0$, on a set of positive measure. Then there exist $\varepsilon, \rho' > 0$, and a compact m-set $E' \subset E$ s.t. for all $x \in E'$, $0 < r < \rho'$,*

$$H^m(E \cap B(x,r)) \geq (\sigma + \varepsilon)(2r)^m .$$

So far, everything is valid for m-sets, but now we specialize to 1-sets for which σ is expected to be $\frac{1}{2}$. The main reason we restrict to 1-sets is the following well-known theorem.

Theorem 6 *If E is a 1-set which is connected, then E is rectifiable.*

It is therefore sufficient, if we want to establish a contradiction, that we construct a connected 1-set which intersects our purely unrectifiable 1-set in a subset of positive measure. It is this theorem which has nothing to do with the conjectured $\frac{1}{2}$ which needs to be replaced in order for our approach to be completely independent of dimension or codimension. We give some suggestions in the closing sections of the paper.

By uniformizing the above theorems and then taking weak limits, one can keep only one (small) constant floating around which is (naturally) ε. Combined with Theorem 4, the problem is then reduced to a really elegant setup which we describe in the next subsection. We remark however that this elegance is deceptive, indeed, the older more classical setups (e.g. [Far 1] which is a modification on that of [Fal]) are precisely of the same difficulty. This elegant setup is just a bit cleaner.

2.2 The Setup Of The Reduced Problem

Suppose E is a purely unrectifiable 1-set for which $\Theta_*^1(E,x) \geq \sigma + \varepsilon$ (think of $\frac{1}{2} + \varepsilon$) on a set of positive measure, then there exists a measure μ whose support F can be written as $F = F_1 \cup F_2$, where F_1, F_2 are disjoint, closed, and with distance attained. Furthermore μ satisfies:

$$\mu(U) \leq diam(U) \text{ for all } U; \tag{1}$$

$$\mu(B(x,r)) \geq (\sigma + \varepsilon) 2r forallx \in F, r > 0. \tag{2}$$

This is all we get from the assumption that $\sigma_1(R^n) \geq \sigma$, with the signature of pure unrectifiability encoded in the fact that we have *two* disjoint F_1, F_2 (with distance attained). In Section 3 we are able for the first time to offer a *completely* systematic description of our program in the right order; previously as in [Far 1],[Far 2], and [Far 3], we had to start from the piece that existed at the time and vaguely envision the other. It is a delight for the author that this is no longer necessary.

3 A Systematic Approach Via Enveloping Loops - Informal Discussion

3.1 The Targeted Global Contradiction

In this section we give an informal description of the two fundamental constructions on which our program (and hence the paper) is based. For simplicity we only focus on how these ideas fit together in a systematic way towards our ultimate goal of establishing a (global) contradiction based on (1) and (2) of Section 2.2. We must emphasize however that the one and (essentially) only reason for the author to persist in pursuit of these ideas is that they were repeatedly exhibited in virtually all our attempts on counterexamples. The main point is that estimates (1) and (2) seem to say nothing significant in basically any given ball by itself, however they actually provide information on the allowed geometry of the set in, or near, such a ball. As one accumulates such information to build the set from information collected from *several* (consecutive) balls, a specific picture starts to materialize, and it is one where (1) and (2) provide a contradiction from a global geometry. We remark however that, in fact, *only* balls where the set in some sense lies on some half of the ball that actually provide useful information.

3.2 Large Scale Enveloping Loops

Let us first idealize the situation and see what we can do. Suppose first that there is a mechanism by which we can obtain a sequence of balls centered at points $\{z_i\}_1^N, \{z_i'\}_1^{N'} \subset F$, with radii less than 1 (assume that $dist\,(F_1, F_2) > 1$ by rescaling) as in Fig. 1, with $|z_i - z_{i+1}| \leq 1$ for all i, and same for the primed sequence. Also assume $z_1 = z_1', z_N = z_{N'}'$. Furthermore, we take the radii r_i, r_i' by recursively placing balls on each sequence in such a way that $B\,(z_i, r_i) \cap B\,(z_j, r_j) = \begin{cases} \text{a singleton, if } i = j \pm 1 \\ \phi, \text{ otherwise.} \end{cases}$, and similarly for the primed sequence. See Fig.1. We may start this construction (assuming we did find the sequences $\{z_i\}_1^N, \{z_i'\}_1^{N'}$ in the first place) from $z_N = z_{N'}$ say. We actually may have to restrict to a subsequence in this construction but this does not affect any results. We can also take $B\,(z_N, r_N) = B\,(z_{N'}', r_{N'}')$, but we now add the assumption that we can also take $B\,(z_1, r_1) = B\,(z_1', r_1')$, i.e. all consecutive balls osculate. Note also that after we start the process we may end up throwing out several points as we mentioned previously. We still denote the sequences by the same names. It is obvious that each chain of balls has balls with disjoint interiors (this can always be done without difficulty). What is far from trivial is how to keep one chain of balls from overlapping with the other one. Suppose however that we can achieve this. Then, except for a possible gap between the last two balls on (only) one of the chains (note that by assuming $B\,(z_1, r_1) = B\,(z_1', r_1')$ in the process above, there is no gap to worry bout) we have:

Figures will be available shortly.

If $U \equiv (\bigcup balls) \cap F$, then by (2) and the triangle inequality

$$\mu(U) \geq \sum diam\,(ball) \times \left(\frac{1}{2} + \varepsilon\right)$$
$$\geq |z_1 - z_N| \times 2\left(\frac{1}{2} + \varepsilon\right).$$

Suppose we can even dream further and hope to construct such a "loop" with the property that

$$|z_1 - z_N| = diam\,(U)\,.$$

Then (1) implies

$$\mu(U) \leq |z_1 - z_N|\,,$$

which would give a "global" contradiction (we wouldn't know which balls really caused the contradiction). See Fig. 1.
Is it really possible to construct such a highly structured loop in an arbitrary closed set. In this paper we succeed in constructing them, leaving the possibility of the overlap between the two chains (actually we can even solve this issue once we fix the gap issue mentioned above). We then give a partial result showing that the "worst" case cannot happen, and that these chains, by and large, stay apart. More precisely we show that one can supplement the sequence by adding more points in order to keep the consecutive distances between points in each sequence smaller than the distances to the points of the other sequence (actually

since we want disjoint balls, we need a bit more, but we actually achieve that anyway). If it hadn't been for the technical (and quite delicate) gap issue, we could have given a complete solution in this paper, but this issue will require further work from us.

The existence of some loop with this structure was, as we mentioned above, exhibited in every attempt on a counterexample. It (surprisingly) turns out to be the simplest construction in our entire program (modulo some topological subtleties). It did however elude us the longest. In retrospect, this seems to have been due to its fundamental entity, in the sense that it turned out to be independent of the problem (or even rectifiability theory) completely, and had to be approached in a very general way. Yet, the understanding of (closed) sets which is deeply rooted in geometrical analysis has (erroneously) given us the illusion that an arbitrary closed set only has topology and holes and that anything beyond that had to come from the underlying problem! Such a point of view has been considered self-evident. Perhaps more important than anything we do in this paper, is that we correct this point of view by showing that these enveloping loops are fundamental to arbitrary closed sets in Euclidean space, and are manifestly global, in contrast to the previous perspective (with holes), which is clearly local. This important fact is one of the reasons we expect these objects to have an impact much further than our present problems.

3.3 Loops Closing At ∞, Small Scale Loops, And The Expected "Hand-Shake"

In this section we deal with a few basic obstructions to the ideas described in Section 3.2, so that we have a more realistic picture in mind for the required properties of these loops. As we go, we will find that we need them to satisfy more properties than may seem reasonable at first sight (in fact more than we ever thought possible till recently). Surprisingly, we will manage to construct them in Section 4 with most of the requirements actually built-in, leaving one basic requirement (gap elimination) of a different nature which we also partially address in Section 4. Let us first eliminate one basic obstacle whose solution is rather easy. Suppose the loops can be constructed in accord with our assumptions as in Section 3.2 in both F_1 and F_2, but that neither F_1 nor F_2 is compact, and these loops "close at ∞", i.e. they never meet or overlap as in Fig. 1. The picture then is as in Fig. 2 and in fact it suffices that the loops close outside $B\left(w_1, \frac{s}{\varepsilon^4}\right)$ for instance, where $w_i \in F_i$, and $|w_1 - w_2| = dist\left(F_1, F_2\right) \equiv s$. In such a case we can even argue more crudely;

$$U \equiv B\left(w_1, \frac{s}{\varepsilon^4}\right) \cap F,$$

and observe that (2) implies

$$\mu\left(U\right) \geq \left(\frac{1}{2} + \varepsilon\right)\left[\frac{4s}{\varepsilon^4} - Cs\right],$$

Figures will be available shortly.

where C is some universal (geometrical constant), as can be seen from 4 chains of balls of length $\frac{4s}{\varepsilon^4} - O(s)$, and each ball carries $\left(\frac{1}{2} + \varepsilon\right) \times$ (diameter of ball) in measure. On the other hand (1) implies that

$$\mu(U) \leq \frac{2s}{\varepsilon^4},$$

which is a contradiction if ε is sufficiently small. We can therefore assume that at least F_1 or F_2 is compact or, more precisely, it is enough to work with a loop that in some sense closes inside some ball of radius less than $\frac{s}{\varepsilon^4}$ say, so at least we can now see that our intuition of closing loops is a natural one (note that standard topological compactification to include the point at ∞ does not transfer easily to issues involving distances or measures and that is why we had to address these issues).

Let us now address another situation. Suppose we were to attach to the loop of Fig. 1 a chain of balls on only one of the original chains of the loop (think of the two chains described in Section 3.2) see Fig. 3. Assume that this is done *away* from the points z_1, z_N and, in fact, so that $diam\,(\text{loop} \cap F)$ is still $|z_1 - z_N|$. Then the description of the contradiction in Section 3.2 is still valid. This seems to give the impression that, if our original loop did not "surround" all of F_1 we can still play the same game. In principle this seems reasonable based on our discussion so far. Yet this was a major obstacle in our investigations because, in any realistic construction of a loop resembling that of Fig. 1, one a priori needs to use almost the same argument to actually build the sequence! Indeed, it always turns out that, as one is attempting to make such a construction, one needs to keep the points of maximal distance of any

Figures will be available shortly.

“sub-loop” (see Fig. 4 for a picture) on the subloop itself, or else basically all constructions fail at some later stage. Such a requirement was originally never insisted upon by the author since it appeared too strong, but it turns out that it is practically necessary and rather easy to build in a natural construction (only in retrospect however). This was a real surprise and immediately resolved significant difficulties. So, in Section 5, we construct an “enveloping” loop, i.e. one which surrounds the set, so that nothing is left out *especially* in diameter computations. In fact, we get for free a Jordan curve which closely approximates the set. Let us see one important application of this property which is needed for the rest of the program, namely how to open “bottle necks”. By that we mean how to show that what seems like a subloop in Fig. 4 can actually be supplemented by more points in the region where we have the loop closing before we reach z_N. See Fig. 5. We are here making an implicit assumption that we have some reasonably big region where the chains stay apart and then they start to overlap possibly all the way to z_N. As we said earlier, it is enough to consider the case where, at the first incidence of overlap, one of the points of maximum distance determining the diameter of this surrounded piece of the set is in one of the balls where the overlap happens (otherwise there is nothing to worry about). Let us simplify the (rather complicated) analysis by placing the centers of the balls in the chains (before the overlap happens) on an ellipse say, and then assume that the remainder of the set is a line segment attached as in Fig. 6. Our aim is to show that (a priori) the only way out of the (by now familiar) contradiction we described, is that the line segment actually “splits” (so it cannot just be a line segment), more precisely that we can continue placing balls in the required way, all the way till we reach z_N. Clearly we can order the

Figures will be available shortly.

points z on the right half of the ellipse in increasing value of the function $D(z) \equiv \max_{w\in \text{ left half of ellipse}} \{|z-w|\}$. Assume a thin ellipse for simplicity here, but that the consecutive distances between points on the top and the bottom chains are small enough to allow disjoint balls all the way up to (say) $\frac{3}{4}$ of the length of the ellipse. The point now is that using the same argument as in Section 3.2, there can never be a first time for the two chains to overlap like this. Indeed, take the first incidence of such overlap. Say between $B(z_{i_0}, r_{i_0}) \cap B\left(z'_{j_0}, r'_{j_0}\right) \neq$ singleton. Then we can order all points lying on the ellipse to the right of z_{i_0}, z'_{j_0} respectively. To each point z on the right $\frac{1}{4}$ piece of the ellipse associate a radius $r \equiv \max\left\{\varepsilon^2, \Psi\right\}$, where $\Psi = \min_w |z-w|$, s.t. $w \in F$ and to the right of z (precisely $D(w) \geq D(z) + \varepsilon^2$). With this construction for both chains take the first incidence of an overlap; by reducing the overlap to a singleton, and then recursively place balls all the way back to z_1. Exactly as argued in Section 3.2, we have a contradiction (once again modulo the gap issue). Clearly, there is nothing special about an ellipse or anything but we did not want to over complicate the geometry here. The conclusion is that we can open bottle necks (hopefully Fig. 5 is suggestive enough of the name).

There are several implicit assumption in this plan but, in fact, only two are significant and they both run hand in hand. Namely, we assumed that there is some region where the chains stay apart (like the ellipse above). Indeed, we need a place to start the process of counting which underlies the whole program. The other assumption is that the gap in the count we mentioned, even in connection with Fig. 1 and which is also relevant here, can be remedied in the count (but not necessarily eliminated in the geometrical picture). Both of these assumptions

Figures will be available shortly.

will actually be treated in Section 5, where we show, for example, that no piece of the loop can completely collapse, e.g. the set is like a line segment in some ball, or just "flat enough". The point is that we want to show some *rigidity* of our loop. More precisely, we supplement this large scale loop (whose existence is proved in Section 5) by a collection of points constructed at finer scales which allow the (supplemented) chains to stay apart, i.e. the consecutive distances between the points on each chain are small enough compared to the distances to the points in the other chain so that disjoint balls can be places in accord with Section 3.2.

We have thus motivated *two* constructions; one of enveloping loops at large scale, and one of finer chains if the set has some thin edges or "tentacles". The thicker we can allow such edges to be and still manage to construct our chains, the closer we are to solving the problem completely, as even a modest level of thickness becomes an immediate consequence of a modest collapse of the large scale loops. We leave the complicated analysis for a future paper and only give a representative result/construction in Section 5. It will be clear to the reader that our proofs can be strengthened in several places but, short of completing the merger between both pieces (and hence finishing the problem), we do not feel that a more complicated paper is warranted. We intend to pursue this merger in a forthcoming paper.

Figures will be available shortly.

4 The Main Results

4.1 General Sequential Enveloping Loops

In this section we give the statement of existence of the sequential enveloping loops in R^2. We do not explicitly present the construction in R^n, but it should be clear to the reader from the construction in Section 5 that a similar (but more elaborate) construction can be made.

Let F be an arbitrary closed set in R^n such that $R^n \backslash F$ contains a ball of radius $> \frac{s}{2}$.

Definition 7 *We will say that $K \subset F$ is an s-component of F if, for every $x, y \in K$, there exists a sequence $\{z_i\}_1^N \subset K$ s.t. $x = z_1, y = z_N$, and $|z_i - z_{i+1}| \leq s$ for all and that K is maximal under such conditions.*

Note that for any $s > 0$, F splits into a countable collection of such components.

Definition 8 *For K an s-component of F, we say that $z \in K \subset F$ is an s-accessibility point if there exists $B(x, r)$, s.t. $B^o(x, r) \subset R^2 \backslash F$, with $s \geq r > \frac{s}{2}$ such that $z \in \partial B(x, r)$, and $B(x, r) \cap K' = \phi$ for all other s-components $K' \subset F$. We will also call such $B(x, r)$ an accessibility ball of scale s at z.*

Note that a simple compactness argument (and the definition of s-components) imply that every s-component K can be accessed by at least one such accessibility ball of scale s. Such accessibility point z will be important since it will determine a certain (enveloping) sequence. For instance suppose K is an annulus of radii $10s$, $20s$ say. It is then intuitive that to envelope K with a sequence of step size s there should be one "outer" and one "inner" sequence. We make this precise:

Definition 9 *Let z be an s-accessibility point with accessibility ball $B(x,r)$. We then say $B(x,r)$ is an inner hole if there exists a finite sequence $\{z_i\}_1^N \subset K$ such that $|z_i - z_{i+1}| \leq s$, and the collection of the line segments $\overline{z_i z_{i+1}}$ define a closed Jordan curve Γ which, in turn, determines a bounded component (of $R^2 \backslash \Gamma$) containing x. Otherwise, we say $B(x,r)$ is an outer hole.*

Our theorems and constructions from now on will deal only with one such K (and one such accessibility point). All other s-components will become invisible in the development. This is because, in every step in the process, we only use balls of radius $\leq s$ (centered at points in K). Our new perspective on the geometry of closed sets in Euclidean space can now be stated (to fix ideas assume the origin is the relevant s-accessibility point):

Theorem 10 *There exist a locally finite sequence $Z = \{z_i\} \subset K$ (finite iff K is either compact or the accessibility ball $B(x,r)$ is an inner hole), and a closed Jordan curve J (p.w. smooth) possibly closing at ∞, s.t.*

1. *$|z_i - z_{i+1}| \leq s$ for all i;*
2. *K is contained in one of the components Ω determined by J;*
3. *For all $z \in J$, there exists $z' \in Z$, s.t. $|z - z'| \leq s$. and the line segment $\overline{zz'} \in \Omega$;*
4. *$dist(z, J) \leq 2s$ for all $z \in Z$;*
5. *A low-resolution convexity property defined below holds.*

See Fig. 7.

Definition 11 *We say that the curve J and the set K satisfy a low-resolution convexity property at scale s if the following property holds:*

suppose $z, z' \in K$, and $|z - z'| \leq s$, and the curve J crosses the line segment $\overline{zz'}$. Suppose Γ_i is an arc of J starting at a point x_i in $\overline{zz'}$, and ending at another point y_i on $\overline{zz'}$, so that $\overline{x_i y_i}$ and Γ_i determine a region G which lies entirely in Ω^c, then we have $\max_{x \in \overline{G}} dist(x, \overline{x_i y_i}) \leq 2s$.

See Fig. 8.

Definition 12 *We will call the sequence in Theorem 10 a sequential enveloping loop. At times, when no confusion may arise, we may call J, or even a chain of disjoint balls of size no more than s placed on Z, the same thing.*

Figures will be available shortly.

Remark 13 *The constant 2 in* (4) *or* (5) *can be improved much better. We do not pursue these issues here as they are not important for our program. The most important properties for us are* (1)-(3) *. Hopefully Fig 7 may already seem natural to the reader from our previous discussion in Section 3. In fact, that was the main obstacle -a global view of the set or, more precisely, the s-components, was the crucial missing idea in our program at least at the fundamental level.*

We prove Theorem 10 in Section 5.

Remark 14 *The reader is advised to take a moment to observe the significant resemblance of the loop J with the loops of homotopy theory. Our loops always give a codimension-*1 *surface in R^n, so they resemble the $(n-1)$-loops in the higher homotopy theory. We stress that the dimension of this surface $(n-1)$ does not impose any restriction on its utility; the dimension of the set determines the count of the balls which may be made disjoint and that is the only relevant object, at least as far as density problems are concerned.*

4.2 Doubling Enveloping Loops

Section 3 and the enveloping loops motivate the following definition:

Definition 15 *We will say that the loop Z of Theorem 10 is doubling at scale s, if we can place balls B with disjoint interiors such that each radius is at most s, and so that if K is compact then*

$$\sum diam\,(B) \geq 2diamK,$$

Figures will be available shortly.

or, if K is not compact, there exists $x \in K$ so that, for every $\delta > 0$, there exists R_δ s.t. if $R \geq R_\delta$,

$$\sum_{B \subset B(x,R)} diam\,(B) \geq 2R\left(1 - \frac{\delta}{2}\right).$$

By the discussion of Section 3 we have:

Theorem 16 *The set F introduced in Section 2.2 cannot have doubling enveloping loops.*

In the next few sections we will reformulate this as sufficient conditions for the set E to satisfy the $\frac{1}{2}$-conjecture. More importantly however, we will give another sufficient condition which is in fact motivated exactly by the possible failure of the loops to be doubling for some geometries. Our precise point is that there is a "hand-shake" between the two constructions and no geometry is omitted.

4.3 A No -Tentacle Theorem For F

In this section we state a simple but very important property for $F = spt\mu$ (see Section 2). This is the result whose importance was motivated in Section 3.3. The result is that F cannot have "tentacles" or "sharp ends". More precisely, in Section 5, we will prove:

Theorem 17 *There does not exist a ball $B\,(x, r)$ with the properties:*

1. $x \in F, r > 0$;

2. After a rotation and a translation (so $x = 0$*),* $B(0, r) \cap F$ *is contained in the half plane* $x^{(1)} \geq 0$ *(the right half plane);*

3. $\beta_F^1(x, r') \leq \frac{1}{8}$ *for* $x \in B\left(0, \frac{r}{4}\right) \cap F$*, and* $\frac{r}{2} < r' < r$.

See Fig 8. We call such a ball $B(0, r)$ a *ball of one-sided narrow containment for* F *with parameter* $\beta = \frac{1}{8}$.

Remark 18 *It is easy to see that property* (3) *would hold if instead we assume* $\beta_F^1(0, r') \leq \frac{1}{10}$ *(say) for* $\frac{r}{4} \leq r' \leq \frac{5}{4}$ *(actually much weaker estimates are sufficient) and this represents roughly two dyadic scales. But it is actually much better to use* (3) *in its stated form since there will be a specific sequence of points like* x *in* (3) *where this condition is relevant and natural. It will also be clear that a much more relaxed condition is also sufficient.*

As we mentioned in Section 3, this is a statement which says that our loop program cannot collapse! We are certainly after more than just that and, in fact, the construction which allows us to prove this theorem will actually provide a small-scale loop inside this strip (implied by (3))! Our ultimate goal is to construct the same kind of small scale loop for a much larger $\beta_F^1(x, r)$, namely around 1 (not 2) is basically what seems to be needed, and constitutes work in progress by the author.

4.4 Spherical Dissection And Small Scale Loops

We start this section by constructing just two points in the small scale loop, so that our theorem will be very natural. The complete construction will be presented in Section 5. Let us assume that there does exist a ball as in Theorem 4.3 (contrary to that theorem). See Fig. 9. For simplicity, let us normalize so that $r \equiv 1$ (for this section only), and recall $x \equiv 0$ via a translation. Consider the ball $B\left(0, \frac{1}{2}\right)$. Let w_1, z_1 be points of maximum distance in $B\left(0, \frac{1}{2}\right) \cap F$. By $(1), (2)$ of Section 2, we must have

$$|z_1 - w_1| > \frac{1}{2},$$

and our strip allows us to label w_1, z_1 s.t. $|w_1| \leq \frac{1}{4}, |z_1| > \frac{1}{4}$ (say). See Fig. 10. We basically forget about $x \equiv 0$ from now on. Now consider the ball $B_1 \equiv B(w_1, |z_1 - w_1|)$. By $(1), (2)$, there exist $w_2, z_2 \in B_1 \cap F$ (labeled s.t. $|w_1 - w_2|, |z_1 - z_2| \leq |w_i - z_i|$) so that

$$|w_2 - z_2| \geq |w_1 - z_1|.$$

Now the points z_1, z_2 are the initial points in two sequences $\left\{z_1^{(i)}\right\}, \left\{z_2^{(j)}\right\}$ which precisely constitute a small scale analogue of our large scale enveloping loop of Section 4.1. This loop is also "enveloping" in some sense but we will not dwell too much on that. The result can be stated in the form:

Figures will be available shortly.

Theorem 19 *Suppose there exists a ball* $B(x,r)$ *contrary to Theorem 17. Let* $z_1^{(0)} \equiv z_1, z_2^{(0)}$, *and* w_1, w_2 *be as above, and let* $a \equiv |z_1 - z_2|$, *then there exist two sequences* $\left\{z_1^{(i)}\right\}_1^\infty, \left\{z_2^{(j)}\right\}_1^\infty \in F$ *with the following properties (let* $D(z) \equiv \max_{w \in B(0,r) \cap F} dist(z,w)$) :

1. $D\left(z_m^{(i)}\right) \geq D\left(z_m^{(i-1)}\right)$ *for all* m, i;
2. $D(z) \leq D\left(z_m^{(i)}\right) + \varepsilon^2$ *for all* $z \in F \cap B\left(z_m^{(i)}, \left|z_m^{(i+1)} - z_m^{(i)}\right|\right)^o$;
3. $\left|z_m^{(i)} - z_m^{(i+1)}\right| \leq \frac{a}{4}$;
4. $\left|z_1^{(i)} - z_1^{(i+1)}\right| + \left|z_2^{(j)} - z_2^{(j+1)}\right| \leq \left|z_1^{(i)} - z_2^{(j)}\right|$ *(this is how the balls from each chain do not overlap with each other);*
5. *the angle* $\widehat{z_1^{(i)} w_m z_2^{(j)}} \leq 30^o$.

See Fig. 11.

Remark 20 *It is a consequence of properties* $(2),(5)$, *and the Pythagorean Theorem, that*

$$diam\left(\bigcup\left(\left\{z_1^{(i)}\right\}_1^{N_1} \cup \left\{z_2^{(j)}\right\}_1^{N_2}\right)\right) = \max\left\{D\left(z_1^{(N_1)}\right), D\left(z_2^{(N_2)}\right)\right\} + \varepsilon^2,$$

Figures will be available shortly.

and, furthermore, this is also true if we even replace $z_m^{(i)}$ *by* $B\left(z_m^i, \left|z_m^i - z_m^{(i+1)}\right|\right)^o \cap F$.

We also wish to comment on what happens as we relax the bound $\frac{1}{8}$ on the beta numbers above, or more precisely, as we shall see, the bound on the angles of the form $\widehat{w_2 z_i w_1}$, or $\widehat{z_1 w_i z_2}$. In fact, the bound on this kind of angle is the only essential restriction in our entire paper. The use of a strip is only to simplify life, but we could just as well use a truncated cone for instance; narrow near w_1, w_2 and wide near z_1, z_2. What happens as the bound on this angle is relaxed significantly is that our construction cannot proceed in the same way as we found z_1, z_2, as we will see in Section 5. Instead, we must then resort to our large scale enveloping loops and now the fact that this angle is large basically says that the "upper" and "lower" chains (of our large scale enveloping loop) in this region actually stay apart, thereby justifying our program as described in Section 3. It is precisely the transitional region between large and small values for these angles that constitutes the remainder of our program. It is rather clear to the author at this point that there are no remaining fundamental difficulties, but only technical (though nontrivial) ones.

4.5 Weak Circle Pairs And A Uniformization Technique

Since both of our fundamental constructions take place at the same scale (or at most a finite number of consecutive scales), we are able to prove the conjecture for sets which are not too far from general. To allow for maximal strength, we introduce a technical notion that allows us to pick up from the same place where

Figures will be available shortly.

Besicovitch left off, in particular this will allow us to verify the conjecture for sets that, frequently often, have pieces whose geometries can be described by either, a doubling loop or, by a tentacle (in the sense of Section 4.3). It will be sufficient to have a countable collection of such pictures occurring at various scales. Obviously however there must be some measure theoretic condition to go with these geometries; or else, if these pictures happen to be associated with pieces of very small measure, they could not have any effect on the density properties. We now introduce a notion which is an elaborate version of Besicovitch's circle pair idea [Be 2],

Definition 21 *Let E be a 1-set, $\eta > 0$. Given $x, y \in E$, we say that the pair of balls $B(x, |x-y|), B(y, |x-y|)$ is a weak circle pair for E at scale $s \equiv |x-y|$ and strength η if*

$$H^1 (B(x,s) \cap B(y,s) \cap E) \leq \eta s.$$

The point of this notion comes from the fact that a connected 1-set is rectifiable, and the weak circle pairs are the places in a purely unrectifiable 1-set where "a cut" or disconnectedness happens with a slight measure theoretic flexibility. More precisely, we make use of the following proposition:

Proposition 22 *Let E be a 1-set and $F \subset E$ also a 1-set. Suppose there exist $C, \eta_0, s_0 > 0$, and $A > 1$, s.t., if $B(x,s), B(y,s)$ is a weak circle pair of strength $\eta \leq \eta_0$, and scale $s \leq s_0$, we can find a ball $B \subset B(x, As)$, with $diam(B) \geq \frac{1}{A} s$, so that $H^1 (E \backslash F) \cap B \geq C diam\ (B)$. Then E contains a rectifiable piece of positive measure.*

For a proof, see [Far 1] which is a modification of a slightly less general statement in [Fal]. Also see [PT].

This proposition assures us that, if E is a purely unrectifiable 1-set and $F \subset E$ is the subset where the density is higher than some σ and we can show a contradiction for F using the geometry and properties of the measure (say) when working near weak circle pairs, then we also have a contradiction for E. Indeed, otherwise we can get a ball like B in the above proposition which would contradict that E is purely unrectifiable. Incidentally, this is also the way that the setup of Section 2.2 can be obtained; producing μ, F etc. We also remark that this notion can be used even in a metric space (see e.g. [PT]).

4.6 Loop-Doubling Near Weak Circle Pairs - Weak Sufficient Conditions

¿From the earlier discussions in Section 4, we know of two (basically opposite or complimentary) geometries which allow a loop that produces the "$\frac{1}{2}$". The first is when the sequential enveloping loop Z is doubling, and the second is when we have a ball of one-sided narrow containment with parameter $\beta \leq \frac{1}{8}$. The latter also produces a loop in the fine-structure of the set (in a proof by contradiction). We will thus refer to a geometry which is of either type as *Loop-Doubling*. At the risk of being repetitive we remind the reader that ultimately there should be no distinction between the two geometries involved; they simply become complimentary parts (or cases) of the *same* construction! We now have:

Definition 23 *We will say that E satisfies a Loop-Doubling condition with parameters β, C near a weak circle pair $B(x,s), B(y,s)$ of scale $s \equiv |y-x|$, if either*

1. *There exists a ball $B(z,r)$ of one-sided narrow containment for E with parameter β, and so that $H^1(B(z,t) \cap E) \geq \frac{1}{C}t$, for all $t \leq \frac{1}{C}s$, $B(z,r) \subset B(x,Cs)$, and $r \geq \frac{1}{C}s$, or*
2. *There exists a sequential enveloping loop Z inside $B(x,Cs)$ which is doubling, and $H^1(E \cap B(z,t)) \geq \frac{1}{C}t$ for all $z \in Z$, and $t \leq \frac{1}{C}s$.*

The only reason for the repeated appearance of measure-theoretic conditions above, is that we want the geometric condition coming from the loops to "stick" to an appropriately chosen F (if it exists in a proof by contradiction). These assumptions basically represent no geometrical restrictions whatsoever, and would automatically disappear once we complete the geometric study. For all intents and purposes we can simply assume that $H^1(E \cap B(x,t)) \geq \frac{1}{C}t$ for all t sufficiently small, a.e. $x \in E$, and avoid such complications all together if needed.

Definition 24 *We say that E satisfies a frequent loop-doubling condition with parameter β near weak circle pairs, if there exist $\eta_0, s_0, C > 0$ s.t., if $B(x,s), B(y,s)$ is a weak circle pair for E with strength $\eta \leq \eta_0, s \leq s_0$, then E satisfies a loop-doubling condition near this pair, with parameters β, C.*

Note that sets which satisfy this property have rather arbitrary geometries in most places and at most scales. However we can still state:

Theorem 25 *Suppose E is a purely unrectifiable 1-set satisfying a frequent loop-doubling condition with parameter $\beta \leq \frac{1}{8}$ near weak circle pairs. Then $\Theta_*^1(E,x) \leq \frac{1}{2}$ a.e.*

We remark first of all that the above theorem significantly generalizes the results of [Far 1],[Far 2],[Far 3] (except for $\beta \leq \frac{1}{8}$ instead of $\frac{1}{4}$ in [Far 2], which is only a temporary restriction for now, as it is known to the author how to surpass this bound but we do not do so here). This is because it removes that older essential flatness hypothesis in those works from all but a countable collection of balls (including scale), and all-together if a doubling enveloping loop happens to be somewhere nearby! In particular, we are able for the first time to show that sets with geometries far from flatness (basically "round-shaped") also satisfy the $\frac{1}{2}$-conjecture. This kind of statement was known to the author all along but the obstacle was that it could not be systematized until a suitable general notion of loops could be found. At last we can state such results in a completely systematic way. It may however seem like an awkward condition to apply since we are requiring that the reader check by hand that a doubling enveloping loop Z exists or not. This however is not a deficiency in the theorem (except of course for the obvious fact that we still need some assumption). The point is that the lesson we learned is that these loops are a natural geometrical object by which sets in Euclidean space should be described. Imposing any other condition to imply the one needed for the loops would significantly reduce the flexibility of the results and their reach. This is heavily pronounced in the present $\frac{1}{2}$-problem. We therefore resist any temptation to alter the statements into less fundamental ones which may favor the convenience of the reader over relevance. In the next section we give a special application in the context of self-similar 1-sets.

4.7 A Special Application For Self-Similar 1-Sets

Consider a self-similar 1-set in the sense of Hutchinson [Hu]. For such sets the condition $H^1(E \cap B(x,r)) \geq Cr$ is valid for all $x \in E$, $r \leq diam(E)$. Such a set is purely unrectifiable iff it is not a line segment. These sets do not seem to offer any simpler grounds to simplify the $\frac{1}{2}$-problem since the latter is basically a problem at one scale! However the theorem we stated in Section 4.6 (Theorem 25) allows a significantly simplified version in this context; since satisfying a Loop-Doubling condition once, implies its validity near every weak circle pair. We thus have:

Theorem 26 *Suppose E is a self-similar 1-set which is not a line segment. Suppose E satisfies a loop-doubling condition with parameter $\beta \leq \frac{1}{8}$, and $C > 0$. Then $\Theta_*^1(E,x) \leq \frac{1}{2}$ a.e.*

We note that the simplification via self-similarity is actually more measure-theoretic than geometric -the geometrical difficulties are still the same.

5 Proofs Of The Main Results

5.1 The Blind-Man-With-Cane Algorithm (BMWCA)

In this section we show that, given a hole in the set of a certain scale, it is possible to walk around the set starting from an accessibility point in such a way that the resulting sequence or its associated curve J do not get "trapped" or "stangled". It turns out that there is a strong and beautiful analogy with an algorithm that is executed in every day life by blind individuals using canes! In honor of those heroic individuals who overcome adversities everyday I would like to refer to our construction as the blind-man-with-a cane-algorithm or BMWCA for short (this sounds like something one might find on a license plate of a BMW in California but we assure the reader that no pun intended here). Indeed, such an individual starts his walk from a place clear of obstacles (a hole in his ambient space). Then he has the hardest time when he can't find a place in his surroundings where he can start "hitting" the surroundings with his cane (i.e. an accessibility point from which to start).

Once an accessibility point is found, this person can start walking around an object that is basically an s-component (for some s determined by his scale and his cane's). He can do this by always keeping the object on one side of his walk by hitting the set one step forward of his location and then stepping with the appropriate angle. He can take a complete round-trip with little trouble. Evidently, he loses orientation when going from one s-component to another, in which case he must depend on some other information source which can see more globally what the geometry of his surroundings is like. In retrospect this is almost exactly the algorithm we use to build the enveloping loops. The simplicity and naturality of this construction and the amount of information it provides especially in the context of the $\frac{1}{2}$-problem truly surprised the author who, for quite sometime, tried to do even much less from within the $\frac{1}{2}$-problem (i.e. using the $\frac{1}{2}$ as a tool) without success. Technically, it is perhaps the easiest piece of our program but, fundamentally, it is the very heart of the problem and should have been the starting point, whereas we actually had to recover it from the small-scale construction described later. In the next section we lay out the construction completely.

5.2 Construction Of The Sequential Enveloping Loops - The Discrete Topology Of Arbitrary Sets

We here embark on the construction (proof of Theorem 4.1). There will be no loss of generality if we simplify matters by rescaling $s \equiv 1$, assuming that the 1-component K lies in the upper half-plane, and that $0 \in K$. In particular, 0 is a 1-accessibility point and we can associate with it a (necessarily) outer hole whose interior (at least) lies in the lower half plane. The general case can actually be handled in basically the same way as we will comment later. We now embark on the construction

5.3 Fine-Structure Loops (Proof of No-Tentacles By Spherical Dissection)

5.4 Removal Of Flatness From Almost Every Point And Scale

6 Concluding Remarks

6.1 The Remaining Task - "The Hand Shake"

For the benefit of the expert reader we summarize exactly what remains of our program which has come rather close to its final goal. Recall first that the point of our large scale sequential enveloping loop is that, as long as we can keep the two chains apart (here we can identify a top and bottom chain for simplicity although there is no restriction from topology here and we can do this in any dimension or codimension), they are sufficient to produce the $\frac{1}{2}$ (or any best bound actually). The intricacy comes when that is not the case at some "end" of the loop. See Fig.10. The geometry and methods are those of Section 4.3 and 4.4. The outcome needed from the spherical dissection technique is that, either an angle like $\widehat{z_1 w_m z_2}$ is large (about 60^o), or else, one like $\widehat{w, z_m w_2}$ is large (about 60^o also). The conclusion is that one such triplet which gives such large angle allows us to use it to improve the information we have for the large scale (enveloping) loop. Indeed, the careful reader would notice that distances between any two points in such a triplet is between $\frac{1}{2}s$ to $2s$, where s is the natural scale of the entire construction (see Section 4). The point is that we are concluding that the enveloping loop can be supplemented by more points to turn it into a doubling loop (which proves $\frac{1}{2}$). In between such outcomes, there is another alternative which is that one can, instead, construct a small-scale loop which again, when considered as a piece of the bigger (enveloping) loop, turns it into a doubling loop in this case also. The author has presently succeeded in doing all of this modulo one technical difficulty which is the gap problem mentioned in Section 3. This issue has to be dealt with in practically every piece of the process and becomes extremely delicate as β approaches its value required here (approximately 1), or more precisely when the angles mentioned above are such that $2\sin\left(\frac{\text{angle}}{2}\right) \approx 1$ (or angle$\approx 60^o$ as stated above).

The author is extremely optimistic that this technical difficulty will be soon resolved. The main tool for this (the only one in fact) is the Pythagorean Theorem which seems to step in just in the appropriate places (so far) to remedy this problem.

6.2 Density Problems and Enveloping Loops - A Universal Perspective

It is becoming increasingly clear to the author that these simple and fundamental objects which we constructed above (sequential enveloping loops in particular) are the precise objects needed for problems of density (basically the $\frac{1}{2}$-problem generalized to all dimensions and codimensions) at least in Euclidean (though possibly also in a Hilbert) space. In fact I will go out on a limb and claim that the best constants for any reasonable density problem associated with a reasonable measure that has properties dependent only on the intrinsic geometry of a set (e.g. its diameter to some power as in H^m) should be computable using these loops. This rather strong claim is based on the fact that the loops provide a canonical way of efficiently packing balls (or some other objects if needed) in a set of a given geometry so, by and large density problems can be seen as discrete optimization problems in terms of loops. We are missing one ingredient in order to make rigorous such a claim at least as far as density problems in the context of rectifiability. Namely, whereas for 1-sets the signature of rectifiability is encoded in the fact that in Section 2 the support of μ can be written as the disjoint union of two closed sets with distance between them attained, in higher dimensions we do not know what the analogue is because connectedness of an m-set does not imply rectifiability. Nevertheless, it is still somehow clear that the best constants have to come from general loop considerations and this technical issue is most likely solvable by, either formulating rectifiability in terms of the enveloping loops themselves which is the broadest view we can hold here or, there is another interesting (and significant) possibility we wish to suggest without making any conjectures on the constants involved.

One classical question which I am not even sure if any one ever thought it can even be answered in a natural way, is to ask what is the analogue for connected m-sets of the rectifiability property only held by 1-sets? Could it be that a natural answer is not too far from us? From examples by P. Mattila and the author, it is clear that one can have a purely unrectifiable$m$-set which is connected and also has a lower bound on the lower density. As one increases this bound however, it seems that such sets become harder to construct. We would like to suggest that there is a specific lower bound on the lower density above which a purely unrectifiable m-set has to be totally disconnected. Such a bound of course exists and must (even without connectedness) be at least a certain (small) distance less than unity [Pr], but here we mean smaller bounds. Indeed, it would be extremely natural if this bound is precisely $\frac{1}{2}$, in which case all m-sets then behave the same since for 1-sets connectedness automatically implies lower density at least $\frac{1}{2}$. This would constitute a very natural answer to this old problem. I suspect that this is not far from true though I do not wish to bet on $\frac{1}{2}$ exactly here. My guess is that the best upper bound on the lower density and the best number for this connectedness implies rectifiability statement might coincide or at least have a relationship between them. but I can't quite make a strong case here. Whether or not we are correct about this, the prospects are quite interesting and we hope that some experts will pursue them.

6.3 A Counter-Conjecture In General Metric Spaces

It is here that I am officially going to go out on a limb. From the essential use of the Pythagorean Theorem, the gap problem (which is really the only remaining difficulty in our program), and how the whole loop construction seems to fall apart without the Pythagorean Theorem. I am finally prepared to do the honorable thing and (strongly) conjecture that $\sigma_1(M) > \frac{1}{2}$ for general metric spaces. Perhaps for a Hilbert space one can hope that $\frac{1}{2}$ is still the correct bound but that requires further investigations which we have not conducted.

6.4 Further Prospects Beyond Densities

As mentioned before, the enveloping loops we constructed can be constructed in any dimension. They always come with a piecewise smooth codimension one surface (since they are constructed from $(n-1)$-spheres), but this is no restriction on their applicability. They are suitable for introducing intrinsic geometrical and topological objects on arbitrary (say closed) sets such as homology, homotopy, curvature, etc., and we hope that some readers may pursue such interesting possibilities which should connect such subjects with analysis in new and deeper ways. We also believe that they will prove very helpful in problems where one starts with sets whose geometry is not a priori known or restricted but that a certain kind of inversion of information is needed to conclude a strong geometry at the end. this is certainly the case with problems involving densities and rectifiability but also with singular integrals and capacities (e.g. analytic, or harmonic Lip_{-1}), where the possibility of using the loops (or at least the global perspective afforded by them) to remove the use of Menger-Curvature in [MMV] and the final characterization of zero analytic capacity or harmonic Lip_{-1} capacities for 1-sets in R^2 in terms of rectifiability [Da],[DM] resp. Though certainly quite successful for 1-sets, the use of the Menger-Curvature has created more questions than it answered (whether for higher dimensional sets or even for more general kernels operating on 1-sets), and we still have outstanding conjectures by Mattila, and David-Semmes for m-sets. In fact, even for 1-sets, the most natural way to examine the set for positivity of analytic capacity is by studying the Cauchy Integral point-wise rather than in an L^2 version as with the Menger-Curvature, and such purely geometrical tests seem to work for 1-sets as well as m-sets (personal communication between the author and P. Mattila). Yet, so far, no purely geometrical and systematic approaches could be found. Any such approach is desperately needed as the results of [Far4] and [Far5] showed that basically there cannot be any reasonable analogue of the Menger-Curvature for m-sets with $m > 1$. The reason we expect the loops to help is that, prior to Menger-Curvature, there was only one basic approach originally due to Mattila [Ma3] which was reiterated from a different angle in [Far6]. Both results however are based on the same old perspective of closed sets (topology and holes) which the present paper generalized to a global one. In those papers both authors “hit” or access the set by local balls and compute the singular integral point wise but have to impose an additional geometrical hypothesis to conclude a contradiction. The author suspects that, by accessing the set by balls surrounding the enveloping loops, using the singular integral computation, and bootstrapping information across scales, we can eliminate the additional hypothesis. This is very sketchy but we hope we can fill in the details in the near future (currently joint efforts with P. Mattila). Finally, we would like to suggest the possibility of their usefulness in projection problems. Our evidence here is rather meager, but the author's intuition is that, by formulating projections in terms of the projections of the loops, one may be able to get a handle on some problem, especially that this can be done at every scale. Of special interest is the characterization of uniform rectifiability in terms of projections ala David-Semmes [DS]. We hope that some experts will study such possibilities also.

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