1 Basic Notions

Notation:
$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \supset \mathbb{Z}_+ = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$
$\mathbb{Q} = \{\text{rational numbers}\}$
$\mathbb{R} = \{\text{real numbers}\} \subset \mathbb{C} = \{\text{complex numbers}\}$.

Principle of Mathematical Induction (PMI): A statement P about $\mathbb{Z}_+$ is true if

(i) P holds for $n = 0$;

and

$$\text{(ii) If } P \text{ holds for all } m < n, \text{ then } P \text{ holds for } n. \tag{*}$$

Inputs for Number Theory:
- Logic
- Algebra
- Analysis (Advanced Calculus)
- Geometry

A slightly different principle from induction:

Well ordering axiom (WOA): Every non-empty subset of $\mathbb{Z}_+$ contains a smallest element.

Note: if S is finite then WOA is obvious and can be checked. *Intuitively*, we often apply it to infinte sets; this is accepting the WOA.

Lemma: WOA$\Rightarrow$PMI (for $\mathbb{Z}_+$).

Proof: Suppose (*) (i), (ii) hold for some property P.
To show: P is true for all non-negative integers.
Prove by contradiction. Suppose P is false. Let S be the subset of $\mathbb{Z}_+$ for which P is false. Since P is assumed to be false S is non-empty. By WOA, $\exists n \geq 0$ such that n is in S, and it is the **smallest** element of S. If $n = 0$, we would get a contradiction by (i). So $n > 0$. Since n is the smallest for which P is false, it is true for all $m < n$. By (ii), P holds for n as well. **Contradiction!** So P holds.

Note: First couple of weeks will be very easy, so use them to learn how to write a proof. (People lose more points on easy problems than hard ones.)

Remark: In fact, PMI and WOA are equivalent. Try to show PMI$\Leftrightarrow$ WOA.

Theorem: (*Euclidean Algorithm*) Let a, b be integers ≥ 1. Then we can write $a = bq + r$ with $q, r \in \mathbb{Z},\ 0 \leq r < b$.

Proof: Put $S = \{a - bn | n \in \mathbb{Z}\} \cap \mathbb{Z}_+$. *Claim*: $S \neq \emptyset$. (Easy) *Reason*: we can take n negative. So by WOA, S has a smallest element r. Since $r \in S$, we can write

$$r = a - bq, \text{ for some } q \in \mathbb{Z}$$

Since $S \subset \mathbb{Z}_+,\ r \geq 0$. Only thing to check: $r < b$. Suppose $r \geq b$. Then let

$$r' = a - b(q+1) = r - b \geq 0 \text{ since } r \geq b.$$

Thus $r' \in S$ and $r' < r$, a contradiction.

Definition: b divides a, written $b|a$, iff $a = bq$ for some $q \in \mathbb{Z}$. If not, write $b \nmid a$.

Definition: An integer $p > 1$ is **prime iff** the only positive integers dividing p are 1 and p.

Examples: 2, 3, 5, 7, 11, 13,... 37,... 691,...

A positive integer which is not a prime is called a **composite** number.

Theorem: Every $n \in \mathbb{N}$ is uniquely written as

$$n = \prod_{i=1}^{r} p_i^{m_i},$$

with each p_i prime and $m_i > 0$.

Proof of unique factorization:

Step 1: Show that any $n \in \mathbb{N}$ is a product of primes.

Proof: If $n = 1$, OK (empty product =1 by convention). So let $n > 1$. If n is a prime, there is nothing to do. So we may assume that n is *composite*. This means that $\exists$ prime p such that $p|n$. So $n = pq$, some $q \geq 1$. Use induction on n. Since $q < n$, by induction q is a product of primes. Hence n is a product of primes.

Step 2: **Uniqueness of factorization**

Suppose this is false. By WOA, $\exists$ smallest n for which it is false. Write $n = p_1 \ldots p_r = q_1 \ldots q_s$ with p_i, q_j primes, $1 \leq i \leq r,\ 1 \leq j \leq s,\ p_i \neq q_j$

for any (i, j). We may assume $p_1 \leq p_2 \leq \cdots \leq p_r$, $q_1 \leq q_2 \leq \cdots \leq q_s$ and $p_1 < q_1$. Now set $n' = p_1 q_2 \ldots q_s < n$. Since p_1 divides n and n', it divides $(n - n')$. We can write

$$n - n' = p_1 \ell_1 \ldots \ell_k \tag{1}$$

for some primes $\ell_1, \ldots, \ell_k$ since $n - n' < n$ and n is the smallest counterexample. We can also write

$$q_1 - p_1 = r_1 r_2 \ldots r_t \tag{2}$$

for primes $r_1, \ldots, r_t$. On the other hand, $n - n' = q_1 \ldots q_s - p_1 q_2 \ldots q_s$, i.e., $n - n' = (q_1 - p_1) q_2 \ldots q_s$. Then

$$n - n' = r_1 r_2 \ldots r_t q_2 \ldots q_s \tag{3}$$

Since $n - n' < n$, and since n is the smallest counterexample, the two factorizations of $n - n'$ given by (1) and (3) must coincide.

$$p_1 \in \{r_1, r_3 \ldots, r_t, q_2, \ldots, q_s\}$$

But $p_1 \neq q_j$; for any j. Thus

$$p_1 = r_i, \text{ for some } i.$$

Then p_1 divides $(q_1 - p_1) \Rightarrow p_1 | q_1$, contradiction!

Analysis enters when we ask questions about the number and distribution of primes.

Theorem. (Euclid) There exist infinitely many primes in $\mathbb{Z}$.

Proof: Suppose not. Then there exist only a finite number of primes; list them as $p_1, p_2, \ldots, p_m$. Put $n = p_1 p_2 \ldots p_m + 1$. If n is prime we get a contradiction since $n > p_m$. So n cannot be prime. Let q be a prime divisor of n. Since $\{p_1, \ldots, p_m\}$ is the set of all primes, q must equal p_j; for some j. Then q divides $n = p_1 \ldots p_m + 1$ and $p_1 \ldots p_m \Rightarrow q | 1$, a contradiction.

Euler's attempted proof. (This can be made rigorous!) Let P be the set of all primes in $\mathbb{Z}$. **Euler's idea**: If P were finite, then $X = \prod_{p \in P} \frac{1}{(1 - \frac{1}{p})} < \infty$.

Lemma.

Let s be any real number > 1. Then

$$\zeta(s) = \prod_{p \in P} \frac{1}{(1 - \frac{1}{p^s})} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(called the "Riemann" zeta function, though Euler studied it a century earlier).

Proof of Lemma. Recall: If $|x| < 1$, then $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ (geometric series). If $s > 1$, $\frac{1}{p^s} < 1$. So $\frac{1}{1-\frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots$ Then

$$\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by unique factorization.

Euler then argued as follows: let $s \to 1$ from right. X$=\lim_{s\to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^s} \to \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. But if P is finite, then X is a finite rational number, a contradiction. (To make this rigorous, we need to be careful about limits and uniform convergence.)

The Prime Number Theorem (PNT)

For any $x \geq 2$, put

$$\pi(x) = \#\{p : \text{ prime} \mid p \leq x\}.$$

What does $\pi(x)$ look like for x very large? The **prime number theorem** (PNT) says:

$$\pi(x) \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

In other words, the fraction of integers in $[1, x]$ which are prime is roughly $\frac{1}{\log x}$ for x large. (Can't prove it in this class.)

Twin Primes These are prime pairs (p, q) with $q = p + 2$.

Examples: (3,5), (5,7), (11, 13),...

Conjecture: There exist infinitely many twin primes.

Stronger conjecture: If $\pi_2(x)$ denotes the number of twin primes $\leq x$, then

$$\pi_2(x) \sim \frac{x}{(\log x)^2} \text{ as } x \to \infty.$$

2 Heuristics on Primes

Let $P = \{\text{primes in } \mathbb{Z}\}$. We saw two proofs of the fact that P is infinite.

Prime Number Theorem (PNT). If $\pi(x) = \#\{p \in P | p \leq x\}$ then $\pi(x) \sim \frac{x}{\log x}$ for x large.

Heuristic reason: Let $F(x) =$ the fraction of positive integers $\leq x$ which are prime. Then $F(x) = \frac{\pi(x)}{x}$. We want to take all $n \leq x$ and then throw out composite numbers. First throw out even numbers, i.e., those divisible by 2.

$$\left\{\begin{matrix}\text{fraction of odd numbers} \\ \text{which are } \leq x\end{matrix}\right\} \sim \frac{1}{2} = \left(1 - \frac{1}{2}\right)$$

$$\text{fraction of numbers which are not divisible by 3} \sim \left(1 - \frac{1}{3}\right)$$

We get

$$F(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)$$

In fact, we should use the bound $\sqrt{x}$ for better accuracy. This way we are off by a factor of 2.

Recall Euler's result:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim \sum_{n \leq x} \frac{1}{n} \sim \int_1^x \frac{1}{t} dt = \log x$$

Consequently,

$$F(x) \sim \frac{1}{\log x}, \text{ and so } \pi(x) \sim \frac{x}{\log x}$$

Twin primes

We are looking for numbers n such that n and $n+2$ are prime.

Put

$$\pi_2(x) = |\{\text{twin primes } \leq x\}|$$

A heuristic argument:

Put

$$F_2(x) = \frac{\pi_2(x)}{x}$$

Again, take all $n \leq x$ and throw out numbers which are not twin primes.

Check:

$$F_2(x) \approx \prod_{p \leq x} \left(1 - \frac{2}{p}\right) \approx \frac{1}{\log^2 x}$$

So one expects:

$$\pi_2(x) \approx \frac{x}{\log^2 x} \quad \leftarrow \text{Not yet proved!}$$

3 More on divisibility and Primes

Proposition 1: Let $a_1, a_2, \ldots, a_n$ be integers. Put

$$M = \{\sum_{i=1}^{n} a_i x_i | x_i \in \mathbb{Z}, \forall i\}.$$

Then $M = d\mathbb{Z}$, for a unique $d \geq 0$. ($d\mathbb{Z}$ is the set of all integers divisible by d.)

Proof. Certainly, $d \in M$. If $M = \{0\}$, take $d = 0$. Otherwise, put $M^+ = \{n \in M | n > 0\}$. Then clearly, M^+ is non-empty since $M \neq \{0\}$, and so by WOA, $\exists$ smallest element, call it d, in M^+. For any n in M, we can write by the Euclidean algorithm: $n = dq + r$, with $q, r \in \mathbb{Z}$, and $0 \leq r < d$.

Note that M is closed under subtraction. So $r = n - dq$ is also in M. If $r = 0$, we are done because then $n = dq$ as desired.

Suppose $r > 0$. Then $r \in M^+$. Since $r < d$, this contradicts the minimality of d. Hence r must be 0, and $n \in d\mathbb{Z}$.

Definition: Let $a_1, \ldots, a_n, d$ be as in Prop. 1. Then d is called the gcd (**greatest common divisor**) of $\{a_i\}$. For brevity, write

$$d = (a_1, \ldots, a_n) = gcd(a_1, \ldots, a_n).$$

Check: $(a_1, (a_2, a_3)) = ((a_1, a_2), a_3)$

Definition: $\{a_i\}$ are mutually relatively prime iff $(a_1, \ldots, a_n) = 1$.

Example: (2,3,9) is mutually relatively prime but not *pairwise* relatively prime.

Proposition 2. $a_1, \ldots, a_n$ are mutually relatively prime iff we can solve the equation

$$\sum_{i=1}^{n} a_i x_i = 1 \tag{*}$$

in integers.

Proof. Suppose $d = (a_1, \ldots, a_n) = 1$. Then by Prop.1, $1 = d \in M = \{\sum_{i=1}^{n} a_i x_i | x_i \in \mathbb{Z}\}$. So (*) can be solved in integers. Conversely, suppose (*) has a solution in integers. Then $1 \in M^+$, and so $d = 1$.

Proposition 3. Let $a, b, c \in \mathbb{Z}$, $(a, b) = 1$. Suppose $a|bc$. Then $a|c$.

Proof. Since $(a, b) = 1$, by Prop.2, $\exists\ x, y \in \mathbb{Z}$. Set $ax + by = 1$. Then $c = c(ax + by) = a(cx) + (bc)y$. Since $a|bc$, a divides the right hand side, hence $a|c$.

Proof of unique factorization in $\mathbb{Z}$.

Existence

As shown before, every $n \geq 1$ is a product of primes.

Uniqueness (second proof)

Let $n > 1$ be the smallest counterexample. So we can write $n = p_1 \ldots p_r = q_1 \ldots q_s$, with p_i, q_j primes and $p_1 \neq q_j$ for any (i, j). So

$$p_1 | n = q_1 \ldots q_s = q_1(q_2 \ldots q_s).$$

Since $p_1 \neq q_1$, $(p_1, q_1) = 1$, and by Prop. 3, $p_1 | (q_2 \ldots q_s)$. Again, since $p_1 \neq q_2$, applying Prop.3 again, $p_1 | (q_3 \ldots q_s)$. Finally get $p_1 | q_s$. So there is no such counterexample.

Third Proof of the Infinitude of Primes in $\mathbb{Z}$ (Polya)

For every $n \geq 1$, put $F_n = 2^{2^n} + 1$, called the nth *Fermat number*.

Lemma. If $n \neq m$, $(F_n, F_m) = 1$.

Proof of Lemma. We may assume $m > n$. Write $m = n + k$, for some $k > 0$. *To show*:

$$(F_n, F_{n+k}) = 1 \quad \text{(for } k > 0.)$$

Suppose $d|F_n$ **and** $d|F_{n+k}$. Put $x = 2^{2^n}$. Then, since

$$F_{n+k} = 2^{2^{n+k}} + 1 = 2^{2^n 2^k} + 1,$$

$$\frac{F_{n+k} - 2}{F_n} = \frac{x^{2^k} - 1}{x + 1}$$
$$= x^{2^k - 1} - x^{2^k - 2} + \cdots - 1 \in \mathbb{Z}$$

$$\Rightarrow F_n | (F_{n+l} - 2) \Rightarrow d|2.$$

But F_n, F_{n+k} are odd. So $d = 1$. Hence the lemma.

Proof of Infinitude of primes

Consider $F_1, F_2, \ldots, F_n \ldots$ By lemma, each F_n is divisible by a prime, call it p_n, not dividing the previous $F_k,\ k < n$. The sequence $\{p_1, p_2, \ldots\}$ is infinite.

One has: $F_1 = 5,\ F_2 = 17,\ F_3 = 257,\ F_4 = 65537$ (Fermat), $F_5 = (641)(6700417), \ldots$

Primes in "Arithmetic Progressions":

Fix $m > 1$, and $a \in \mathbb{Z}$ such that $(a, m) = 1$.

Theorem (Dirichlet) $\exists$ infinitely many primes p which are $\equiv a \pmod m$.

We cannot possibly prove it in this class. But we can prove the following:

Baby Lemma $\exists$ infinitely many primes p which are $\equiv 3 \pmod 4$.

Proof: Suppose $\exists$ only a finite number of such primes, say 3, p_1, p_2, $\cdots$, p_r.

Consider

$$N = 4p_1\, p_2\ \cdots p_r + 3.$$

By unique factorization in $\mathbb{Z}$ we can write $N = q_1 q_2 \cdots q_s$, with the q_j's being primes.

Claim 1: Some q_j must be $\equiv 3 (mod\, 4)$.

Indeed, every q_j is an odd prime as N is odd, and moreover if $q_j \equiv 1 \pmod 4$ $\forall_j$, then N will also be $\equiv 1 \pmod 4$, contradiction! Hence Claim 1.

Say $q_1 \equiv 3 \pmod 4$.

Claim 2: $q_1 \notin \{3, p_1, \cdots, p_r\}$.

Indeed, if $q_1 = 3$, then $3|N$, and since $N = 4p_1 \cdots p_r + 3$, 3 must divide $4p_1 \cdots p_r, \rightarrow\leftarrow$. So $q_1 \neq 3$. Suppose $q_1 = p_i$ for some $1 \leq i \leq r$. Then $p_i \mid N$, and since $N = 4p_1 \cdots p_r + 3$, $p_i \mid 3$, $\rightarrow\leftarrow$. So $q_1 \neq p_i$. Hence Claim 2.

So we have produced a new prime $q_1 \equiv 3 (\operatorname{mod} 4)$ which is not in the original list, $\rightarrow\leftarrow$.
Remark: There is no such simple argument to prove Dirichlet's theorem for primes $\equiv 1 (\operatorname{mod} 4)$. We can try to start the same way by assuming that we have a finite list of primes $\equiv 1 (\operatorname{mod} 4)$, say $p_1, p_2, \cdots, p_r$, and we can consider $N = 4p_1 \cdots p_r + 1$. Factor N as $q_1 \cdots q_s$. Now the analog of Claim 1 will in general fail as the product of an even number of numbers congruent to 3 (mod 4) is 1 (mod 4). However, we will prove the infinitude of such primes later after studying squares mod p.

Earlier we saw a *heuristic reason* for expecting there to be an infinite number of **twin primes**, e.g. $\{3, 5\}, \{5, 7\}, \{11, 13\}, \cdots$
Expectation:

$$\pi_2(x) := \# \begin{pmatrix} \text{twin primes} \\ \leq x \end{pmatrix} \approx \quad C \frac{x}{\log^2 x}, \qquad \text{as } x \to \infty.$$

This means $\pi_2(x) - \frac{cx}{\log^2 x}$ goes to 0 as x goes to ∞.

This twin prime problem is closely related to the **Goldbach problem**, which asks if every even number ≥ 4 is a sum of 2 primes.
Best known result: (Chen)

$$2n = a_1 + a_2, \text{ with } a_i \text{ prime or a product of 2 primes.}$$

A similar heuristic reason makes one expect that there are infinitely many primes p of the form $n^2 + 1$.
Best known result: (Iwaniec)

$$\exists \text{ an infinite of sequence } \{m_1, m_2, \cdots\}$$

such that

(i)

$$m_j = n_j^2 + 1, \qquad \forall j$$

and for every j,

(ii)

$$m_j \text{ is a prime or a product of 2 primes}$$

The proof is quite hard and beyond the scope of our class.

4 Pythagorean Triples

Problem:

Find all $x, y \in \mathbb{N}$ such that

$$x^2 + y^2 = z^2 \tag{1}$$

If $d = (x, y, z) > 1$, then $(\frac{x}{d}, \frac{y}{d}, \frac{z}{d})$ is another solution, called the **primitive solution**.

For primitive solutions, we may assume that x is odd and y is even.

The Geometric Method

Solving (1) in integers amounts to solving the following in rational numbers:

$$X^2 + Y^2 = 1 \tag{2}$$

Geometrically, (2) is the equation of the unit circle in $\mathbb{R}^2$ with center at $O = (0, 0)$. Try to parametrize the circle.

One can try as in calculus to set

$$X = \cos\theta,\ Y = \sin\theta.$$

This turns out to be *terrible* for number theory. A better way is to consider the parametrization

$$X = \frac{1 - t^2}{1 + t^2}, \quad Y = \frac{2t}{1 + t^2}$$

This is *ingenious* as this only involves rational functions. If $t \in \mathbb{Q}$, then $X, Y \in \mathbb{Q}$. Of course

$$X^2 + Y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2} = 1$$

As $t \to \infty$ (along rationals) then

$$X = \frac{1 - t^2}{1 + t^2} \to -1$$

So we are only missing one solution, $(-1, 0)$, which we will remember.

Check: If $X, Y \in \mathbb{Q}$, then $t \in \mathbb{Q}$. (Show: $t = \frac{Y}{1+X}$.)

So the rational solutions of (2) are obtained by setting

$$X = \frac{1-t^2}{1+t^2}, \quad Y = \frac{2t}{1+t^2},$$

with $t \neq \pm 1, 0$, together with $(\pm 1, 0)$ and $(0, \pm 1)$.

Write $t = \frac{u}{v}$, $u, v \in \mathbb{Z}$. Then

$$X = \frac{u^2 - v^2}{u^2 + v^2}, \quad Y = \frac{2uv}{u^2 + v^2}$$

It follows that the non-zero solutions in $\mathbb{Z}$ of (1) are given by

$$x = u^2 - v^2, \ y = 2uv, \ z = u^2 + v^2$$

with

$$u \neq \pm v, \ u, v \neq 0$$

To get primitive solutions, it is convenient to put

$$m = u + v, \ n = u - v$$

$$x = (u+v)(u-v) = mn, \ y = \frac{m^2 - n^2}{2}, \ z = \frac{m^2 + n^2}{2}$$

For primitive solutions, take m, n odd ≥ 1, $m > n$, with $(m, n) = 1$. Check that these are all the primitive solutions.

5 Basic Notions

Notation:
$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \supset \mathbb{Z}_+ = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$
$\mathbb{Q} = \{\text{rational numbers}\}$
$\mathbb{R} = \{\text{real numbers}\} \subset \mathbb{C} = \{\text{complex numbers}\}$.

Principle of Mathematical Induction (PMI): A statement P about $\mathbb{Z}_+$ is true if

(i) P holds for $n = 0$;

and

(ii) If P holds for all $m < n$, then P holds for n. (*)

Inputs for Number Theory:

Logic
Algebra
Analysis (Advanced Calculus)
Geometry

A slightly different principle from induction:

Well ordering axiom (WOA): Every non-empty subset of $\mathbb{Z}_+$ contains a smallest element.

Note: if S is finite then WOA is obvious and can be checked. *Intuitively*, we often apply it to infinte sets; this is accepting the WOA.

Lemma: WOA$\Rightarrow$PMI (for $\mathbb{Z}_+$).

Proof: Suppose (*) (i), (ii) hold for some property P.
To show: P is true for all non-negative integers.
Prove by contradiction. Suppose P is false. Let S be the subset of $\mathbb{Z}_+$ for which P is false. Since P is assumed to be false S is non-empty. By WOA, $\exists n \geq 0$ such that n is in S, and it is the **smallest** element of S. If $n = 0$, we would get a contradiction by (i). So $n > 0$. Since n is the smallest for which P is false, it is true for all $m < n$. By (ii), P holds for n as well. **Contradiction!** So P holds.

Note: First couple of weeks will be very easy, so use them to learn how to write a proof. (People lose more points on easy problems than hard ones.)

Remark: In fact, PMI and WOA are equivalent. Try to show PMI$\Leftrightarrow$ WOA.

Theorem: (*Euclidean Algorithm*) Let a, b be integers ≥ 1. Then we can write $a = bq + r$ with $q, r \in \mathbb{Z},\ 0 \leq r < b$.

Proof: Put $S = \{a - bn | n \in \mathbb{Z}\} \cap \mathbb{Z}_+$. *Claim*: $S \neq \emptyset$. (Easy) *Reason*: we can take n negative. So by WOA, S has a smallest element r. Since $r \in S$, we can write

$$r = a - bq, \text{ for some } q \in \mathbb{Z}$$

Since $S \subset \mathbb{Z}_+,\ r \geq 0$. Only thing to check: $r < b$. Suppose $r \geq b$. Then let

$$r' = a - b(q + 1) = r - b \geq 0 \text{ since } r \geq b.$$

Thus $r' \in S$ and $r' < r$, a contradiction.

Definition: b divides a, written $b|a$, iff $a = bq$ for some $q \in \mathbb{Z}$. If not, write $b \nmid a$.

Definition: An integer $p > 1$ is **prime iff** the only positive integers dividing p are 1 and p.

Examples: 2, 3, 5, 7, 11, 13,... 37,... 691,...

A positive integer which is not a prime is called a **composite** number.

Theorem: Every $n \in \mathbb{N}$ is uniquely written as

$$n = \prod_{i=1}^{r} p_i^{m_i},$$

with each p_i prime and $m_i > 0$.

Proof of unique factorization:

Step 1: Show that any $n \in \mathbb{N}$ is a product of primes.

Proof: If $n = 1$, OK (empty product =1 by convention). So let $n > 1$. If n is a prime, there is nothing to do. So we may assume that n is *composite*. This means that $\exists$ prime p such that $p|n$. So $n = pq$, some $q \geq 1$. Use induction on n. Since $q < n$, by induction q is a product of primes. Hence n is a product of primes.

Step 2: **Uniqueness of factorization**

Suppose this is false. By WOA, $\exists$ smallest n for which it is false. Write $n = p_1 \dots p_r = q_1 \dots q_s$ with p_i, q_j primes, $1 \leq i \leq r$, $1 \leq j \leq s$, $p_i \neq q_j$ for any (i, j). We may assume $p_1 \leq p_2 \leq \cdots \leq p_r$, $q_1 \leq q_2 \leq \cdots \leq q_s$ and $p_1 < q_1$. Now set $n' = p_1 q_2 \dots q_s < n$. Since p_1 divides n and n', it divides $(n - n')$. We can write

$$n - n' = p_1 \ell_1 \dots \ell_k \tag{3}$$

for some primes $\ell_1, \dots, \ell_k$ since $n - n' < n$ and n is the smallest counterexample. We can also write

$$q_1 - p_1 = r_1 r_2 \dots r_t \tag{4}$$

for primes $r_1, \dots, r_t$. On the other hand, $n - n' = q_1 \dots q_s - p_1 q_2 \dots q_s$, i.e., $n - n' = (q_1 - p_1) q_2 \dots q_s$. Then

$$n - n' = r_1 r_2 \dots r_t q_2 \dots q_s \tag{5}$$

Since $n - n' < n$, and since n is the smallest counterexample, the two factorizations of $n - n'$ given by (1) and (3) must coincide.

$$p_1 \in \{r_1, r_3 \ldots, r_t, q_2, \ldots, q_s\}$$

But $p_1 \neq q_j$; for any j. Thus

$$p_1 = r_i, \text{ for some } i.$$

Then p_1 divides $(q_1 - p_1) \Rightarrow p_1 | q_1$, contradiction!

Analysis enters when we ask questions about the number and distribution of primes.

Theorem. (Euclid) There exist infinitely many primes in $\mathbb{Z}$.

Proof: Suppose not. Then there exist only a finite number of primes; list them as $p_1, p_2, \ldots, p_m$. Put $n = p_1 p_2 \ldots p_m + 1$. If n is prime we get a contradiction since $n > p_m$. So n cannot be prime. Let q be a prime divisor of n. Since $\{p_1, \ldots, p_m\}$ is the set of all primes, q must equal p_j; for some j. Then q divides $n = p_1 \ldots p_m + 1$ and $p_1 \ldots p_m \Rightarrow q|1$, a contradiction.

Euler's attempted proof. (This can be made rigorous!) Let P be the set of all primes in $\mathbb{Z}$. **Euler's idea**: If P were finite, then $X = \prod_{p \in P} \frac{1}{(1 - \frac{1}{p})} < \infty$.

Lemma.

Let s be any real number > 1. Then

$$\zeta(s) = \prod_{p \in P} \frac{1}{(1 - \frac{1}{p^s})} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(called the "Riemann" zeta function, though Euler studied it a century earlier).

Proof of Lemma. Recall: If $|x| < 1$, then $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ (geometric series). If $s > 1$, $\frac{1}{p^s} < 1$. So $\frac{1}{1-\frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots$ Then

$$\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by unique factorization.

Euler then argued as follows: let $s \to 1$ from right. X=$\lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^s} \to \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. But if P is finite, then X is a finite rational number, a contradiction. (To make this rigorous, we need to be careful about limits and uniform convergence.)

The Prime Number Theorem (PNT)

For any $x \geq 2$, put

$$\pi(x) = \#\{p : \text{ prime} \mid p \leq x\}.$$

What does $\pi(x)$ look like for x very large? The **prime number theorem** (PNT) says:

$$\pi(x) \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

In other words, the fraction of integers in $[1, x]$ which are prime is roughly $\frac{1}{\log x}$ for x large. (Can't prove it in this class.)

Twin Primes These are prime pairs (p, q) with $q = p + 2$.

Examples: (3,5), (5,7), (11, 13),...

Conjecture: There exist infinitely many twin primes.

Stronger conjecture: If $\pi_2(x)$ denotes the number of twin primes $\leq x$, then

$$\pi_2(x) \sim \frac{x}{(\log x)^2} \text{ as } x \to \infty.$$

2 Heuristics on Primes

Let $P = \{\text{primes in } \mathbb{Z}\}$. We saw two proofs of the fact that P is infinite.

Prime Number Theorem (PNT). If $\pi(x) = \#\{p \in P | p \leq x\}$ then $\pi(x) \sim \frac{x}{\log x}$ for x large.

Heuristic reason: Let $F(x) =$ the fraction of positive integers $\leq x$ which are prime. Then $F(x) = \frac{\pi(x)}{x}$. We want to take all $n \leq x$ and then throw out composite numbers. First throw out even numbers, i.e., those divisible by 2.

$$\left\{\begin{matrix}\text{fraction of odd numbers} \\ \text{which are } \leq x\end{matrix}\right\} \sim \frac{1}{2} = \left(1 - \frac{1}{2}\right)$$

$$\text{fraction of numbers which are not divisible by 3} \sim \left(1 - \frac{1}{3}\right)$$

We get

$$F(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)$$

In fact, we should use the bound $\sqrt{x}$ for better accuracy. This way we are off by a factor of 2.

Recall Euler's result:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim \sum_{n \leq x} \frac{1}{n} \sim \int_1^x \frac{1}{t} dt = \log x$$

Consequently,

$$F(x) \sim \frac{1}{\log x}, \text{ and so } \pi(x) \sim \frac{x}{\log x}$$

Twin primes

We are looking for numbers n such that n and $n+2$ are prime.

Put

$$\pi_2(x) = |\{\text{twin primes } \leq x\}|$$

A heuristic argument:

Put

$$F_2(x) = \frac{\pi_2(x)}{x}$$

Again, take all $n \leq x$ and throw out numbers which are not twin primes.

Check:

$$F_2(x) \approx \prod_{p \leq x} \left(1 - \frac{2}{p}\right) \approx \frac{1}{\log^2 x}$$

So one expects:

$$\pi_2(x) \approx \frac{x}{\log^2 x} \quad \leftarrow \text{Not yet proved!}$$

3 More on divisibility and Primes

Proposition 1: Let $a_1, a_2, \ldots, a_n$ be integers. Put

$$M = \{\sum_{i=1}^{n} a_i x^i | x_i \in \mathbb{Z}, \forall i\}.$$

Then $M = d\mathbb{Z}$, for a unique $d \geq 0$. ($d\mathbb{Z}$ is the set of all integers divisible by d.)

Proof. Certainly, $D \in M$. If $M = \{0\}$, take $d = 0$. Otherwise, put $M^+ = \{n \in M | n > 0\}$. Then clearly, M^+ is non-empty since $M \neq \{0\}$, and so by WOA, $\exists$ smallest element, call it d, in M^+. For any n in M, we can write by the Euclidean algorithm: $n = dq + r$, with $q, r \in \mathbb{Z}$, and $0 \leq r < d$.

Note that M is closed under subtraction. So $r = n - dq$ is also in M. If $r = 0$, we are done because then $n = dq$ as desired.

Suppose $r > 0$. Then $r \in M^+$. Since $r < d$, this contradicts the minimality of d. Hence r must be 0, and $n \in d\mathbb{Z}$.

Definition: Let $a_1, \ldots, a_n, d$ be as in Prop. 1. Then d is called the gcd (**greatest common divisor**) of $\{a_i\}$. For brevity, write

$$d = (a_1, \ldots, a_n) = gcd(a_1, \ldots, a_n).$$

Check: $(a_1, (a_2, a_3)) = ((a_1, a_2), a_3)$

Definition: $\{a_i\}$ are mutually relatively prime iff $(a_1, \ldots, a_n) = 1$.

Example: (2,3,9) is mutually relatively prime but not *pairwise* relatively prime.

Proposition 2. $a_1, \ldots, a_n$ are mutually relatively prime iff we can solve the equation

$$\sum_{i=1}^{n} a_i x_i = 1 \tag{*}$$

in integers.

Proof. Suppose $d = (a_1, \ldots, a_n) = 1$. Then by Prop.1, $1 = d \in M = \{\sum_{i=1}^{n} a_i x^i | x^i \in \mathbb{Z}\}$. So (*) can be solved in integers. Conversely, suppose (*) has a solution in integers. Then $1 \in M^+$, and so $d = 1$.

Proposition 3. Let $a, b, c \in \mathbb{Z}$, $(a, b) = 1$. Suppose $a|bc$. Then $a|c$.

Proof. Since $(a, b) = 1$, by Prop.2, $\exists\ x, y \in \mathbb{Z}$. Set $ax + by = 1$. Then $c = c(ax + by) = a(cx) + (bc)y$. Since $a|bc$, a divides the right hand side, hence $a|c$.

Proof of unique factorization in $\mathbb{Z}$.

Existence

As shown before, every $n \geq 1$ is a product of primes.

Uniqueness (second proof)

Let $n > 1$ be the smallest counterexample. So we can write $n = p_1 \ldots p_r = q_1 \ldots q_s$, with p_i, q_j primes and $p_1 \neq q_j$ for any (i, j). So

$$p_1 | n = q_1 \ldots q_s = q_1(q_2 \ldots q_s).$$

Since $p_1 \neq q_1$, $(p_1, q_1) = 1$, and by Prop. 3, $p_1|(q_2 \ldots q_s)$. Again, since $p_1 \neq q_2$, applying Prop.3 again, $p_1|(q_3 \ldots q_s)$. Finally get $p_1|q_s$. So there is no such counterexample.

Third Proof of the Infinitude of Primes in $\mathbb{Z}$ (Polya)

For every $n \geq 1$, put $F_n = 2^{2^n} + 1$, called the nth *Fermat number*.

Lemma. If $n \neq m$, $(F_n, F_m) = 1$.

Proof of Lemma. We may assume $m > n$. Write $m = n + k$, for some $k > 0$. *To show*:

$$(F_n, F_{n+k}) = 1 \quad \text{(for } k > 0.)$$

Suppose $d|F_n$ **and** $d|F_{n+k}$. Put $x = 2^{2^n}$. Then, since

$$F_{n+k} = 2^{2^{n+k}} + 1 = 2^{2^n 2^k} + 1,$$

$$\begin{aligned} \frac{F_{n+k} - 2}{F_n} &= \frac{x^{2^k} - 1}{x + 1} \\ &= x^{2^k - 1} - x^{2^k - 2} + \cdots - 1 \in \mathbb{Z} \end{aligned}$$

$$\Rightarrow F_n | (F_{n+l} - 2) \Rightarrow d|2.$$

But F_n, F_{n+k} are odd. So $d = 1$. Hence the lemma.

Proof of Infinitude of primes

Consider $F_1, F_2, \ldots, F_n \ldots$ By lemma, each F_n is divisible by a prime, call it p_n, not dividing the previous $F_k,\ k < n$. The sequence $\{p_1, p_2, \ldots\}$ is infinite.

One has: $F_1 = 5,\ F_2 = 17,\ F_3 = 257,\ F_4 = 65537$ (Fermat), $F_5 = (641)(6700417), \ldots$

Primes in "Arithmetic Progressions":

Fix $m > 1$, and $a \in \mathbb{Z}$ such that $(a, m) = 1$.

Theorem (Dirichlet) $\exists$ infinitely many primes p which are $\equiv a \pmod{m}$.

We cannot possibly prove it in this class. But we can prove the following:

Baby Lemma $\exists$ infinitely many primes p which are $\equiv 3 \pmod{4}$.

Proof: Suppose $\exists$ only a finite number of such primes, say 3, $p_1,\ p_2,\ \cdots,\ p_r$.

Consider

$$N = 4p_1 p_2 \cdots p_r + 3.$$

By unique factorization in $\mathbb{Z}$ we can write $N = q_1 q_2 \cdots q_s$, with the q_j's being primes.

Claim 1: Some q_j must be $\equiv 3 (mod\, 4)$.

Indeed, every q_j is an odd prime as N is odd, and moreover if $q_j \equiv 1 \pmod{4}\ \forall_j$, then N will also be $\equiv 1 \pmod{4}$, contradiction! Hence Claim 1.

Say $q_1 \equiv 3 \pmod{4}$.

Claim 2: $q_1 \notin \{3, p_1, \cdots, p_r\}$.

Indeed, if $q_1 = 3$, then $3|N$, and since $N = 4p_1 \cdots p_r + 3$, 3 must divide $4p_1 \cdots p_r, \rightarrow\leftarrow$. So $q_1 \neq 3$. Suppose $q_1 = p_i$ for some $1 \leq i \leq r$. Then $p_i \mid N$, and since $N = 4p_1 \cdots p_r + 3$, $p_i \mid 3$, $\rightarrow\leftarrow$. So $q_1 \neq p_i$. Hence Claim 2.

So we have produced a new prime $q_1 \equiv 3 \pmod 4$ which is not in the original list, $\rightarrow\leftarrow$.
Remark: There is no such simple argument to prove Dirichlet's theorem for primes $\equiv 1 \pmod 4$. We can try to start the same way by assuming that we have a finite list of primes $\equiv 1 \pmod 4$, say $p_1, p_2, \cdots, p_r$, and we can consider $N = 4p_1 \cdots p_r + 1$. Factor N as $q_1 \cdots q_s$. Now the analog of Claim 1 will in general fail as the product of an even number of numbers congruent to 3 (mod 4) is 1 (mod 4). However, we will prove the infinitude of such primes later after studying squares mod p.

Earlier we saw a *heuristic reason* for expecting there to be an infinite number of **twin primes**, e.g. $\{3, 5\}$, $\{5, 7\}$, $\{11, 13\}, \cdots$
Expectation:

$$\pi_2(x) := \# \begin{pmatrix} \text{twin primes} \\ \leq x \end{pmatrix} \approx \quad C\,\frac{x}{\log^2 x}, \qquad \text{as } x \to \infty.$$

This means $\pi_2(x) - \frac{cx}{\log^2 x}$ goes to 0 as x goes to ∞.
This twin prime problem is closely related to the **Goldbach problem**, which asks if every even number ≥ 4 is a sum of 2 primes.
Best known result: (Chen)

$$2n = a_1 + a_2, \text{ with } a_i \text{ prime or a product of 2 primes.}$$

A similar heuristic reason makes one expect that there are infinitely many primes p of the form $n^2 + 1$.
Best known result: (Iwaniec)

$$\exists \text{ an infinite of sequence } \{m_1, m_2, \cdots\}$$

such that
(i)

$$m_j = n_j^2 + 1, \qquad \forall j$$

and for every j,
(ii)

$$m_j \text{ is a prime or a product of 2 primes}$$

The proof is quite hard and beyond the scope of our class.

4 Pythagorean Triples

Problem:

Find all $x, y \in \mathbb{N}$ such that

$$x^2 + y^2 = z^2 \tag{1}$$

If $d = (x, y, z) > 1$, then $(\frac{x}{d}, \frac{y}{d}, \frac{z}{d})$ is another solution, called the **primitive solution**.

For primitive solutions, we may assume that x is odd and y is even.

The Geometric Method

Solving (1) in integers amounts to solving the following in rational numbers:

$$X^2 + Y^2 = 1 \tag{2}$$

Geometrically, (2) is the equation of the unit circle in $\mathbb{R}^2$ with center at $O = (0, 0)$. Try to parametrize the circle.

One can try as in calculus to set

$$X = \cos\theta, \ Y = \sin\theta.$$

This turns out to be *terrible* for number theory. A better way is to consider the parametrization

$$X = \frac{1-t^2}{1+t^2}, \quad Y = \frac{2t}{1+t^2}$$

This is *ingenious* as this only involves rational functions. If $t \in \mathbb{Q}$, then $X, Y \in \mathbb{Q}$. Of course

$$X^2 + Y^2 = \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = 1$$

As $t \to \infty$ (along rationals) then

$$X = \frac{1-t^2}{1+t^2} \to -1$$

So we are only missing one solution, $(-1, 0)$, which we will remember.

Check: If $X, Y \in \mathbb{Q}$, then $t \in \mathbb{Q}$. (Show: $t = \frac{Y}{1+X}$.)

So the rational solutions of (2) are obtained by setting

$$X = \frac{1-t^2}{1+t^2}, \quad Y = \frac{2t}{1+t^2},$$

with $t \neq \pm 1, 0$, together with $(\pm 1, 0)$ and $(0, \pm 1)$.

Write $t = \frac{u}{v}$, $u, v \in \mathbb{Z}$. Then

$$X = \frac{u^2 - v^2}{u^2 + v^2}, \quad Y = \frac{2uv}{u^2 + v^2}$$

It follows that the non-zero solutions in $\mathbb{Z}$ of (1) are given by

$$x = u^2 - v^2, \; y = 2uv, \; z = u^2 + v^2$$

with

$$u \neq \pm v, \; u, v \neq 0$$

To get primitive solutions, it is convenient to put

$$m = u + v, \; n = u - v$$

$$x = (u+v)(u-v) = mn, \; y = \frac{m^2 - n^2}{2}, \; z = \frac{m^2 + n^2}{2}$$

For primitive solutions, take m, n odd ≥ 1, $m > n$. Check that these are all the primitive solutions.

5 Linear Equations

Basic problem: Fix $a_1, \ldots, a_n \in \mathbb{Z}$, $n > 0$. Consider the equation:

$$a_1 x_1 + \ldots a_n x_n = \vec{a} \cdot \vec{x} = m, \tag{*}$$

where $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{x} = (x_1, \ldots, x_n)$. Determine if (*) can be solved **in integers**. If so, determine all the solutions.

These are the simplest Diophantine Equations.

Earlier, we proved that, given $a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, $\exists!$ positive integer d, the **greatest common devisor**, such that we can solve

$$a_1 x_1 + \ldots a_n x_n = m$$

if m is a multiple of d, and that the set

$$M = \{a_1 x_1 + \ldots a_n x_n > 0 | x_1, \ldots, x_n \in \mathbb{Z}\}$$

is simply $d\mathbb{Z}$. Moreover, d is the smallest number in $M^+ = \{r \in M | r > 0\}$, which exists by the WOA.

Consequently we have

Lemma 1. (*) can be solved iff m is a multiple of gcd ($\{a_i\}$).

So the basic problem comes down to determining all solutions of $a \cdot x = dN$, for any $N \geq 1$.

Suppose **n=1**; then it is trivial. We have:

$$a_1 \neq 0, \; d = gcd = |a_1|,$$

and we need to solve

$$a_1 x_1 = |a_1| N \tag{$*_N$}$$

But there is a **unique** solution, namely:

$$x_1 = sgn(a_1) N$$

n=2:

First look at case **gcd=1**, **N=1**.

$$a_1 x_1 + a_2 x_2 = 1 \tag{$*_1$}$$

By Lemma 1 there exists a solution, call it (u_1, u_2). Suppose (v_1, v_2) is another solution. Then

$$a_1 u_1 + a_2 u_2 = 1 \tag{1}$$

$$a_1 v_1 + a_2 v_2 = 1 \tag{2}$$

Multiply (1) by v_1; (2) by u_1:

$$\begin{array}{l} a_1 u_1 v_1 + a_2 u_2 v_1 = v_1 \\ \underline{a_1 u_1 v_1 + a_2 u_1 v_2 = u_1} \\ a_2 (v_1 u_2 - u_1 v_2) = v_1 - u_1 = k \end{array}$$

Do same with (1) times v_2, (2) times u_2 to get:

$$a_1 \underbrace{(u_1v_2 - u_2v_1)}_{-k} = (v_2 - u_2)$$

So

$$v_1 = u_1 + ka_2, \quad v_2 = u_2 - ka_1.$$

(u_1, u_2) is a **particular solution** which we use to generate all solutions.

Conversely, for **any** integer k,

$$(u_1 + ka_2,\ u_2 - ka_1)$$

is a solution of $\vec{a} \cdot \vec{x} = 1$.

If gcd $(a_1, a_2) = 1$, then we can solve $a_1x_1 + a_2x_2 = 1$ in integers. Moreover, if (u_1, u_2) is a particular solution, then any other solution is of the form $(u_1 + ka_2, u_2 - ka_1),\ k \in \mathbb{Z}$.

n=2, **d >1**, **N=1**:

$$a_1x_1 + a_2x_2 = d \qquad (*_1)$$

Since $d =$gcd(a_1, a_2), $d|a_1$ and $d|a_2$. Put $b_i = \frac{a_i}{d}$. Then (*) becomes

$$b_1x_1 + b_2x_2 = 1 \text{ with } (b_1, b_2) = 1.$$

So if (u_1, u_2) is a particular solution, every solution is of the form

$$\left(u_1 + k\frac{a_2}{d},\ u_2 - k\frac{a_1}{d}\right).$$

This finishes the $n = 2$ case. We summarize the results in the following

Proposition *Let a_1, a_2 be non-zero integers, and let d be their gcd. Then the equation*

$$a_1x_1 + a_2x_2 = m$$

is solvable in integers iff m is divisible by d. Moreover, if (u_1, u_2) is any particular solution, then the set of all solutions is parametrized by $\mathbb{Z}$, and for each $r \in \mathbb{Z}$, the corresponding solution is given by

$$x_1 = u_1 + r\frac{a_2}{d}, \quad \text{and} \quad x_2 = u_2 - r\frac{a_1}{d}.$$

$\mathbf{n}, \mathbf{a}, \mathbf{N}$ arbitrary: (general case)

It will be good to understand the example at the end of the section (for $n = 3$). The rest of the section may be difficult and is included here for completeness.

Definition:

$$M_n(\mathbb{Z}) = \{a = (a_{ij}) : \ n \times n - \text{matrices with } a_{ij} \in \mathbb{Z} \ \forall i, j\}.$$

$$I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$GL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : \det(A) = \pm 1\}$$

The equation of interest is

$$(a_1, \ldots, a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = Nd \tag{$*_N$}$$

Lemma 1. *Let $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n - \{0\}$ with $d = \gcd(a_1, \cdots, a_n)$. Then $\exists\, C \in GL_n(\mathbb{Z})$ such that $aC = de_n = (0, \cdots, 0, d)$.*

Proof. $n = 1$: $\quad d = |a_1|$, so we can take $C = (sgn(a_1))$. Now let $n > 1$, and assume Lemma by induction for $m < n$. If $a_1 = \cdots = a_{n-1} = 0$ we can take

$$C = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \text{sgn } (a_n) \end{array} \right).$$

So we may suppose that $a' := (a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1} - \{0\}$.

Let $d' = \gcd(a_1, \cdots, a_{n-1})$. By the inductive hypothesis, $\exists\, C' \in GL_{n-1}(\mathbb{Z})$ such that $a' C' = (0, \cdots, d') \in \mathbb{Z}^{n-1}$.

Let

$$A = \left(\begin{array}{c|c} C' & 0 \\ \hline 0 & 1 \end{array} \right) \in GL_n\,(\mathbb{Z}).$$

Then $aA = (0, \cdots, 0, d', a_n)$. Clearly, $d = \gcd(d', a_n)$, and $\exists\, x, y \in \mathbb{Z}$ such that $d'x + a_n\, y = d$.

Put

$$B = \begin{pmatrix} a_n/d & x \\ -d'/d & y \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Then $(d', a_n)\, B = (0, d)$.

Put

$$C = A \left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & B \end{array} \right) \in GL_n\,(\mathbb{Z}).$$

Then

$$\begin{aligned} aC = (aA) \left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & B \end{array} \right) &= (0, \cdots, 0, d', a_n) \left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & B \end{array} \right) && (3) \\ &= (0, \cdots, 0, d). && (4) \end{aligned}$$

Theorem 5.1. *Let $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n - \{0\}$ with* gcd *equal to d.*

Let C be the matrix given by Lemma. Pick any $N \in \mathbb{Z}$. Then we have:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}^n$$

is a solution of $\sum_{i=1}^n a_i\, x_i = Nd$ if and only if $\exists m_1, \cdots, m_{n-1} \in \mathbb{Z}$ such that

$$x = \sum_{i=1}^{n-1} m_i C^i + NC^n$$

where C^j denotes $(\forall j)$ the j-th column of C.

Proof.

Let $y = x - NC^n$.

Then

$$
\begin{array}{c}
a \cdot x = Nd \Leftrightarrow a{\cdot}y = 0 \\
\Updownarrow \\
aC(C^{-1}y) = (0, \cdots, 0, d)(C^{-1}y) = 0 \\
\Updownarrow \\
C^{-1}y = m = \begin{pmatrix} m_1 \\ 1 \\ 1 \\ m_{n-1} \\ 0 \end{pmatrix}, \text{ for some } m_i \in \mathbb{Z}, 1 \le i \le n-1 \\
\Updownarrow \\
y = Cm = \sum_{i=1}^{n-1} m_i C^i \\
\Updownarrow \\
x = Cm + NC_n.
\end{array}
$$

Example: Find all the integral solutions of

$$5x + 7y + 11z = 2. \qquad (*)$$

Put $a = (5,\ 7,\ 11)$. Then the gcd of the coordinates of a is 1. By Lemma, we can find a 3×3 - integral matrix C of determinant ± 1 such that $aC = (0,\ 0,\ 1)$. The proof of Lemma gives a recipe for finding C. First solve $5x + 7y = 1$. Since $1 = \gcd(5, 7)$, this can be solved, and a solution (by inspection) is given by $x = -4$, $y = 3$. Put $C' = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$. Next we have to solve $d'u + 11v = 1$, where $d' = \gcd(a_1, a_2) = 1$. A solution is given by $u = 1$, $v = 0$. Let $B = \begin{pmatrix} 11 & 1 \\ -1 & 0 \end{pmatrix}$.

Then the proof of Lemma says that

$$C = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{C'}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right).$$

Matrix multiplication gives

$$C = \begin{pmatrix} 7 & -44 & -4 \\ -5 & 33 & 3 \\ 0 & -1 & 0 \end{pmatrix}.$$

By the Theorem, the complete set of integral solutions of (*) is given by:

$$\begin{bmatrix} x = 7m - 44n - 8 \\ y = -5m + 33n + 6 \\ z = -n \end{bmatrix} \text{ where } m,\, n \in \mathbb{Z}$$

6 Congruences

Fix an integer $m > 1$. We say that two integers a, b are **congruent modulo** m iff $m|(a - b)$.

Remark: If we had done this for $m = 1$, then any pair a, b would be congruent mod 1.

If a, b are congruent mod m, we write

$$a = b \pmod{m}$$

Modular arithmetic:

If a is any integer, we can use the Euclidean algorithm to write

$$a = mq + r, \text{ with } 0 \leq r < m$$

Then $m|(a - r)$, so $a \equiv r \pmod{m}$.

Consequently, we can partition $\mathbb{Z}$ into m blocks, one for each integer r, with $0 \leq r < m$. Suppose B_r is the block corresponding to r. Then, for **any** a in B_r, $a \equiv r \pmod{m}$. Note: $B_0 = \{\dots, -2m, -m, 0, m, 2m, \dots\}$, $B_1 = \{\dots, -2m + 1, -m + 1, 1, m + 1, 2m + 1, \dots\}$, etc.

If $m = 2$, this partition will yield even and odd integers; the even integers are $\equiv 0 \pmod{2}$ and the odd integers are $\equiv 1 \pmod{2}$.

These blocks are called **congruence classes modulo** m. There are exactly m classes. We write $\mathbb{Z}/m$ for $\{B_0, B_1, \dots B_{m-1}\}$.

Definition: A **set of representatives** for $\mathbb{Z}/m$ is a subset $S = \{x_0, x_1, \dots, x_{n-1}\}$ of $\mathbb{Z}$ such that $x_r \in B_r$ for each $r = 0, 1, \dots, m - 1$.

m=3:

a	r (mod 3)
0	0
1	1
2	2
3	0
4	1
5	2
6	0

Note: There is a **natural choice** for S, namely $S_0 = \{0, 1, \ldots, m-1\}$, called the **standard** or **usual** set of representatives.

So for $m = 3$, we can use

$$S_0 = \{0, 1, 2\}$$

or

$$S_1 = \{9, 16, -1\}$$

as a set of representatives.

Claim:

One has addition, subtraction, 0, and multiplication in $\mathbb{Z}/m$, just like in $\mathbb{Z}$.

Proof. Consider B_i, B_j. Look at $i + j$. By Euclidean algorithm,

$$i + j = qm + r_{i+j},$$

for some r_{i+j} with $0 \leq r_{i+j} < m$. We put

$$B_i + B_j = B_{r_{i+j}}$$

Similarly, $B_i - B_j = B_{r_{i-j}}$, if $i - j = q'm + r_{i-j}$, with $0 \leq r_{i-j} < m$. B_0 is the "zero" of $\mathbb{Z}/m$, because

$$B_0 + B_i = B_i = B_i + B_0$$

Multiplication

$$B_i B_j = ?$$

Write $ij = bm + r_{ij}$, $0 \leq r_{ij} < m$. Put $B_i B_j = B_{r_{ij}}$. Note that

$$B_1 B_j = B_j, \text{ for any } j.$$

So B_1 is the "one" element. Also have distributive and associative laws just like in $\mathbb{Z}$.

Definition: If $a \in \mathbb{Z}$, write $a \pmod{m}$ to denote the block it belongs to. If $a, b \in \mathbb{Z}$, we write $a+b \pmod{m}$ for any element of $B_i + B_j$, if $a \in B_i,\ b \in B_j$. Similarly, $ab \pmod{m}$ is defined.

Remark. In $\mathbb{Z}$ the only numbers we can divide by, i.e., which have "multiplicative inverses", are ± 1. The situation is better in $\mathbb{Z}/m$. In fact, when m is a prime p, all the non-zero elements of $\mathbb{Z}/m$ are invertible (mod m).

7 Linear Equations mod m

Given $a, c \in \mathbb{Z}$, we want to solve

$$ax \equiv c \pmod{m} \tag{*}$$

Note that we can solve the "congruence" (I) iff we can solve

$$ax + my = c \qquad \star$$

with $x, y \in \mathbb{Z}$.

We have looked at $\star$ before.

Recall:

(i) For $\star$ to have a solution in integers, it is necessary and sufficient to have c be divisible by the gcd, say d, of a, m.

(ii) Let u, v satisfy

$$\left(\frac{a}{d}\right) u + \left(\frac{m}{d}\right) v = 1 \qquad \star'$$

This is possible as $(\frac{a}{d}, \frac{m}{d}) = 1$.

All the solutions for $\star'$ are obtained by first finding one solution, say (u_0, v_0) and writing the general solution as

$$(u, \mathrm{v}) = \left(u_0 + k\frac{m}{d}, \mathrm{v}_0 - k\frac{a}{d}\right)$$

for any $k \in \mathbb{Z}$.

So the general solution of $\star$ is given by

$$\begin{aligned}(x, y) &= \left(c\left(u_0 + \frac{km}{d}\right),\ c\left(\mathrm{v}_0 - \frac{ka}{d}\right)\right) \\ &= \left(cu_0 + k\frac{c}{d}m,\ c\mathrm{v}_0 - k\frac{c}{d}a\right)\end{aligned}$$

So the general solution to (*) is given by

$$x = cu_0 + k\left(\frac{c}{d}\right)m$$

Suppose x, x' are both solutions of (*) mod m. Then

$$a(x - x') \equiv 0 \bmod m,$$

so

$$m|a(x - x').$$

Since $d = gcd(a, m)$ we need

$$\frac{m}{d}|(x - x')$$

Example. $m = 6,\ a = 4$

$$4(x - x') \equiv 0 \pmod 6,\ d = 2 \Leftrightarrow 3|(x - x')$$

So

$$(x - x') \equiv 0 \text{ or } 3 \pmod 6$$

In general, if $(a, m) = d$, then

$$a(x - x') \equiv 0 \pmod m \Rightarrow x - x' \text{ is divisible by } \frac{m}{d}$$

There exists exactly d distinct solutions of (*) mod m. So we have

Lemma. $ax \equiv c \pmod m$ has solutions if

$$d = gcd(a, m) \mid c.$$

When $d|c$, there are d distinct solutions mod m.

Corollary: $ax \equiv 1 \pmod m$ can be solved iff $(a, m) = 1$. Moreover, the solution is unique in this case.

Definition: If $(a, m) = 1$, we call the unique $x \pmod{m}$ such that $ax \equiv 1 \pmod{m}$ the **inverse** of a mod m.

Often, people write it as $a' \pmod{m}$.

Example. $m = 7,\ a = 2,\ a' = 4 \pmod{7}$.

Recall

$$\mathrm{S}_0 = \{0, 1, \ldots, m-1\}$$

is a set of reps. for $\mathbb{Z}/m$. (It is the standard set of reps.)

Definition:

$$(\mathbb{Z}/m)^* = \{\text{Invertible elements of } \mathbb{Z}/m\}$$

$$\varphi(m) = \#(\mathbb{Z}/m)^*$$

Explicitly,

$$\varphi(m) = |\{a \in \{0, 1, \ldots, m-1\} \ |(a, m) = 1\}|.$$

8 Euler's φ-function

The function φ introduced above is called Euler's totient function. Note: If m is a prime p, then $\varphi(p) = p - 1$.

Theorem. Fix any $m \geq 1$. Then, for any integer a relatively prime to m, we have

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Corollary (Fermat's Little Theorem). For any prime p, and for any a not divisible by p,

$$a^{p-1} \equiv 1 \pmod{p}.$$

This is *very* useful for computations.

Example: Compute $11^{470} \pmod{37}$.

Idea: Since 37 is a prime, by Fermat's little theorem,

$$a^{36} \equiv 1 \pmod{37}.$$

Hence

$$a^{r+36b} \equiv a^r \pmod{37}.$$

Write, using the Euclidean algorithm,

$$\begin{aligned} 470 &= 36b + r, \;\; 0 \le r < 37 \\ &= 36 \cdot 13 + 2 \\ \Rightarrow 11^{470} &\equiv 11^2 \pmod{37} \\ &\equiv 10 \pmod{37}. \end{aligned}$$

Proof of Theorem. Let

$$\mathrm{S} = \{r_0, \ldots, r_{n-1}\}$$

be a set of reps. for $\mathbb{Z}/m$, and let $(a,m) = 1$. Consider

$$\mathrm{S}' = \{ar_0, ar_1, \ldots, ar_{m-1}\}.$$

Claim. S$'$ is another set of reps for $\mathbb{Z}/m$.

To show the claim, we need to prove

$$ar_i \neq ar_j \pmod{m}, \;\text{for } i \neq j.$$

Suppose $ar_i = ar_j$, for some $i \neq j$. Then

$$a(r_i - r_j) \equiv 0 \pmod{m},$$

i.e., $m|a(r_i - r_j)$. Since $(a,m) = 1,\; m|(r_i - r_j)$, but this contradicts the fact that S is a set of reps. for $\mathbb{Z}/m$. Hence the claim.

So S and S$'$ are both sets of reps for $\mathbb{Z}/m$. In other words, for each congruence class B_i and m, $\exists!$ number in $B_i\cap$ S and in $B_i\cap$ S$'$. Consequently, the product of all the numbers in S coprime to m will be congruent (mod m) to the product of all the numbers in S$'$ coprime to m'.

Moreover, if r_i is coprime to m, so is ar_i. So

$$\prod_{\substack{r_i \in \mathrm{S} \\ (r_i,m)=1}} (ar_i) \equiv \prod_{\substack{r_i \in \mathrm{S} \\ (r_i,m)=1}} r_i \bmod m)$$

$$\Rightarrow a^{\varphi(m)} \underbrace{\left(\prod_{\substack{r_i \in s \\ (r_i,m)=1}} r_i \right)}_{=b, \text{ say}} \equiv \left(\prod_{\substack{r_i \in s \\ (r_i,m)=1}} r_i \right) \pmod{m}$$

$$\Rightarrow a^{\varphi(m)}b \equiv b \pmod{m}, \text{ with } (b,m)=1.$$
$$\Rightarrow m \mid (a^{\varphi(m)}-1)b.$$

Since $(b,m)=1$,

$$m|a(^{\varphi(m)}-1), \text{ i.e., } a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Warning: Little Fermat says that $a^{p-1} \equiv 1 \pmod{p}$, for any prime p **and** $1 \leq a < p$. It might happen that $\exists m \geq 1$ which is **not** a prime and a such that

$$a^{m-1} \equiv 1 \pmod{m}.$$

For example, consider $m = 340 = (11)(31)$, and $a = 2$.

$$2^{340} \equiv 2^{11-1}34 \equiv 1 \pmod{11}$$

by Little Fermat. Also

$$2^{340} \equiv 2^{(31-1)11} \cdot 2^{10} \equiv 2^{10} \pmod{31}$$

Clearly, if m is a prime p, then $\varphi(m) = p-1$. It is of great importance to have a formula for computing $\varphi(m)$ even when m is not a prime. To this end we prove the following

Theorem Let $m > 1$. Write $m = \prod_{i=1}^{r} p_i^{a_i}$, with $p_1, \cdots, p_r$ primes and $a_1, \cdots, a_r$ positive integers. Then

$$\varphi(m) = \prod_{i=1}^{r} p_i^{a_i-1}(p_i-1) \tag{a}$$

and

$$m = \sum_{d|m} \varphi(d). \tag{b}$$

Proof: (a) **Step 1:** *Show* $\varphi(n_1n_2) = \varphi(n_1)\varphi(n_2)$ *if* n_1, n_2 *are relatively prime.*

Proof of Step 1:

$$\begin{aligned}\varphi(n_1n_2) &= \#\{y \in \{1, 2, \cdots, n_1n_2 - 1\} \mid (y,\, n_1n_2) = 1\} \\ &= \#\{a_in_1 + b_jn_2 \mid (a_in_1 + b_jn_2,\, n_1n_2) = 1, a_i \bmod n_2,\, b_j \bmod n_1\}.\end{aligned}$$

But since $(n_1, n_2) = 1$, we have

$$(a_in_1 + b_jn_2,\, n_1n_2) = 1 \Longleftrightarrow \begin{pmatrix}(a_in_1 + b_jn_2,\, n_1) = 1 \\ \text{and} \\ (a_in_1 + b_jn_2,\, n_2) = 1\end{pmatrix}$$

Also, $(a_in_1 + b_jn_2, n_1) = 1$ iff $(b_jn_2, n_1) = 1$, that is iff $(b_j, n_1) = 1$.

Similarly, $(a_in_1 + b_jn_2, n_2) = 1$ iff $(a_i, n_2) = 1$.

Consequently,

$$\begin{aligned}\varphi(n_1n_2) &= \#\{a_in_1 + b_jn_2 \mid (a_i, n_2) = 1, (b_j, n_1) = 1\} \\ &= \varphi(n_1)\,\varphi(n_2).\end{aligned}$$

Hence we have achieved Step 1.

Step 2: *If p is a prime and $a > 0$, then show: $\varphi(p^a) = p^{a-1}(p - 1)$.*

Proof of Step 2:

$$\varphi(p^a) = \#\{b \in \{0, \cdots, p^a-1\} \mid p \nmid b\} = p^a - \#\{b \in \{0, 1, \cdots, p^a\} \mid p \mid b\} = p^a - p^{a-1},$$

which proves the assertion.

Step 3: *Proof of the general case.*

By step 1, we have

$$\text{If} \quad m = \prod_{i=1}^{r} p_i^{a_i}, \text{then} \quad \varphi(m) = \prod_{i=1}^{r} \varphi\left(p_i^{a_i}\right)$$

This is so because $(p_i^{a_i}, p_j^{a_j}) = 1$ if $i \neq j$. Now part (a) of the Theorem follows by Step 2.

(b): $m = \prod_{d|m}^{r} p_r^{a_i}$. So every positive divisor d of m is of the form $m = \prod_{i=1}^{r} p_i^{b_i}$ with $0 \leq b_i \leq a_i$. So

$$\sum_{d|m} \varphi(d) = \sum_{\{(b_1,\cdots,b_r)|0\leq b_i\leq a_i,\forall i\}} \varphi\left(\prod_{i=1}^{r} p_i^{b_i}\right).$$

By part (a) this equals

$$\sum_{\{(b_1,\dots,b_r)|0\le b_i\le a_i,\forall i\}}^{r} \varphi(p_i^{b_i}),$$

with $\varphi(p_i^{b_i})$ being $p^{b_i} - p_i^{b_i-1}$ (resp. 1) if $b_i > 0$ (resp. $b_i = 0$). Exchanging the sum and the product, and noting that

$$\sum_{\{(b_1,\dots,b_r)|0\le b_i\le a_i,\forall i\}}^{r} \varphi(p_i^{b_i}) \;=\; p_i^{a_i},$$

we get

$$\sum_{d|m} \varphi(d) \;=\; \prod_{i=1}^{r} p_i^{a_i} \;=\; m.$$

9 Linear congruences revisited

Theorem. Fix $m > 1$. Let $a, c \in \mathbb{Z}$. Put $d = gcd(a, m)$. Then the congruence

$$ax \equiv c \pmod{m} \tag{*}$$

has a solution $x \pmod{m}$ iff $d|c$. Moreover, when $d|c$, all d solutions are of the form

$$x \equiv \frac{cu_0 + mk}{d} \pmod{m},$$

with $k \in \mathbb{Z}$, where (u_0, v_0) is a solution of $au + mv = d$.

We already proved (*) has a solution $x \pmod{m}$ iff $d|c$. So let $d|c$. Let (u_0, v_0) be a solution of

$$au + mv = d. \tag{**}$$

Multiply by c, get

$$acu_0 + mcv_0 = cd,$$

i.e.,

$$a\left(\frac{cu_0}{d}\right) + m\left(\frac{cv_0}{d}\right) = c$$

$$\Rightarrow a\left(\frac{cu_0}{d}\right) \equiv c \pmod{m}$$
$$\Rightarrow x \equiv \frac{cu_0}{d} \pmod{m} \text{ is a solution of (*).}$$

Recall that we get all the solutions of (**) by taking

$$(u,v) = \left(u_0 + \frac{km}{d},\ v_0 - \frac{kc}{d}\right),$$

as k runs over $\mathbb{Z}$. So the general solution of (*) is given by

$$x \equiv \frac{cu_0}{d} + \frac{km}{d} \equiv \frac{cu_0 + km}{d} \pmod{m}$$

Corollary: $ax \equiv 1 \pmod{m}$ has a solution iff $(a,m) = 1$. In this case, $\exists!$ solution, the multiplicative inverse of a mod m, and denoted $a' \pmod{m}$.

We knew before that a has a multiplicative inverse if $(a,m) = 1$. This corollary replaces the if by iff.

Definition:

$$(\mathbb{Z}/m)^* = \{a \in \mathbb{Z}/m | (a,m) = 1\}.$$

Note: By corollary, $(\mathbb{Z}/m)^*$ is precisely the subset of $\mathbb{Z}/m$ consisting of elements which have multiplicative inverses mod m.
Recall:

$$\begin{aligned}\varphi(m) &= |(\mathbb{Z}/m)^*|.\\ &= |\{a \in \{0,1,\dots,m-1\} | (a,m) = 1\}|.\end{aligned}$$

In the previous section we proved the following:

Theorem: (Euler) For any $a \in \mathbb{Z}$ with $(a,m) = 1$,

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Corollary. (Fermat's Little Theorem)

$$m = p \text{ (prime)},\ p|a \Rightarrow a^{p-1} \equiv 1 \pmod{p}.$$

Remark. Fermat's Little Theorem says that

$$x^{p-1} \equiv 0 \pmod{p}$$

has $p-1$ solutions mod p, namely

$$x \equiv 1, 2, \ldots, p-1 \pmod{p}$$

$$\Rightarrow a^p - a \equiv 0 \pmod{p},\ \forall a = 1, 2, \ldots, p-1.$$

This is also true for

$$a \equiv 0 \pmod{p}.$$

So,

$$x^p - x \equiv 0 \pmod{p}$$

has p solutions mod p. On the other hand,

$$x^p \equiv 0 \pmod{p}$$

has only one solution, namely $x \equiv 0 \pmod{p}$. In other words, if $a \neq 0$ (mod p), then a^p cannot be 0 (mod p).

Claim. If $ab \equiv 0 \pmod{p}$, then either a or b must be $\equiv 0 \pmod{p}$.

Proof of Claim. Suppose $a \not\equiv 0 \pmod{p}$. Then

$$a \in (\mathbb{Z}/p)^*,$$

and so $\exists a'$ such that $a'a \equiv 1 \pmod{p}$. Multiple both sides of $ab = 0$ (mod p) by a' to get $(aa')b \equiv 0 \pmod{p}$, giving

$$b \equiv 0 \pmod{p}.$$

Conclusion: $\mathbb{Z}/p$ has no "zero divisors."

Note: If m is any integer > 1 which is **not** a prime, then $\mathbb{Z}/m$ has zero divisors.

Proof. Since m is composite, we can write $m = m_1, m_2$ with $m_1, m_2 > 1$. then

$$m_1 m_2 \equiv 0 \pmod{m},$$

but neither m_1 nor m_2 is $\equiv 0 \pmod{m}$.

Moral: Congruences modulo a prime p are nicer to study. They have much more structure.

10 Number of solutions modulo a prime

Theorem *(Lagrange) Fix a prime p and integer $n \geq 1$. Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with coefficients $a_i \in \mathbb{Z}$, such that some a_j is prime to p. Then the congruence*

$$f(x) \equiv 0 \, (mod \, p) \tag{1}$$

has at most n solutions mod p.

Proof: Suppose $n \equiv 1$. Then the congruence is $a_1 x \equiv -a_0 \, (\mathrm{mod}\, p)$. By hypothesis, either a_1 or a_0 is not divisible by p. The former case must happen as otherwise we would have $0 \equiv -a_0 \pmod p$, implying a_0 is also $\equiv 0 \,(\mathrm{mod}\ p)$, leading to a contradiction. Thus a_1 is invertible mod p; let a_1' be such that $a_1' \, a_1 \equiv 1 (\mathrm{mod}\, p)$. Multiplying $a_1 x \equiv -a_0 \,(\mathrm{mod}\, p)$ by a_1', get

$$(a_1' a_1)x \equiv x \equiv -a_1' a_0 (\mathrm{mod}\, p)$$

Thus we get a unique solution, and the Theorem is O.K. for $n = 1$.

Now let $n > 1$, and assume by induction that the Theorem holds for all $k < n$. Suppose (1) has no solutions mod p. Then there is nothing to prove. So we may assume that there is at least one solution, say $x \equiv x_1 \,(\mathrm{mod}\, p)$. Then we get

$$f(x_1) \equiv 0 \,(\mathrm{mod}\, p). \tag{2}$$

Subtracting (2) from (1), we get

$$f(x) - f(x_1) \equiv a_n(x^n - x_1^n) + a_{n-1}(x^{n-1} - x_1^{n-1}) + \cdots a_1(x - x_1) \equiv 0 \,(\mathrm{mod}\, p).$$

But for any $k \geq 1$, $(x - x_1) \mid (x^k - x_1^k)$, so $f(x) - f(x_1) = (x - x_1)g(x)$, where $g(x)$ is a polynomial in x of degree $k - 1$. Thus, $f(x) - f(x_1) \equiv 0 \,(\mathrm{mod}\, p)$ holds iff

$$(x - x_1)g(x) \equiv 0 \,(\mathrm{mod}\, p). \tag{3}$$

Then **either** $x - x_1 \equiv 0$ **or**

$$g(x) \equiv 0 \,(\mathrm{mod}\, p) \tag{4}$$

The coefficients of g cannot all be $\equiv 0 \pmod p$, for otherwise $f(x)$ would be congruent to 0 (mod p). Since the degree of g is $< n$, we then have by the inductive hypothesis, that the number of solutions of (4) mod p is bounded above by $n-1$. Then the number of solutions mod p of (1) is $\leq 1+n-1 = n$.

11 Remarks on Fermat's Last Theorem and an approach of Gauss

Recall the Fermat equation $x^n + y^n = z^n$. For $n = 2$, this leads to Pythogorean triples and we classified all the solutions in this case.

Theorem (A. Wiles) ('97): For $n \geq 3$, $x^n + y^n = z^n$ has no positive integral solutions.

There is no way we can prove this magnificient result in this class.

Note: To prove this, it suffices to prove in the cases where $n = 4$ and when $n = p$, where p is any odd prime.

Reason: If $m|n$, then any solution of $u^n + v^n = w^n$ will give a solution for m, namely $(u^{n/m})^m + (v^{n/m})^m = (w^{n/m})^m$.

Moreover, for any $n \geq 3$, n will be divisible by 4 or by an odd prime p.

We also proved in the first week that $x^4 + y^4 = z^4$ has no integral solutions for. (In fact, we showed Fermat's result that $x^4 + y^4 = w^2$ has no integral solutions.) Consequently, the key fact needed to be proven is that $x^p + y^p = z^p$ has no solution for any odd prime.

This gets split into two cases:

$$\begin{aligned} \text{Case I:} &\quad p \nmid xy\,z. \\ \text{Case II:} &\quad p \mid xy\,z. \end{aligned}$$

Proposition (Gauss). Suppose the congruence

$$(*) \qquad x^p + y^p \equiv (x+y)^p \pmod{p^2}$$

has no *non-trivial* solutions, i.e. with none of x, y, $x + y \equiv 0 \,(\mathrm{mod}\, p)$. Then Case I of FLT holds for p, i.e.

$$\nexists\, x, y, z \in \mathbb{Z}_{>0}, \quad p \nmid x\,y\,z, \text{ such that } x^p + y^p = z^p.$$

Note:

$$(x+y)^p = \sum_{j=p}^{p} \binom{p}{j} x^j y^{p-j}, \qquad \binom{p}{j} = \frac{p!}{(p-j)!j!}$$

If $j \neq 0$ or p, then $\binom{p}{j}$ is divisible by p. Since $(x+y)^p = x^p + y^p + \sum_{j=1}^{p-1} \binom{p}{j} x^j y^{p-j}$, we get

$$(x+y)^p \equiv x^p + y^p \pmod{p}.$$

Proof of Prop.

Suppose we have positive integers x, y, z, with $p \nmid xyz$, such that $x^p + y^p = z^p$. We have just seen that $x^p + y^p \equiv (x+y)^p \pmod{p}$, so $z^p \equiv (x+y)^p \pmod{p}$.

Moreover, we have the Little Fermat Theorem, which says that $x^p \equiv x \pmod{p}$, $z^p \equiv z \pmod{p}$, $y^p \equiv y \pmod{p}$, and $(x+y)^p \equiv x+y \pmod{p}$. Consequently, $z \equiv x+y \pmod{p}$, i.e. $z = x+y+mp$, for some $m \in \mathbb{Z}$.

Since $x^p + y^p = z^p$, we get

$$\begin{aligned} x^p + y^p = (x+y+mp)^p &= \sum_{i=0}^{p} \binom{p}{i} (x+y)^i \, (mp)^{p-i} \\ &= (mp)^p + p(x+y)(mp)^{p-1} + \cdots + p(x+y)^{p-1}(mp) + (x+y)^p. \end{aligned}$$

Therefore $x^p + y^p \equiv (x+y)^p \pmod{p^2}$

Difficulty:

If $p \equiv 1 \pmod{3}$, one can always solve the congruence $x^p + y^p \equiv (x+y)^p \pmod{p^2}$. So Gauss's Proposition doesn't help us. On the other hand, when $p \equiv 2 (\mathrm{mod}\, 3)$, for many small primes, $x^p + y^p \equiv (x+y)^p \pmod{p^2}$ has no solution.

Still, there are primes $p \equiv 2 \pmod{3}$ for which $\exists$ solutions to this congruence. This happens for 13 primes less than 1000. For example, when $p = 59$, $1^{59} + 3^{59} \equiv 4^{59} \pmod{59^2}$.

12 Mersenne Primes and Perfect Numbers

Basic idea: try to construct primes of the form $a^n - 1$; $a, n \geq 1$. e.g.,

$2^1 - 1 = 3$ but $2^4 - 1 = 3 \cdot 5$
$2^3 - 1 = 7$
$2^5 - 1 = 31$
$2^6 - 1 = 63 = 3^2 \cdot 7$
$2^7 - 1 = 127$
$2^{11} - 1 = 2047 = (23)(89)$
$2^{13} - 1 = 8191$

Lemma: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$

Corollary: $(x-1)|(x^n - 1)$

So for $a^n - 1$ to be prime, we need $a = 2$.
Moreover, if $n = md$, we can apply the lemma with $x = a^d$. Then

$$(a^d - 1)|(a^n - 1)$$

So we get the following

Lemma If $a^n - 1$ is a prime, then $a = 2$ and n is prime.

Definition: A *Mersenne prime* is a prime of the form

$$q = 2^p - 1, \ p \text{ prime}.$$

Question: are they infinitely many Mersenne primes?
Best known: The 37th Mersenne prime q is associated to $p = 3021377$, and this was done in 1998. One expects that $p = 6972593$ will give the next Mersenne prime; this is close to being proved, but not all the details have been checked.
Definition: A positive integer n is *perfect* iff it equals the sum of all its (positive) divisors $< n$.

Definition: $\sigma(n) = \sum_{d|n} d$ (divisor function)

So u is perfect if $n = \sigma(u) - n$, i.e. if $\sigma(u) = 2n$.
Well known example: $n = 6 = 1 + 2 + 3$
Properties of σ:

1. $\sigma(1) = 1$
2. n is a prime *iff* $\sigma(n) = n + 1$
3. If p is a prime, $\sigma(p^j) = 1 + p + \cdots + p^j = \frac{p^{j+1}-1}{p-1}$
4. (Exercise) If $(n_1, n_2) = 1$ then $\sigma(n_1)\sigma(n_2) = \sigma(n_1 n_2)$ "multiplicativity".

Consequently, if

$$n = \prod_{j=1}^{r} p_i^{e_j}, \ e_j \geq 1 \ \ \forall j, \ p_j \text{ prime},$$

$$\sigma(n) = \prod_{j=1}^{r} \sigma(p_j^{e_j}) = \prod_{j=1}^{r} \left(\frac{p^{e_j+1} - 1}{p - 1} \right)$$

Examples of perfect numbers: $\begin{cases} 6=1+2+3 \\ 28=1+2+4+7+14 \\ 496 \\ 8128 \end{cases}$

Questions:

1. Are there infinitely many perfect numbers?
2. Is there any odd perfect number?

Note:

6=(2)(3), 28=(4)(7), 496=(16)(31), 8128=(64)(127)

They all look like

$$2^{n-1}(2^n - 1),$$

with $2^n - 1$ prime (i.e., Mersenne).

Theorem (Euler) Let n be a positive, *even* integer. Then

$$n \text{ is perfect} \Leftrightarrow n = 2^{p-1}(2^p - 1), \text{ for a prime } p, \text{ with } 2^p - 1 \text{ a prime.}$$

Corollary. There exists a bijection between even perfect numbers and Mersenne primes.

Proof of Theorem. ($\Leftarrow$) Start with $n = 2^{p-1}q$, with $q = 2^p - 1$ a Mersenne prime. To show: n is perfect, i.e., $\sigma(n) = 2n$. Since $2^{p-1}q$, and since $(2^{p-1}, q) = 1$, we have

$$\sigma(n) = \sigma(2^{p-1})\sigma(q) = (2^p - 1)(q + 1) = q2^p = 2n.$$

($\Rightarrow$): Let n be a even, perfect number. Since n is even, we can write

$$n = 2^j m, \text{ with } j \geq 1, \ m \text{ odd } \neq n$$

.

$$\Rightarrow \sigma(n) = \sigma(2^j)\sigma(m) = (2^{j+1} - 1)\sigma(m)$$

Since n is perfect,

$$\sigma(n) = 2n = 2^{j+1} m$$

Get

$$2^{j+1}m = \underbrace{(2^{j+1}-1)}_{\text{odd}}\sigma(m)$$

$\Rightarrow$

$$2^{j+1}|\sigma(m);$$

so

$$r2^{j+1} = \sigma(m) \tag{1}$$

for some $r \geq 1$

Also

$$2^{j+1}m = (2^{j+1}-1)r2^{j+1},$$

so

$$m = (2^{j+1}-1)r \tag{2}$$

Suppose $r > 1$. Then

$$m = (2^{j+1}-1)r$$

will have $1, r$ and m as 3 distinct divisors. (Explanation: by hypothesis, $1 \neq r$. Also, $r = m$ iff $j = 0$ iff $n = m$, which will then be odd!)
Hence

$$\begin{aligned}\sigma(m) \geq& 1 + r + m\\ =&1 + r + (2^{j+1}-1)r\\ =&1 + 2^{j+1}r\\ =&1 + \sigma(m)\end{aligned}$$

Contradiction!

So $r = 1$, and so (1) and (2) become

$$\sigma(m) = 2^{j+1} \tag{1'}$$

$$m = 2^{j+1} - 1 \tag{2'}$$

Since $n = 2^j m$, we will be done if we prove that m is a prime. It suffices to show that $\sigma(m) = m + 1$. But this is clear from (1') and (2').

$M_n = 2^n - 1$ Mersenne number. Define numbers S_n recursively by setting $S_n = S_{n-1}^2 - 2$, and $S_1 = 4$.

Theorem: (Lucas-Lehmer Primality Test) Suppose for some $n \geq 1$ that M_n divides S_{n-1}. Then M_n is prime.

Proof. (Very clever) Put $\alpha = 2 + \sqrt{3}$, $\beta = 2 - \sqrt{3}$. Note that $\alpha + \beta = 4$, $\alpha\beta = 1$. So $S_1 = \alpha + \beta$.

Lemma. For any $n \geq 1$, $S_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}}$.

Proof of Lemma: $n = 1:\ S_1 = \alpha + \beta = 4$. So let $n > 1$, and assume that the lemma holds for $n - 1$. Since

$$S_n = S_{n-1}^2 - 2$$

we get (by induction)

$$S_n = (\alpha^{2^{n-1}} + \beta^{2^{n-1}})^2 - 2$$

Note:

$$\begin{aligned}(\alpha^k + \beta^k)^2 &= \alpha^{2k} + 2\alpha^k\beta^k + \beta^{2k} \\ &= \alpha^{2k} + \beta^{2k} + 2, \text{ as } \alpha\beta = 1.\end{aligned}$$

So we get (setting $k = 2^{n-2}$)

$$S_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}} + 2 - 2.$$

Hence the lemma.

Proof of Theorem (continued): Suppose $M_n | S_{n-1}$. Then we may write $rM_n = S_{n-1}$, some positive integer. By the lemma, we get

$$rM_n = \alpha^{2^{n-2}} + \beta^{2^{n-2}} \tag{3}$$

Multiply (3) by $\alpha^{2^{n-2}}$ and subtract 1 to get:

$$\alpha^{2^{n-1}} = rM_n\alpha^{2^{n-2}} - 1 \tag{4}$$

Squaring (4) we get

$$\alpha^{2^n} = (rM_n\alpha^{2^{n-2}} - 1)^2 \tag{5}$$

Suppose M_n is not a prime. Then $\exists$ a prime ℓ dividing M_n, $\ell \leq \sqrt{M_n}$. Let us work in the number system

$$R = \{a + b\sqrt{3} | a, b \in \mathbb{Z}\}$$

Check: R is closed under addition, subtraction, and multiplication (it is what one calls a ring). Equations (4) and (5) happen in R. Define $R/\ell = \{a, b\sqrt{3} | a, b \in \mathbb{Z}/\ell\}$.

Note: $|R/\ell| = \ell^2$

We can view α, β as elements of R/ℓ. Since $\ell | M_n$, (4) becomes the following congruence in R/ℓ:

$$\alpha^{2^{n-1}} \equiv -1 \ (mod\ \ell) \tag{6}$$

Similarly, (5) says

$$a^{2^n} \equiv 1 \ (mod\ \ell)$$

Put

$$X = \{\alpha^j \bmod \ell | 1 \leq j \leq 2^n\}.$$

Claim $|X| = 2^n$.

Proof of claim. Suppose not. Then $\exists j, k$ between 1 and 2^n, with $j \neq k$, such that $\alpha^j \equiv \alpha^k \pmod{\ell}$.

If r denotes $|j - k|$, then $0 < r < 2^n$ and $\alpha^r \equiv 1 \pmod{\ell}$. Let d denote the gcd of r and 2^n, so that $ar + b2^n = d$ for some $a, b \in \mathbb{Z}$. Then we have

$$\alpha^d = \alpha^{ar + b2^n} = (\alpha^r)^a \cdot (\alpha^{2^n})^b \equiv 1 \pmod{\ell}.$$

But since $d | 2^n$, d is of the form 2^m for some $m < n$, and $\alpha^d \equiv 1 \pmod{\ell}$ contradicts $\alpha^{2^{n-1}} \equiv -1 \pmod{\ell}$. Hence the claim.

So $|X| \leq \ell^2 - 1$, i.e., we need $2^n \leq \ell^2 - 1$.

Since

$$\ell \leq \sqrt{M_n}, \ \ell^2 - 1 < M_n = 2^n - 1.$$

$\Rightarrow 2^n < 2^n - 1$, a contradiction!

So M_n is prime.

13 RSA Encryption

The mathematics behind the very successful RSA encryption method is very simple and uses mainly Euler's congruence for any $N \geq 1$:

$$b^{\varphi(N)} \equiv 1 \pmod{N}$$

if $(b, N) = 1$. (When N is a prime, this is Fermat's little theorem.)

Imagine that a person X wants to send a carefully encrypted message to another person Y, say. X will look in a directory which publishes the *public key* of various people including Y. The public key of Y will be a pair (e, N) of positive integers, where N will be a large number which is a product of 2 distinct primes p and q. The point is that the directory will contain no information on the factorization of N. For large enough N it will become impossible (virtually) to factor N. The number e will be chosen mod N and it will be prime to $\varphi(N)$.

The person X will first represent his/her *plain text* message by a numeral a (which can be done in many ways). For simplicity, suppose that a is prime to N. X will then raise a to the power e mod N and send the message as b. So

$$b \equiv a^e \pmod{N}.$$

If someone intercepts the message, he or she will be unable to recover a from b without knowing the factorization of N. So it is secure. On the other hand, the recipient of the message, namely Y, will be able to decode (decrypt) the message as follows. He/she will pick a number d (*decryption constant*) such that

$$de \equiv 1 \pmod{(p-1)(q-1)}.$$

Y can do this because he/she knows the prime factors p, q and because e is prime to $\varphi(N)$; observe that since p and q are distinct primes and $N = pq$, one has

$$\varphi(N) = \varphi(p)\varphi(q) = (p-1)(q-1).$$

So by applying Euler's congruence mod N, we get

$$b^d \equiv a^{ed} \equiv a^{1+c(p-1)(q-1)} \equiv a \pmod{N}.$$

Thus Y recovers a.

Note that if someone does not have the factorizatino of N, he/she cannot decrypt the message.

14 Primitive roots mod p and Indices

Fix an odd prime p, and $x \in \mathbb{Z}$. By little Fermat:

$$x^{p-1} \equiv 1 \ (mod\ p) \text{ if } x \not\equiv 0 \ (mod\ p)$$

E.g.

	x	x^2	x^3	x^4
	1	1	1	1
$p = 5$	2	-1	3	1
	3	-1	2	1
	4	1	-1	1

2 and 3 are called "primitive roots mod 5" since no smaller power than $p - 1$ is $\equiv 1$.

Definition: Let $x \in \mathbb{Z}$, $p \not| x$. Then the *exponent* of x (relative to p) is the smallest integer r among $\{1, 2, \ldots, p-1\}$ such that $x^r \equiv 1 \pmod p$. One writes $r = e_p(x)$.

When $p = 5$, $e_5(1) = 1$, $e_5(2) = 4 = e_5(3)$, $e_5(4) = 2$.

Definition: x is a *primitive root mod* p iff $e_p(x) = p - 1$.

Again, when $p = 5$, 2 and 3 are primitive roots.

Claim: For any x prime to p,

$$e_p(x) | (p - 1).$$

Proof: Since $1 \leq e_p(x) \leq p - 1$, by definition, it suffices to show that

$$d = \gcd(e_p(x), p - 1) \geq e_p(x).$$

Suppose $d < e_p(x)$. Since d is the gcd of $e_p(x)$ and $p-1$, we can find $a, b \in \mathbb{Z}$ such that $ae_p(x) + b(p - 1) = d$. Then

$$x^d = x^{ae_p(x)+b(p-1)} = (x^{e_p(x)})^a (x^{p-1})^b$$

But

$$x^{p-1} \equiv 1 \pmod p \text{ by Little Fermat,}$$

and

$$x^{e_p(x)} \equiv 1 \pmod p \text{ by definition of } e_p(x).$$

Thus

$$x^d \equiv 1 \pmod{p}$$

Since we are assuming that $d < e_p(x)$, we get a contradiction as $e_p(x)$ is the smallest such number in $\{1, 2, \ldots, p-1\}$.

$\Rightarrow d \geq e_p(x)$.

Since $d = \gcd(e_p(x), p-1)$, $d | e_p(x) \Rightarrow d = e_p(x)$. Hence the claim.

Two natural questions

1. Are these primitive roots mod p?

2. If so, how many?

For $p = 5$, the answers are (1) yes, and (2) two.

Theorem: Fix an odd prime p. Then
(i) $\exists$ primitive roots mod p
(ii) $\#\{$primitive roots mod $p = \varphi(p-1)$.

Proof: For every (positive) divisord of $p-1$, put

$$\psi(d) = \#\{x \in \{1, \ldots, p-1\} | e_p(x) = d\}$$

Both (i) and (ii) will be proved if we show

$$\psi(p-1) = \varphi(p-1). \tag{*}$$

We will in fact show that

$$\psi(d) = \varphi(d) \quad \forall d | (p-1)$$

Every x in $\{1, \ldots, p-1\}$ has an exponent, and by the claim above this exponent is a divisor of d. Consequently

$$(p-1) = \sum_{d|(p-1)} \psi(d) \tag{1}$$

Recall that we proved last week

$$p-1 = \sum_{d|(p-1)} \varphi(d) \tag{2}$$

Consequently,

$$\sum_{d|(p-1)} \psi(d) = \sum_{d|(p-1)} \varphi(d) \tag{3}$$

It suffices to show that

$$\psi(d) \le \varphi(d) \quad \forall d|(p-1) \tag{A}$$

Proof of (A): Pick any $d|(p-1)$. If $\psi(d) = 0$, we have nothing to prove. So assume that $\psi(d) \neq 0$. Then

$$\exists a \in \{1, \ldots, p-1\} \text{ such that } e_p(a) = d.$$

Consider

$$Y = \{1, a, \ldots, a^{\mathrm{d}-1}\}$$

Then $(\mathrm{d}^j)^\alpha \equiv 1 \pmod{p}$. Further, Y supplies d distinct solutions to the congruence

$$x^d \equiv 1 \pmod{p}.$$

We proved earlier (LaGrange) that, given any polynomical $f(x)$ with integral coef's f degree n, there are at most n solutions mod p of $f(x) \equiv 0 \pmod{p}$. So $x^d - 1 \equiv 0 \pmod{p}$ has at most d solutions mod p. Consequently, Y is exactly the set of solutions to this congruence and $\#Y = d$. Hence

$$\psi(d) = \#\{a^j \in Y | e_p(a^j) = d\}.$$

Proof of claim: Let $r = \gcd(j, d)$. Then, by the proof of the earlier claim,

$$e_p(a^j) = \frac{d}{r}.$$

So $r = 1$ iff $e_p(a^j) = d$. This proves the claim.

Thanks to the claim, we have:

$$\psi(d) = \#\left\{ a^j \in Y \middle| \begin{array}{c} j \in \{0, 1, \ldots, d-1\} \\ (j, d) = 1 \end{array} \right\} \le \varphi(d) \text{ for all } d|(p-1).$$

In fact we see that $\psi(d) = 0$ or $\varphi(d)$, which certainly proves (A), and hence the Theorem.

2 is a primitive root module the following primes < 100:

$$3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83$$

Artin's Conjecture

There are infinitely many primes with 2 as a primitive root.

More generally, for any non-square a, are there infinitely many primes with a a prime root?

Claim:

$$e_p(a^j) = d \text{ iff } (j, d) = 1.$$

This cannot be true if a is a perfect square. Indeed if $a = b^2$, since $b^{(p-1)} \equiv 1 \pmod{p}$, if $p \not| b$, we have

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$

So, for any odd $p \not| a$, $e_p(a)|(\frac{p-1}{2})$. Similarly, $a = -1$ is a bad case, because

$$(-1)^2 = 1 \text{ and } e_p(-1) = 2 \text{ or } 1,\ \forall p \text{ odd}.$$

So we are led to the following

Generalized Artin Conjecture. Let a be an integer which is not -1 and not a perfect square. Then $\exists$ infinitely many primes such that $e_p(a) = p - 1$.

Here is a positive result in this direction:

Theorem: (Gupta, Murty, and Heath-Brown) There are at most three pairwise relatively prime a's for which there are possibly a finite number of primes such that $e_p(a) = p - 1$.

Problem: no one has any clues as to the nature and size of these three possible exceptions, or whether they even exist. Is 2 an exception?

Indices

Fix an odd prime y and a primitive root a mod p. We can consider

$$Y = \{a^j | 0 \leq j < p - 1\}.$$

Then each element of Y is in $(\mathbb{Z}/p)^*$ and we get $p - 1$ distinct elements. But $\#(\mathbb{Z}/p)^* = p - 1$. So Y gives a set of reps. for $(\mathbb{Z}/p)^*$.

Consequently, given any integer b prime to p, we can find a *unique* $j \in \{0, 1, \ldots, p-2\}$ such that $b \equiv a^j \pmod{p}$.

This (unique) j is called the **index** of b mod p relative to a, written $I_p(b)$ or $I(b)$. Properties: $I(ab) = I(a+b)$, $I(ka) = kI(a)$.

15 Squares mod p

Fix a prime p.

Basic question: Given a, how can we determine if $\exists b \in \mathbb{Z}$ such that $a \equiv b^2 \pmod{p}$?

Trivial case if $p|a$, take $b \equiv 0$. So from now on take $(a, p) = 1$.

p=3		p=5		p=7	
x	x^2	x	x^2	x	x^2
1	1	1	1	1	1
2	1	2	-1	2	4
		3	-1	4	2
		-1=4	1	5	4
				6	1
$1 \equiv d_p$		$1, 4$ as mod 5		$1, 2, 4$ as mod p	
$2 \not\equiv$		$2, 3 \not\equiv$		$3, 5, 6 \not\equiv$	

Guess

$$\# \text{ of squares in } (\mathbb{Z}/p)^* = \# \text{ of non-squares in } (\mathbb{Z}/p)^*$$

p odd, $p \not| a$.

Definition: the Legendre symbol of a mod p is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \equiv \text{ and } p \\ -1, & \text{if } a \not\equiv \text{ mod } p \end{cases}$$

We say a is a *quadratic residule* mod p if it is a , otherwise a quadratic *non-residue*. (Some would allow a to be divisible by p and set $(\frac{a}{p}) = 0$ if $p|a$.)

Lemma: the guess is on the money.

Proof: Let $S = \{1, 2, \ldots, p-1\}$. We know that S is a set of reps. for $(\frac{\mathbb{Z}}{p})^*$. Put

$$T = \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$$

and

$$T^2 = \{b^2 | b \in T\}$$

Claim 1: $\#(T^2 \bmod p) = \frac{p-1}{2}$, i.e., if $b, c \in T$, $b \neq c$, then $b^2 \not\equiv c^2 \pmod{p}$. Indeed, if $p^2 \equiv c^2 \pmod{p}$ then $b = \pm c \pmod{p}$. This cannot happen as, $\forall b \in T$, $\exists! b'$ in $S - T$ such that $b' \equiv -b \pmod{p}$, unless $b = c$.

Claim 2: $T^2 \equiv S^2 \pmod{p}$

Proof: Let $a \in S - T$. Then $\exists! a' \in T$ such that $a' \equiv a \pmod{p}$. Then $a^2 \equiv (a')^2 \pmod{p}$. Hence $a^2 \in T^2 \bmod p \Rightarrow$ the equare of any elt. of S is in $T^2 \bmod p$. Hence the claim.

But $\#\{\text{quad res. mod } p = \#S^2 \pmod{p}$. By claims 1 and 2, there is $\frac{p-1}{2} \Rightarrow \#\{\text{non}\} = p - 1 - \frac{p-1}{2} = \frac{p-1}{2}$.

Corollary of Lemma: Let p be an odd prime. then

$$\sum_{a \in (\frac{\mathbb{Z}}{p})^*} \left(\frac{a}{p}\right) = 0.$$

Proof:

$$\begin{aligned} \text{LHS} &= \sum_{\text{quad res}} \underbrace{\left(\frac{a}{p}\right)}_{1} + \sum_{\text{quad non-res}} \underbrace{\left(\frac{a}{p}\right)}_{-1} \\ &= 1\#\{\text{quad res.}\} - 1\#\{\text{quad non-res.}\} \\ &= \frac{p-1}{2} - \frac{p-1}{2} = 0. \end{aligned}$$

Lemma: Let a, b be integers prime to p. Then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

Proof:

Case 1: a, b are both $q, r, m, p.$, i.e. $a \equiv a_1^2$, $b \equiv b^2 \pmod{p}$ for some a, b. Hence $ab \equiv (a_1 b_1)^2 \pmod{p}$, and $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p}) = 1 \cdot 1$.

Case 2: $(\frac{a}{p}) = 1$, $(\frac{b}{p} = -1$. Suppose $(\frac{ab}{p}) = 1$. Then $\exists c$ such that $ab \equiv c^2$. Since $(\frac{a}{p}) = 1$, $\exists a_1$ such that $a_1^2 \equiv a \ (\div p)$. $\Rightarrow a_1^2 b \equiv c \div p$.

Since $p \not| a_1$, a_1 is invertible mod p, i.e., $\exists a_2$ such that $a_1 a_2 \equiv 1$. Then $a_1^2 a_1^2 \equiv 1$.

$$\Rightarrow b \equiv a_2^2 c^2 \pmod{p} \Rightarrow \left(\frac{b}{p}\right) = 1.$$

So $(\frac{ab}{p}) = -1$ when $(\frac{a}{p}) = 1$ and $(\frac{b}{p}) = -1$.

Case (iii) $(\frac{a}{p}) = -1$, $(\frac{b}{p}) = 1$ same as (ii).

Case (iv) $(\frac{a}{p}) = (\frac{b}{p}) - 1$ (Try this!)

Lemma 3 (Wilson's Theorem) For any prime p, $(p-1)! \equiv -1 \pmod{p}$.

Proof: If $p = 2$, both sides $= 1 \pmod{2}$, done. So assume p odd. Look at $S = \{1, \ldots, p-1\}$, set of resp. $forall a \in S$, let a' be the unique elt. of S such that $aa' = 1 \pmod{p}$.

$a = a'$ iff $a^2 = 1 \pmod{p}$, i.e., iff $a = 1$ or $a = p - 1$. So,

$$\forall a \in \{2, \ldots, p-2\} a' \not\supset a \text{ and } a' \in \{2, \ldots, p-1\}.$$

$$\begin{aligned} \Rightarrow (2)(3) \cdot (p-2) \equiv 1 \pmod{p}. \\ \Rightarrow (p-1)! \equiv 1(p-1) \pmod{p} \\ \equiv -1 \pmod{p}. \end{aligned}$$

Proposition (Euler's criterion) Let p be an odd prime, and let $a \in \mathbb{Z}$ with $(a, p) = 1$. Then

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$$

Recall that the Little Fermat theorem says that

$$a^{p-1} \equiv +1 \pmod{p} \text{ since } p \not| a;$$

so $a^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$.

Corollary of Proposition (Strict multiplicativity)

$$\left(\frac{ap}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right), \quad \forall a, b \in \mathbb{Z} \text{ with } p \not| ab.$$

Proposition $\Rightarrow$ Corollary 1: By Euler,

$$\begin{aligned}\left(\frac{ab}{p}\right) &= (ab)^{\frac{p-1}{2}} \\ &\equiv \left(a^{\frac{p-1}{2}}\right)\left(b^{\frac{p-1}{2}}\right) \\ &= \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).\end{aligned}$$

Corollary 2 of Proposition: If $p =$ odd prime, -1 is a square mod p iff $p \equiv 1 \pmod 4$.

Proposition$\Rightarrow$Corollary 2: By Euler, $(\frac{-1}{p}) = 1$ iff $(-1)^{\frac{p-1}{2}} \equiv 1 \pmod p$. Since p is odd, $p \equiv 1 \pmod 4$ are -1 (mod 4).

$p \equiv 1 \pmod 4$:
$p = 4m+1$, some $m \in \mathbb{Z}$:

$$\Rightarrow (-1)^{\frac{p-1}{2}} = (-1)^{2m} = 1$$

$p \equiv -1 \pmod 4$:
$p = 4m-1$:

$$(-1)^{\frac{p-1}{2}} = (-1)^{-1} \equiv -1 \pmod p.$$

Proof of proposition: By Fermat, $a^{p-1} \equiv 1 \pmod p$. Since p is odd, $\frac{p-1}{2} \in \mathbb{Z}$ and we can factor:

$$\underbrace{a^{p-1} - 1}_{\equiv 0 \text{ by Fermat}} = \left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} - 1\right)$$

$$\Rightarrow \left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} + 1\right) \equiv 0 \bmod p$$

$$\Rightarrow a^{\frac{p-1}{2}} \equiv \pm 1 \pmod p.$$

Now suppose a is a square mod p. Then $\exists b$ such that $a \equiv b^2 \pmod p$. So

$$a^{\frac{p-1}{2}} \equiv (b^2)^{\frac{p-1}{2}} \equiv b^{p-1} \equiv 1 \pmod p.$$

So:

$$\left(\frac{a}{p}\right) = 1 \Rightarrow a^{\frac{p-1}{2}} \equiv 1 \pmod p.$$

On the other hand, the congruence $X^{\frac{p-1}{2}} - 1 \equiv 0 \pmod p$ has at most $\frac{p-1}{2}$ solutions mod p by Lagrange. We have just proved that, given any quadratic residue a mod p,

$$a^{\frac{p-1}{2}} \equiv 1 \pmod p,$$

i.e., a is a solution of

$$X^{\frac{p-1}{2}} - 1 \equiv 0 \text{ (mod } p.$$

By lemma 1, there exists exactly $\frac{p-1}{2}$ quadratic residues mod p. Consequently,

$$X^{\frac{p-1}{2}} - 1 \equiv 0 \pmod p$$

has exactly $\frac{p-1}{2}$ solutions, and each of them is a quadratic residue mod p. In other words, if a is a quad. non-residue mod p, then a is not a solution of $X^{\frac{p-1}{2}} \equiv 0 \pmod p$.

$$\Rightarrow a^{\frac{p-1}{2}} \equiv -1 \pmod p \equiv \left(\frac{a}{p}\right) \pmod p$$

if $a \not\equiv \pmod p$.

To summarize, we have the following properties of $(\frac{\cdot}{p})$:

(i) $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ *Product formula*

(ii) $(\frac{-1}{p} \equiv (-1)^{\frac{p-1}{2}} \pmod p$, i.e., -1 is a square $\pmod p$ iff $p \equiv 1 \pmod 4$.

Remark:

Thanks to (i) and the unique factorization in $\mathbb{Z}$, in order to find $(\frac{a}{p})$ for any $a,\ (a,p)=1$, we need only know

$$\left(\frac{-1}{p}\right),\ \left(\frac{2}{p}\right),\ \text{and}\ \left(\frac{q}{p}\right),\ q \neq p \text{ an odd prime.}$$

We have already found a formula for $(\frac{-1}{p})$.

As an application of (ii) we will prove the following, special case of Dirichlet's theorem:

Proposition: There are infinitely many primes p which are congruent to 1 modulo 4.

Earlier we proved that there exists infinitely many primes $\equiv 3 \pmod 4$ in the following way: Suppose there exists a finite number of such primes. List them as $3, p_1, \ldots, p_r$. Consider

$$N = 4p_1 \ldots p_r + 3.$$

Factor N as $q_1, \ldots z_s$, q_j prime for all j. Since N is odd, each q_j is an odd prime. Moreover, since $N \equiv e \pmod 4$, since q_j must be $\equiv 1$ [3?] $\pmod 4$. But this q_j cannot be among $\{3, p_1, \ldots, p_r\}$.

Suppose we tried this for primes $\equiv 1 \pmod 4$. Assume there exists only finitely many such primes $p_1, \ldots, p_m$. Put $N = 4p_1 \ldots p_m + 1$. Factor N as $q_1 \ldots q_s$. Since N is odd, each q_j is an odd prime. But, if s is even, we cannot hope to say that some q_j must be $\equiv 1 \pmod 4$. The method breaks down.

Proof of Proposition: Now we try again using (ii). Again start by assuming there exists only a finite number of primes $\equiv 1 \pmod 4$, say $p_1, \ldots, p_m$. Let $N = 4(p_1 p_2 \ldots p_m)^2 + 1$. Factor N as $q_1 \ldots q_k$, q_j prime for all j. Evidently, each q_j is an odd prime because N is odd.

Claim:

$$\textit{Every } q_j \text{ is } \equiv 1 \pmod 4.$$

Proof of Claim: Pick any odd prime q_j dividing N. Then, since $N = (2p_1 \ldots p_m)^2 + 1$, we get $-1 \equiv b^2 \pmod{q_j}$, where $b = 2p_1 \ldots p_m$. By the criterion (ii), -1 is a square mod q_j iff $q_j \equiv 1 \pmod 4$. Hence the claim.

So q_j is a prime which is $\equiv 1 \pmod 4$, and it cannot be among $\{p_1, \ldots, p_m\}$ because if $p_1 = q_j$ for some i, we will get $1 \equiv 0 \pmod{q_j}$, a contradiction, proving the proposition.

Remark: This proof tells us a way to generate new primes which are $\equiv 1 \pmod 4$ from known ones. Here are some simple examples:

1. Start with 5, and consider $N = 4(5)^2 + 1 = 101$; this is a prime.
2. Start with 13, and consider $N = 4(13)^2 + 1$. Then $N = 677$, also prime.
3. Start with 17. $N = 4(17)^2 + 1 = 1157 = (13)(89)$. Note: 13 and 89 are both $\equiv 1 \pmod 4$.

Next Question: When is 2 a square mod p? To answer this question, Gauss proved a very useful lemma:

Proposition A (Gauss' Lemma) Fix a, prime to p. Let S be a subst of $\mathbb{N}$ such that $S \cup (-S)$ is a set of reps. for $(\mathbb{Z}/p)^*$. Given any $\mathrm{s} \in S$, we can then write $a\mathrm{s} \equiv e_{\mathrm{s}}(a)\mathrm{s}_a \pmod p$, where $\mathrm{s} \in S$ and $e_{\mathrm{s}}(a) \in \{\pm 1\}$. Then

$$\left(\frac{a}{p}\right) = \prod_{\mathrm{s}\in S} e_{\mathrm{s}}(a).$$

Proof: Let s, s$'$ be distinct numbers in S. Then

$$a\mathrm{s} \not\equiv a\mathrm{s}' \pmod p, \text{ i.e., } \mathrm{s}_a \not\equiv \mathrm{s}'_a.$$

Hence the map $S \to S$ given by $\mathrm{s} \to \mathrm{s}_a$ has to be a bijection, i.e., 1-1 and out. (This is also called a pem. or a rearrangement of S.) We get

$$\begin{aligned}
\underbrace{\prod_{\mathrm{s}\in S}(a\mathrm{s})}_{a^{\frac{p-1}{2}}\prod_{\mathrm{s}\in S}\mathrm{s}} &\equiv \prod_{\mathrm{s}\in S} e_{\mathrm{s}}(a)\mathrm{s}_a \pmod p \\
&\equiv \left(\prod_{\mathrm{s}\in S} e_{\mathrm{s}}(a)\right)\left(\prod_{\mathrm{s}\in S}\mathrm{s}_a\right) \pmod p \\
&\equiv \prod_{\mathrm{s}\in S} \mathcal{S} \pmod p
\end{aligned}$$

So $a^{\frac{p-1}{2}}(\prod_{\mathrm{s}\in S}\mathrm{s}) \equiv (\prod_{\mathrm{s}\in S} e_{\mathrm{s}}(a))(\prod_{\mathrm{s}\in S}\mathrm{s}) \pmod p \equiv 0 \bmod p$

Cancelling $(\prod_{\mathrm{s}\in S}\mathcal{S})$, which is invertible mod p from each side, get

$$a^{\frac{p-1}{2}} \equiv \prod_{\mathcal{S}\in S} e_{\mathcal{S}}(0)$$

Done because

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \bmod p.$$

Remark: people very often take S to be the "canonical" half set of reps for $(\mathbb{Z}/p)^*$, namely $S = \{1, 2, \ldots, \frac{p-1}{2}\}$.

Formulation (II) of Gauss' Lemma: Let $S = \{1, 2, \ldots, \frac{p-1}{2}\}$. For each $j \in S$, find the smallest positive residue $\bar{a}_j$ of a_j mod p. This is well defined, and

$$\bar{a}_j \in \{1, 2, \ldots, p-1\}.$$

Let
$$k = \#\{j \in S | \bar{a}_j \notin S\}.$$
Then Gauss' Lemma says
$$\left(\frac{a}{p}\right) = (-1)^k.$$
Corollary of Gauss' lemma:
$$\left(\frac{2}{p}\right) = (-1)^{n(p)},$$
$n(p)$ is the number of integers s such that
$$\frac{p-1}{4} < \mathrm{s} < \frac{p-1}{2}.$$
Explicitly,
$$\left(\frac{2}{p}\right) = \begin{cases} 1, & p = \pm 1 \pmod 8 \\ -1, & \text{if } p = \pm 5 \pmod 8 \end{cases}$$
Proof. Apply Gauss' lemma to $S = \{1, 2, \ldots, \frac{p-1}{2}\}$ with $a = 2$. Then
$$e_{\mathrm{s}}(2) = \begin{cases} 1, & \text{if } 2\mathrm{s} \leq \frac{p-1}{2} \\ -1, & \text{otherwise} \end{cases}$$
Since $(\frac{2}{p} = \prod_{\mathrm{s}\in S} e_{\mathrm{s}}(a) \pmod p)$, $(\frac{2}{p}) = (-1)^{n(p)}$. The rest follows.

Definition: If $x \in \mathbb{R}$, its integral part $[x]$ is the largest integer $\leq x$.

Proposition (Formulation III of Gauss' Lemma) Let p odd prime, and $a \in \mathbb{Z}$ with $p \not| a$. Then
$$\left(\frac{a}{p}\right) = (-1)^t, \text{ where } t = \sum_{j=1}^{(p-1)/2} [\frac{ja}{p}].$$
Proof: For every $j \in \{1, 2, \ldots, \frac{p-1}{2}\}$ it is easy to see that
$$a_j = q_j p + \bar{a}_j, \text{ with } 0 < \bar{a}_j < p.$$
Easy exercise:
$$q_j = \left[\frac{a_j}{p}\right].$$

So $\bar{a}_j = a_j - [\frac{a_j}{p}]$.

Summing over all the j's from 1 to $\frac{p-1}{2}$, we get

$$\sum_{j=1}^{\frac{p-1}{2}} a_j = \sum_{j=1}^{\frac{p-1}{2}} \left[\frac{a_j}{p}\right] p + \sum_{i-1}^{k} r_i + \sum_{i=1}^{k'} \ell_i, \tag{1}$$

where $k' = \frac{p-1}{2} - k$, $\{r_i\}$ = residues $\bar{a}_j$ *not* in S, $\{\ell_i\}$ = residues in S.

Also

$$\sum_{j=1}^{\frac{(p-1)}{2}} j = \sum_{i=1}^{k}(p - r_i) - \sum_{i=1}^{k} \ell_i. \tag{2}$$

Subtracting equation (2) from equation (1), we get

$$\begin{aligned}(a-1)\sum_{j=1}^{\frac{(p-1)}{2}} &= p\left(\sum_{j=1} \left[\frac{ja}{p}\right] - k\right) + 2\sum_{i=1}^{k} r, \\ &= \frac{1}{2}\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right) = \frac{p^2-1}{8}\end{aligned}$$

Thus

$$\underbrace{(a-1)}_{\text{even since } a \text{ is odd}} \left(\frac{p^2-1}{8}\right) = \sum_{j=1}^{\frac{(p^2-1}{2}} \left[\frac{ja}{p}\right] - k \pmod 2$$

Consequently, k has the same parity as

$$\sum_{j=1}^{\frac{(p-1)}{2}} \left[\frac{ja}{p}\right].$$

Review: p prime, $a \in \mathbb{Z}$, $p \not| a$:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & a \equiv \text{ mod } p \\ -1, & a \not\equiv \text{ mod } p \end{cases}$$

(Some also define $\frac{a}{p}$ for all $\mathbb{Z}$ by setting $(\frac{a}{p}) = 0$ if $p|a$.)

$p=2$: Everything is a square mod p. So assume p odd from now on. One has the multiplicativity property

$$\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \tag{*}$$

This follows from Euler's result that

$$\left(\frac{a}{p}\right)\equiv a^{\frac{p-1}{2}} \pmod{p}.$$

Note: Since p is odd, if $a^{\frac{p-1}{2}}\equiv b^{\frac{p-1}{2}} \pmod{p}$, for some a,b prime to p, then $(\frac{a}{b})=(\frac{b}{p})$. $(*)$ reduces finding $(\frac{a}{p})$ to the three cases

(i) $a=-1$

(ii) $a=2$

(iii) $a=q$, an odd prime $\neq p$

We have already proved

(i)

$$\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}=\begin{cases}1, & \text{if } p\equiv 1 \pmod{4}\\ -1, & \text{if } p\equiv -1 \pmod{4}\end{cases}$$

(ii)

$$\left(\frac{2}{p}\right)=\begin{cases}1, & \text{if } p\equiv \pm 1 \pmod{8}\\ -1, & \text{if } p\equiv \pm 5 \pmod{8}\end{cases}$$

(iii) q: odd prime $\neq p$.

$$\left(\frac{q}{p}\right)=?$$

16 The Quadratic Reciprocity Law

Fix an odd prime p. If q is another odd prime, a fundamental question, as we saw in the previous section, is to know the sign $\left(\frac{q}{p}\right)$, i.e., whether or not q is a square mod p. This is a very hard thing to know in general. But Gauss noticed something remarkable, namely that knowing $\left(\frac{q}{p}\right)$ is equivalent to knowing $\left(\frac{p}{q}\right)$; they need not be equal however. He found the precise law which governs this relationship, called the *Quadratic Reciprocity Law.* Gauss was very proud of ths result and gave several proofs. We will give one of his proofs, which incidentally introduces a very basic, ubiquitous sum in Mathematics called the *Gauss sum.* We will also give an alternate proof, which is in some sendse more clever than the first, due to Eisenstein.

Theorem (Gauss) (Quadratic reciprocity) *Let p, q be distinct odd primes. Then*

$$\left(\frac{q}{p}\right) = (-1)^{\frac{(q-1)(p-1)}{4}} \left(\frac{p}{q}\right).$$

Explicitly,

$$(q/p)/(p/q) = \begin{cases} 1, & \textit{if } p \textit{ or } q \textit{ is } \equiv 1 \textit{ (mod 4)} \\ -1, & \textit{if } p \textbf{ and } q \textit{ are } \equiv 3 \textit{ (mod 4)} \end{cases}$$

This theorem is very useful in computations.

Example

$$\left(\frac{37}{691}\right) = 3$$

It is not easy to compute $(37)^{\frac{691-1}{2}}$ (mod 691).

Better to use then:

$$\begin{aligned} \left(\frac{37}{691}\right) &= \underbrace{(-1)^{\left(\frac{37-1}{2}\right)\left(\frac{691-1}{2}\right)}}_{1} \left(\frac{691}{37}\right) \\ &= \left(\frac{691}{37}\right); \ \frac{691}{37} = 18\,\frac{25}{37} \\ &= \left(\frac{25}{37}\right) = \left(\frac{5}{37}\right)^2 = 1 \end{aligned}$$

Proof # 1 of theorem: p, q odd primes, $p \neq q$. Put

$$\xi = e^{\frac{2\pi i}{q}} \in \mathbb{C}$$

Then

$$\xi^q = 1, \text{ but } \xi^m \neq 1 \text{ if } m < q.$$

ξ is called a primitive qth root of unity in $\mathbb{C}$. All powers of ξ will be on the unit circle. In fact, we get a regular q-gon by converting the point

$$1, \xi, \ldots, \xi^{q-1}$$

cyclotong = "circle division"

Put $R = \{\alpha = a_0 + a_1\xi + \cdots + a_{\xi-1}\xi^{q-1} | a_0, a_1, \ldots, a_{q-1} \in \mathbb{Z}\}$. Clearly, $R \supset \mathbb{Z}$, hence R has 0×1. Let

$$\alpha = \sum_{i=1}^{q-1} a_i q^{\xi}, \ \beta = \sum_{i=1}^{q-1} b_i q^{\xi}$$

be in R. Then

$$\alpha \pm \beta = \sum_{i=1}^{q-1} (a_i + b_i)\xi^i \in R.$$

Since $\xi^q = 1$, given any $n \in \mathbb{Z}$ we can write $n = \ell q + r$, $0 \leq r \leq q - 1$ by Euclidean algorithm in $\mathbb{Z}$, and conclude that

$$\xi^n = \xi^r.$$

So R contains all the integral powers of ξ. Then it also contains finite integral linear combinations of such powers. Consequently,

$$\alpha\beta \in R \text{ if } \alpha, \beta \in R.$$

So R is very much like $\mathbb{Z}$. It is a q-dimensional analog of $\mathbb{Z}$. This allows us to define the divisibility in R. To be precise, if $\alpha, \beta \in R$, we say that β divides α, $\beta|\alpha$ iff $\exists \gamma \in R$ such that $\alpha = \beta\gamma$.

In particular, $R \ni p$, and it makes sense to ask if p divides some number in R.

Definition: Let $\alpha, \beta \in R$. We say that

$$\alpha \equiv \beta \pmod{p} \text{ iff } p|(a - b) \text{ in } R.$$

This allows us to do "congruence arithmetic" mod p in R.

To study $(\frac{q}{p})$, Gauss introduced the following "Gauss Sum":

$$S_q = \sum_{a \bmod q} \left(\frac{a}{q}\right) \xi^a.$$

Clearly, $S_q \in R$.

Aside (Not part of proof of Quad. Recip., but interesting)

$$S_q = \sum_{a=1}^{\frac{q-1}{2}} \left(\left(\frac{a}{q}\right) \xi^a + \underbrace{\left(\frac{-a}{q}\right) \xi^{-a}}_{\left(\frac{-1}{q}\right)\left(\frac{a}{q}\right)\bar{\xi^a}} \right)$$

So if

$$\left(\frac{-1}{q}\right) = 1, \; S_q = \sum_{a=1}^{\frac{q-1}{2}} \left(\frac{a}{q}\right) (\xi^a + \bar{\xi^a}) \in R \cap \mathbb{R}$$

and if

$$\left(\frac{-1}{q}\right) = -1, \; S_q \in R \cap i\mathbb{R}$$

pure read or im.

Lemma 1:

$$S_q^2 = (-1)^{\frac{q-1}{2}} q$$

Proof of Lemma 1:

$$\begin{aligned}
S_q^2 &= \left(\sum_{a \bmod q} \left(\frac{a}{q}\right) \xi^a \right) \left(\sum_{b \bmod q} \left(\frac{b}{q}\right) \xi^b \right) \\
&= \sum_{a \bmod q} \sum_{b \bmod q} \left(\frac{a}{q}\right) \left(\frac{b}{q}\right) \xi^a \xi^b \\
&= \sum \sum \left(\frac{ab}{q}\right) \xi^{a+b} \\
&= \sum_{c \bmod q} \xi^c \left(\sum_{a \bmod q} \left(\frac{a(c-a)}{q}\right) \right)
\end{aligned}$$

So

$$\begin{aligned}
S_q^2 &= \sum_{c \bmod q} \xi^c \sum_{a \bmod q} \left(\frac{ac - a^2}{q}\right) \\
&= \sum_{c \bmod q} \xi^c \sum_{a \bmod q} \left(\frac{-a^2(1 - a'c)}{q}\right),
\end{aligned}$$

where $a'a \equiv 1 \pmod{q}$.

But

$$\left(\frac{-a^2(1-a'c)}{q}\right) = \underbrace{\left(\frac{-1}{q}\right)}_{(-1)^{\frac{q-1}{2}}} \underbrace{\left(\frac{a^2}{q}\right)}_{=1 \text{ as } a \not\equiv 0 \bmod q} \left(\frac{1-a'c}{q}\right)$$

$$\Rightarrow S_q^2 = (-1)^{\frac{q-1}{2}} \sum_{c \bmod q} \xi^c f(c),$$

where

$$f(c) = \sum_{a \bmod q} \left(\frac{1-a'c}{q}\right) \quad a \not\equiv 0 \bmod q$$

$f(c) = ?$

$\mathbf{c \equiv 0}$ (mod q):

$$f(0) = \sum_{\substack{a \bmod q \\ a \not\equiv 0 \pmod{q}}} \left(\frac{1}{q}\right)$$

$$\Rightarrow f(0) = q - 1$$

$\mathbf{c \not\equiv 0}$ (mod q): Note that, in this case, the set

$$\{1 - a'c | a \bmod q,\ a \not\equiv 0 \bmod q\}$$

runs over elements of $\mathbb{Z}/q - \{1\}$ exactly once. Indeed, given any $b \in \mathbb{Z}/q$, $b \not\equiv 1 \pmod{q}$, we can solve $(a' + b \equiv 1 \pmod{q})$, and the solution is unique.

Therefore,

$$f(c) = \sum_{\substack{b \bmod q \\ b \not\equiv 1 \pmod{q}}} \left(\frac{b}{q}\right).$$

We proved earlier that

$$\sum_{b \bmod q} \left(\frac{b}{q}\right) = 0$$

so

$$f(c) = \left(\frac{1}{q}\right) = -1,$$

when $c \not\equiv 0 \pmod{q}$.

Consequently

$$S_q^2 = (-1)^{\frac{q-1}{2}} \left[(q-1) + (-1) \sum_{\substack{c \bmod q \\ c \not\equiv 0 \bmod q}} \xi^c \right]$$

Claim: $\sum_{c \bmod q} \xi^c = 0$.

Proof of claim:

$$\sum_{c \bmod q} \xi^c = \sum_{(c-1) \bmod q} \xi^c = \sum_{c \bmod q} \xi^{c+1} = \xi \sum_{c \bmod q} \xi^c$$

$$\Rightarrow \underbrace{(1-\xi)}_{\neq 0} \sum_{c \bmod q} \xi^c = 0 \Rightarrow \sum_{c \bmod q} \xi^c = 0 \text{ as claimed.}$$

Proof 2 of claim:

$$\begin{aligned} \sum_{c \bmod q} \xi^c = 1 + \xi + \cdots + \xi^{q-1} &= \frac{1-\xi^q}{1-\xi} \\ &= 0 \text{ since } \xi^q = 1. \end{aligned}$$

By claim,

$$\begin{aligned} S_q^2 &= (-1)^{\frac{q-1}{2}} \left((q-1) + \underbrace{(-1)(0-1)}_{+1} \right) \\ &= (-1)^{\frac{q-1}{2}} q. \end{aligned}$$

This proves Lemma 1.

Lemma 1: $S_q^2 = (-1)^{\frac{q-1}{2}} q$

Lemma 2: $S_q^{p-1} \equiv (\frac{p}{q}) \pmod p$
(This happens in $R \bmod p$)

Proof of Lemma 2:

$$\begin{aligned} S_q^p &= \left(\sum_{a \bmod q} \left(\frac{a}{q} \right) \xi^a \right)^p \\ &= \sum_{a \bmod q} \left(\frac{a}{q} \right)^p \xi^{ap} + pw, w \in R. \end{aligned}$$

$(\frac{a}{p})^q = (\frac{a}{p})$ because $(\frac{a}{q}) = \pm 1$ and p is odd

In other words,

$$S_q^p \equiv \sum_{a \bmod q} \left(\frac{a}{q}\right) \xi^{ap} \pmod{p}.$$

Since $p \neq q$, p is invertible mod q, and the map $a \mapsto ap$ is a permutation of $\mathbb{Z}/q$, also $ap \equiv 0 \pmod{q}$ iff $a \equiv 0 \pmod{q}$. so the sum over a mod q can be replaced with the sume over ap mod q. Write b for ap mod q. Then

$$a \equiv bp' \pmod{q}, \text{ where } pp' \equiv 1 \bmod q).$$

$$\Rightarrow S_q^p \equiv \sum_{b \bmod q} \left(\frac{bp'}{q}\right) \xi^b \pmod{p} \tag{*}$$

But

$$\left(\frac{bp'}{q}\right) = \left(\frac{b}{q}\right)\left(\frac{p'}{q}\right).$$

Since $p'p \equiv 1 \pmod{q}$,

$$\left(\frac{p'}{q}\right)\left(\frac{p}{q}\right) = \left(\frac{1}{q}\right) = 1 \Rightarrow \left(\frac{p'}{q}\right) = \left(\frac{p}{q}\right)$$

So

$$\left(\frac{bp'}{q}\right) = \left(\frac{b}{q}\right)\left(\frac{p}{q}\right).$$

So $(*)$ gives

$$S_q^p \equiv \left(\frac{p}{q}\right) \underbrace{\sum_{b \bmod q} \left(\frac{b}{q}\right) \xi^b}_{S_q} \pmod{p}$$

$$\Rightarrow S_q^{p-1} \equiv \left(\frac{p}{q}\right) \pmod{p}$$

This is justified because

$$S_q \not\equiv 0 \pmod{p},$$

which follows from lemma 1.

Proof of Theorem: Compute S^{p-1} in 2 different ways. On the one hand, by lemma 1,

$$\begin{aligned} S^{p-1} = (S^2)^{\frac{p-1}{2}} &= \left((-1)^{\frac{q-1}{2}} q\right)^{\frac{p-1}{2}} \\ &\underset{\text{Euler}}{\equiv} \left(\frac{(-1)^{\frac{q-1}{2}} q}{p}\right) \pmod p \\ \Rightarrow S^{p-1} &\equiv \left(\frac{-1}{p}\right)^{\frac{q-1}{2}} \left(\frac{q}{2}\right) \pmod p, \end{aligned}$$

i.e.,

$$S^{p-1} \equiv (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} \left(\frac{q}{p}\right) \pmod p.$$

On the other hand, by lemma 2,

$$S^{p-1} \equiv \left(\frac{p}{q}\right) \pmod p.$$

So, putting them together we get

$$\left(\frac{p}{q}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} \left(\frac{q}{p}\right).$$

Last time, gave a proof of Quadratic Reciprocity law. More precisely we proved:

Theorem (Gauss) Let p, q be distinct, odd primes. Then

$$\left(\frac{p}{q}\right) (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} \left(\frac{q}{p}\right).$$

Example:

Check if 29 is a square mod 43: 29 and 43 are distinct odd primes, so by definition $29 \equiv \pmod{43}$ iff $\left(\frac{29}{43}\right) = 1$. by QRL,

$$\begin{aligned}
\left(\frac{29}{43}\right) &= (-1)^{\frac{28(42)}{4}} \left(\frac{43}{29}\right) = \left(\frac{43}{29}\right) \\
&= \left(\frac{14}{29}\right) = \left(\frac{2}{29}\right)\left(\frac{7}{29}\right) \\
\left(\frac{2}{29}\right) &= -1 \text{ as } 29 \equiv 5 \pmod 8 \\
\left(\frac{29}{43}\right) &= -\left(\frac{7}{29}\right)_{\text{QRL}} = -(-1)^{\frac{6(28)}{4}} \left(\frac{29}{7}\right) \\
&= -\left(\frac{29}{7}\right) = -\left(\frac{1}{7}\right) = -1
\end{aligned}$$

So $29 \not\equiv \pmod{43}$.

Remark: QRL tells you a way to know

1. whether q is a square mod p or not. But when it is a square, it gives no procedure to find the square root.
2. One can use QRL to check whether a number is a prime, similar to the way one uses Fermat's little theorem. For example, one can show that $m = 1729$ is not a prime by looking at

$$y \stackrel{\text{def}}{\equiv} 11^{864} \pmod{1729}$$

Note: $864 = \frac{1729-1}{2}$. So, if m is a prime, $y \equiv \left(\frac{11}{1729}\right) \pmod m$.

Since $1729 \equiv 1 \pmod 4$, by QRL,

$$\left(\frac{11}{1729}\right) = \left(\frac{1729}{11}\right) = \left(\frac{2}{11}\right) = -1$$

as $11 \equiv 3 \pmod 8$. on the other hand, one can check using PARI, or by successively squaring mod $m = 1729$, that

$$11^{864} \equiv 1 \pmod m.$$

(This is part of a homework problem.) Get a contradiction! So the only possibility is that 1729 is not a prime (which is easy to verify directly as $1729 = 13 \cdot 133 = 13 \cdot 7 \cdot 17$). But this method is helpful, when it works, for larger numbers.

A histoical remark: G.H.Hardy went to see Ramanujan, when the latter was dying of TB in England. Then Ramanujan asked Hardy if the number of the taxicab Hardy came in was an interesting number. Hardy said "No, not interesting, just 1729". Ramanujan replied immediately, saying, "On the contrary, the number *is* interesting because it is the first number which can be written as a sum of 2 cubes in two different ways". (Indeed we have

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

)

A second proof of quadratic recip. (Eisenstein) (Eisenstein's trignometric lemma)

Lemma: Let n be a positive, odd integer. Then

$$\frac{\sin nx}{\sin x} = (-4)^{\frac{n-1}{2}} \prod_{j=1}^{\frac{(n-1)}{2}} \left(\sin^2 x - \sin^2 \frac{2\pi j}{n} \right)$$

Proof: Up to us. Hint: treat as a polynomial in $\sin x$:

Example:

$n = 3$

$$\begin{aligned}
\text{LHS} &= \frac{\sin 3x}{\sin x} = \frac{\sin(2x+x)}{\sin x} \\
&= \frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x} \\
&= \frac{2 \sin x \cos^2 x + (1 - 2\sin^2 x) \sin x}{\sin x} \\
&= 2(1 - \sin^2 x) + (1 - 2\sin^2 x) = 3 - 4\sin^2 x \\
\text{RHS} &= -4(\sin^2 x - \left(\underbrace{\sin \frac{2\pi}{3}}_{\sqrt{3}/2} \right)^2 = -4 \left(\sin^2 x - \frac{3}{4} \right) \\
&= 3 - 4\sin^2 x.
\end{aligned}$$

Sketch of proof of lemma: Use induction on n to show that

$$\frac{\sin nx}{\sin x} = f_n(\sin^2 x),$$

where f_n is a polynomial in $\sin^2 x$ of degree $\frac{n-1}{2}$.

$$(f_0(t) = 1,\ f_3(t) = 3 - 4t, \dots)$$

On the other hand, the RHS of lemma is also of the form $g_n(\sin^2 x)$, where g_n is the explicitly given polynomial in $\sin^2 x$ of degree $\frac{n-1}{2}$.

So it suffices to show that f_n and g_n have the same roots and that the leading coefficient of f_n is $(-4)^{\frac{n-1}{2}}$. So when we use induction on n, check that the leading coefficient is $(-4)^{\frac{(n-1)}{2}}$ and that its roots are

$$\left\{ \sin^2 \frac{2\pi j}{n} \middle| 1 \le j \le \frac{n-1}{2} \right\}.$$

Alternatively, check the constant coefficient by checking at $x \to 0$.

Recall Gauss' lemma:

$$\left(\frac{q}{p}\right) = \prod_{s \in S} e_s(q)$$

where $S = \{1, 2, \dots, \frac{p-1}{2}\}$ and $e_x(q) \in \{\pm 1\}$ defined by

$$qs = e_s(q)s',\ \text{with } s' \in S.$$

Applying $\sin(\frac{2\pi}{p})$, we get

$$\begin{aligned} \sin\left(\frac{2\pi qs}{p}\right) &= \sin\left(\frac{2\pi e_s(q)s'}{p}\right) \\ &= e_s(q) \sin\left(\frac{2\pi s'}{p}\right) \end{aligned}$$

since sin is an odd function. So

$$e_s(q) = \frac{\sin\left(\frac{2\pi qs}{p}\right)}{\sin\left(\frac{2\pi s'}{p}\right)}$$

By Gauss' lemma,

$$\begin{aligned} \left(\frac{q}{p}\right) &= \prod_{s \in S} \frac{\sin\left(\frac{2\pi qs}{p}\right)}{\sin\left(\frac{2\pi s'}{p}\right)} \\ &= \frac{\prod_{s \in S} \sin\left(\frac{2\pi qs}{p}\right)}{\prod_{s \in S} \sin\left(\frac{2\pi s'}{p}\right)}. \end{aligned}$$

Note the map $S \mapsto S'$ is a permutation of S. So,

$$\prod_{s \in S} \sin\left(\frac{2\pi s'}{p}\right) = \prod_{s \in S} \sin\left(\frac{2\pi s}{p}\right)$$

$$\Rightarrow \left(\frac{q}{p}\right) = \prod_{i=1}^{\frac{(p-1)}{2}} \frac{\left(\sin\frac{2\pi i q}{p}\right)}{\sin\frac{2\pi i}{p}} \tag{1}$$

Applying Eisenstein's trig. lemma with $n = q$ and sub. in (3), we get

$$\left(\frac{q}{p}\right) = (-4)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} \prod_{i-1}^{\frac{(p-1)}{2}} \prod_{i-1}^{\frac{(q-1)}{2}} \left(\sin^2\left(\frac{2\pi i}{p}\right) - \sin^2\left(\frac{2\pi j}{p}\right)\right)$$

Can get everything we need from this without computing the sines:

Reversing the roles of p and q, we get

$$\left(\frac{p}{q}\right) = (-4)^{\frac{q-1}{2}} \prod_{i-1}^{\frac{p-1}{2}} \prod_{i-1}^{\frac{q-1}{2}} \left(\sin^2\left(\frac{2\pi j}{p}\right) - \sin^2\left(\frac{2\pi i}{p}\right)\right)$$

Comparing (3) and (4), we see that

$$\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)}{2}\frac{(q-1)}{2}} \left(\frac{p}{q}\right)$$

17 Sums of two squares

$$n = a^2 + b^2;\ a, b \geq 0,\ n \geq 1$$

Note:

For all integers a, b, we have

$$a^2 + b^2 \equiv 0, 1 \text{ or } 2 \pmod 4$$

Indeed, $a, b = 0, 1, 2, 3 \pmod 4 \Rightarrow a^2, b^2 \equiv 0, 1 \pmod 4 \Rightarrow a^2 + b^2 \equiv 0, 1, 2 \pmod 4$. So the numbers congruent to 3 mod 4 *cannot* be written as sums

$1 = 1^2 + 0^2$		$16 = 4^2 + 0^2$		$\underline{31 = --}$	$\leftarrow$
$\underline{2 = 1^2 + 1^2}$		$\underline{17 = 4^2 + 1^2}$		$32 = 4^2 + 4^2$	
$\underline{3 = --}$	$\leftarrow$	$18 = 3^2 + 3^2$		33=–	
$4 = 2^2 + 0^2$		$\underline{19 = --}$	$\leftarrow$	$34 = 5^2 + 3^2$	
$\underline{5 = 2^2 + 1^2}$		$20 = 4^2 + 2^2$		35=–	$\leftarrow$
6=–		21=–		$36 = 6^2 + 0^2$	
$\underline{7 = --}$	$\leftarrow$	22=–		$\underline{37 = 6^2 + 1^2}$	
$8 = 2^2 + 2^2$		$\underline{23 = --}$	$\leftarrow$	38=–	
$9 = 3^2 + 0^2$		24=–		39=–	$\leftarrow$
$10 = 3^2 + 1^2$		$25 = 5^2 + 0^2 = 4^2 + 3^2$		$40 = 6^2 + 2^2$	
$\underline{11 = --}$	$\leftarrow$	$26 = 5^2 + 1^2$		$\underline{41 = 5^2 + 4^2}$	
12=–		27=–	$\leftarrow$	42=–	
$\underline{13 = 3^2 + 2^2}$		28=–		$\underline{43 = --}$	$\leftarrow$
14=–		$\underline{29 = 5^2 + 2^2}$			
15=–	$\leftarrow$	30=–			

of 2 squares. It appears from this table that if p is an odd prime, we may write $p = a^2 + b^2$ iff $p \not\equiv 3 \bmod 4$.

Lemma A: If m, n are sums of 2 squares, then so is their product mn.

Proof: Use the identity $(A^2 + B^2)(x^2 + y^2) = (Ax + By)^2 + (Ay - Bx)^2$

Proposition A. Let p be a prime congruent to 1 mod 4. Then p is a sum of two squares in $\mathbb{Z}$.

Proof of Proposition A. First we claim that there exists integers A, B, m, with $1 \le m < p$, such that

$$mp = A^2 + B^2 \tag{1}$$

Indeed, since $p \equiv 1 \pmod 4$, $(\frac{-1}{p}) = 1$ and so we can find $n \in \mathbb{Z}$ such that $n^2 \equiv -1 \pmod p$. It was proved earlier that the set $T : \{1, 2, \ldots, \frac{p-1}{2}\}$ is a set of representatives for the **squares** in $(\mathbb{Z}/p)^*$. Hence we may choose $n \in T$ such that

$$n^2 + 1 = mp,$$

for some integer $m \ge 1$. Since $n < \frac{p}{2}$, we have:

$$m = \frac{1}{p}(n^2 + 1) < \frac{1}{p}\left(\frac{p^2}{4} + 1\right) < p,$$

which proves the claim.

Now there may be more than one m for which (1) holds. (Of course (A, B) will depend on m.). So we may, and we will, choose m to be the **smallest** integer ≥ 1 for which (1) holds. Of course, $m < p$. We are done if $m = 1$, so we will assume that $m > 1$ and derive a contradiction.

Find $x, y \in \mathbb{Z} \cap [-\frac{m}{2}, \frac{m}{2}]$ such that $x \equiv A \bmod m$, $y \equiv B \bmod m$.

Then

$$x^2 + y^2 = km, \text{ for some integer } k \geq 1, \tag{2}$$

$$\text{since } A^2 + B^2 \equiv\ 0 \bmod m.$$

By construction,

$$x^2 + y^2 \leq \frac{m^2}{4} + \frac{m^2}{4} = \frac{m^2}{2} = \frac{m}{2} \cdot m.$$

So $k < m$. Applying the identity proving Lemma 1, we obtain

$$\begin{aligned}(x^2 + y^2)(A^2 + B^2) &= km \cdot mp = m^2kp \\ &= (Ax + By)^2 + (Ay - Bx)^2.\end{aligned}$$

Notice that

$$Ay \equiv xy \equiv xB \pmod{m}.$$

So

$$m^2 | (Ay - Bx)^2,$$

and this gives

$$m^2 | (Ax + By)^2.$$

Hence $m | (Ax + By)$, and

$$\left(\frac{Ax + By}{m}\right)^2 + \left(\frac{Ay - Bx}{m}\right)^2 = kp. \tag{3}$$

Since $k < m$, and (3) gives a contradiction to the minimality of m.

Example: $p = 41$, $9^2 = 81 \equiv -1 \pmod{p}$

Start with $9^2 + 1^2 = 2 \cdot 41$, $x, y \in \mathbb{Z} \cap [-1, 1]$ such that $x \equiv 9 \pmod{2}$, $y \equiv 1 \pmod{2}$. Pick $x = y = 1$,

$$\frac{Ax + By}{m} = \frac{9 \cdot 1 + 1 \cdot 1}{2} = 5$$

$$\frac{Ay - Bx}{m} = \frac{9 \cdot 1 - 1}{2} = 4$$

This gives:

$$41 = 5^2 + 4^2.$$

Proposition C. Let p be a prime $\equiv 3$ mod 4. Then no integer n divisible precisely by an **odd** power of p can be written as a sum of two squares.

Theorem Let $n \geq 1$ be an integer. Then n can be written as a sum of two squares **iff** every prime $\equiv 3$ (mod 4) occurs to a even power in its prime factorization.

Proof of Theorem (modulo Proposition C)
($\Rightarrow$): This is because Proposition C says that any prime congruent to 3 mod 4 has to occur to an even power r in n.
($\Leftarrow$): Let $r = p_1 p_2 \ldots p_m q_1^{2n} \ldots q_\ell^{2n_\ell}$, with $p_i \equiv 1$ mod 4, $q_j \equiv 3$ mod 4. By Prop. B, p_i is an sum of two squares, and $q_j^{2n_j} = (q_j^{n_1})^2 + 0^2$. Thus n is a product of numbers which are sums of two squares, and we are done by applying Lemma A.

Proof of Proposition C: Let $p \equiv 3$ (mod 4) be a prime. Suppose

$$n = a^2 + b^2, \text{ with } p^{2s+1} \| n.$$

Let $d = (a, b)$, so that $d^2 | (a^2 + b^2) = n$. Hence

$$\left(\frac{n}{d}\right)^2 = \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2, \text{ if } m = \frac{n}{d}, \ x = \frac{a}{d}, \ y = \frac{b}{d}.$$

So we get

$$m = x^2 + y^2, \text{ with } \gcd(x, y) = 1,$$

and

$$p^{2s+1} \| m.$$

In particular, $p | m$, but p does not divide both x and y. But if $p | x$, as $m = x^2 + y^2$, $p | y^2$, and so $p | y$. Consequently, $p \not| xy$.

It follows, since $(p, x) = 1$, that

$$Ax - Bp = t$$

is solvable in $\mathbb{Z}$ for all t. Take $t = y$ to get $Ax \equiv y$ (mod p).

Then

$$0 \equiv x^2 + y^2 \equiv x^2(A^2 + 1) \text{ (mod } p).$$

Since $p \not| x$, get:

$$A^2 + 1 \equiv 0 \pmod{p}.$$

But $(\frac{-1}{p}) = -1$ as $p \equiv 3 \bmod 4$, giving a contradiction.

Questions:

1. What if one considers sums of k squares with $k > 2$, e.g., $7 = 2^2 + 1^2 + 1^2 + 1^2$.

   In Section 19, we will prove that any positive integer can be written as a sum of four squares.

2. If $n = a^2 + b^2$, in how many ways can one write n as a sum of two squares?

   Example: $25 = 5^2 + 0^2 = 4^2 + 3^2$

   $65 = 8^2 + 1^2 = 7^2 + 4^2$

   Note in general that

$$\begin{aligned}(x^2 + y^2)(A^2 + B^2) &= (xA + yB)^2 + (xB - yA)^2 \\ &= (xA - yB)^2 + (xB + yA)^2\end{aligned}$$

   Example:

$$\begin{aligned}25 = 5 \cdot 5 &= +(2^2 + 1)(2^2 + 1) \\ &= (x \cdot 2 + 1 \cdot 1)^2 + (2 \cdot 1 - 1 \cdot 2)^2 = 5^2 + 0^2 \\ &= (2 \cdot 2 - 1 \cdot 1)^2 + (2 \cdot 1 - 1 \cdot 2)^2 = 3^2 + 4^2\end{aligned}$$

   When do these two ways of writing it coincide?

They do iff we have

$$(xA + yB)^2 = (xA - yB)^2$$

or

$$(xA + yB)^2 = (xB + yA)^2$$

First case:

Square both sides to get

$$xyAB = 0 \text{ i.e., at least one of } x, y, A, B \text{ is zero..}$$

Second case: Here we get

$$x^2A^2 + y^2B^2 = y^2A^2 + x^2B^2$$

$$\Leftrightarrow x^2(A^2 - B^2) + y^2(B^2 - A^2) = 0$$
$$\Leftrightarrow (x^2 - y^2)(A^2 - B^2) = 0$$
$$\Leftrightarrow x = y \text{ or } A = B$$

Claim: If $p \equiv 1 \pmod 4$ is a prime, then $p = a^2 + b^2$ uniquely.

Indeed, suppose $p = a^2 + b^2 = c^2 + d^2$, for $a, b, c, d \in \mathbb{Z}$. Then

$$a^2d^2 - b^2c^2 = (a^2 + b^2)d^2 - (c^2 + d^2)b^2 = p(d^2 - b^2)$$

$\Rightarrow ad \equiv bc \pmod p$, or $ad \equiv -bc \pmod p$.

Clearly $0 < a, b, c, d < \sqrt{p}$. So

$$ad \equiv bc, \text{ or } ad = p - bc.$$

If $ad = p - bc$

$$\begin{aligned} p^2 &= (a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2 \\ &= p^2 + (ac - bd)^2 \Rightarrow ac = bd \end{aligned}$$

Hence $a|bd$, and $\gcd(a, b) = 1$. $\Rightarrow a|d$. Also $d|ac$, and $\gcd(c, d) = 1$, so $d|a$. So $a = \pm d$, so $a = d$. $\Rightarrow b = c$.

If $ad = bc$, we find that $a = c,\ b = c$, and also $c = d$. Now the uniqueness assertion follows.

18 Gaussian Integers

Definition: $\mathbb{Z}[i] \subseteq \mathbb{C} = \{a + ib : \ a, b \in \mathbb{Z}\}$

Elements of $\mathbb{Z}[i]$ are called Gaussian integers, which can be added, subtracted and multiplied. But we cannot divide in $\mathbb{Z}[i]$. For example, $\frac{1}{1+i} = \frac{1}{2}(1 - i) \notin \mathbb{Z}[i]$. Note: if $\alpha\beta = 0 \Rightarrow \alpha = 0$ or $\beta = 0$. Define the norm

$$N : \mathbb{Z}[i] \to \mathbb{Z}_+$$

by

$$\alpha = a + bi \mapsto a^2 + bi = (a + bi)(c - bi) = \alpha\bar{\alpha}$$

The complex conjugation map $\alpha \mapsto \bar{\alpha}$ satisfies:

$$\overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta},\ \overline{\alpha\beta} = \bar{\alpha}\cdot\bar{\beta}.$$

So

$$N(\alpha\beta) = \alpha\beta\overline{\alpha\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = N(\alpha)N(\beta)$$

Notice that in $\mathbb{C}$, $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$

Definition: α, β in $\mathbb{Z}[i]$. Say $\alpha|\beta$ **iff** $\beta = \alpha \cdot \gamma$, some $\gamma \in \mathbb{Z}[i]$.

Definition: A **unit** in $\mathbb{Z}[i]$ is an element α in $\mathbb{Z}[i]$ such that $\alpha\beta = 1$ for some $\beta \in \mathbb{Z}[i]$. If α is a unit in $\mathbb{Z}[i]$, say $\alpha\beta = 1$,

$$N(\alpha\beta) = N(1) = 1 = N(\alpha)N(\beta).$$

If $\alpha = a + bi,\ a, b \in \mathbb{Z}$,

$$(a^2 + b^2) = N(\beta) = 1$$

Hence

$$a = 0,\ b = \pm 1,\ \text{or } a = \pm 1,\ b = 0.$$

This means $\alpha = \pm 1$ or $\pm i$. Put

$$D = \{a + bi : a \geq 1,\ b \geq 0\}$$

$\alpha \sim \beta$ ("associated") **iff** $\alpha = u\beta$ for some unit u in $\mathbb{Z}[i]$.

If $\alpha \neq 0$, there is exactly one associate of α in D, the normalized associate.

$\pi \in \mathbb{Z}[i]$ is called a **Gaussian prime** if its only divisors are units and its associates.

Question: What are the Gaussian primes?
$(1+i)(1-i) = 2$ so $(1 \pm i)|2$. Hence 2 is **not** a Gaussian prime. $1+i$, $2+i$ are Gaussian primes, so is $1+2i$ because it is an associate of $2+i$. (*Conjecture*: $a + ib$ is Gaussian prime iff $(a, b) = 1$.)

Unsolved Problem: If you are allowed only steps of bounded size, is it possible to walk to ∞ stepping only on Gaussian primes?

Euclidean algorithm: Recall the norm function

$$\begin{aligned} N : \mathbb{Z}[i] &\to \mathbb{Z} \\ a + bi &\mapsto a^2 + b^2 \\ \alpha &\mapsto \alpha\bar{\alpha}, \end{aligned}$$

which is multiplicative, i.e.,

$$N(\alpha\beta) = N(\alpha)N(\beta)$$

Given $\alpha, \beta \in \mathbb{Z}[i],\ \beta \neq 0,\ \exists$ [unique] $\rho, \kappa \in \mathbb{Z}[i]$ such that $\alpha = \kappa\beta + \rho$ **and** $0 \neq N(\rho) \leq \frac{N(\beta)}{2}$.

Proof: $\forall x \in \mathbb{R}$, let round(x) = closest integer to x. Then $|x - \text{round}(x)| \leq \frac{1}{2}$. Choose round$(\frac{1}{2}) = 1$ and let round$(x + iy)$ = round(x)+$i$round(y).

Let $z = \frac{\alpha}{\beta} = \mathbb{C}$.

Let κ = round(z).

$$\begin{aligned} N(z - \kappa) &= N(z - \text{round}(z)). \\ &= N((x - \text{round}(x)) + i(y - \text{round}(y))) \\ &= (x - \text{round}(x))^2 + (y - \text{round}(y))^2 \leq \frac{1}{2} \end{aligned}$$

$$\text{Since } \frac{\alpha}{\beta} = \kappa + \left(\frac{\alpha}{\beta} - \kappa\right),$$

$$\alpha = \beta\kappa + \rho,$$

$$\text{with } \rho = (\alpha - \beta\kappa),\ 0 \leq N(\rho).$$

$$\text{Then } z - \kappa = \frac{\alpha}{\beta} - \kappa = \frac{\alpha - \kappa\beta}{\beta}, \text{ and}$$

$$N(z - \kappa) = \frac{N(\alpha - \kappa\beta)}{N(\beta)} = \frac{N(\rho)}{N(\beta)} \leq \frac{1}{2}.$$

Corollary: The ring $\mathbb{Z}[i]$ has unique factorization into Gaussian primes.

Proof: Similar to the proof in $\mathbb{Z}$, with $\gcd(\alpha, \beta)$ being defined using the Euclidean algorithm.

Now investigate what Gaussian primes look like.

$$N(3 + i) = \underbrace{9 + 1}_{\text{[sum of squares]}} = 10 = 2 \cdot 5$$

[Notice relatioship to sums of squares!] So 3+i must be divisible by something of norm 2 and something of norm 5. $2 + i$, $2 - i$ has norm 5, while $1 + i$ has norm 2.

$$(2 + i)(1 + i) = 2 + 3i - 1 = 1 + 3i$$

$$(2-i)(1+i) = i+3$$

Theorem: Let p be a prime of $\mathbb{Z}$. If p is not a Gaussian prime then $p = \pi\overline{\pi}$, π, $\overline{\pi}$ Gaussian primes. ($\pi \not\sim \overline{\pi}$ if p is odd). Also, p has no other divisors. Moreover, p is not a Gaussian prime iff

$$p = 2 = (1+i)^2$$

or

$$p \equiv 1 \pmod 4.$$

Consequently if $p \equiv 3 \pmod 4$, p is a Gaussian prime.

Conversely, every Gaussian prime π is either a rational prime $\equiv 3$ (mod 4) or its norm is a rational prime $\not\equiv 3 \pmod 4$. In the latter case, $N(\pi) = 2$ iff $\pi \sim \bar{\pi}$.

Proof: By unique factorization, we may write $p = w\pi_1 \ldots \pi_m$, with w a unit, and the π_j's Gaussian primes.

$$N(p) = p\bar{p} = p^2 = \prod_{j=1}^{m} N(\pi_j).$$

Thus $\exists$ unique j such that $N(\pi_j) = p^2$. Then $m = 1$ and $p = w\pi_1$. Consequently, p is a Gaussian prime. So if $p \neq$ Gaussian prime, then none of the $N(\pi_j)$'s are p^2. So

$$p = \pi_1\pi_2$$

with π_1, π_2 Gaussian primes, $N(\pi_i) = p$. Since $\pi_1, \pi_2 \notin \mathbb{Z}$, and $\pi_1\pi_2 \in \mathbb{Z}$, $\pi_2 = \overline{\pi_1}$.

Assume p is odd. Then $\pi \sim \bar{\pi}$ means $\pi = a + bi \sim a - bi$. The associates of π are $\pm(a+ib)$ and $\pm(a+ib)$. This is because the units in $\mathbb{Z}[i]$ are ± 1, $\pm i$. Then $a - ib = \gamma(a+ib)$, where $\gamma \in \{1-1, i, -i\}$. If $\gamma = 1$, $p = a^2$, if $\gamma = -1$, $p = b^2$; and if $\gamma = \pm i$, $p = 2a^2$. None of these is a possibility as p is an odd prime. Thus π, $\bar{\pi}$ are not associates, and$p = \pi\bar{\pi}$, with π Gaussian prime of norm p. When $p = 2$, we have $2 = N(1+i) = (1+i)(1-i)$, and $1 - i = -i(1+i)$.

We have yet to show that an odd rational prime p is **not** a Gaussian prime precisely when $p \equiv 1 \pmod 4$. But we have just shown that p must be of the form $N(\pi)$ for a Gaussian prime π when p is not itself a Gaussian prime. Then $\exists x, y \in \mathbb{Z}$ such that

$$p = x^2 + y^2.$$

As we have seen in the previous section, this implies, as derived, that $p \equiv 1 \pmod 4$. But we can also check this directly. Modulo 4, the square of any integer must be 0 or 1. Then $p = x^2 + y^2$ must be 0 or 1 mod 4. Since p is odd, it must be 1 mod 4.

Now let π be any Gaussian prime, which is not in $\mathbb{Q}$. We have to show that $N(\pi) = p$ with $p \equiv 1 \pmod 4$ or $p = 2$. Since $N(\pi)$ is an integer ≥ 1, and since $N(\pi)$ cannot be 1 as π is not a unit, there must be some (rational) prime q dividing $N(\pi)$. Write $N(\pi) = q_1 q_2 \dots q_r$, with each q_j a rational prime. Now since $N(\pi) = \pi\bar{\pi}$, and since π is a Gaussian prime, viewing $\pi\bar{\pi} = q_1 q_2 \dots q_r$ as an equation in $\mathbb{Z}[i]$, we see that π must divide some q_j, call it p. By what we proved above, p must be the norm of some Gaussian prime π_1. Then π divides $p = \pi_1\overline{\pi_1}$. So π must divide π_1 or $\overline{\pi_1}$, say it divides π_1. Then $\pi \sim \pi_1$, and we will have $p = u\pi\bar{\pi}$, for some unit u. But both p and $\pi\bar{\pi}$ are real and positive, so u must be 1. The rest is clear.

19 Sums of Four Squares

The following result of Lagrange is surprising at first; it had been predicted earlier, i.e., before Lagrange, by Fermat.

Theorem (Lagrange) Every positive integer n is a sum of four squares.

Proof. We may take n to be > 1, as $1 = 0^2 + 0^2 + 0^2 + 1^2$. We need two lemmas.

Lemma 1 Let $x_j, y_j \in \mathbb{Z}$, with $1 \geq j \geq 4$. Then

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_1^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2,$$

where $z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, $z_2 = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$, $z_3 = x_1y_3 + x_3y_1 + x_4y_2 + x_2y_4$, and $z_4 = x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2$.

The checking of this, called Euler's identity, is straightforward and is left as an exercise.

Lemma 2 Let p be an odd prime. Then $\exists$ integers x, y, m, with $1 \leq m < p$, such that $mp = x^2 + y^2 + 1$.

Proof of Lemma 2: Put $T = \{1 \leq j \leq \frac{p-1}{2}\}$. Then we have seen earlier that the squares of elements of T are pairwise unequal, i.e., the set $T_1 = \{x^2 | x \in T\}$ has cardinality $\frac{p-1}{2}$. But the set $T_2 = \{-1 - y^2 | y \in T\}$ has the same property. As p is odd, there are exactly $\frac{p-1}{2}$ squares in $(\mathbb{Z}/p)^*$. So we must have $T_1 \equiv T_2 \pmod p$. Consequently, $\exists x, y \in T$ such that $x^2 + y^2 + 1 = mp$, for some integer m, which is evidently ≥ 1. Moreover, as $x, y \in T$, x^2, y^2 are bounded from above by $\frac{p^2}{4}$. Hence

$$mp = x^2 + y^2 + 1 < \frac{p^2}{2} + 1 < p^2,$$

which implies that $m < p$.

This proves the Lemma.

Proof of Lagrange's theorem (cont.)

Thanks to Lemma 1, and the fact that $2 = 1^2 + 1^2 + 0^2 + 0^2$, it suffices to prove the Theorem for odd primes. Pick any odd prime p. Let m_0 be the smallest integer with $1 \leq m_0 < p$ such that

$$m_0 p = x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{1}$$

for some $x_1, x_2, x_3, x_4 \in \mathbb{Z}$. By Lemma 2, $\exists$ such an m_0. If $m_0 = 1$ we are done, so **assume not** and derive a contradiction.

Claim: m_0 is odd.

Proof of Claim Suppose m_0 is even. Then either all the x_i's are even or all of them are odd, or exactly half of them are odd. In the third case we may, after renumbering the x_j, assume that x_1, x_2 are even and x_3, x_4 are odd. It follows from (1) that

$$\frac{m_0 p}{2} = \left(\frac{x_1 + x_2}{2}\right) + \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{x_3 + x_4}{2}\right)^2 + \left(\frac{x_3 - x_4}{2}\right)^2 \tag{2}$$

Then as $\frac{x_1 \pm x_2}{2}$ and $\frac{x_3 \pm x_4}{2}$ are integers, we get a contradiction to the minimality of m_0. Hence the claim.

Proof of Theorem (cont.)

So m_0 is odd and ≥ 3. Let us write $(\forall j)$

$$x_j = y_j + a_j m_0, \tag{3}$$

with $a_j \in \mathbb{Z}$ chosen such that $|y_j| < \frac{m_0}{2}$ (check that this can be done; the oddness of m_0 is essential).

Since $m_0 < p$, not all the x_j can be divisible by m_0. Consequently,

$$\sum_{j=1}^{4} y_j^2 > 0 \tag{4}$$

We also have

$$\sum_{j=1}^{4} y_j^2 < 4\left(\frac{m_0}{2}\right)^2 = m_0^2. \tag{5}$$

But (1) + (3) implies that

$$\sum_{j=1}^{4} y_j^2 \equiv 0 \pmod{m_0}. \tag{6}$$

This means we have

$$0 < y_1^2 + y_2^2 + y_3^2 + y_4^2 = km_0, \tag{7}$$

with $0 < k < m_0$.

Applying Lemma 1 with the z_j defined by the $x_j + y_j$, we get

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = m_0^2 kp. \tag{8}$$

But

$$z_1 = \sum_{j=1}^{4} x_j y_j = \sum_{j=1}^{4} x_j(x_j - a_j m_0) \equiv \sum_{j=1}^{4} x_j^2 \equiv 0 \pmod{m_0}$$

Similarly, $z_2 \equiv z_3 \equiv z_4 \equiv 0 \pmod{m_0}$. So $z_j = m_0 w_j$, with $w_j \in \mathbb{Z},\ \forall j \leq 4$. Substituting this in (8) we get

$$kp = w_1^2 + w_2^2 + w_3^2 + w_4^2, \tag{9}$$

with $1 \leq k < m_0 < p$.

Since this contradicts the minimality of m_0, we get the derived contradiction.

20 Approximation by rationals (Diophantine approximation)

We all know that real numbers can be approximated by the rationals $\frac{p}{q}$ (with $(p, q) = 1$). But how well can this be done? And does it depend on the nature of x? We will try to answer these questions now.

For any $x \in \mathbb{R}$, let $[x]$ denote its integral part.

Theorem 1 (Dirichlet) Let x be an irrational number. There $\exists$ infinitely many rationals $\frac{p}{q}$ (with $(p,q)=1$) such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Remark 1: This theorem is false for the rationals. Indeed, suppose4 $x = \frac{a}{b}$ and $\frac{p}{q} \neq \frac{a}{b}$ a rational with $|q| > |b|$. Then $|x - \frac{p}{q}| = |\frac{aq-bp}{bq}| \geq \frac{1}{bq} > \frac{1}{q^2}$. Let $I = [[x]-1, [x]+1]$. If $\frac{p}{q} \notin I$, then clearly, $|x - \frac{p}{q} > 1$. We are done now, because the number of $\frac{p}{q}$ in I with $|q| \leq |b|$ is finite.

Proof of Theorem 1. This is done by first establishing the following result, also due to Dirichlet:

Proposition Let $x, t \in \mathbb{R}$ with $t > 1$. Then $\exists p, q \in \mathbb{Z}$ such that $1 \leq q < t$ and $|qx - p| \leq \frac{1}{t}$.

Proof of Proposition Let $\{x\}$ denote the fractional part of x, lying in $[0,1)$. Suppose t is an integer > 1. Then the $t+1$ numbers $0, 1, \{x\}, \{2x\}, \ldots, \{(t-1)x\}$ lie in $[0,1]$, and so the difference between some pair among these must be in absolute value bounded by t^{-1}. Then $\exists m_1, m_2, n_1, n_2 \in \mathbb{Z}$ with $0 \leq m_i \leq t-1,\ i = 1,2$, and $m_1 \neq m_2$, such that $|(m_1 x - n_1) - (m_2 x - n_2)| \leq \frac{1}{t}$. We may assume that $m_1 > m_2$. Then the Proposition is satisfied by taking $p = n_1 - n_2$ and $q = m_1 - m_2$. Done if $t \in \mathbb{Z}$. Suppose $t \notin \mathbb{Z}$. Then $t' = [t]+1$ is an integer > 1, and $\exists p, q$ with $1 \leq q < t'$ and $|qx - p| \leq \frac{1}{t'}$. Evidently we have: $1 \leq q < t$ and $|qx - p| < \frac{1}{t'}$. Hence the proposition.

Proposition $\Rightarrow$ Theorem 1: Since x is irrational, the bound $|qx - p| \leq \frac{1}{t}$ can hold, for a fixed (p,q), only for bounded values of t, say for $t \leq t + - = t_0(p,q)$. Hence, as $t \to \infty$, there will be infinitely many distinct coprime integers (p,q) as in Proposition, giving rise to infintely many $\frac{p}{q}$ satifying $|x - \frac{p}{q}| < \frac{1}{q^2}$.

Theorem 2 (Hurwitz) Let x be irrational. Then $\exists$ infinitely many $\frac{p}{q}$ with $(p,q) = 1$ such that $|x - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$.

Remark 2: Hurwitz's theorem is the best possible. Indeed, suppose x is a real **quadratic** irrational and suppose $\exists$ infinitely many $\frac{p}{q} \in \mathbb{Q}$ with $(p,q) = 1$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{Cq^2} \tag{*}$$

for some $C > 1$. Let $f(X) = aX^2 + bX + c$ be the integral polynomial with root x. Then $f(X) = a(X - x)(X - x')$ (over $\mathbb{R}$) where x' is the conjugate root. For every $\frac{p}{q}$ satisfying $(*)$ we have

$$\frac{q}{q^2} \leq \left| f\left(\frac{p}{q}\right)\right| = \left| x - \frac{p}{q}\right| \cdot \left| a\left(x' - \frac{p}{q}\right)\right| < \frac{1}{Cq^2}\left| a\left(x' - x + x - \frac{p}{q}\right)\right|$$
$$< \frac{\sqrt{D}}{Cq^2} + \frac{|a|}{C^2q^4},$$

where $D = b^2 - 4ac = a^2(x - x')^2$. It follows that $C \leq \sqrt{D}$. In the special case when $x = \frac{\sqrt{5}-1}{2}$, $f(x) = x^2 + x - 1$, we have $D = 5$, and so $C \leq \sqrt{5}$.

Definition $x \in \mathbb{R}$ is an **algebraic number** iff $\exists\ f(X) \in \mathbb{Q}(X)$ such that $f(x) = 0$. It is **transcendental** if it is not algebraic.

Fact π, e are not **algebraic**.

Definition An algebraic number x has degree d if d is the minimum of the degrees of polynominals $f(x)$ such that $f(x) = 0$.

Theorem 3 (Liouville) Suppose x is a real algebraic number of degree d. Then $\exists\ c = c(x) > 0$ such that

$$\left| x - \frac{p}{q}\right| > \frac{c(x)}{q^d},$$

for all rational numbers $\frac{p}{q} \neq x$, with $(p, q) = 1$.

Corollary: $\alpha := \sum_{n\geq 1} \frac{1}{2^{n!}}$ is transcendental.

Indeed, let us put $p_m = 2^{m!} \sum_{n=1}^{m} \frac{1}{2^{n!}}$ and $q_m = 2^{m!}$. Then

$$\left| x - \frac{p_m}{q_m}\right| = \sum_{n>m} \frac{1}{2^{n!}} < \frac{2}{2^{(m+1)!}} = \frac{2}{q_m^{m+1}}$$

Hence for any d and any constant $c > 0$ we have

$$\left| x - \frac{p_m}{q_m}\right| < \frac{c}{q_m^d}$$

for all large enough m.

So x can't be algebraic of any degree d. Done.

Proof of Theorem 3 Let $f(x)$ be the (minimal) polynominal of x with deg $f = d_0$, coprime coefficients, and positive leading coefficient. Taylor's formula gives

$$\left| f\left(\frac{p}{q}\right) \right| = \left| \sum_{n=1}^{d} \left(\frac{p}{q} - x\right)^n \frac{1}{n!} f^{(n)}(x) \right| < \frac{1}{c(x)} \left| \frac{p}{q} - x \right|$$
$$\text{if } \left| \frac{p}{q} - x \right| \leq 1.$$

Let $\frac{p}{q}$ be a rational such that $\frac{p}{q} \neq x$. Then $f(\frac{p}{q}) \neq 0$ by the minimality of f. So $|f(\frac{p}{q})| \leq \frac{1}{q^d}$. So we get

$$\frac{1}{q^d} < \frac{1}{c(x)} \left| \tfrac{p}{q} - x \right|$$

if $|\frac{p}{q} - x| \leq 1$. The Theorem is of course obvious if $|\frac{p}{q} - x| > 1$.