4 Pythagorean Triples

Problem:
Find all \(x, y \in \mathbb{N}\) such that
\[
x^2 + y^2 = z^2 \tag{1}
\]

If \(d = (x, y, z) > 1\), then \((\frac{x}{d}, \frac{y}{d}, \frac{z}{d})\) is another solution, called the **primitive solution**.

For primitive solutions, we may assume that \(x\) is odd and \(y\) is even.

**The Geometric Method**

Solving (1) in integers amounts to solving the following in rational numbers:
\[
X^2 + Y^2 = 1 \tag{2}
\]

Geometrically, (2) is the equation of the unit circle in \(\mathbb{R}^2\) with center at \(O = (0, 0)\). Try to parametrize the circle.

One can try as in calculus to set
\[
X = \cos \theta, \quad Y = \sin \theta.
\]

This turns out to be *terrible* for number theory. A better way is to consider the parametrization
\[
X = \frac{1 - t^2}{1 + t^2}, \quad Y = \frac{2t}{1 + t^2}
\]

This is *ingenious* as this only involves rational functions. If \(t \in \mathbb{Q}\), then \(X, Y \in \mathbb{Q}\). Of course
\[
X^2 + Y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2} = 1
\]

As \(t \to \infty\) (along rationals) then
\[
X = \frac{1 - t^2}{1 + t^2} \to -1
\]

So we are only missing one solution, \((-1, 0)\), which we will remember.

**Check:** If \(X, Y \in \mathbb{Q}\), then \(t \in \mathbb{Q}\). (Show: \(t = \frac{Y}{1+X}\).)
So the rational solutions of (2) are obtained by setting

\[ X = \frac{1 - t^2}{1 + t^2}, \quad Y = \frac{2t}{1 + t^2}, \]

with \( t \neq \pm 1, 0 \), together with \((\pm 1, 0)\) and \((0, \pm 1)\).

Write \( t = \frac{u}{v} \), \( u, v \in \mathbb{Z} \). Then

\[ X = \frac{u^2 - v^2}{u^2 + v^2}, \quad Y = \frac{2uv}{u^2 + v^2} \]

It follows that the non-zero solutions in \( \mathbb{Z} \) of (1) are given by

\[ x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2 \]

with

\[ u \neq \pm v, \quad u, v \neq 0 \]

To get primitive solutions, it is convenient to put

\[ m = u + v, \quad n = u - v \]

\[ x = (u + v)(u - v) = mn, \quad y = \frac{m^2 - n^2}{2}, \quad z = \frac{m^2 + n^2}{2} \]

For primitive solutions, take \( m, n \) odd \( \geq 1 \), \( m > n \). Check that these are all the primitive solutions.