4 Pythagorean Triples

Problem:

Find all $x, y \in \mathbb{N}$ such that

$$x^2 + y^2 = z^2 \tag{1}$$

If $d = (x, y, z) > 1$, then $(\frac{x}{d}, \frac{y}{d}, \frac{z}{d})$ is another solution, called the **primitive solution**.

For primitive solutions, we may assume that x is odd and y is even.

The Geometric Method

Solving (1) in integers amounts to solving the following in rational numbers:

$$X^2 + Y^2 = 1 \tag{2}$$

Geometrically, (2) is the equation of the unit circle in $\mathbb{R}^2$ with center at $O = (0, 0)$. Try to parametrize the circle.

One can try as in calculus to set

$$X = \cos\theta, \ Y = \sin\theta.$$

This turns out to be *terrible* for number theory. A better way is to consider the parametrization

$$X = \frac{1 - t^2}{1 + t^2}, \quad Y = \frac{2t}{1 + t^2}$$

This is *ingenious* as this only involves rational functions. If $t \in \mathbb{Q}$, then $X, Y \in \mathbb{Q}$. Of course

$$X^2 + Y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2} = 1$$

As $t \to \infty$ (along rationals) then

$$X = \frac{1 - t^2}{1 + t^2} \to -1$$

So we are only missing one solution, $(-1, 0)$, which we will remember.

Check: If $X, Y \in \mathbb{Q}$, then $t \in \mathbb{Q}$. (Show: $t = \frac{Y}{1+X}$.)

So the rational solutions of (2) are obtained by setting

$$X = \frac{1-t^2}{1+t^2}, \quad Y = \frac{2t}{1+t^2},$$

with $t \neq \pm 1, 0$, together with $(\pm 1, 0)$ and $(0, \pm 1)$.

Write $t = \frac{u}{v},\ u, v \in \mathbb{Z}$. Then

$$X = \frac{u^2-v^2}{u^2+v^2}, \quad Y = \frac{2uv}{u^2+v^2}$$

It follows that the non-zero solutions in $\mathbb{Z}$ of (1) are given by

$$x = u^2 - v^2,\ y = 2uv,\ z = u^2 + v^2$$

with

$$u \neq \pm v,\ u, v \neq 0$$

To get primitive solutions, it is convenient to put

$$m = u + v,\ n = u - v$$

$$x = (u+v)(u-v) = mn,\ y = \frac{m^2-n^2}{2},\ z = \frac{m^2+n^2}{2}$$

For primitive solutions, take m, n odd $\geq 1,\ m > n$. Check that these are all the primitive solutions.