2 Heuristics on Primes

Let $P = \{\text{primes in } \mathbb{Z}\}$. We saw two proofs of the fact that P is infinite.

Prime Number Theorem (PNT). If $\pi(x) = \#\{p \in P | p \leq x\}$ then $\pi(x) \sim \frac{x}{\log x}$ for x large.

Heuristic reason: Let $F(x) =$ the fraction of positive integers $\leq x$ which are prime. Then $F(x) = \frac{\pi(x)}{x}$. We want to take all $n \leq x$ and then throw out composite numbers. First throw out even numbers, i.e., those divisible by 2.

$$\left\{\begin{matrix}\text{fraction of odd numbers} \\ \text{which are } \leq x\end{matrix}\right\} \sim \frac{1}{2} = \left(1 - \frac{1}{2}\right)$$

$$\text{fraction of numbers which are not divisible by 3} \sim \left(1 - \frac{1}{3}\right)$$

We get

$$F(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)$$

In fact, we should use the bound $\sqrt{x}$ for better accuracy. This way we are off by a factor of 2.

Recall Euler's result:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim \sum_{n \leq x} \frac{1}{n} \sim \int_1^x \frac{1}{t} dt = \log x$$

Consequently,

$$F(x) \sim \frac{1}{\log x}, \text{ and so } \pi(x) \sim \frac{x}{\log x}$$

Twin primes

We are looking for numbers n such that n and $n+2$ are prime.

Put

$$\pi_2(x) = |\{\text{twin primes } \leq x\}|$$

A heuristic argument:

Put

$$F_2(x) = \frac{\pi_2(x)}{x}$$

Again, take all $n \leq x$ and throw out numbers which are not twin primes.

Check:

$$F_2(x) \approx \prod_{p \leq x} \left(1 - \frac{2}{p}\right) \approx \frac{1}{\log^2 x}$$

So one expects:

$$\pi_2(x) \approx \frac{x}{\log^2 x} \quad \leftarrow \text{Not yet proved!}$$