

13 RSA Encryption

The mathematics behind the very successful RSA encryption method is very simple and uses mainly Euler's congruence for any $N \geq 1$:

$$b^{\varphi(N)} \equiv 1 \pmod{N}$$

if $(b, N) = 1$. (When N is a prime, this is Fermat's little theorem.)

Imagine that a person X wants to send a carefully encrypted message to another person Y , say. X will look in a directory which publishes the *public key* of various people including Y . The public key of Y will be a pair (e, N) of positive integers, where N will be a large number which is a product of 2 distinct primes p and q . The point is that the directory will contain no information on the factorization of N . For large enough N it will become impossible (virtually) to factor N . The number e will be chosen mod N and it will be prime to $\varphi(N)$.

The person X will first represent his/her *plain text* message by a numeral a (which can be done in many ways). For simplicity, suppose that a is prime to N . X will then raise a to the power e mod N and send the message as b . So

$$b \equiv a^e \pmod{N}.$$

If someone intercepts the message, he or she will be unable to recover a from b without knowing the factorization of N . So it is secure. On the other hand, the recipient of the message, namely Y , will be able to decode (decrypt) the message as follows. He/she will pick a number d (*decryption constant*) such that

$$de \equiv 1 \pmod{(p-1)(q-1)}.$$

Y can do this because he/she knows the prime factors p, q and because e is prime to $\varphi(N)$; observe that since p and q are distinct primes and $N = pq$, one has

$$\varphi(N) = \varphi(p)\varphi(q) = (p-1)(q-1).$$

So by applying Euler's congruence mod N , we get

$$b^d \equiv a^{ed} \equiv a^{1+c(p-1)(q-1)} \equiv a \pmod{N}.$$

Thus Y recovers a .

Note that if someone does not have the factorization of N , he/she cannot decrypt the message.