1 Mersenne Primes and Perfect Numbers

Basic idea: try to construct primes of the form $a^n - 1$; $a, n \geq 1$. e.g.,
$2^1 - 1 = 3$ but $2^4 - 1 = 3 \cdot 5$
$2^3 - 1 = 7$
$2^5 - 1 = 31$
$2^6 - 1 = 63 = 3^2 \cdot 7$
$2^7 - 1 = 127$
$2^{11} - 1 = 2047 = (23)(89)$
$2^{13} - 1 = 8191$

Lemma: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$

Corollary: $(x-1)|(x^n - 1)$

So for $a^n - 1$ to be prime, we need $a = 2$.
Moreover, if $n = md$, we can apply the lemma with $x = a^d$. Then

$$(a^d - 1)|(a^n - 1)$$

So we get the following

Lemma If $a^n - 1$ is a prime, then $a = 2$ and n is prime.

Definition: A *Mersenne prime* is a prime of the form

$$q = 2^p - 1, \ p \text{ prime.}$$

Question: are they infinitely many Mersenne primes?
Best known: The 37th Mersenne prime q is associated to $p = 3021377$, and this was done in 1998. One expects that $p = 6972593$ will give the next Mersenne prime; this is close to being proved, but not all the details have been checked.
Definition: A positive integer n is *perfect* iff it equals the sum of all its (positive) divisors $< n$.

Definition: $\sigma(n) = \sum_{d|n} d$ (divisor function)

So u is perfect if $n = \sigma(u) - n$, i.e. if $\sigma(u) = 2n$.
Well known example: $n = 6 = 1 + 2 + 3$
Properties of σ:

1. $\sigma(1) = 1$

2. n is a prime *iff* $\sigma(n) = n + 1$

3. If p is a prime, $\sigma(p^j) = 1 + p + \cdots + p^j = \frac{p^{j+1}-1}{p-1}$

4. (Exercise) If $(n_1, n_2) = 1$ then $\sigma(n_1)\sigma(n_2) = \sigma(n_1 n_2)$ "multiplicativity".

Consequently, if

$$n = \prod_{j=1}^{r} p_i^{e_j}, \; e_j \geq 1 \;\; \forall j, \; p_j \text{ prime},$$

$$\sigma(n) = \prod_{j=1}^{r} \sigma(p_j^{e_j}) = \prod_{j=1}^{r} \left(\frac{p^{e_j+1} - 1}{p - 1} \right)$$

Examples of perfect numbers: $\begin{cases} 6=1+2+3 \\ 28=1+2+4+7+14 \\ 496 \\ 8128 \end{cases}$

Questions:

1. Are there infinitely many perfect numbers?

2. Is there any odd perfect number?

Note:
6=(2)(3), 28=(4)(7), 496=(16)(31), 8128=(64)(127)
They all look like

$$2^{n-1}(2^n - 1),$$

with $2^n - 1$ prime (i.e., Mersenne).

Theorem (Euler) Let n be a positive, *even* integer. Then

$$n \text{ is perfect} \Leftrightarrow n = 2^{p-1}(2^p - 1), \text{ for a prime } p, \text{ with } 2^p - 1 \text{ a prime.}$$

Corollary. There exists a bijection between even perfect numbers and Mersenne primes.

Proof of Theorem. ($\Leftarrow$) Start with $n = 2^{p-1}q$, with $q = 2^p - 1$ a Mersenne prime. To show: n is perfect, i.e., $\sigma(n) = 2n$. Since $2^{p-1}q$, and since $(2^{p-1}, q) = 1$, we have

$$\sigma(n) = \sigma(2^{p-1})\sigma(q) = (2^p - 1)(q + 1) = q2^p = 2n.$$

($\Rightarrow$): Let n be a even, perfect number. Since n is even, we can write

$$n = 2^j m, \text{ with } j \geq 1, \; m \text{ odd } \neq n$$

.

$$\Rightarrow \sigma(n) = \sigma(2^j)\sigma(m) = (2^{j+1} - 1)\sigma(m)$$

Since n is perfect,

$$\sigma(n) = 2n = 2^{j+1}m$$

Get

$$2^{j+1}m = \underbrace{(2^{j+1} - 1)}_{\text{odd}}\sigma(m)$$

$\Rightarrow$

$$2^{j+1} | \sigma(m);$$

so

$$r2^{j+1} = \sigma(m) \tag{1}$$

for some $r \geq 1$

Also

$$2^{j+1}m = (2^{j+1} - 1)r2^{j+1},$$

so

$$m = (2^{j+1} - 1)r \tag{2}$$

Suppose $r > 1$. Then

$$m = (2^{j+1} - 1)r$$

will have $1, r$ and m as 3 distinct divisors. (Explanation: by hypothesis, $1 \neq r$. Also, $r = m$ iff $j = 0$ iff $n = m$, which will then be odd!)
Hence

$$\begin{aligned} \sigma(m) \geq & 1 + r + m \\ = & 1 + r + (2^{j+1} - 1)r \\ = & 1 + 2^{j+1}r \\ = & 1 + \sigma(m) \end{aligned}$$

Contradiction!

So $r = 1$, and so (1) and (2) become

$$\sigma(m) = 2^{j+1} \tag{1'}$$

$$m = 2^{j+1} - 1 \tag{2'}$$

Since $n = 2^j m$, we will be done if we prove that m is a prime. It suffices to show that $\sigma(m) = m + 1$. But this is clear from (1') and (2').

$M_n = 2^n - 1$ Mersenne number. Define numbers S_n recursively by setting $S_n = S_{n-1}^2 - 2$, and $S_1 = 4$.

Theorem: (Lucas-Lehmer Primality Test) Suppose for some $n \geq 1$ that M_n divides S_{n-1}. Then M_n is prime.

Proof. (Very clever) Put $\alpha = 2 + \sqrt{3}$, $\beta = 2 - \sqrt{3}$. Note that $\alpha + \beta = 4$, $\alpha\beta = 1$. So $S_1 = \alpha + \beta$.

Lemma. For any $n \geq 1$, $S_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}}$.

Proof of Lemma: $n = 1:$ $S_1 = \alpha + \beta = 4$. So let $n > 1$, and assume that the lemma holds for $n - 1$. Since

$$S_n = S_{n-1}^2 - 2$$

we get (by induction)

$$S_n = (\alpha^{2^{n-1}} + \beta^{2^{n-1}})^2 - 2$$

Note:

$$\begin{aligned}(\alpha^k + \beta^k)^2 &= \alpha^{2k} + 2\alpha^k\beta^k + \beta^{2k} \\ &= \alpha^{2k} + \beta^{2k} + 2, \text{ as } \alpha\beta = 1.\end{aligned}$$

So we get (setting $k = 2^{n-2}$)

$$S_n = \alpha^{2^{n-1}} + \beta^{2^{n-1}} + 2 - 2.$$

Hence the lemma.

Proof of Theorem (continued): Suppose $M_n | S_{n-1}$. Then we may write $rM_n = S_{n-1}$, some positive integer. By the lemma, we get

$$rM_n = \alpha^{2^{n-2}} + \beta^{2^{n-2}} \tag{3}$$

Multiply (3) by $\alpha^{2^{n-2}}$ and subtract 1 to get:

$$\alpha^{2^{n-1}} = rM_n\alpha^{2^{n-2}} - 1 \tag{4}$$

Squaring (4) we get

$$\alpha^{2^n} = (rM_n\alpha^{2^{n-2}} - 1)^2 \tag{5}$$

Suppose M_n is not a prime. Then $\exists$ a prime ℓ dividing M_n, $\ell \leq \sqrt{M_n}$. Let us work in the number system

$$R = \{a + b\sqrt{3} | a, b \in \mathbb{Z}\}$$

Check: R is closed under addition, subtraction, and multiplication (it is what one calls a ring). Equations (4) and (5) happen in R. Define $R/\ell = \{a, b\sqrt{3} | a, b \in \mathbb{Z}/\ell\}$.

Note: $|R/\ell| = \ell^2$

We can view α, β as elements of R/ℓ. Since $\ell | M_n$, (4) becomes the following congruence in R/ℓ:

$$\alpha^{2^{n-1}} \equiv -1 \ (mod\ \ell) \tag{6}$$

Similarly, (5) says

$$a^{2^n} \equiv 1 \ (mod\ \ell)$$

Put

$$X = \{\alpha^j \bmod \ell | 1 \leq j \leq 2^n\}.$$

Claim $|X| = 2^n$.

Proof of claim. Suppose not. Then $\exists j, k$ between 1 and 2^n, with $j \neq k$, such that $\alpha^j \equiv \alpha^k \pmod{\ell}$.

If r denotes $|j - k|$, then $0 < r < 2^n$ and $\alpha^r \equiv 1 \pmod{\ell}$. Let d denote the gcd of r and 2^n, so that $ar + b2^n = d$ for some $a, b \in \mathbb{Z}$. Then we have

$$\alpha^d = \alpha^{ar+b2^n} = (\alpha^r)^a \cdot (\alpha^{2^n})^b \equiv 1 \pmod{\ell}.$$

But since $d | 2^n$, d is of the form 2^m for some $m < n$, and $\alpha^d \equiv 1 \pmod{\ell}$ contradicts $\alpha^{2^{n-1}} \equiv -1 \pmod{\ell}$. Hence the claim.

So $|X| \leq \ell^2 - 1$, i.e., we need $2^n \leq \ell^2 - 1$.

Since

$$\ell \leq \sqrt{M_n}, \ \ell^2 - 1 < M_n = 2^n - 1.$$

$\Rightarrow 2^n < 2^n - 1$, a contradiction!

So M_n is prime.