LAPLACIAN PATH MODELS

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1. Introduction and results

1.1. **Laplacian growth.** In this paper we study two growth models in the complex plane – the needle and the geodesic η-models, defined below.

Let $\{K_t\}$, $t \geq t_0$, be a growing family of connected sets: K_{t_0} is the initial configuration, and $K_s \subset K_t$ for $s < t$. The growth is localized at a finite number of points $a_j(t)$, $1 \leq j \leq d$, so that $K_t \setminus K_{t_0}$ consists of d disjoint smooth Jordan arcs (”*arms*”) from K_{t_0} to $a_j(t)$; the points $a_j(t)$ are the *”tips”* of the arms at time t. We call such a family $\{K_t\}$ a *radial chain with d arms*. We will also consider the family of domains $\Omega_t = \hat{\mathbb{C}} \setminus K_t$ and the (Loewner) chain of the Riemann maps

$$\varphi_t : \Delta \equiv \{|z| > 1\} \to \Omega_t, \qquad (\infty \mapsto \infty, \quad \varphi_t'(\infty) > 0). \tag{1.1}$$

The dynamics of the tips $a_j(\cdot)$ is described in terms of the Laplacian fields ∇G_t, where $G_t = \log|\varphi_t^{-1}|$ is the Green function of Ω_t with pole at infinity. Let $\zeta_j(t) \in \partial\Delta$ be the points (called *poles*) corresponding to the tips :

$$\varphi_t : \zeta_j(t) \mapsto a_j(t). \tag{1.2}$$

The common assumption for the two models will be the following: the velocities of the tips at time t are proportional to some fixed power η of the magnitude of the field ∇G_t, so

$$|\dot{a}_j(t)| \ \div \ |\varphi_t''(\zeta_j(t))|^{-\eta}. \tag{1.3}$$

The notation $x_j \div y_j$ means $[x_1 : x_2 : \cdots : x_d] = [y_1 : y_2 : \cdots : y_d]$. Since the Laplacian field is infinite at the tips, i.e. $\varphi_t'(\zeta_j(t)) = 0$, we take second derivatives in (1.3). We will refer to (1.3) as the *η-equation*; in the literature it is sometimes called the Laplacian growth equation with parameter η. If $\eta = 1/2$, then (1.3) means $|\dot{a}_j| \div \omega_j$, where ω_j's are harmonic measures of infinitesimally small arcs of equal length at the tips of the arms. Even if one is only interested in this particular case, it is often helpful, as we will see later, to consider other values of η.

In addition to velocities, we need to specify the geometry of the arms.

Needle model. A set K is a *needle configuration* if it consists of several straight line segments (”*needles*”) with a common endpoint at the origin. A (radial) *needle* or *starlike η-process* is a chain of needle configurations satisfying the η-equation. Any such process is related to a set of d fixed directions; the tip points move along the fixed rays with velocities given by (1.3). The process is then described (up to a time change) by an autonomous system of ordinary differential equations in a phase

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space of dimension $d-1$, and so the initial lengths of the needles determine the evolution uniquely.

Geodesic model. A radial chain $\{K_t\}$ is called *geodesic* if the tips move along the field lines. More precisely, it is required that at a point $a_j(t)$, the j-th arm be second order tangential to the hyperbolic geodesic (external ray) from $a_j(t)$ to ∞ in Ω_t. A *geodesic η-process* is a geodesic chain satisfying the η-equation. The evolution is determined by the initial configuration because the geodesic condition prescribes the curvature of the arms. It is not difficult to prove an appropriate global existence theorem using, for instance, an approach with Loewner's equation.

The above definitions do not depend on time parametrization. We will usually identify chains which differ only by some time change.

The main theme of this paper will be stability analysis of stationary solutions. A chain $\{K_t\}$ is called *stationary* if $K_t = \lambda(t)K_{t_0}$ for some positive increasing function $\lambda(t)$, which means that the shape of configurations K_t does not change in time. It is clear that stationary chains are starlike, and therefore they are in one-to-one correspondence (up to a time change) with needle configurations. We will use the term *stationary solution* for stationary η-processes as well as for the corresponding needle configuration.

1.2. **Needle dynamics.** The needle η-model with $\eta = 1/2$ is similar to a model suggested independently by Meakin [8] and Rossi [10] as a simplified, non-branching version of the DLA process, see also [7]. In [10], one considers the strip $\{0 \le x \le N, \quad y \ge 0\}$ on the lattice $\mathbb{Z}^2$, and identifies the sides $x = 0$ and $x = L$. The initial N trapping sites are located at the bottom $\{y = 0\}$ of the strip. Random walkers are launched, one at a time, from a site chosen at random near $y = \infty$. When a walker arrives at a trapping site, it occupies it and becomes part of the aggregate; the site immediately above the one where the walker has landed becomes a trapping site. In this way vertical needles are grown. One disregards the walker if it reaches a site already occupied. (If the aggregation on the sides of the needles is allowed, we would just get an ordinary DLA process [14].)

One can also consider a *radial* version of the Meakin-Rossi model so that the needles are grown along the straight lines which pass through the origin at equal angles; each time the length of a needle is increased by 1. Initially many needles compete to trap the walkers. Then more and more needles are left behind and (almost) stop growing while the competition continues between the longer needles until only few needles survive.

The initial, stochastic stage of the Meakin-Rossi model is perhaps the most interesting. We can give a non-trivial (but not an optimal) bound for the doubling time; this will appear elsewhere. In this paper we rather concentrate on the long term behavior, and replace the stochastic mechanism with a differential equation thus obtaining the starlike η-model with η=1/2. The dynamics turns out to be quite simple.

Theorem 1. *Consider a starlike η-process and let $l_j(t)$ denote the lengths of the needles. Assume $\sum_k l_k(t) \to \infty$. Then there exist limits*

$$l_j^* = \lim_{t\to\infty} \frac{l_j(t)}{\sum_k l_k(t)},$$

and the limiting configuration (with lengths l_j^*) is a stationary solution.

Note that some (or most) of the lengths l_j^* can be zero.

By definition, a stationary solution of the starlike η-model is (asymptotically) stable if it is (asymptotically) stable for the system of differential equations describing the model. Geometrically, stability means that if we slightly change the initial configuration K_{t_0} of a stationary solution without changing the directions of the needles, then the shape of configurations K_t does not change much in the course of the process. One can show that the limiting configurations arising in the Meakin-Rossi model have to be stable.

In the case of d-fold symmetry of directions and $\eta = 1/2$, it is claimed in [6] that five needles of equal length are asymptotically stable, and that six needles of equal length are stable as well as six-needle configurations with long needles alternating with short ones. In general, see Example 1 in Section 3.2, the symmetric configuration with d needles is asymptotically stable with respect to the starlike η-model if $\eta < 2(d-2)^{-1}$ and d is even, or if $\eta < 2d(d^2-2d-1)^{-1}$ and d is odd. For example, three symmetric needles are asymptotically stable for $\eta < 3$, four symmetric needles are asymptotically stable for $\eta < 1$, five needles for $\eta < 5/7$, six needles for $\eta < 1/2$ etc. The case of non-symmetric stationary configurations is less studied. Some examples can be found in Section 4 of [12].

If a needle configuration is a stationary solution of the *geodesic* η-model, then it is also a solution of the starlike model, and so the term "stability" may have two meanings. To avoid confusion, we will sometimes use the terms *G-stability* and $*$-*stability* respectively. In all known examples, G-stability is a stronger property, though we don't have a proof of this fact in the general case. We also don't know if the following statement holds for all $\eta > 0$.

Theorem 2. *If $\eta > 1/2$, then the number of needles in a $*$-stable stationary solution of the geodesic η-model is bounded by a finite constant depending only on η.*

1.3. Geodesic model.

The geodesic model arises as a limiting case ($\delta \to 0$) of the following version of the radial Meakin-Rossi model, see Sections 3.1–3.2 of [12]: instead of prolonging the arms of the cluster along the straight lines, each time we add a δ-segment of the corresponding external ray.

Another interpretation (which may be not completely satisfactory) of geodesic η-processes with $\eta = 1/2$ is to consider them as generalized, "finger" solutions of the zero-tension Hele-Shaw free boundary problem after the time at which the classical solution blows up, see, e.g., [5].

The geodesic model is more difficult than the starlike model because the phase space is infinite dimensional. The competition between the arms remains the main feature of the process. The competition is stronger for larger η's: if $\eta = \infty$, then typically there is a sole winner, but if $\eta = 0$ then all arms survive. See [12] for the case $\eta = -1$. Computer simulations (which we reproduce from Selander's thesis [12]) give some idea of the asymptotic behavior of the geodesic model in the case $\eta = 1/2$.

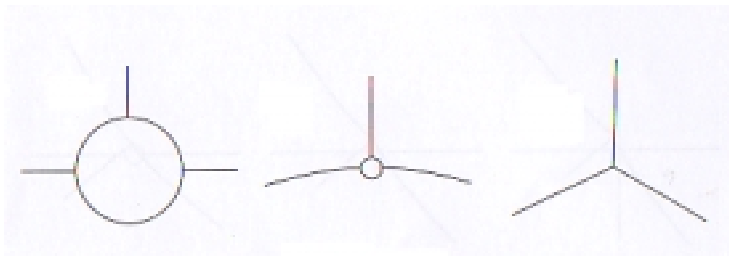

Figure 1

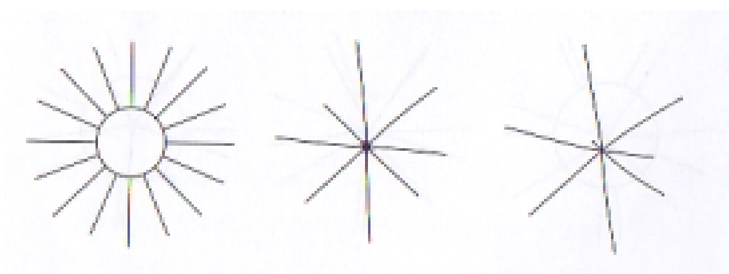

Figure 2

In Figure 1, the initial configuration (left picture) has 3 arms and the process converges to the symmetric state (right picture); the middle picture shows some intermediate state of the process. In Figure 2, we start with 16 arms. After some time it becomes clear that only half of them have a chance to survive, and a little later we see that probably one or two more arms will also dissappear. There are many more pictures in [12], and they all suggest that the process always converges to some steady state: some arms dissappear and the others straighten out.

Geodesic model can be described in terms of Loewner's equation, see [12] and the next section for details. Also see [4] for a general discussion of the role of Loewner's equation in aggregation models. The family of Riemann maps (1.1) corresponding to a radial chain with d arms satisfy the equation

$$\dot{\varphi}_t(z) = \varphi_t'(z) \cdot z \sum_{j=1}^{d} \frac{z + \zeta_j(t)}{z - \zeta_j(t)} \; \mu_j(t),$$

where the poles $\zeta_j(t)$ are given by (1.2), and $\mu_j(t)$ are some non-negative functions. In terms of these functions, the η-equation (1.3) has the form

$$\mu_j(t) \;\div\; |\varphi_t''(\zeta_j(t))|^{-(1+\eta)}. \tag{1.4}$$

The Loewner chain $\{\varphi_t\}$ is geodesic if and only if it satisfies the following *geodesic condition*:

$$\forall k, \qquad \dot{\zeta}_k(t) = -\zeta_k(t) \sum_{j \neq k} \frac{\zeta_k(t) + \zeta_j(t)}{\zeta_k(t) - \zeta_j(t)} \, \mu_j(t).$$

A radial chain is stationary iff the parameter functions of the (normalized) Loewner equation are constant:

$$\zeta_j(t) \equiv \zeta_j, \qquad \frac{\mu_j(t)}{\|\mu(t)\|} \equiv \sigma_j,$$

where $\|\mu\| = \sum \mu_j$. In particular, a stationary chain is geodesic if it satisfies the *stationary geodesic condition*:

$$\forall k, \qquad \sum_{j \neq k} \frac{\zeta_k + \zeta_j}{\zeta_k - \zeta_j} \sigma_j = 0. \tag{1.5}$$

One can show that for any collection of positive numbers $\{\sigma_j\}$ with $\sum \sigma_j = 1$, the system (1.5) has a unique (up to a rotation) solution $\{\zeta_j\}$, see [12]. Thus one can identify the set of stationary geodesic d-arm chains with the $d-1$ dimensional simplex

$$\mathcal{G}_d = \left\{ \sigma \in \mathbb{R}^d : \ \sum \sigma_j = 1, \ \sigma_j > 0 \right\}. \tag{1.6}$$

Stationary solutions of the geodesic η-model are determined by the $d-1$ equations in (1.4), so generically they are isolated points in $\mathcal{G}_d$.

Stability of stationary solutions is defined in terms of Loewner's equation, see the definition in Section 5. Roughly speaking asymptotic stability means that if we slightly perturb the Loewner parameters of a stationary solution σ in a finite time interval and then run the geodesic η-process, then $\mu_j(t) \to \sigma_j$ as $t \to \infty$. We discuss stability analysis in Section 5; the results are stated in Theorems 4 and 5 below.

There is another aspect of the stability problem. We can identify the boundary points of $\mathcal{G}_d$ in $\mathbb{R}^d$ with degenerate stationary chains (some of the needles have zero length). Considering small perturbations of such degenerate solutions we come to the question of survival and disappearance of arms. This will be discussed in Section 4 and the result is stated in Theorem 3.

1.4. **Chordal chains.** Along with radial chains and models, we will consider their chordal counterparts. A *chordal chain* $\{K_t\}$ has d arms growing from the origin in the slit plane $\mathbb{C} \setminus \mathbb{R}_+$, where $\mathbb{R}_+$ is the positive real axis. The corresponding family of the Riemann maps

$$\varphi_t : \ \mathbb{H} \equiv \{\Im z > 0\} \ \to \ \Omega_t \equiv \mathbb{C} \setminus (\mathbb{R}_+ \cup K_t)$$

with normalization $\varphi_t(z) = z^2 + O(1)$ as $z \to \infty$ satisfies the Loewner equation

$$\dot{\varphi}_t(z) = \varphi_t'(z) \sum_{j=1}^{d} \frac{\mu_j(t)}{x_j(t) - z},$$

where $x_j(t) \in \mathbb{R}$ denote the preimages of the tips $a_j(t)$. The chain is geodesic if the tips "follow" hyperbolic geodesics (wrt Ω_t) from $a_j(t)$ to ∞. In terms of the Loewner equation, this is expressed by the chordal *geodesic condition*

$$\dot{x}_k(t) \;=\; \sum_{j \neq k} \frac{\mu_j(t)}{x_k(t) - x_j(t)}.$$

The needle and the geodesic η-processes are defined by the η-*equation*

$$|\dot{a}_j(t)| \div |\varphi_t''(x_j(t))|^{-\eta} \qquad \text{or, equivalently,} \qquad \mu_j(t) \div |\varphi_t''(x_j(t))|^{-1-\eta}.$$

We can think of these models as representing Laplacian growth (or branching) at the tip of an infinitely long needle.

The parameter functions $\mu_j(t)$ are constant in the case of stationary chains, $\mu_j(t) \equiv \mu_j$. As in the radial case the numbers $\sigma_j = \mu_j/\|\mu\|$ determine a stationary geodesic chain uniquely up to a time change, so again we can identify the set of stationary geodesic d-arm chains with the simplex $\mathcal{G}_d$, see (1.6)

The chordal and radial models have similar features but the chordal case in somewhat simpler from the technical point of view (no trigonometry). We will do our computations mostly in the chordal case. Another reason to consider the chordal version is that the simplest non-trivial cases in the stability and survival problems come with fewer needles than in the radial case. We borrowed the name "chordal" from [11]; this adjective reflects the fact that chordal arms are growing from one boundary point of $\mathbb{C} \setminus \mathbb{R}_+$, the origin, to another boundary point, infinity.

1.5. **Survival and disappearance of arms.** It is clear that every geodesic chordal process with one arm converges to a trivial stationary solution. Is this solution attracting with respect to the geodesic η-model with two arms? We give an answer in the following theorem.

Theorem 3. *Consider a chordal geodesic η-process with two arms.*

(i) If $\eta < 1$, then both arms survive: $\mu_j(t) \not\to 0$ as $t \to \infty$.

(ii) This is not always true if $\eta > 1$.

In fact, in the case $\eta > 1$ the trivial stationary solution with one arm is a local attractor for the geodesic dynamics. The precise statement is this. Let $\{\varphi_t\}_{t \geq 0}$ be a normalized ($\mu_1 + \mu_2 \equiv 1$) geodesic chain with two arms, and let $K_0 = \{0\}$. Suppose also that $\{\varphi_t\}$ satisfies the η-equation for $t \geq 1$. Then the smallness of $\mu_1(\cdot)$ on $[0, 1]$, $\mu_1(\cdot) < \text{const}(\eta)$, implies the disappearance of the first arm: $\mu_1(t) \to 0$. We don't know whether the one-arm stationary solution is a global attractor.

In the case of d-arm stationary solutions, it is natural to expect that the right condition for the disappearance of any additional small arms should be

$$\eta > \frac{\alpha_{\max}}{1 - \alpha_{\max}},$$

where $2\pi\alpha_{\max}$ is the maximal angle between the needles (including $\mathbb{R}_+$ in the chordal case), see the discussion in Section 4.1.

1.6. **Stability of stationary solutions.** We first consider the simplest case – the *chordal model with two arms.* There is a 1-parameter family of geodesic stationary chains parametrized by the interval

$$\mathcal{G}_2 = \{\sigma(s) = (s, 1-s) : \ 0 < s < 1\}.$$

If s is small, then $l_1 \ll l_2$ and the needles are almost perpendicular. If s is close to $1/2$, then the lengths l_j of the needles are almost equal. The symmetric configuration $\sigma(1/2)$ has both needles of the same length making an angle $2\pi/5$. The configuration $\sigma(1-s)$ is symmetric to $\sigma(s)$.

If $s \neq 1/2$, then there is a unique number $\eta_c(s)$ such that the chain $\sigma(s)$ is a stationary solution of the geodesic η-model with $\eta = \eta_c(s)$. The chain $\sigma(1/2)$ is a solution for all η. In this way we obtain a function $\eta_c(s)$, $s \neq 1/2$, ("c" is for "chordal"). One can easily see that $\eta_c(s) \to 1$ as $s \to 0$. The result of Theorem 3 suggests that $\eta_c < 1$, and in fact $\eta_c(\cdot)$ decreases on $(0, 1/2)$ from 1 to a limit

$$\eta_c(1/2) := \lim_{s \to 1/2} \eta_c(s),$$

the exact value of which is

$$\frac{-2\left(35 + 3\sqrt{5}\,\log(\frac{3-\sqrt{5}}{2}) - 6\sqrt{5}\,\log(\frac{1+\sqrt{5}}{2})\right)}{-55 + 6\sqrt{5}\,\log(\frac{3-\sqrt{5}}{2}) - 12\sqrt{5}\,\log(\frac{1+\sqrt{5}}{2})} = .54656... \tag{1.7}$$

The graph of the function η_c is shown in Figure 4 in the next section where we also provide some details of computation.

It follows that for $\eta_c(1/2) < \eta < 1$ there are two (up to a reflection) stationary solutions: $\sigma(1/2)$ and $\sigma(s)$ with $\eta_c(s) = \eta$. There are also unstable degenerate solutions corresponding to $s = 0$ or $s = 1$. One can guess that "by continuity", the solution with $s \neq 1/2$ should be a local attractor for the corresponding η-model with two arms, and that the symmetric configuration $\sigma(1/2)$ should be unstable. For $\eta < \eta_c(1/2)$ the symmetric configuration should become stable (again, for the model with two arms) because there are no other solutions except degenerate which are unstable. In Section 5 we will justify this answer.

Theorem 4. *If $\eta < 1$, $\eta \neq \eta_c(1/2)$, then the chordal geodesic η-model with two arms has a unique (asymptotically) stable stationary solution. The stable configuration is symmetric if $\eta < \eta_c(1/2)$ and asymmetric if $\eta_c(1/2) < \eta < 1$.*

Are the stable solutions of Theorem 4 also stable for the model with more than two arms? The answer should be "no": as the graphs in Figure 4 show, we have $\eta_c(s) < \alpha_{\max}/(1-\alpha_{\max})$ for all s. If we believe that the process with two arms always converges to a stationary solution, then it follows that at least three arms survive in any chordal geodesic η-process with $\eta < 1$ (unless we have started with just two arms).

We can apply similar methods in the *radial case.* The 2-arm geodesic model is trivial: for any η, every solution converges to a symmetric 2-needle configuration; this follows from the geodesic condition. By the maximal angle criterion, this trivial solution has to be a local attractor for any number of arms if $\eta > 1$, but as we'll see below, for $\eta < 2.118...$ it is definitely not a global attractor.

Consider the case of *three radial arms.* We first observe that a geodesic stationary chain $\sigma \in \mathcal{G}_3$ is a solution for some η-model if and only if two or three of the numbers σ_j coincide. It does not seem easy to verify this statement by elementary means – we rely on computer calculations. As soon as we know this fact, the computation is quite similar to the chordal case. We consider geodesic configurations of the form $\sigma(s) = (s, s, 1-2s)$, $0 < s < 1/2$, so that the first two needles are symmetric with respect to to the line of the third needle. The 3-fold symmetric configuration $\sigma(1/3)$ is a solution for any η.

As above we introduce the function $\eta_r(s)$, $s \neq 1/3$, such that $\sigma(s)$ is a solution for the $\eta(s)$-model. It is easy to see that $\eta_r(s) \to +\infty$ as $s \to 0$ and that $\eta_r(s) \to 1$ as $s \to 1$. In fact the function η_r decreases on $(0, 1/2) \setminus \{1/3\}$ and has a limit

$$\eta_r(1/3) := \lim_{s \to 1/3} \eta_r(s) = \frac{15 - 4\log 2}{3 + 4\log 2} = 2.11815\ldots. \tag{1.8}$$

See Figure 3 in the next section for the graph of η_r. It follows that if $\eta > 1$, then there are two stationary solutions: $\sigma(1/2)$ and $\sigma(s)$ with $\eta_r(s) = \eta$ (plus the degenerate solution corresponding to $s = 0$). If $\eta < \eta_r(1/3)$, then $\sigma(1/3)$ is the only non-degenerate solution. The solutions $\sigma(s)$ with s close to 0 are unstable because this is essentially the chordal case with $\eta > 1$, see Theorem 4.

Theorem 5. *Consider the the radial geodesic η-model with 3 arms.*
(i) The symmetric configuration $\sigma(1/3)$ is asymptotically stable if $\eta < \eta_r(1/3)$, and unstable if $\eta > \eta_r(1/3)$.
(ii) There are no other non-degenerate stable stationary solutions.

2. Preliminaries

2.1. **Loewner's equation.** If $\{K_t\}$ is a radial chain with d arms, then the corresponding Loewner chain $\varphi(z,t) = \varphi_t(z)$ of the Riemann maps (1.1) satisfies the Loewner equation

$$\dot{\varphi}(z,t) = \varphi'(z,t) \cdot z \sum_{j=1}^{d} \frac{z + \zeta_j(t)}{z - \zeta_j(t)}\ \mu_j(t). \tag{2.1}$$

We will write $\|\mu(t)\| = \sum \mu_j(t)$, so the equation for logarithmic capacity $c(t) = \operatorname{cap} K_t \equiv \varphi_t'(\infty)$ is

$$\dot{c}(t) = \|\mu(t)\|\ c(t). \tag{2.2}$$

The chain is *normalized* if $\|\mu(t)\| \equiv 1$.

In the opposite direction, we can use the Loewner equation to define radial chains. Given parameter functions $\mu_j(t)$, $\zeta_j(t)$, and given a univalent function φ_{t_0}, the initial value problem: (2.1) with $\varphi(\cdot, t_0) = \varphi_{t_0}(\cdot)$, has a unique solution for $t \geq t_0$, and all the functions φ_t are univalent, see [1]. It is known that if the parameter functions are sufficiently smooth, then the solution is a radial chain with smooth arms.

Let us explain how the geodesic model is related to the Loewner equation.

The tip points $a_j(t) = \varphi_t(\zeta_j(t))$ move with velocities

$$|\dot{a}_j| = 2\mu_j |\varphi''(\zeta_j)| \tag{2.3}$$

as the following computation shows:

$$\begin{aligned}\dot{a}_j(t) &= \frac{d}{dt}[\varphi_t(\zeta_t)] = \dot{\varphi}_t(\zeta_t) + \varphi_t'(\zeta_t)\dot{\zeta}_t, \quad \zeta_t := \zeta_j(t),\\ &= \lim_{r\to 1} \dot{\varphi}_t(z) + 0, \qquad z := r\zeta_t,\\ &= \lim_{r\to 1} \frac{\varphi_t'(z)}{z-\zeta_t} z(z+\zeta_t)\mu_j(t) = 2\zeta_t^2 \mu_j(t)\varphi_t''(\zeta_t);\end{aligned}$$

this argument can be made rigorous in the case of smooth parameter functions.

Following [3], we introduce the *radial beta-numbers*

$$\beta_j(t) \equiv \beta(\Omega_t; a_j(t), \infty) = \frac{2c(t)}{|\varphi_t''(\zeta_j(t))|}. \tag{2.4}$$

These numbers characterize the concentration of harmonic measure at the tip points in a scale invariant way. If we think of logarithmic capacity as a function of the arm lengths, $c = c(l_1, \dots, l_d)$, then

$$\sum \frac{\partial c}{\partial l_j} \dot{l}_j = \dot{c} \overset{(2.2)}{=} \sum \mu_j c = \sum \frac{c}{2|\varphi''(\zeta_j)|} \cdot 2\mu_j |\varphi''(\zeta_j)| = \sum \frac{\beta_j}{4} |\dot{a}_j|,$$

and so

$$\beta_j = 4 \frac{\partial c}{\partial l_j}. \tag{2.5}$$

By (2.3) and (2.4), the η-equation (1.3) can be expressed in the form

$$\mu_j \div \beta_j^{1+\eta}. \tag{2.6}$$

Next we claim that the Loewner chain (2.1) is geodesic if and only the parameter functions satisfy

$$\dot{\zeta}_k = -\zeta_k \sum_{j\neq k} \frac{\zeta_k + \zeta_j}{\zeta_k - \zeta_j}\ \mu_j. \tag{2.7}$$

If we denote $\zeta_j = e^{i\theta_j}$, then the geodesic condition has the form

$$\dot{\theta}_k \ = \ \sum_{j\neq k} \mu_j \cot \frac{\theta_k - \theta_j}{2}. \tag{2.8}$$

One can derive (2.7) as follows. If we freeze all the arms except k in an infinitesimally small time interval $(t, t+\varepsilon)$, then geodesic growth means $\dot{\zeta}_k(t) = 0$. Taking the contribution of arms $j \neq k$ into account, we change the curvature of the external ray to $a_k(t)$ infinitesimally. We also get the motion (2.7) of the k-th pole, which follows from the inverse Loewner equation

$$\dot{g} = -g \sum \frac{g+\zeta_j}{g-\zeta_j}\ \mu_j,$$

where $g = \{g_t\}$ is the inverse chain, $g_t = \varphi_t^{\circ -1}$.

The *chordal case* is similar. We consider the maps $\varphi_t : \mathbb{H} \to \Omega_t$ in a Loewner chain

$$\dot\varphi_t(z) = \varphi_t'(z) \sum \frac{\mu_j(t)}{x_j(t) - z},$$

with normalization

$$\varphi_t(z) = z^2 - T(t) + o(1) \qquad (z \to \infty).$$

The quantity $T(t) = T(\varphi_t)$ is the Schwarz derivative of φ_t at infinity. It plays the same role as logarithmic capacity in the radial case. We have $T(t) \geq 0$, and $\dot T(t) = 2\|\mu(t)\|$.

The *η-equation* is $\mu_j \div \beta_j^{1+\eta}$, where β_j are the *chordal beta-numbers*,

$$\beta_j(t) \equiv \beta(\Omega_t; a_j(t), \infty) = \frac{2}{|\varphi''(x_j)|} = \frac{\partial T}{\partial l_j}, \tag{2.9}$$

see [3]. Note that $\beta = 1$ in the one-arm case $\varphi(z) = z^2$.

A chordal chain is geodesic if

$$\dot x_k \;=\; \sum_{j \neq k} \frac{\mu_j}{x_k - x_j}. \tag{2.10}$$

Selander [12] observed that if all μ_j's are equal (which is the case when the chain satisfies the η-equation with $\eta = -1$), then (2.8) and (2.10) are exactly the well-known systems of Sutherland and Calogero–Moser which appear in several important applications, see [2], [9], [13].

2.2. **Stationary chains and starlike functions.** A radial chain $\{\varphi_t\}$ with d arms is stationary if it has the form

$$\varphi_t(z) = \lambda(t)\psi(z) \tag{2.11}$$

for some increasing positive function $\lambda(t)$ and a univalent function ψ mapping Δ onto the complement of a needle configuration. In terms of Loewner's equation, the chain is stationary iff the normalized Loewner measures are constant:

$$\zeta_j(t) \equiv \zeta_j, \qquad \frac{\mu_j(t)}{\|\mu(t)\|} \equiv \sigma_j.$$

The β-numbers are also constant, $\beta_j(t) \equiv \beta_j$, and by (2.5) we have

$$\sigma_j = \frac{\beta_j l_j}{4c}, \tag{2.12}$$

where l_j's and c are the lengths and the capacity of a needle configuration representing the stationary chain.

We denote by σ the normalized Loewner measure, $\sigma = \sum \sigma_j \delta_{\zeta_j}$, and by S^σ its Schwarz integral,

$$S^\sigma(z) = \sum \frac{\zeta_j + z}{\zeta_j - z}\sigma_j,$$

so the stationary Loewner equation is

$$\dot\varphi_t(z) = \lambda(t)\varphi_t'(z) z S^\sigma(z). \tag{2.13}$$

The function ψ in (2.11) is starlike:

$$\Re\,[z\psi'/\psi] > 0 \qquad \text{in} \quad \Delta.$$

If α is the corresponding (probability) Herglotz measure:

$$z\psi'/\psi = S^{\alpha}, \tag{2.14}$$

then from (2.13) we find

$$\frac{\dot{\lambda}\psi}{\lambda\psi'} = \frac{\dot{\varphi}}{\varphi'} = \|\mu_t\| z S^{\sigma},$$

and therefore

$$S^{\sigma} S^{\alpha} = 1. \tag{2.15}$$

Since σ is pure point with d atoms, then so is α, $\alpha = \sum_{\nu=1}^{d} \alpha_\nu \ \delta_{\eta_\nu}$, and the integration of (2.14) gives the familiar formula

$$\psi(z) = cz^{-1} \prod (z-\eta_\nu)^{2\alpha_\nu},$$

where the points η_ν are preimages of the origin and the numbers $2\pi\alpha_\nu$ are the angles between the needles.

Given Loewner parameters c, σ_j, ζ_j of a needle configuration we can find parameters α_ν, η_ν and l_j as follows. If we denote

$$P(z) = (z-\zeta_1)\dots(z-\zeta_d), \qquad Q(z) = (z-\eta_1)\dots(z-\eta_d),$$

then the identity (2.15) implies

$$\sum_j \sigma_j \frac{z+\zeta_j}{z-\zeta_j} = \frac{Q(z)}{P(z)}, \qquad \sum_\nu \alpha_\nu \frac{z+\eta_\nu}{z-\eta_\nu} = \frac{P(z)}{Q(z)}. \tag{2.16}$$

Given σ_j, ζ_j, we obtain $Q(z)$ from the first equation and then find the roots η_ν. Equating the residues in the second equation, we get

$$\alpha_\nu = \frac{P(\eta_\nu)}{2\eta_\nu Q'(\eta_\nu)}, \qquad l_j = c\prod_\nu |\zeta_j - \eta_\nu|^{2\alpha_\nu}. \tag{2.17}$$

Chordal case. Let $\psi : \mathbb{H} \mapsto \mathbb{C} \setminus (K \cup \mathbb{R}_+)$ be the Riemann map onto the complement of a d-needle chordal configuration K, and let $\psi(z) = z^2 - T + \dots$ at ∞. Then

$$\psi(z) = \prod_{\nu=0}^{d} (z-t_\nu)^{2\alpha_\nu}, \qquad \sum \alpha_\nu t_\nu = 0,$$

where the points t_ν's are the preimages of the origin and the numbers $2\pi\alpha_\nu$ are the corresponding angles between the needles (including $\mathbb{R}_+$). Consider the parameter functions $\mu_j(t) \equiv \mu_j$ and $x_j(t) \equiv \sqrt{t} x_j$ of a stationary Loewner chain

$$\varphi_t(z) \equiv t\psi(t^{-1/2} z);$$

the points x_j's are the preimages of the tips of K under ψ. Define the probability measures

$$\alpha = \sum_\nu \alpha_\nu \delta_{t_\nu} \qquad \text{and} \qquad \sigma = \sum_j \sigma_j \delta_{x_j}, \quad (\sigma_j := \mu_j/\|\mu\|).$$

The analogue of (2.15) is the relation

$$TC^{\sigma} + (C^{\alpha})^{-1} + z = 0, \tag{2.18}$$

where C^σ and C^α are the Cauchy integrals of the respective measures, e.g.

$$C^\sigma(z) = \sum_j \frac{\sigma_j}{x_j - z}.$$

The equation (2.18) follows by comparing the Loewner equation

$$\frac{\dot\varphi_t}{\varphi_t'} = \frac{1}{\sqrt{t}} \cdot \frac{\psi(t^{-1/2}z)}{\psi'(t^{-1/2}z)} - \frac{z}{2t},$$

(at $t = 1$) and the Herglotz formula

$$\frac{\psi'}{\psi} = -2C^\alpha.$$

Rewriting (2.18) in the form

$$z - T\sum_j \frac{\sigma_j}{z - x_j} = \frac{Q(z)}{P(z)}, \qquad \sum_\nu \frac{\alpha_\nu}{z - t_\nu} = \frac{P(z)}{Q(z)},$$

where

$$P(z) = (z - x_1)\dots(z - x_d), \qquad Q(z) = (z - t_0)\dots(z - t_d),$$

we can determine, as in the radial case, the parameters α_ν and l_j of the needle configuration from the Loewner parameters σ_j, x_j, and T:

$$\alpha_\nu = \frac{P(t_\nu)}{Q'(t_\nu)}, \quad \text{and} \quad l_j = \prod_\nu |x_j - t_\nu|^{2\alpha_\nu}. \tag{2.19}$$

Let us also mention the formula

$$\frac{1}{\alpha_\nu} = 1 + \sum_j \frac{T\sigma_j}{(t_\nu - x_j)^2}, \tag{2.20}$$

which follows by differentiating (2.18) and setting $z = t_\nu$, and the formula for the beta-numbers

$$\beta_j l_j = T\sigma_j, \tag{2.21}$$

which follows from (2.9)

2.3. **Stationary solutions.** We want to describe stationary solutions in the simplest non-trivial cases of the radial and the chordal geodesic η-models. Even in these simplest examples some computations involve analysis of rather complicated elementary and algebraic functions. It seems reasonable to us to accept standard computer calculations as proofs, though with a reasonable effort one can certainly find formal proofs of the general properties mentioned below.

Recall that a stationary solution is a needle configuration satisfying (i) the η-equation

$$l_j^{1+\eta} \div \sigma_j^\eta, \tag{2.22}$$

see (2.6) and (2.12), (2.21), and (ii) the stationary geodesic condition (1.5) in the radial case or its chordal counterpart

$$\forall k, \qquad x_k = T\sum_{j\neq k} \frac{\sigma_j}{x_k - x_j}. \tag{2.23}$$

Let us denote

$$\Sigma_{ij} = \log\frac{\sigma_i}{\sigma_j}, \qquad L_{ij} = \log\frac{l_i}{l_j}.$$

Three radial needles. Given $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, we first find the poles $\zeta_j = e^{i\theta_j}$,

$$\zeta_1 = e^{-i\beta}, \quad \zeta_2 = e^{i\alpha}, \quad \zeta_3 = -1, \qquad (0 < \alpha, \beta < \pi),$$

from the stationary geodesic condition (1.5):

$$\sum_{j\neq k} \sigma_j \cot\frac{\theta_k - \theta_j}{2} = 0.$$

Considering this as a linear system for σ_j's with the matrix

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix},$$

where

$$a = \tan\frac{\alpha}{2}, \quad b = \tan\frac{\beta}{2}, \quad c = \cot\frac{\alpha+\beta}{2} = \frac{1-ab}{a+b},$$

we observe that A has rank two, and that the vector (a, b, c) belongs to the kernel of A, and therefore $[\sigma_1 : \sigma_2 : \sigma_3] = [a : b : c]$, so

$$\tan\frac{\alpha}{2} = \frac{\sigma_1}{\sqrt{\sigma_1\sigma_2 + (1-\sigma_3)\sigma_3}}, \qquad \tan\frac{\beta}{2} = \frac{\sigma_2}{\sqrt{\sigma_1\sigma_2 + (1-\sigma_3)\sigma_3}}. \tag{2.24}$$

If σ is an η-solution for some $\eta \neq -1$, then $L_{12}\Sigma_{13} - L_{13}\Sigma_{12} = 0$. Computer calculation shows that this is possible only if $\sigma_1 = \sigma_2$, or $\sigma_1 = \sigma_3$, or $\sigma_2 = \sigma_3$.

It remains to consider the case

$$\sigma = \sigma(s) \equiv (s, s, 1-2s) \in \mathcal{G}_3, \qquad s \in (0, 1/2),$$

cf. [12], Section 4.7. If $s \neq 1/3$, then $\sigma(s)$ is a stationary solution for the η-model with

$$\eta = \eta_r(s) \equiv \frac{L_{13}(\sigma)}{\Sigma_{13}(\sigma) - L_{13}(\sigma)}, \qquad \sigma = \sigma(s),$$

see (2.22). It is not difficult to find an explicit formula:

$$\eta_r(s) = \frac{(1-2s)\log 8 - (4-6s)\log(1-2s) + (5-6s)\log s - \log(1-s)}{-(1-2s)\log 8 - \log(1-2s) + \log(1-s)}. \tag{2.25}$$

Proof. We will write ζ for ζ_1, and denote

$$x = \Re\zeta = \frac{1-2s}{1-s}.$$

Let

$$\eta_1 = 1,\ \eta_2 = \eta,\ \eta_3 = \bar\eta; \qquad \alpha_1 = \alpha,\ \alpha_2 = \alpha_3 = \beta.$$

Then we have

$$\frac{|\zeta-\eta||\zeta-\bar\eta|}{|1+\eta|^2} = \frac{2s}{1-2s}, \qquad \alpha = \frac{2-x}{4+x}. \tag{2.26}$$

Indeed, equating the residues at $z = \zeta_3 = -1$ in the first equation (2.16), we get

$$-2\sigma_3 = \frac{Q(-1)}{|1+\zeta|^2} = \frac{-2|1+\eta|^2}{|1+\zeta|^2},$$

and therefore

$$|1+\eta|^2 = \sigma_3|1+\zeta|^2.$$

From the second equation (2.16), we similarly have

$$|1-\zeta|^2 = \alpha|1-\eta|^2.$$

It follows that

$$4 = |1+\eta|^2 + |1-\eta|^2 = \sigma_3|1+\zeta|^2 + \alpha^{-1}|1-\zeta|^2$$

and so

$$\alpha = \frac{2-2x}{4-\sigma_3(2+2x)} = \frac{2-x}{4+x}.$$

Finally, we equate the residues at $z = \zeta$ in the first equation (2.16) and obtain

$$\frac{Q(\zeta)}{(1+\zeta)(\zeta-\bar{\zeta})} = 2\zeta\sigma,$$

so $(\zeta-\eta)(\zeta-\bar{\eta}) = 2\sigma(1+\zeta)^2$. This proves (2.26). Using the expression (2.17) for the lengths, we derive

$$l/l_3 = (1-x)x^{-2\beta}2^{2\beta-\alpha},$$

and

$$\eta_r(s) = \frac{(4+x)\log(1-x) - (2+2x)\log x + 3x\log 2}{(x-2)\log x - 3x\log 2}, \qquad x = \frac{1-2s}{1-s}.$$

□

The analysis of the elementary function (2.25) gives the following result.

Proposition 1. *The function $\eta_r(s)$ extends to a smooth, strictly decreasing function on $(0, 1/2)$ with $\eta_r(0) = \infty$, $\eta_r(1/2) = 1$, and with $\eta_r(1/3)$ given by (1.8).*

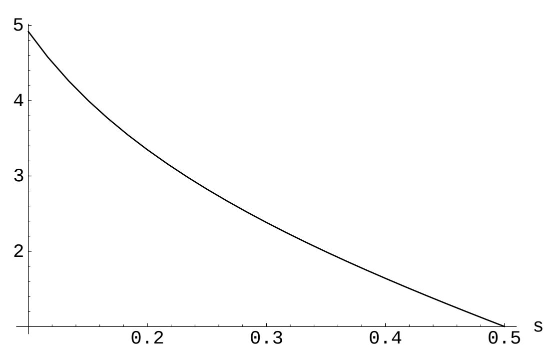

Figure 3

In addition to $\eta_r(\cdot)$, we will also need the function $\tilde{\eta}(\cdot)$, which is defined in the following statement. For $\sigma \in \mathcal{G}_3$, denote (cf. (2.22))

$$\eta_{12}(\sigma) = \frac{L_{12}(\sigma)}{\Sigma_{12}(\sigma) - L_{12}(\sigma)}.$$

Proposition 2. *There exists a limit*

$$\tilde{\eta}(s) = \lim_{\varepsilon \to 0} \eta_{12}(s+\varepsilon, s-\varepsilon, 1-2s), \qquad (0 < s < 1/2).$$

We have $\tilde{\eta}(s) < \eta_r(s)$ *for* $0 < s < 1/3$*, and* $\tilde{\eta}(s) > \eta_r(s)$ *for* $1/3 < s < 1/2$.

This can be explained as follows. If s is small, then $\eta_r(s)$ is large by Proposition 1. We also have $l_3 \gg l_1 = l_2$, which is essentially a chordal situation, and so $\tilde{\eta}(s) \approx \eta_c(1/2) < 1$ by Proposition 3 below. On the other hand, if s is close to $1/2$, then we have essentially a two-needle radial case, and $\tilde{\eta}(s)$ is large. It is possible to express $\tilde{\eta}$ in terms of elementary functions but we used a computer for the verification of the statement.

Two chordal needles. Assuming normalization $T = 1$, the poles of the geodesic needle configuration

$$\sigma(s) = (s, 1-s) \in \mathcal{G}_2, \qquad s \in (0, 1/2),$$

are $x_1 = 1 - s$ and $x_2 = -s$, see (2.23). We determine the points t_ν's by solving the cubic equation

$$Q(z) = z^3 - (1-2s)z^2 - (1+s-s^2)z + (1-2s) = 0,$$

and find the angles α_ν and the lengths l_ν of the needles from the equations (2.19). For instance, in the symmetric case $s = 1/2$, we have $P(z) = z^2 - 1/4$, $Q(z) = z^3 - (5/4)z$, and so the middle angle (corresponding to $t_\nu = 0$) is $2\pi/5$. For $s \neq 1/2$, the function $\eta_c(s)$ is given by the formula

$$\eta_c(s) = \frac{L_{12}(\sigma)}{\Sigma_{12}(\sigma) - L_{12}(\sigma)}, \qquad \sigma = \sigma(s).$$

Proposition 3. *The function* $\eta_c(\cdot)$ *extends to a smooth function, which is strictly decreases on* $(0, 1/2)$ *and strictly increases on* $(1/2, 1)$*. We have* $\eta_c(0) = \eta_c(1) = 1$*, and the value of* $\eta_c(1/2)$ *is given by* (1.7).

The graph of $\eta_c(\cdot)$ is shown in Figure 4 (the lower curve) together with the graph of the function

$$a(s) = \frac{\alpha_{\max}(s)}{1 - \alpha_{\max}(s)}, \qquad (0 < s < 1),$$

where $2\pi\alpha_{\max}(s)$ is the maximal angle between the needles (including $\mathbb{R}_+$) in the configuration $\sigma(s)$. We have $\eta_c(s) < a(s)$ for all s.

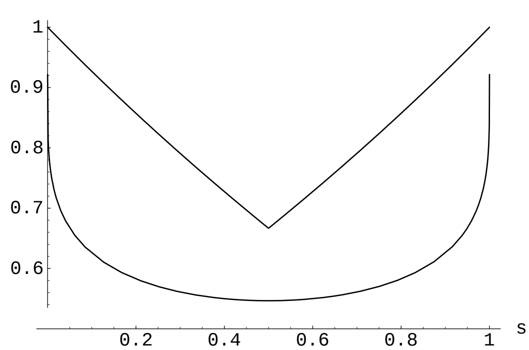

Figure 4

3. Needle dynamics

In this section we study the needle η-model. The evolution of a d-needle configuration with given fixed angles $2\pi\alpha_\nu$ is described by the differential equation

$$\dot{y} = p(y),$$

where $y = \{y_j\}$ is the vector of lengths, $\beta(y)$ the vector of β-numbers, and

$$p_j(y) = \frac{\beta_j^\eta(y)}{\beta_1^\eta(y) + \cdots + \beta_d^\eta(y)}. \tag{3.1}$$

It will be convenient to identify similar configurations and consider the flow

$$\dot{l} = p(l) - l, \tag{3.2}$$

on the $(d-1)$-dimensional manifold of equivalence classes

$$\mathcal{L} \;=\; \left\{ l \in \mathbb{R}^d : \; \sum l_j = 1, \; l_j > 0 \right\}.$$

(If $\|y\| = y_1 + \cdots + y_d$ and $l = \|y\|^{-1} y$, then $p(l) = p(y)$ and $\dot{l} = \|y\|^{-1} p - l$. Changing the time scale, we get (3.2).) The flow extends in a natural way to the boundary of $\mathcal{L}$ in $\mathbb{R}^d$, which we identify with the set of degenerate configurations (certain needles become extinct). Critical points of the flow (3.2) on clos $\mathcal{L}$ are configurations satisfying

$$p(l) = l, \quad \text{i.e.} \quad l_j \div \beta_j^\eta,$$

where some of the l_j's and β_j's can be zero.

3.1. **Lyapunov function.**

A remarkable property of the needle η-model is the existence of a global Lyapunov function. To establish this fact we will write

$$\gamma_{jk} = -\frac{\partial \beta_j}{\partial y_k} \qquad \text{for} \quad j \neq k.$$

Since $\beta(\lambda y) = \beta(y)$ for all $\lambda > 0$, we have

$$\frac{\partial \beta_k}{\partial y_k} = \sum_{j \neq k} \gamma_{jk} \, \frac{y_j}{y_k}. \tag{3.3}$$

It is important that the matrix γ_{jk} is symmetric. As usual, $x_j \in \mathbb{R}$ or $\zeta_j \in \partial\Delta$ is the notation for the preimages of the tips of the needles. In the radial case we denote by θ_{jk} the angle between ζ_j and ζ_k.

Proposition. *For $j \neq k$, we have*

$$\gamma_{jk} = \frac{\beta_k \beta_j}{(x_k - x_j)^2} \qquad \text{in the chordal case,} \tag{3.4}$$

$$\gamma_{jk} = \frac{\cot^2(\theta_{jk}/2)}{4c} \, \beta_j \beta_k, \qquad \text{in the radial case.} \tag{3.5}$$

Proof. Let us explain the first formula. Consider configurations with the same angles and with lengths $\{y_j\}$ and $\{\tilde{y}_j\}$, where $\tilde{y}_k = y_k + \delta$ and $\tilde{y}_j = y_j$ for $j \neq k$. We assume $\delta > 0$ to be infinitesimally small. Let φ and $\tilde{\varphi}$ be the normalized Riemann maps. We have $\tilde{\varphi} = \varphi \circ \tau$, where τ maps $\mathbb{H}$ onto the a slit halfplane, the slit being an almost a vertical line segment with an endpoint at x_k. Let ε be the length of the slit. Without loss of generality, we can assume $x_k = 0$. Then $\tau(z) \approx \sqrt{z^2 - \varepsilon^2}$, and so $\varepsilon^2 \approx \beta_k \delta$. If $j \neq k$, then $x_j = \tau(\tilde{x}_j)$, and since the inverse of τ is the map $z \mapsto \sqrt{z^2 + \varepsilon^2}$, we have

$$\tilde{\beta}_j = \beta_j |(\tau^{\circ -1})(x_j)|^2 \approx \beta_j \left| 1 + \frac{\varepsilon^2}{x_j^2} \right|^{-1}.$$

It follows that

$$\tilde{\beta}_j - \beta_j \approx -\frac{\beta_k \beta_j}{(x_k - x_j)^2} \delta,$$

which proves the statement. The proof in the radial case is similar. □

An alternative (and more accurate) argument is indicated in the Remark 1 below.

Theorem. *The function* $\varphi(l) = \sum \beta_k^{1+\eta}(l)$ *is a Lyapunov function of the flow* (3.2) *on* clos $\mathcal{L}$*:*

$$\dot{\varphi} = (1+\eta) \sum_i \beta_i^\eta \sum \sum\nolimits_{j \neq k} \gamma_{jk} l_j l_k \left(\frac{p_k}{l_k} - \frac{p_j}{l_j} \right)^2,$$

where the sum is taken over all pairs (j,k) *with* $l_j \neq 0$*,* $l_k \neq 0$*. In particular, every orbit of the flow tends to a stationary solution (maybe degenerate).*

Proof. By (3.1), we have

$$\dot{\varphi} = (1+\eta) \sum_k \beta_k^\eta \dot{\beta}_k = (1+\eta) \sum_i \beta_i^\eta \sum_k p_k \dot{\beta}_k,$$

and by (3.2) and (3.3),

$$\begin{aligned} \dot{\beta}_k &= \frac{\partial \beta_k}{\partial l_k} \dot{l}_k + \sum_{j \neq k} \frac{\partial \beta_k}{\partial l_j} \dot{l}_j \\ &= \sum_{j \neq k} \gamma_{jk} \left(\frac{l_j}{l_k} \dot{l}_k - \dot{l}_j \right) \\ &= \frac{1}{l_k} \sum_{j \neq k} \gamma_{jk} (l_j p_k - p_j l_k). \end{aligned}$$

Using the symmetry of the matrix γ_{jk}, we get

$$\begin{aligned} \sum_k p_k \dot{\beta}_k &= \frac{1}{2} \sum \sum\nolimits_{j \neq k} \gamma_{jk} \left(\frac{p_k}{l_k} - \frac{p_j}{l_j} \right) (l_j p_k - p_j l_k) \\ &= \frac{1}{2} \sum \sum\nolimits_{j \neq k} \gamma_{jk} l_j l_k \left(\frac{p_k}{l_k} - \frac{p_j}{l_j} \right)^2. \end{aligned}$$

□

Remark 1. The formulae (3.4) and (3.5) are valid for general configurations with rectifiable, not necessarily straight arms. This is a consequence of the following fact concerning Loewner chains of several time variables. The chordal case is mentioned in [11]; we will state the radial version. Suppose the arms are parametrized by real variables $s_1, \ldots, s_d$. Consider logarithmic capacity as a function $c = c(s_1, \ldots, s_d)$. Then

$$\partial_k \partial_j c = \frac{1}{c} \left(\frac{\zeta_j + \zeta_k}{\zeta_j - \zeta_k} \right)^2 \partial_j c \partial_k c, \qquad (k \neq j). \tag{3.6}$$

If the arms are rectifiable, then we can use the arclength parametrization. In this case, $\partial_j c = 4^{-1} \beta_j$ by (2.5), and so (3.6) implies (3.5).

Proof. To prove (3.6), we repeat the computation in [11]. Denote $T = \log c$. Then $\partial_j T = \mu_j$. The inverse Loewner chain g satisfies the system of equations

$$\partial_j g = -g \frac{g + \zeta_j}{g - \zeta_j} \partial_j T. \tag{3.7}$$

Applying the j-th equation to the tip of the k-th needle, we get

$$\partial_j \zeta_k = -\zeta_k \frac{\zeta_k + \zeta_j}{\zeta_k - \zeta_j} \partial_j T. \tag{3.8}$$

Let us write ζ for ζ_j and η for ζ_k, and note that

$$\frac{\partial}{\partial z} \left(z \frac{z + \zeta}{z - \zeta} \right) = 1 - \frac{2\zeta^2}{(z - \zeta)^2}, \qquad \frac{\partial}{\partial \zeta} \left(z \frac{z + \zeta}{z - \zeta} \right) = \frac{2z^2}{(z - \zeta)^2}.$$

Differentiating (3.7) and using (3.8), we have

$$g^{-1} \partial_k \partial_j g = -X \frac{g + \zeta}{g - \zeta} + Y \left[\frac{g + \eta}{g - \eta} \left(1 - \frac{2\zeta^2}{(g - \zeta)^2} \right) + \frac{\zeta + \eta}{\zeta - \eta} \frac{2g\zeta}{(g - \zeta)^2} \right], \tag{3.9}$$

where $X = \partial_k \partial_j T$ and $Y = \partial_k T \partial_j T$. Equating (3.9) to a similar expression for $g^{-1} \partial_j \partial_k g$ (just interchange ζ and η), we find $X = 4\zeta\eta(\zeta - \eta)^{-2} Y$, which is exactly (3.6). □

Finally we want to emphasize the fact that for general Loewner chains with several growth points, there are no simple formulas like (3.4), (3.5) describing the "diagonal" term $\partial \beta_k / \partial y_k$.

Remark 2. Here is a simple application of the Lyapunov function constructed in the theorem. Claim: for any radial needle configurations we have $\sum \beta_j \leq 4$. Proof: starting with a given configuration, we run the needle η-process with $\eta = 0$ until we reach an equilibrium. The sum of β-numbers only increases during the process. At equilibrium all lengths are equal, let's say to 1. By (2.5), $\sum \beta_j = 4c \leq 4$ because the capacity of the unit disc is 1.

It is tempting to apply similar considerations to establish that $\sum \beta_j^2 \leq 1$ for an arbitrary *chordal* needle configurations, (this is a special case of Brennan's conjecture, cf. [3]). According to the theorem, it's enough to verify the inequality only for stable stationary solutions of the η-model with $\eta = 1$. See Corollary 3.3 for the case of geodesic solutions.

3.2. **Stability.** By definition, a stationary solution of the needle η-model is (asymptotically) stable if it is (asymptotically) stable in the sense of the differential equation (3.2). We will apply the standard linearization method. It is interesting that the trace of the linearized vector field has a simple expression in terms of Loewner's parameters. Let us denote

$$T_{jk} = \begin{cases} \frac{1}{4c}\cot^2\frac{\theta_{jk}}{2}, & \text{in the radial case,} \\ \frac{1}{(x_k-x_j)^2}, & \text{in the chordal case.} \end{cases}$$

Proposition. *Let $l \in \mathcal{L}$ be a critical point of (3.2) with parameter η. Then*

$$\frac{\partial p}{\partial l}(l) = \eta M, \tag{3.10}$$

where the matrix $M = \|m_{jk}\|$ is given by

$$m_{jk} \ = \ -l_j\beta_k T_{jk} - l_j\sum_{\nu\neq k}(\beta_\nu - \beta_k)l_\nu T_{\nu k}, \quad (j\neq k); \qquad m_{kk} \ = \ -\sum_{j\neq k} m_{jk}.$$

In particular, we have (in the radial and chordal cases respectively)

$$\text{trace } \frac{\partial p}{\partial l}(l) \ = \ \begin{cases} \eta\sum\sum\ (\sigma_j+\sigma_k)\ \cot^2\frac{\theta_{jk}}{2} \\ \eta\sum\sum\ (\sigma_j+\sigma_k)\ \frac{T}{(x_k-x_j)^2} \end{cases}. \tag{3.11}$$

Proof. Denote $m = \sum \beta_j^\eta$. Then

$$\frac{\partial \beta_j^\eta}{\partial l_k} \ = \ -\eta m\ \beta_k l_j\ T_{jk}, \qquad (\text{at } l). \tag{3.12}$$

Indeed,

$$\frac{\partial \beta_j^\eta}{\partial l_k} \ = -\eta\beta_j^{\eta-1}\gamma_{jk} \ = \ -\eta\beta_j^\eta\beta_k T_{jk} = -\eta m l_j\beta_k T_{jk}$$

because $\beta_j^\eta = m l_j$ at the critical point. Similarly, we show

$$\frac{\partial \beta_k^\eta}{\partial l_k} \ = \ \eta m\sum_{j\neq k}\ \beta_j l_j\ T_{jk}. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\frac{\partial m}{\partial l_k} \ = \ \eta m\sum_{j\neq k}\ (\beta_j - \ \beta_k) l_j\ T_{jk}. \tag{3.14}$$

By (3.12) and (3.14), we have the following expression at l:

$$\begin{aligned} \frac{\partial p_j}{\partial l_k} \ &= \ \frac{1}{m}\left[\frac{\partial\beta_j^\eta}{\partial l_k} - l_j\frac{\partial m}{\partial l_k}\right] \\ &= \ -\eta l_j\ \left[\beta_k\ T_{jk} + \sum_\nu\ (\beta_\nu - \ \beta_k) l_\nu\ T_{\nu k}\right], \end{aligned}$$

and similarly,

$$\frac{\partial p_k}{\partial l_k} \ = \ \eta\ \left[\sum_{j\neq k}\beta_j l_j\ T_{jk} - \sum_{j\neq k}\ (\beta_j - \ \beta_k) l_j l_k\ T_{jk}\right].$$

□

Example 1: symmetric configurations. Consider d radial needles of equal length with equal angles. Clearly, such configurations satisfy the geodesic condition and are stationary solutions for all η-models.

Theorem. *The d-fold symmetric needle configuration is asymptotically $*$-stable if*

$$\eta \ < \ \begin{cases} 2(d-2)^{-1}, & \text{if } d \text{ is even,} \\ 2d(d^2-2d-1)^{-1}, & \text{if } d \text{ is odd.} \end{cases}$$

It is not $*$-stable if η is greater than the number to the right.

In particular, three symmetric needles are $*$-stable for all $\eta < 3$, but they are G-stable only for $\eta < 2.11...$ as we claimed in Theorem 5.

Proof. We have

$$l_j = l, \quad \beta_j = \beta, \quad 4c = d\beta l,$$

and so

$$m_{jk} = -\frac{1}{d}\cot^2\frac{\theta_{jk}}{2}, \qquad (j \neq k).$$

The configuration is $*$-stable if $\eta < d/\lambda_{\max}(A)$, where $\lambda_{\max}(A)$ is the largest eigenvalue of the matrix

$$a_{jk} = -\cot^2\frac{|k-j|\pi}{d}, \quad (k \neq j); \qquad a_{kk} = -\sum_{j\neq k} a_{jk}.$$

It remains to show that

$$\lambda_{\max}(A) = \begin{cases} 2^{-1}d(d-2), & \text{if } d \text{ is even,} \\ 2^{-1}(d^2-2d-1), & \text{if } d \text{ is odd.} \end{cases}$$

This must be well known in elementary algebra. For the convenience of the reader we include a proof. Denote $\omega = e^{2\pi i/d}$, and

$$a_\nu = \frac{1+\Re\,\omega^\nu}{1-\Re\,\omega^\nu}, \quad (1 \le \nu \le d-1), \qquad a_0 = \sum_{\nu=1}^{d-1} a_\nu.$$

Since A is a circulant matrix, $A = \mathrm{Circ}\,\{a_0, -a_1, -a_2, \ldots, -a_{d-1}\}$, the eigenvalues are the numbers

$$\sum_{\nu=1}^{d-1} a_\nu(1-\Re\,\omega^{k\nu}), \qquad (0 \le k \le d-1).$$

Note that

$$\frac{1+\Re\,\zeta}{1-\Re\,\zeta}(1-\Re\,\zeta^k) \ = \ \sum_{l=-k}^{k}\alpha_l\zeta^l, \qquad (|\zeta|=1),$$

with

$$\alpha_0 = 2k-1, \quad \text{and} \quad \sum_{l=-k}^{k}\alpha_l = 2k^2,$$

and therefore

$$\sum_{\nu=1}^{d-1} a_\nu(1-\Re\,\omega^{k\nu}) = (2k-1)d - 2k^2, \quad (1 \le k \le d-1).$$

The maximal eigenvalue corresponds to $k = d/2$ if d is even, and to $k = d \pm 1/2$ if d is odd. $\square$

Example: two and three needles. (i) A *two needle chordal* configuration satisfying the η-equation is $*$-stable or unstable according as

$$\eta < \frac{(x_2 - x_1)^2}{T} \qquad \text{or} \qquad \eta > \frac{(x_2 - x_1)^2}{T}.$$

This is immediate from the trace formula (3.11) since one of the eigenvalues of M is zero. For configurations which also satisfy the geodesic condition, we have $T = (x_2 - x_1)^2$, and so the criterion for $*$-stability is $\eta < 1$. Comparing this criterion with the statement of Theorem 4, we see that asymmetric stationary solutions of the geodesic model are simultaneously $*$- and G-stable but the symmetric solution can be $*$-stable and G-unstable.

(ii) Consider a *radial three needle* configuration such that needles 1 and 2 are symmetric with respect to the line of needle 3. We can choose the Riemann map so that the Loewner parameters are

$$\zeta_1 = e^{-i\alpha},\ \zeta_2 = e^{i\alpha},\ \zeta_3 = -1; \qquad \sigma_1 = \sigma_2 = s,\ \sigma_3 = 1 - 2s.$$

Denote

$$A = \tan^2 \frac{\alpha}{2}, \qquad B = \cot^2 \alpha = \frac{(1-A)^2}{4A}.$$

Simple computations show that the set of eigenvalues of the matrix M in (3.10),

$$M = \begin{pmatrix} a+c & -a & -b \\ -a & a+c & -b \\ -c & -c & 2b \end{pmatrix}, \qquad (a := -m_{12} = -m_{21}, \quad \text{etc.}),$$

is $\{0,\ 2b+c,\ 2a+c\} = \{0,\ A,\ \sigma_3 A + 2sB\}$. If the configuration also satisfies the geodesic condition then by (2.24), $A = s/(2-3s)$, so $\sigma_3 A + 2sB = \sigma_3$, and the $*$-stability criterion is

$$\eta < \min\left\{\frac{1}{A},\ \frac{1}{\sigma_3}\right\} = \min\left\{\frac{2-3s}{s},\ \frac{1}{1-2s}\right\}.$$

Comparing the function to the right with the function $\eta_r(s)$, see Section 2.3, we conclude that for some values of η, (namely for $1 < \eta < 2.43...$), the geodesic configurations $\sigma = (s, s, 1-2s)$ with $\eta_r(s) = \eta$ are $*$-stable. We claimed in Theorem 5 that such configurations are not G-stable.

3.3. **Finiteness Theorem.** We will estimate, in terms of η, the number of needles in a $*$-stable stationary solution of the geodesic η-model, and will prove Theorem 2. As we have just seen, G-stability implied $*$-stability in all our examples. If this is true in general, then Theorem 2 would give the same bounds for the number of arms in an stable stationary solution of the geodesic model.

Let us consider the chordal case. The computation is based on the trace formula (3.11), which implies that a stationary solution with d needles is $*$-unstable if

$$\sum_k \sum_{j<k} \frac{T(\sigma_j + \sigma_k)}{|x_j - x_k|^2} \ > \ \frac{d-1}{\eta}. \tag{3.15}$$

Proof of Theorem 2 (in the chordal case). Assume $T = 1$, and denote $T_{jk} = |x_j - x_k|^{-2}$. If $j < k$, then from the stationary geodesic condition we find

$$\begin{aligned}(\sigma_j + \sigma_k)T_{jk} &= 1 + \sum_{i=1}^{j-1} \sigma_i \sqrt{T_{ij}T_{ik}} - \sum_{i=j+1}^{1} \sigma_i \sqrt{T_{ij}T_{ik}} + \sum_{i=k+1}^{d} \sigma_i \sqrt{T_{ij}T_{ik}} \\ &\geq 1 + \sum_{i=1}^{j-1} \sigma_i T_{ik} - \frac{1}{2} \sum_{i=j+1}^{1} \sigma_i (T_{ij} + T_{ik}) + \sum_{i=k+1}^{d} \sigma_i T_{ij}.\end{aligned}$$

Denote

$$\begin{aligned}&\Sigma_1 = (\sigma_1 + \sigma_2)T_{12} + (\sigma_2 + \sigma_3)T_{23} + \cdots + (\sigma_{d-1} + \sigma_d)T_{d-1,d}, \\ &\Sigma_2 = (\sigma_1 + \sigma_3)T_{13} + (\sigma_2 + \sigma_4)T_{24} + \cdots + (\sigma_{d-2} + \sigma_d)T_{d-2,d}, \\ &\ldots \\ &\Sigma_{d-1} = (\sigma_1 + \sigma_d)T_{1d}.\end{aligned}$$

Then we have

$$\begin{aligned}&\Sigma_1 \geq (d-1) + \Sigma_2 + \cdots + \Sigma_{d-1}, \\ &\Sigma_2 \geq (d-2) - \frac{1}{2}\Sigma_1 + \Sigma_3 + \cdots + \Sigma_{d-1}, \\ &\ldots \\ &\Sigma_3 \geq (d-3) - \frac{1}{2}\Sigma_1 - \frac{1}{2}\Sigma_2 + \Sigma_4 + \cdots + \Sigma_{d-1}, \\ &\ldots \\ &\Sigma_{d-1} \geq 1 - \frac{1}{2}\Sigma_1 - \cdots - \frac{1}{2}\Sigma_{d-2}.\end{aligned}$$

Fix $\nu = \nu(d) < d$ to be specified later, and consider only the first ν inequalities:

$$\begin{aligned}\Sigma_1 - \ \Sigma_2 - \cdots - \Sigma_\nu &\geq d - 1 \\ \frac{1}{2}\Sigma_1 + \ \Sigma_2 - \cdots - \Sigma_\nu &\geq d - 2 \\ &\ldots \\ \frac{1}{2}\Sigma_1 + \frac{1}{2}\Sigma_2 + \cdots + \Sigma_\nu &\geq d - \nu\end{aligned}$$

Multiplying the first equation by 1, the second by 4, the third by 4^2, ..., the last by $4^{\nu-1}$, we obtain

$$\left(\frac{1}{3} + \frac{1}{6}4^\nu\right)\Sigma \geq \frac{1}{9}4^d - \frac{d}{3} - \frac{1}{9},$$

where $\Sigma = \Sigma_1 + \cdots + \Sigma_\nu$. In this computation we used the formulae

$$1 + \frac{1}{2}(4 + \cdots + 4^{\nu-1}) = \frac{1}{3} + \frac{1}{6}4^\nu,$$

and

$$\begin{aligned}&(d-1) + 4(d-2) + 4^2(d-3) + \cdots + 4^{\nu-1}(d-\nu) \\ &= \frac{d4^\nu - d}{3} - \frac{(3\nu - 1)4^\nu + 1}{9} \\ &= \frac{(3d - 3\nu + 1)4^\nu - (3d+1)}{9}.\end{aligned}$$

Thus

$$\Sigma \geq \frac{2}{3}\ \frac{(3d-3\nu+1)4^\nu-(3d+1)}{2+4^\nu},$$

and according to (3.15), the configuration is $*$-unstable if

$$\eta > \eta(d,\nu) := \frac{3}{2}\ \frac{(2+4^\nu)(d-1)}{(3d-3\nu+1)4^\nu-(3d+1)} \qquad \text{for some } \nu < d. \tag{3.16}$$

Observe that if $d \gg \nu \gg 1$, then

$$\eta(d,\nu) \sim \frac{3}{2}\ \frac{4^\nu d}{3d4^\nu} = \frac{1}{2},$$

and so given $\eta > 1/2$, for all sufficiently large d we can find $\nu < d$ such that the inequality (3.16) holds. $\diamondsuit$

The expression for $\eta(d,\nu)$ in the proof of the theorem provides an explicit bound for the number of needles in a $*$-stable stationary solution of the geodesic η-model. Let $N_c(\eta)$ denote by the maximal number of needles in the chordal case. Applying (3.16) with $\eta(d,1) = 1$, we get

Corollary. $N_c(\eta) = 1$ *for* $\eta > 1$.

As we mentioned, this statement somewhat supports Brennan's conjecture. From (3.16) we can also find

$$\begin{aligned} N_c(\eta) &= 3 \quad \text{for} \quad 9/11 < \eta < 1;\\ N_c(\eta) &\leq 4 \quad \text{for} \quad \eta > 3/4;\\ N(\eta) &\leq 5 \quad \text{for} \quad \eta > 5/7, \qquad \text{etc.,} \end{aligned}$$

because $\eta(4,2) = 9/11$, $\eta(5,2) = 3/4$, $\eta(6,2) = 5/7, \ldots$

Similar estimates can be done in the radial case. For example, one can show that radial stationary solutions with 3 or more needles are $*$-unstable if $\eta > 3$. As in the chordal case, the finiteness result holds for $\eta > 1/2$. We do not know whether it holds for all $\eta > 0$.

4. Survival and disappearance of arms

4.1. **The maximal angle criterion.** Suppose $\sigma \in \mathcal{G}_d$, see (1.6), is a stable stationary solution for the geodesic η-process with d arms. Considering $\mathcal{G}_d$ as a boundary set of $\mathcal{G}_{d+1}$, one can ask whether σ is stable with respect to the process with $d+1$ arms. In particular, if we add a small additional arm to σ, will this arm survive or dissappear as a result of the η-process?

It seems clear that the answer should be the following. Let $2\pi\alpha_{\max}$ be the maximal angle between the needles (including $\mathbb{R}_+$ in the chordal case). Then the configuration σ is "stable" (i.e. any small additional arm disappears) if

$$\eta > \frac{\alpha_{\max}}{1-\alpha_{\max}}, \tag{4.1}$$

and "unstable" if we have "$<$" in (4.1)

Here is a non-rigorous argument for the chordal version. In the first approximation, let us assume that the new arm is a straight line segment and let us replace the geodesic evolution with the needle dynamics keeping the original angles α_ν constant as well as the angles α' and α'' that the new needle makes inside some α_μ, $\alpha'+\alpha''=\alpha_\mu$. Normalize the needle process so that the sum of lengths of all needles is one at any time and consider the flow (3.2). Instead of time, we can take the Loewner parameter $\sigma_* = \varepsilon \ll 1$ of the new needle for an independent variable, and consider the functions $x_j = x_j(\varepsilon)$, $x_* = x_*(\varepsilon)$, etc. We use the star index for the functions corresponding to the new needle. The preimages of the origin are $t_\nu(\varepsilon)$ for $\nu \neq \mu$, and $t'_*(\varepsilon)$, $t''_*(\varepsilon)$. As $\varepsilon \to 0$, we recover the parameters $x_j(0)$ and $t_\nu(0)$ of the original configuration, and we also have

$$t_\mu(0) = t'_*(0) = t''_*(0) = x_*(0).$$

According to (3.2), the length $l_* = l_*(\varepsilon)$ of the new needle satisfies the differential equation

$$\dot{l}_* = p_* - l_*, \qquad p_*(\varepsilon) := \frac{\beta_*^\eta}{\beta_*^\eta + \sum \beta_j^\eta},$$

and so the new needle will dissappear if

$$p_* \ll l_* \quad \text{as} \quad \varepsilon \to 0.$$

By (2.19), we have

$$l_* \asymp |x_* - t'_*|^{2\alpha'} |x_* - t''_*|^{2\alpha''},$$

and from (2.20) we conclude

$$\frac{1}{\alpha'} - \frac{1}{\alpha_\mu} = \frac{T\varepsilon}{|x_* - t'_*|^2} + o(1).$$

It follows that $|x_* - t'_*|^2 \asymp \varepsilon$, $|x_* - t''_*|^2 \asymp \varepsilon$, and $l_* \asymp \varepsilon^{\alpha_\mu}$. On the other hand, we have

$$p_* \asymp \beta_*^\eta \asymp (l_*^{-1}\varepsilon)^\eta \asymp \varepsilon^{(1-\alpha_\mu)\eta}.$$

Thus the new needle will dissappear if $(1-\alpha_\mu)\eta > \alpha_\mu$ and will survive if $(1-\alpha_\mu)\eta < \alpha_\mu$.

As we already mentioned, the needle dynamics is not a faithful approximation of the geodesic mode but the fact that the functions $p_*(\varepsilon)$ and $l_*(\varepsilon)$ scale with different exponents should be the same in both models (in contrast to the situation of Section 5). The goal of this section is to provide a rigorous proof of the maximal angle criterion in the simplest (two arm chordal) case. We will only prove the survival part of Theorem 1. The reasoning for the second part is completely similar.

4.2. **Inverse Loewner's equation.** Let φ_t be a chordal Loewner chain with two arms. Our first step is to express the numbers $\beta_j(t)$ in terms of the parameter functions $x_j(t)$ and $\mu_j(t)$. As usual this is done by means of the inverse equation

$$\dot{h}_s = -A(h_s, s) = \frac{\mu_1(s)}{h_s - x_1(s)} + \frac{\mu_2(s)}{h_s - x_2(s)}$$

for the transition maps $h_s = \varphi_s^{\circ -1}\varphi_t$, $(0 \le s \le t)$. From now on, t is fixed and dot means the s-derivative. Denote

$$w_j(s) := h_s(x_j(t)),$$

so h_s maps $\mathbb{H}$ onto the halfplane minus slits joining $x_j(s)$ and $w_j(s)$. Clearly, $\varphi_s(w_j(s)) = a_j(t)$, the tips at time t, and

$$\dot{w}_1(s) = \frac{\mu_1}{w_1 - x_1} + \frac{\mu_2}{w_1 - x_2}, \quad w_1(t) = x_1(t),$$
$$\dot{w}_2(s) = \frac{\mu_1}{w_2 - x_1} + \frac{\mu_2}{w_2 - x_2}, \quad w_2(t) = x_2(t).$$

Lemma. *Suppose the chain starts with $\varphi_0(z) = z^2$ and satisfies the geodesic condition. Then*

$$\beta_2(t) = \exp\left\{ -\int_0^t \Re \left[\frac{1}{(w_2 - x_1)^2} + \frac{1}{(w_2 - x_1)(x_2 - x_1)} \right] \mu_1(s)\ ds \right\}.$$

Proof. Denote

$$\kappa(s) = h_s''(x_2(t)).$$

Differentiating the inverse equation twice, we get

$$\frac{\dot{\kappa}}{\kappa} = -A'(w_2) = -\frac{\mu_1}{(w_2 - x_1)^2} - \frac{\mu_2}{(w_2 - x_2)^2}.$$

Since

$$\begin{aligned} \dot{w}_2 - \dot{x}_2 &= \frac{\mu_1}{w_2 - x_1} + \frac{\mu_2}{w_2 - x_2} - \frac{\mu_1}{x_2 - x_1} \\ &= \frac{\mu_2}{w_2 - x_2} - \frac{\mu_1 (w_2 - x_2)}{(w_2 - x_1)(x_2 - x_1)}, \end{aligned}$$

we have

$$\frac{\dot{\kappa}}{\kappa} + \frac{\dot{w}_2 - \dot{x}_2}{w_2 - x_2} = -\frac{\mu_1}{(w_2 - x_1)^2} - \frac{\mu_1}{(w_2 - x_1)(x_2 - x_1)}.$$

Integrating from 0 to t, we obtain

$$\log\left[\kappa(0) w_2(0)\right] = \int_0^t \left(\frac{1}{(w_2 - x_1)^2} + \frac{1}{(w_2 - x_1)(x_2 - x_1)} \right) \mu_1(s)\ ds. \tag{4.2}$$

Here we used the fact that $x_2(0) = 0$, and that

$$\lim_{s \to t} \left[\kappa(s)(w_2(s) - x_2(s))\right] = 1.$$

To see the latter, we note that for s close to t, the map h_s is essentially $z \mapsto \sqrt{z^2 - \varepsilon^2}$ in the coordinates with $x_2(t) = 0$, where ε is the length of the corresponding slit. Then $w_2(s) \sim i\varepsilon$ and $h_s''(0) = 1/(i\varepsilon)$. Since $\varphi_t = h_0^2$, we have $\varphi_t''(x_2(t)) = 2 w_2(0) \kappa(0)$, and

$$\beta_2 = \frac{2}{|\varphi_t''(x_2(t))|} = \frac{1}{|w_2(0)||\kappa(0)|}.$$

The statement now follows from (4.2) □

The same argument shows that in the d-arm case, we have

$$\beta_j(t) = \exp\left\{ -\Re \int_0^t \sum_{k \neq j} \mu_k \left[\frac{1}{(w_j - x_k)^2} + \frac{1}{(w_j - x_k)(x_j - x_k)} \right] ds \right\}. \tag{4.3}$$

The rest of the section is devoted to the proof of the following fact which corresponds to the "survival" part of Theorem 3.

Theorem. *Suppose that for all $t > t_0$, a geodesic Loewner chain satisfies the equations*

$$\mu_1(t) = \frac{\beta_1(t)^\gamma}{\beta_1(t)^\gamma + \beta_2(t)^\gamma}, \qquad \mu_2(t) = \frac{\beta_2(t)^\gamma}{\beta_1(t)^\gamma + \beta_2(t)^\gamma},$$

where γ is some fixed constant. Then $\gamma \geq 2$ if $\mu_2 \to 0$ as $t \to \infty$.

In the proof we assume for simplicity that $x_1(0) = x_2(0) = 0$, so $\Delta(s) := x_2(s) - x_1(s) = \sqrt{2s}$ by the geodesic condition. We only need to study w_2, therefore denote $w_2 = w$ (and in general delete the index 2) and set

$$z = u + iv := w_2 - x_2.$$

The function z satisfies

$$\dot{z} = (1-\mu)\left(\frac{1}{z+\Delta} - \frac{1}{\Delta}\right) + \frac{\mu}{z}, \qquad z(t) = 0.$$

Taking the real and imaginary parts, we have

$$\dot{v} = \left(-\frac{1-\mu}{|z+\Delta|^2} - \frac{\mu}{|z|^2}\right) v, \tag{4.4}$$

and

$$\dot{u} = -\frac{(1-\mu)|z|^2}{\Delta|z+\Delta|^2} - \left(\frac{1-\mu}{|z+\Delta|^2} - \frac{\mu}{|z|^2}\right) u. \tag{4.5}$$

Let us mention some basic properties:

- $v(s) \searrow 0$ on $(0, t)$;
- $v(s)^2 \sim 2\int_s^t \mu(\tau)d\tau$ as $s \to t$;
- $\dot{u}(t) = 0$ and $\ddot{u}(t) > 0$;
- $u > 0$ on $(0, t)$. (This is true because $u > 0$ for $s \to t$, and because by (4.5) we have $\dot{u}(s) < 0$ if $u(s) = 0$.)

4.3. **Main lemma.** Denote

$$M(s) = s^{-1}\left(\int_s^t \mu(\tau)2\tau d\tau\right)^{1/2}.$$

We will repeatedly use the fact that the functions $M(s)$ and $sM(s)$ are decreasing. Also observe that

$$v(s) \leq \sqrt{s}M(s). \tag{4.6}$$

Indeed, if we denote $V = v^2$, then by (4.4) we have

$$\dot{V} = -\frac{1-\mu}{|z+\Delta|^2}2V - \frac{2\mu V}{|z|^2} \geq -\frac{V}{s} - 2\mu, \tag{4.7}$$

and so $(sV)^\cdot \geq -2\mu s$, $V(t) = 0$, which gives (4.6)

Lemma. *If $M(s) \ll 1$, then $u(s) \ll v(s) \sim \sqrt{s}M(s)$.*

The precise meaning of this statement is that given $\varepsilon > 0$, there is $\delta > 0$ such that if $M < \delta$, then $u \le \varepsilon v$, and $v \le \sqrt{s}M \le (1+\varepsilon)v$. A different way to state the lemma is to say that $u \ll v$ as long as $v \ll \Delta$. In this form, the statement is almost obvious from the geodesic nature of the growth, but a rigorous proof still requires some effort.

Proof. Consider the following decreasing function:

$$\varepsilon(s) = \sup_{\sigma \ge s} \frac{u(\sigma)}{v(\sigma)}.$$

We first show that

$$\text{if} \quad M \le 1/2 \ \text{ on } (s,t), \quad \text{then} \quad \varepsilon(s) \le 2M(s). \tag{4.8}$$

To see this, rewrite (4.5) with obvious notation $\dot{u} = -F(\sigma) - g(\sigma)u$. Then we have

$$u(s) = \int_s^t F(\sigma) \ \exp\left\{\int_s^\sigma g(\tau)d\tau\right\} \ d\sigma,$$

where

$$\int_s^\sigma g(\tau)d\tau \le \int_s^\sigma \frac{d\tau}{\Delta^2} = \frac{1}{2}\log\frac{\sigma}{s},$$

and

$$F(\sigma) \le \frac{(1+\varepsilon(\sigma)^2)v(\sigma)^2}{(2\sigma)^{3/2}},$$

so that

$$\begin{aligned} u(s) \ &\le \ \int_s^t \frac{(1+\varepsilon(\sigma)^2)v(\sigma)^2}{(2\sigma)^{3/2}} \ \frac{\sigma^{1/2}}{s^{1/2}} \ d\sigma \\ &\le \ v(s) \ \frac{(1+\varepsilon(s)^2)}{2(2s)^{1/2}} \int_s^t \frac{v(\sigma)}{\sigma} \ d\sigma. \end{aligned}$$

Since

$$\begin{aligned} s^{-1/2}\int_s^t \frac{v(\sigma)}{\sigma}d\sigma &\le s^{-1/2}\int_s^t \frac{\sigma M(\sigma)}{\sigma^{3/2}}d\sigma \qquad \text{by (4.6)} \\ &\le \sqrt{s}M(s)\int_s^\infty \frac{d\sigma}{\sigma^{3/2}} \ = \ 2M(s), \end{aligned}$$

It follows that if $\varepsilon(s) \le 1$, then $u(s)/v(s) < 2M(s)$, which also shows that $\varepsilon(s) \le 1$ if $M \le 1/2$ on (s,t).

Next we claim that *there is an absolute constant* $c > 0$ *such that*

$$M(s) < c \quad \Rightarrow \quad v(s) \ge \frac{1}{2}\sqrt{s}M(s), \tag{4.9}$$

and also that

$$M(s) \ll 1 \quad \Rightarrow \quad v \sim \sqrt{s}M(s). \tag{4.10}$$

The lemma clearly follows from (4.8), (4.9), and (4.10).

To prove (4.9) we rewrite the equation in (4.7) in the form

$$\dot{V} = -\frac{V}{s} - 2\mu + f \tag{4.11}$$

with

$$\begin{aligned} |f| &\le \left|\frac{2V}{|z+\Delta|^2} - \frac{2V}{\Delta^2}\right| + \frac{2\mu V}{|z+\Delta|^2} + \left|\frac{2\mu V}{|z|^2} - 2\mu\right| \\ &\preceq v^2\frac{v^2+u\Delta}{\Delta^4} + \frac{\mu v^2}{\Delta^2} + \mu\varepsilon^2 \\ &\preceq M^4 + \varepsilon M^3 + \mu M^2 + \mu\varepsilon^2 \qquad \text{by (4.6)} \\ &\preceq M^4 + \mu M^2, \end{aligned}$$

where we used the symbol $\preceq$ for inequalities with absolute constants. It follows that if c is small enough, then $f \le M^3 + \mu$, and integrating (4.11) we have

$$\begin{aligned} sv^2(s) &= 2\int_s^t \mu(\tau)\tau d\tau - \int_s^t \tau f(\tau) d\tau \\ &\ge \int_s^t \mu(\tau)\tau d\tau - \int_s^t \tau M^3(\tau) d\tau \\ &\frac{1}{2}s^2M^2(s) - s^3M^3(s)\int_s^\infty \tau^{-2}d\tau \\ &\ge \frac{1}{4}s^2M^2(s). \end{aligned}$$

We prove (4.10) by a similar argument. □

4.4. **Proof of Theorem.** Suppose $\mu_2 \to 0$ and suppose $\gamma < 2$. We want to get a contradiction. Take a small constant $c > 0$ and define a function $a = a(t)$ by the equation $M(a) = c$. Then we have

$$\begin{aligned} s < a \quad &\Rightarrow \quad v(s) > v(a) \sim c\sqrt{a} > c\sqrt{s}, \\ s > a \quad &\Rightarrow \quad v(s) \sim \sqrt{s}M(s) < c\sqrt{s}. \end{aligned}$$

Note that in the latter case, we also have $|z| \ll \Delta$ according to the main lemma. Recall

$$-\log\beta_2(t) = \int_0^t \Re\left[\frac{1}{(z+\Delta)^2} + \frac{1}{\Delta(z+\Delta)}\right](1-\mu(s))\, ds.$$

We first observe

$$\begin{aligned} \int_0^a \Re\left[\frac{1}{(z+\Delta)^2} + \frac{1}{\Delta(z+\Delta)}\right] ds \;&\preceq\; \int_0^a \frac{ds}{v^2(s)} + \int_0^a \frac{ds}{\sqrt{s}v(s)} \\ &\le \frac{1}{a}\int_0^a ds + \frac{1}{\sqrt{a}}\int_0^a \frac{ds}{\sqrt{s}} \;=\; O(c^{-2}). \end{aligned}$$

On the other hand, for $s > a$ we have

$$\Re\left[\frac{1}{(z+\Delta)^2} + \frac{1}{\Delta(z+\Delta)}\right] \sim \frac{2}{\Delta^2} = \frac{1}{s},$$

and it follows that

$$\beta_2(t) \succeq \text{const}\frac{a(t)}{t},$$

with constant depending on c. In the same way we prove that $\beta_1(t) \succeq 1$ because $\mu_1 \to 1$. It now follows that

$$\mu_2(t) \sim \left(\frac{a(t)}{t}\right)^\gamma,$$

and so $a(t) = o(t)$. To finish the proof we will show that if $a(t) = o(t)$ and $\gamma < 2$, then the following two inequalities are contradictory:

$$a(t)^2 \geq \text{const} \int_{a(t)}^{t} \mu(s) s ds,$$

$$\mu(s) \geq \text{const} \left(\frac{a(s)}{s} \right)^{\gamma}.$$

We can of course combine them to get

$$a(t)^2 \geq \text{const} \int_{a(t)}^{t} a(s)^{\gamma} s^{1-\gamma} ds.$$

Denote

$$A_n = \min\{a(t) : \ 2^{n-1} < t < 2^n\},$$

and let t_n be the minimum point, $A_n = a(t_n)$. Since $A_n \ll 2^n$ (by assumption), we have

$$A_n^2 \geq \text{const} \int_{2^{n-2}}^{2^{n-1}} a(s)^{\gamma} s^{1-\gamma} ds \geq \text{const}\ A_{n-1}^{\gamma}\ 2^{n(2-\gamma)}.$$

Let $\varepsilon_n := 2^{-n} A_n$, so that $\varepsilon_n \to 0$. Since $\gamma < 2$, we have $\varepsilon_n^2 \geq \text{const}\ \varepsilon_{n-1}^{\gamma} \geq 2\varepsilon_{n-1}^2$, which is a contradiction. $\diamondsuit$

5. Stability analysis

In this section we study stability properties of stationary solutions of the geodesic model and prove Theorems 4 and 5. Let $\sigma \in \mathcal{G}_d$ be a stationary solution of the η-model. Consider a small perturbation of the Loewner parameters in the time interval $[0, 1]$:

$$\mu_j(t) = \sigma_j + \varepsilon_j(t), \qquad (0 < t < 1).$$

Together with the geodesic condition, the functions $\mu_j(t)$ determine a d-arm Loewner chain φ_t for $t \in [0, 1]$. Let us take the configuration φ_1 as an initial condition to run the (normalized) η-process for $t > 1$, and let $\mu_j(t)$, $t > 1$, be the corresponding parameter functions of the process, so the functions $\varepsilon_j(t) = \mu_j(t) - \sigma_j$ are also defined for $t > 1$. We will write $\|\varepsilon\|_{(0,t)}$ for the $L^\infty(0, t)$-norm. By definition, a stationary solution σ is *stable* if given $c > 0$, there exists a $\delta > 0$ such that

$$\|\varepsilon\|_{(0,1)} < \delta \qquad \Rightarrow \qquad \|\varepsilon\|_{(0,\infty)} < c,$$

and σ is *asymptotically stable* if, in addition,

$$\|\varepsilon\|_{(0,1)} < \delta \qquad \Rightarrow \qquad \lim_{t \to \infty} \varepsilon(t) = 0.$$

5.1. **Linearization of Loewner's equation.** Consider a general geodesic Loewner chain $\{\varphi_t\}_{t>0}$ with $\mu_j(t) = \sigma_j + \varepsilon_j(t)$. Denote the β-numbers by $\beta_j(t)$ and write β_j^σ for the β-numbers of the stationary chain. We want to linearize the formula (4.3) for the β-numbers and to estimate the kernels (at $s = 0$ in particular).

Lemma. *Given $\sigma \in \mathcal{G}_d$, there are continuous functions*

$$h_{jk} = h_{jk}^\sigma : \ (0, 1] \to \mathbb{R}, \qquad |h_{jk}(s)| \asymp \frac{1}{\sqrt{s}} \quad \text{as } s \to 0,$$

such that

$$\beta_j(t) = \beta_j^\sigma + \sum_k \int_0^1 h_{jk}(s)\varepsilon_k(st)ds + O\left(\int_0^1 |\varepsilon(st)|^2 \frac{ds}{\sqrt{s}}\right),$$

where the O-bound depends only on σ.

We will occasionally use the vector notation and write $h * \varepsilon$ for the multiplicative convolution operation.

Proof. It is enough to prove the statement for $t = 1$, for it then follows for arbitrary t by the scaling properties of stationary chains. Namely, let us fix t and observe that in the chordal case, the chain

$$\tilde{\varphi}_s(z) = t^{-1}\varphi_{st}(\sqrt{t}z), \qquad 0 < s < 1,$$

is driven by the functions

$$\tilde{\mu}_j(s) = \mu_j(st), \qquad \tilde{x}_j(s) = t^{-1/2}x_j(st).$$

Indeed,

$$\tilde{\varphi}_s'(z) = t^{-1/2}\varphi_{st}'(\sqrt{t}z),$$

and so the poles $\tilde{x}_j(s)$ of $\tilde{\varphi}_s$ are as stated. We also have

$$\partial_s\tilde{\varphi}_s(z) = \dot{\varphi}_{st}'(\sqrt{t}z),$$

and by the Loewner equation,

$$\sum_j \frac{\tilde{\mu}_j(s)}{\tilde{x}_j(s) - z} \ = \ \sum_j \frac{\sqrt{t}\ \mu_j(st)}{x_j(st) - \sqrt{t}z},$$

so $\tilde{\mu}_j(s) = \mu_j(st)$. The chain $\tilde{\varphi}_s$ satisfies the geodesic condition:

$$\partial_s\tilde{x}_j(s) = \sqrt{t}\dot{x}_j(ts) = \sum_{k\neq j} \frac{\mu_k(st)}{t^{-1/2}x_j(st) - t^{-1/2}x_k(st)} = \sum_{k\neq j} \frac{\tilde{\mu}_k(s)}{\tilde{x}_j(s) - \tilde{x}_k(t)}.$$

Finally, observe $\tilde{\mu}(s) = \sigma + \varepsilon(st)$. Applying the identity $\tilde{\varphi}_s''(z) = \varphi_{st}''(\sqrt{t}z)$ at $z = \tilde{x}_j(s)$, we get

$$\beta(t) = \tilde{\beta}(1) = \beta^\sigma + \int_0^1 h(s)\varepsilon(st)ds + O\left(\int_0^1 |\varepsilon(st)|^2 \frac{ds}{\sqrt{s}}\right),$$

provided that the estimate for $t = 1$ is known.

Let us now prove the statement for $t = 1$. For simplicity we indicate the argument in the two arms chordal case. (If there are more than two arms, we also have to consider the variation in the geodesic condition.) Changing time,

$$d\tilde{t} = (\mu_1(t) + \mu_2(t))\ dt,$$

we can consider a normalized geodesic chain with

$$\mu_1(t) = \sigma_1 + \varepsilon(t), \qquad \mu_2(t) = \sigma_2 - \varepsilon(t).$$

By Lemma 4.2, we have

$$-\log \beta_1(1) = \Re \int_0^1 \left[\frac{1}{(z+\Delta)^2} + \frac{1}{\Delta(z+\Delta)} \right] \mu_2(s) ds, \tag{5.1}$$

where $\Delta(s) = \sqrt{2s}$, and $z = z(s)$ satisfies the inverse equation

$$\dot{z} = \frac{\mu_1}{z} - \frac{\mu_2 z}{\Delta(z+\Delta)}, \qquad z(1) = 0.$$

We will write $Z(s)$ for the solution of the corresponding inverse equation with $\varepsilon(t) \equiv 0$; note that $|Z(s)| \asymp \sqrt{1-s}$. Define the function $\zeta = z - Z$. From (5.1) we find

$$\log \frac{\beta_1(1)}{\beta_1^\sigma} = \int_0^1 \varepsilon A + \int_0^1 \zeta B + \dots, \tag{5.2}$$

where

$$A = \Re \left[\frac{1}{(Z+\Delta)^2} + \frac{1}{\Delta(Z+\Delta)} \right] = \frac{a}{\Delta} + \text{bdd}, \qquad a = \Re Z(0)^{-1} > 0, \tag{5.3}$$

$$B = \Re \left[\frac{2\sigma_2}{(Z+\Delta)^3} + \frac{\sigma_2}{\Delta(Z+\Delta)^2} \right] = \frac{\text{const}}{\Delta} + \text{bdd},$$

("bdd" means "a bounded function"), and the error term in (5.2) is

$$O\left(\int_0^1 \Delta^{-1}(|\varepsilon|^2 + |\zeta|^2) \right).$$

Next we notice that ζ satisfies the equation

$$\dot{\zeta} = -f\zeta + g, \qquad \zeta(1) = 0, \tag{5.4}$$

with

$$g = \left(\frac{1}{Z} + \frac{Z}{\Delta(Z+\Delta)} \right) \varepsilon + O\left(\frac{|\varepsilon||\zeta|}{|Z|^2} + \frac{|\zeta|^2}{|Z|^3} \right), \tag{5.5}$$

and

$$f = \frac{\sigma_1}{Z^2} + \frac{\sigma_2}{(Z+\Delta)^2} = \frac{\dot{Z}}{Z} + \frac{\sigma_2}{\Delta(Z+\Delta)} + \frac{\sigma_2}{(Z+\Delta)^2}. \tag{5.6}$$

Solving (5.4) for ζ, we have

$$\zeta(s) = -\int_s^1 g(\tau) e^{\int_s^\tau f} d\tau. \tag{5.7}$$

From (5.6), we derive the estimate

$$e^{\int_s^\tau f} d\tau \asymp \left| \frac{Z(\tau)}{Z(s)} \right|,$$

which we use together with (5.5) and (5.7) to show that

$$\zeta(s) \preceq \frac{1}{|Z(s)|} \int_s^1 \frac{|\varepsilon|}{\Delta}.$$

Returning to (5.2), we see that

$$\int_0^1 \zeta B = \int_0^1 \varepsilon C + \dots$$

with a *bounded* kernel C and an error term

$$O\left(\int_0^1 \Delta^{-1}(|\varepsilon|^2 + |\zeta|^2) + |Z|^{-1}|\zeta|^2\right) = O\left(\int_0^1 \Delta^{-1}|\varepsilon|^2\right).$$

□

Remark. It follows from the proof (namely, from the inequality $a > 0$ in (5.3) and from the boundedness of the kernel C) that $h_{jk}(0) = -\infty$ for $j \neq k$. The values $h_{jk}(1)$ are also negative: if we use the notation $\gamma_{jk} = -\partial\beta_j/\partial l_k$ of Section 3.1, then

$$h_{jk}(1) \;=\; -\frac{T\gamma_{jk}}{\beta_k^\sigma} \overset{(3.4)}{=} -\frac{T\beta_j^\sigma}{(x_j - x_k)^2}.$$

Thus one can expect $h_{jk}(s) < 0$ for all s if $j \neq k$. This seems to be intuitively clear: the increase in μ_k results in the decrease of β_j. We will need this fact in connection with the stability criterion which we discuss next.

Proposition. *If $d = 2$ in the chordal case, or if $d = 3$ and $\sigma_1 = \sigma_2$ in the radial case, then the kernels $h_{jk}^\sigma(s)$, $j \neq k$, are negative.*

The computer assisted verification is based on the following approach. Let $\varphi_t(z) = t\varphi(t^{-1/2}z)$ be the stationary solution, and let φ_t be the normalized geodesic chain with $\mu_1 = \sigma_1 + \varepsilon$, where $\varepsilon(t) \equiv \sigma_2$ for $t \in (1, 1+\alpha)$ and 0 otherwise. The statement follows if we can show that for an infinitesimally small $\alpha > 0$ we have

$$\forall t, \qquad |\varphi_t''(x_t)| < |\varphi_t''(X_t)|,$$

where $X_t = X_1(t)$ and $x_t = x_1(t)$ are the poles. Representing the chains in the form

$$\varphi_t = \varphi_1 \circ [\varphi_1^{\circ -1} \circ \varphi_{1+\alpha}] \circ [\varphi_{1+\alpha}^{\circ -1} \circ \varphi_t] \equiv \varphi \circ h \circ \psi, \qquad \varphi_t = \varphi_{1+\alpha} \circ \Psi,$$

we observe that $|\psi_t''(x_t)| = |\Psi_t''(X_t)|$, so the inequality to verify is

$$1 - \frac{|\varphi'(h(p_t))| \cdot |h'(p_t)|}{|\varphi_{1+\alpha}'(P_t)|} > 0, \tag{5.8}$$

where $p_t = \psi(x_t)$ and $P_t = \Psi(X_t)$. Since α is small, we can use the approximation

$$h(z) \approx \sqrt{(z - x_1)^2 - 2\alpha} + x_1,$$

to express the main term (as $\alpha \to 0$) in the left hand side of (5.8) in terms of elementary functions and the explicit function $t \mapsto P_t$.

5.2. **A stability criterion.** Let us return to the situation described in the beginning of this section. Denote

$$q_j(t) = \frac{\beta_j^{1+\eta}(t)}{\beta_1^{1+\eta}(t) + \cdots + \beta_d^{1+\eta}(t)},$$

so $\mu(t) = q(t)$ for $t > 1$ by the definition of a (normalized) η-process. By the last lemma, we have

$$q(t) = q^\sigma + (k * \varepsilon)(t) + O\left(\int_0^1 |\varepsilon(st)|^2 \frac{ds}{\sqrt{s}}\right), \tag{5.9}$$

where k is some kernel depending on η and σ, namely,

$$k_{j\nu} = (1+\eta)\sum_k q_k q_j \left(\frac{h_{j\nu}}{\beta_j} - \frac{h_{k\nu}}{\beta_k}\right). \tag{5.10}$$

Since $q(t) - q^\sigma = \mu(t) - \sigma = \varepsilon(t)$ for $t > 1$, the function $\varepsilon(t)$ satisfies the equation

$$\varepsilon(t) = (k * \varepsilon)(t) + O\left(\int_0^1 |\varepsilon(st)|^2 \frac{ds}{\sqrt{s}}\right), \qquad t \geq 1. \tag{5.11}$$

We will only use this equation in a scalar case. The following lemma will be applied to establish asymptotic stability of stationary solutions.

Lemma. *Suppose $\varepsilon(t)$ is a real-valued function satisfying*

$$\varepsilon(t) = \int_0^1 k(s)\varepsilon(st)ds + a(t)\int_0^1 \frac{\varepsilon^2(st)}{\sqrt{s}}ds, \quad (t \geq 1), \qquad \textit{with} \quad |a(t)| \leq A. \tag{5.12}$$

Suppose also

$$\int_0^1 |k(s)|ds < 1,$$

Then if $\varepsilon_0 > 0$ is small enough and $\|\varepsilon\|_{(0,1)} < \varepsilon_0$, then $|\varepsilon(t)| < \varepsilon_0$ for all $t > 0$, and $\varepsilon(t) \to 0$.

Proof. Choose ε_0 so that $\int_0^1 |k| + 2A\varepsilon_0 < 1$. If there is a point $t > 1$ such that $\varepsilon_0 = |\varepsilon(t)| > |\varepsilon(\tau)|, \qquad (\forall\, \tau < t)$, then

$$\varepsilon_0 = |\varepsilon(t)| < \varepsilon_0 \int_0^1 |k| + A\varepsilon_0^2 \int_0^1 \frac{ds}{\sqrt{s}} = \varepsilon_0 (\int_0^1 |k| + 2A\varepsilon_0) < \varepsilon_0,$$

a contradiction. Thus $\varepsilon(t) < \varepsilon_0$ for all t.

Let us show that $\varepsilon(t) \to 0$. We derive this from the inequality

$$|\varepsilon(t)| \leq \int_0^1 K(s)\varepsilon(st)ds, \qquad (t \geq 1), \tag{5.13}$$

where $K(s) = k(s) + s^{-1/2}A\varepsilon_0$. Note that $\rho := \int_0^1 K < 1$. Define

$$\varepsilon_n = \max\{|\varepsilon(t)|,\ 2^{n-1} \leq t \leq 2^n\},$$

and choose t_n with

$$|\varepsilon(t_n)| = \varepsilon_n.$$

Choose also some slowly growing function $N = N(n)$, e.g. $N(n) = [\sqrt{n}]$. By (5.13), we have

$$\begin{aligned}\varepsilon_n \le \varepsilon_n \int_{2^{n-1}/t_n}^{1} K + \varepsilon_{n-1} \int_{2^{n-2}/t_n}^{2^{n-1}/t_n} K + \cdots + \varepsilon_{n-N} \int_{2^{n-N-1}/t_n}^{2^{n-N}/t_n} K \\ + \int_0^{2^{n-N-1}/t_n} K \\ := a_0(n)\varepsilon_n + \cdots + a_N(n)\varepsilon_{n-N} + O\left(2^{-N/2}\right).\end{aligned}$$

Since

$$a_0 + \cdots + a_N \le \rho < 1,$$

we have

$$\frac{a_1 + \cdots + a_N}{1 - a_0} \le \frac{\rho - a_0}{1 - a_0} < \rho,$$

and hence

$$\varepsilon_n \le \rho \max_{1 \le j \le N} \varepsilon_{n-N} + C 2^{-N/2}.$$

Let the maximum be assumed for $j = j_1$. Writing N_1 for $N(n)$, N_2 for $N(n - j_1)$, etc., we have

$$\begin{aligned}\varepsilon_n &= \rho \varepsilon_{n-j_1} + O\left(2^{-N_1/2}\right), \\ \varepsilon_{n-j_1} &= \rho \varepsilon_{n-j_2} + O\left(2^{-N_2/2}\right), \\ &\cdots\end{aligned}$$

and

$$\varepsilon_n \le \rho^p \varepsilon_0 + C \sum_{\nu=1}^{p} 2^{-N_\nu/2} \rho^\nu$$

with $p = p(n) \to \infty$ as $n \to \infty$. It follows that $\varepsilon_n \to 0$. □

The next lemma will be used to establish instability of stationary solutions.

Lemma. *Suppose* $\varepsilon(\cdot)$ *satisfies the relation* (5.12)*. Suppose also that the kernel* $k(s)$ *is positive, continuous on* $(0, 1]$*,* $k(s) \asymp s^{-1/2}$ *as* $s \to 0$*, and*

$$\int_0^1 k(s) ds > 1.$$

Then there is a constant $c > 0$ *such that no matter how small the norm* $\|\varepsilon\|_{(0,1)}$ *is, we have* $\varepsilon(t) > c$ *for some* t *provided that* $\varepsilon(\cdot)$ *is non-negative on* $[0, 1]$.

Proof. Let us choose $c > 0$ such that the function

$$K(s) = k(s) - \frac{Ac}{\sqrt{s}}$$

is positive on $(0, 1)$ and $\int_0^1 K > 1$. This can be done because $\sqrt{s} K(s) > \text{const} > 0$ by assumption. Suppose $\varepsilon(\cdot)$ is non-negative on $(0, 1)$. Assuming $\varepsilon(t) < c$ for all t, we will arrive at a contradiction.

Note that $\varepsilon(\cdot) > 0$ on $[1, \infty)$ for if $t \ge 1$ is the first zero, then

$$0 = \varepsilon(t) \ge \int_0^1 k(s)\varepsilon(st) ds - Ac \int_0^1 \frac{\varepsilon(st) ds}{\sqrt{s}} = \int_0^1 K(s)\varepsilon(st) ds > 0.$$

It follows that the inequality

$$\varepsilon(t) \geq \int_0^1 K(s)\varepsilon(st)ds \tag{5.14}$$

holds for all $t > 0$. Fix $\lambda \in (0,1)$ such that $Q := \int_\lambda^1 K > 1$. Then by (5.14),

$$\varepsilon(t) \geq Q \min_{[\lambda t, t]} \varepsilon(\cdot).$$

Let $t_1 \in [\lambda t, t]$ be the minimum point. Let t_2 be the minimum point of $\varepsilon(\cdot)$ on $[\lambda t_1, t_1]$, etc. Then

$$\varepsilon(t) \geq Q\varepsilon(t_1) \geq Q^2\varepsilon(t_2) \geq \ldots,$$

and so for large t we have $\varepsilon(t) \geq c$. □

Remark. The linear part $\varepsilon = k * \varepsilon$ of the equation (5.9) has the form

$$\varepsilon(t) - (V\varepsilon)(t) = g(t), \qquad (t > 1),$$

where the operator V is defined by the formula

$$(V\varepsilon)(t) = \frac{1}{t}\int_1^t \varepsilon(s)k\left(\frac{s}{t}\right)ds,$$

and $g(t) = t^{-1}\int_0^1 \varepsilon(s)k(t^{-1}s)ds$ is a given function. We get a traditional form of the Wiener-Hopf equation after we change the variables $t = e^x$, $s = e^y$. It may be interesting to take a look at the stability problem from the point of view of the general theory of Wiener-Hopf equations.

Now we turn to the proof of Theorems 4 and 5. We will use the notation and results of Section 2.3. We first consider the case of *two chordal* arms.

5.3. **Proof of Theorem 4.** Consider the family

$$\sigma(s) = (s, 1-s) \in \mathcal{G}_2, \qquad 0 < s < 1,$$

of stationary geodesic chains with 2 arms in the chordal case. Denote

$$Q = Q(s,\eta) = \frac{\beta_1^{1+\eta}(s)}{\beta_1^{1+\eta}(s) + \beta_2^{1+\eta}(s)}.$$

By the definition of the function $\eta_c(\cdot)$, we have the identity

$$Q(s, \eta_c(s)) \equiv s, \tag{5.15}$$

which holds for all s including $s = 1/2$ provided that $\eta_c(1/2)$ is defined as a limit. Differentiating (5.15), we get

$$\partial_s Q(s, \eta_c(s)) + \eta_c'(s) \cdot \partial_\eta Q(s, \eta_c(s)) \equiv 1. \tag{5.16}$$

Note that

$$\log\frac{P}{1-P} = (1+\eta)B, \qquad B := \log\frac{\beta_1}{\beta_2},$$

and therefore

$$\left[\frac{1}{Q} + \frac{1}{1-Q}\right]\partial_s Q = (1+\eta)B' > 0, \quad (0 < s < 1), \tag{5.17}$$

$$\left[\frac{1}{Q}+\frac{1}{1-Q}\right]\partial_\eta Q \;=\; B\begin{cases}<0, & \text{if} \quad s<1/2,\\ >0, & \text{if} \quad s>1/2.\end{cases} \tag{5.18}$$

Suppose $\sigma(s)$ is a solution of the η-model, i.e. $Q(s,\eta)=s$. We want to apply the criterion of the previous section. Consider some perturbation $\varepsilon_j(t)$ of σ_j as described in the definition of stability. In the normalized case we have just one unknown function:

$$\varepsilon_1=\varepsilon(t), \qquad \varepsilon_2=-\varepsilon(t),$$

and so the equation (5.11) reduces to the scalar equation

$$\varepsilon=m*\varepsilon+\dots$$

with a positive kernel $m=m^{s,\eta}=k_{11}-k_{12}>0$, see (5.10) and Proposition 5.1. Thus the stability criterion is $\int_0^1 m<1$. Observe now that

$$\int_0^1 m=\partial_s Q(s,\eta).$$

This follows if we apply (5.9) to constant functions $\varepsilon_j(t)$ and use the definition of Q. We now consider two cases.

If $s\neq 1/2$, then $\eta=\eta_c(s)$, and by (5.16), the stability condition is

$$\eta_c'(s)\cdot\partial_\eta Q(s,\eta_c(s))>0.$$

This inequality is always true because of (5.18) and the properties of $\eta_c(\cdot)$, see Section 2.3. Hence all asymmetric stationary solutions are asymptotically stable.

If $s=1/2$, then η can be arbitrary, but in any case $Q=1/2$. Using (5.16) and (5.17) we see that the condition for stability is

$$\frac{1+\eta}{4}B'(1/2)=\partial_s Q(1/2,\eta)<1.$$

Applying (5.17) again, we have

$$\frac{1+\eta_c(s)}{4}B'(1/2)=\partial_s Q(1/2,\eta_c(1/2))=1,$$

and so η has to be less than $\eta_c(1/2)$ for the solution $\sigma(1/2)$ to be stable. ♢

Finally we turn to the case of *three radial* arms.

5.4. **Proof of Theorem 5.** Consider the family

$$\sigma(s)=(s,s,1-2s)\in\mathcal{G}_3, \qquad 0<s<1/2,$$

of stationary geodesic chains. Suppose $\sigma(s)$ is a solution of the geodesic η-model. Linearizing the η-equation in a neighborhood of $\sigma(s)$, we get the system (5.11):

$$\varepsilon=\varepsilon*k+\dots, \qquad k=k^{s,\eta}.$$

By symmetry, we have

$$k_{11}=k_{22}, \quad k_{12}=k_{21}, \quad k_{13}=k_{23}, \quad k_{31}=k_{32},$$

and so the system is

$$\begin{cases}\varepsilon_1-\varepsilon_2=(\varepsilon_1-\varepsilon_2)*(k_{11}-k_{12})+\dots\\ \varepsilon_3=(\varepsilon_1+\varepsilon_2)*k_{31}+\varepsilon_3*k_{33}+\dots\end{cases} \tag{5.19}$$

In the normalized case ($\varepsilon_1+\varepsilon_2+\varepsilon_3=0$) we can take $\varepsilon_1-\varepsilon_2$ and ε_3 for unknown functions and separate the variables in the linear part of the system:

$$\begin{cases}\varepsilon_1-\varepsilon_2=(\varepsilon_1-\varepsilon_2)*a+\dots\\ \varepsilon_3=\varepsilon_3*b+\dots\end{cases}$$

where the scalar kernels $a=k_{11}-k_{12}$ and $b=k_{33}-k_{31}$ are positive by Proposition 5.1. From the argument of Section 5.2 it follows that the stability criterion is

$$\int_0^1 a<1 \qquad \& \qquad \int_0^1 b<1. \tag{5.20}$$

Note that if $s=1/3$, then there is an additional symmetry, and we have $a=b$.

The next step is to express the integrals in terms of the function

$$Q(s_1,s_2;\eta)=q_1(s_1,s_2,1-s_1-s_2;\eta)\equiv\frac{\beta_1^{1+\eta}}{\beta_1^{1+\eta}+\beta_2^{1+\eta}+\beta_3^{1+\eta}},$$

where β_j's are the beta-numbers of the geodesic stationary chain $(s_1,s_2,1-s_1-s_2)\in\mathcal{G}_3$. We claim

$$\int_0^1 a=\partial_1Q-\partial_2Q, \qquad \int_0^1 b=\partial_1Q+\partial_2Q, \tag{5.21}$$

where the derivatives are computed at $(s,s;\eta)$. Indeed, for any constant δ we have

$$q_3(s+\delta,s+\delta,1-2s-2\delta;\eta)=1-2Q(s+\delta,s+\delta). \tag{5.22}$$

From the second equation in (5.19) with $\varepsilon_1=\varepsilon_2=\delta$ and $\varepsilon_3=-2\delta$ we see that the left hand side in (5.22) is equal to

$$1-2s+\delta(K_{31}+K_{32}-2K_{33})+\dots, \qquad K_{ij}:=\int_0^1 k_{ij},$$

Since the right hand side is $1-2s-2\delta(\partial_1Q+\partial_2Q)+\dots$, we get

$$\partial_1Q+\partial_2Q=K_{33}-K_{31}=\int_0^1 b.$$

The proof of the formula for $\int_0^1 a$ is similar.

Differentiating the identity $Q(s,s;\eta_r(s))=s$, which holds for all s including $s=1/3$, we get

$$(\partial_1Q+\partial_2Q)(s,s,\eta_r(s))+\eta_r'(s)\partial_\eta Q(s,s,\eta_r(s))=1, \tag{5.23}$$

and in particular,

$$(\partial_1Q+\partial_2Q)(1/3,1/3;\eta_r(1/3))=1. \tag{5.24}$$

Let us analyze the condition $\int_0^1 b<1$. If $s\neq1/3$, then $\sigma(s)$ is an solution for the geodesic model with $\eta=\eta_r(s)$, and by (5.21), (5.23) we have

$$\int_0^1 b=(\partial_1Q+\partial_2Q)(s,s;\eta_r(s))=1-\eta_r'(s)\partial_\eta Q(s,s;\eta_r(s)).$$

This is less than one iff $\eta_r'(s)\partial_\eta Q(s,s;\eta_r(s)) > 0$, and since $\eta_r'(s) < 0$ for all s, we have

$$\int_0^1 b < 1 \quad \Longleftrightarrow \quad B_{13} \equiv \log\frac{\beta_1}{\beta_3} < 0, \quad \text{i.e.} \quad s < 1/3. \tag{5.25}$$

Here we used the identity $\partial_\eta Q = Q(1-2Q)B_{13}$, which is obtained by differentiating with respect to η the identity

$$(1+\eta)B_{13}(s,s,1-2s) = \log\frac{Q(s,s;\eta)}{1-2Q(s,s;\eta)}. \tag{5.26}$$

If $s = 1/3$, then η can be arbitrary, and $\int_0^1 b < 1$ iff

$$(\partial_1 Q + \partial_2 Q)(1/3, 1/3; \eta) < 1 = (\partial_1 Q + \partial_2 Q)(1/3, 1/3; \eta_r(1/3)),$$

which holds iff $\eta < \eta_r(1/3)$. To see the latter, we use the formula

$$(\partial_1 Q + \partial_2 Q)(1/3, 1/3; \eta) = \frac{1+\eta}{9}\frac{d}{ds}B_{13}(s,s,1-2s)|_{s=1/3},$$

which is obtained by differentiating (5.26) with respect to s, and note that the derivative in the right hand side is positive. This completes the proof of the theorem in the symmetric case ($s = 1/3$) since then $a = b$, as we mentioned.

To finish the proof in the non-symmetric case, by (5.25) it is enough to show

$$\int_0^1 a > 1 \qquad \text{if} \qquad 0 < s < 1/3. \tag{5.27}$$

The proof of (5.27) is based on a fact stated in Section 2.3. Fix $s < 1/3$ and write η for $\eta_r(s)$. Consider the functions

$$B(\delta) = B_{12}(s+\delta, s-\delta), \quad \Sigma(\delta) = \log\frac{s+\delta}{s-\delta}, \qquad (\delta \in \mathbb{R}),$$

and for $\delta \neq 0$ define $E(\delta)$ by the equation

$$(1+E(\delta))B(\delta) = \Sigma(\delta). \tag{5.28}$$

By Proposition 2 in Section 2.3,

$$E(0) = \lim_{\delta\to 0} E(\delta) = \tilde{\eta}(s) < \eta.$$

Also note that $E(\cdot)$ is a positive even smooth function, in particular $E'(0) = 0$. Differentiating (5.28) and taking the limit as $\delta \to 0$, we get

$$(1+E(0))B'(0) = \Sigma'(0). \tag{5.29}$$

The computation of $\int_0^1 a$ goes as follows. If we differentiate the identity

$$\log\frac{Q(s+\delta, s-\delta;\ \eta)}{Q(s-\delta, s+\delta;\ \eta)} \ = \ (1+\eta)\ B(\delta)$$

with respect to δ and set $\delta = 0$, then we have

$$\frac{2}{s}\ (\partial_1 Q - \partial_2 Q)(s,s;\eta) \ = \ (1+\eta)\ B'(0)$$

Since $2/s = \Sigma'(0)$, we conclude

$$\int_0^1 a \overset{(5.20)}{=} \frac{(1+\eta)B'(0)}{\Sigma'(0)} \overset{(5.29)}{=} \frac{1+\eta}{1+E(0)} \ > \ 1.$$

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