BOUND STATES OF DISCRETE SCHRÖDINGER OPERATORS WITH SUPER-CRITICAL INVERSE SQUARE POTENTIALS

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Abstract. We consider discrete one-dimensional Schrödinger operators whose potentials decay asymptotically like an inverse square. In the super-critical case, where there are infinitely many discrete eigenvalues, we compute precise asymptotics of the number of eigenvalues below a given energy E as this energy tends to the bottom of the essential spectrum.

1. Introduction

This paper is concerned with discrete one-dimensional Schrödinger operators in $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. That is,

$$Hu(n) = -\Delta u(n) + V(n)u(n), \quad \Delta u(n) = u(n+1) - 2u(n) + u(n-1),$$

where we impose a Dirichlet boundary condition, $u(0) = 0$.

If $V(n) \to 0$ as $n \to \infty$, then zero is the bottom of the essential spectrum of H. We are interested in the discrete spectrum of H below zero. Thus, for $E \geq 0$, we define

$$N_E(H) = \dim \operatorname{Ran} P_H((-\infty, -E]),$$

where P_H is the family of spectral projections associated with H by the spectral theorem. It is well known that $V(n) \sim -n^{-2+\varepsilon}$ produces finitely many eigenvalues if $\varepsilon < 0$ and infinitely many if $\varepsilon > 0$ and so inverse square decay is critical for the existence of infinitely many discrete eigenvalues below the essential spectrum. Furthermore, by the discrete analogue of Kneser's theorem, the discrete spectrum below zero of the operator H with potential $V(n) = -cn^{-2}$ is finite when $c \leq \frac{1}{4}$ and infinite when $c > \frac{1}{4}$ (see, e.g., [2, 8, 9]).

Our goal is to study the behavior of $N_E(H)$ as $E \downarrow 0$ in the super-critical case $c > \frac{1}{4}$. This question is also motivated by recent results on a connection between singular spectrum embedded in the essential spectrum and the discrete spectrum of a given Schrödinger operator. See [1, 3, 5, 6] and especially [4, Sect. 2].

In the continuous case, Kirsch and Simon carried out an investigation of $N_E(H)$ for super-critical inverse square potentials [7] (see also [10] for extensions). We prove the discrete analogue of their result. On the one hand, this case is more relevant to the question raised in [4, Sect. 2]. On the other hand, the proof of Kirsch and Simon uses some arguments that do not carry over directly to the discrete case: They scale the spatial variable and use exact solvability of the Euler differential

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equation. Spatial scaling is not possible in the discrete case and, while there exists a discrete Euler equation, it is not symmetric.

Theorem 1. *Suppose*

$$V(n) = -\frac{c}{n^2} + W(n), \qquad c > \frac{1}{4},$$

where W *is a decaying sequence such that* $N_0(-\Delta + \gamma W) < \infty$ *for all* $\gamma \in \mathbb{R}$. *Then*

$$\lim_{E \downarrow 0} \frac{N_E(-\Delta + V)}{-\ln(E)} = \frac{1}{2\pi}\sqrt{c - \tfrac{1}{4}} \tag{1.1}$$

Remarks. (i) We say that a sequence W is decaying if $W(n) \to 0$ as $n \to \infty$.
(ii) The hypothesis on W is satisfied, for example, if $\sum_{n>0} n|W(n)| < \infty$. See [12, Thm. 5.10].
(iii) The whole-line case can be reduced to the half-line case by Dirichlet decoupling.
(iv) For perturbations of the form $V(n) = \frac{c}{n^2} + W(n)$, an analogous result holds near the top of the essential spectrum.

2. Proof of Theorem 1

As a preparation we state the discrete analog of Proposition 5 from [7]. The proof is analogous.

Lemma 2. *Let* V, W *be decaying sequences. Then for every* $E > 0$ *and* $0 < \varepsilon < 1$ *we have*

$$N_E\left(-\Delta + V + W\right) \leq N_E\left(-\Delta + \tfrac{1}{1-\varepsilon} V\right) + N_E\left(-\Delta + \tfrac{1}{\varepsilon} W\right),$$
$$N_E\left(-\Delta + V + W\right) \geq N_E\left(-\Delta + (1-\varepsilon) V\right) - N_E\left(-\Delta - \tfrac{1-\varepsilon}{\varepsilon} W\right).$$

Now we come to the proof of our main theorem. We start with

$$V_c(n) = -\frac{c}{n^2}$$

and replace it by $V_{E,c}$ which is just $V_c - E$ on $\{n : V_c(n) \leq -E\}$ and equal to V otherwise. To investigate the asymptotics of $N_E(-\Delta + V_{E,c})$ we split our domain into two parts by cutting at $\sqrt{\frac{c}{E}}$. For the first part, we will compute the asymptotics of N_E directly. The remaining part does not contribute to N_E. Then we use Lemma 2 to show that N_E has the same asymptotics for $V_{E,c}$ and $V = V_c + W$.

Lemma 3. *We have*

$$\lim_{E \downarrow 0} \frac{N_E(-\Delta + V_{E,c})}{-\ln(E)} = \frac{1}{2\pi}\sqrt{c - \tfrac{1}{4}}.$$

Proof. We first decompose $-\Delta + V_{E,c}$ into two parts by imposing an additional Dirichlet boundary condition at $\lfloor\sqrt{\frac{c}{E}}\rfloor$. Since this constitutes a rank-one resolvent perturbation it will not affect the limit. By the choice of our cut point, the part with $n > \lfloor\sqrt{\frac{c}{E}}\rfloor$ does not contribute and by oscillation theory (see e.g. [11] or [12, Ch. 4]) it suffices to count the number of sign flips of some solution of $(-\Delta + V_{E,c})u = -Eu$ on $(1, \sqrt{\frac{c}{E}})$, that is, the number of sign flips of some solution of $(-\Delta + V_c)u = 0$ on $(1, \sqrt{\frac{c}{E}})$.

Unfortunately, $(-\Delta + V_c)u = 0$ is not explicitly solvable, but

$$\tilde{u}_c(n) = \sqrt{n} \exp\left(\mathrm{i}\sqrt{c - \tfrac{1}{4}}\ln(n)\right)$$

solves $(-\Delta + \tilde{V}_c)\tilde{u} = 0$ with the complex-valued potential

$$\tilde{V}_c(n) = \frac{\Delta \tilde{u}_c(n)}{\tilde{u}_c(n)} = -\frac{c}{n^2} + O(\frac{1}{n^3}).$$

Moreover, it is straightforward to check (cf. [12, Lemma 7.10], resp. [8]) that $-\Delta u + V_c u = 0$ has a solution u_c which asymptotically looks like $\tilde{u}_c(n)$. Taking the real part of u_c, we see that the number of sign flips behaves to leading order like $-\frac{1}{2\pi}\sqrt{c - \frac{1}{4}}\ln(E)$. □

Let us prove the upper bound in (1.1). By Lemma 2,

$$\begin{aligned} N_E(-\Delta + V_c + W) &= N_E(-\Delta + (V_c - \chi_{(1-\varepsilon)E,c}) + N_E(\chi_{(1-\varepsilon)E,c} + W)) \\ &\le N_E(-\Delta + V_{E,c/(1-\varepsilon)}) + N_E(-\Delta + \tfrac{1}{\varepsilon}(\chi_{(1-\varepsilon)E,c} + W)), \end{aligned}$$

where $\chi_{E,c} = E\chi_{(0,\sqrt{c/E})}$ and χ_Ω is the characteristic function of the set Ω. Using

$$N_E(-\Delta + \tfrac{1}{\varepsilon}(\chi_{(1-\varepsilon)E,c} + W)) \le N_0(-\Delta + \tfrac{1}{\varepsilon}W),$$

the assumption on W, and Lemma 3, we see that

$$\limsup_{E\downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \le \frac{1}{2\pi}\sqrt{\tfrac{c}{1-\varepsilon} - \tfrac{1}{4}}$$

for every $0 < \varepsilon < 1$, that is,

$$\limsup_{E\downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \le \frac{1}{2\pi}\sqrt{c - \tfrac{1}{4}}. \tag{2.1}$$

It remains to show the lower bound in (1.1). By Lemma 2,

$$\begin{aligned} N_E(-\Delta + V_c + W) &= N_E(-\Delta + (V_c - \chi_{E/(1-\varepsilon),c}) + N_E(\chi_{E/(1-\varepsilon),c} + W)) \\ &\ge N_E(-\Delta + V_{E,(1-\varepsilon)c}) - N_E(-\Delta - \tfrac{1-\varepsilon}{\varepsilon}(\chi_{E/(1-\varepsilon),c} + W)) \end{aligned}$$

Observe that it suffices to show that the second summand does not contribute to the limit. Invoking Lemma 2 a second time we have

$$N_E(-\Delta - \tfrac{1-\varepsilon}{\varepsilon}(\chi_{E/(1-\varepsilon),c} + W)) \le N_E(-\Delta - \tfrac{1}{\varepsilon}\chi_{E/(1-\varepsilon),c}) + N_E(-\Delta - \tfrac{1-\varepsilon}{\varepsilon^2}W).$$

The second term is bounded for fixed ε as $E \downarrow 0$ by assumption and it remains to investigate the first one. As before we impose a Dirichlet boundary condition at $\lfloor\sqrt{\frac{c(1-\varepsilon)}{E}}\rfloor$ and we need to count the sign flips of the solution of $-\Delta u - \frac{E}{\varepsilon(1-\varepsilon)}u = -Eu$ on $(0, \sqrt{\frac{c(1-\varepsilon)}{E}})$. Since this equation is explicitly solvable we obtain

$$N_E(-\Delta - \tfrac{1}{\varepsilon}\chi_{E/(1-\varepsilon),c}) = \sqrt{c(1 - \varepsilon - \tfrac{1}{\varepsilon})} + O(E).$$

Hence

$$\liminf_{E\downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \ge \frac{1}{2\pi}\sqrt{(1-\varepsilon)c - \tfrac{1}{4}}$$

for every $0 < \varepsilon < 1$ and thus,

$$\liminf_{E\downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \ge \frac{1}{2\pi}\sqrt{c - \tfrac{1}{4}}. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain the assertion of the theorem. □

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