LYAPUNOV EXPONENTS OF CONTINUOUS SCHRÖDINGER COCYCLES OVER IRRATIONAL ROTATIONS

KRISTIAN BJERKLÖV, DAVID DAMANIK, AND RUSSELL JOHNSON

Abstract. We consider the Lyapunov exponent of those continuous $\mathrm{SL}(2,\mathbb{R})$-valued cocycles over irrational rotations that appear in the study of Schrödinger operators and prove generic results related to large coupling asymptotics and uniform convergence.

1. Introduction

We consider Schrödinger cocycles of the form

$$A_{f,E}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

over irrational rotations $\omega \mapsto \omega + \alpha$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let us denote the associated Lyapunov exponent by $\gamma_f(E)$, that is,

$$\gamma_f(E) = \inf_{n \geq 1} \frac{1}{n} \int_{\mathbb{T}} \log \| A^n_{f,E}(\omega) \| \, d\omega, \tag{2}$$

where $A^n_{f,E}(\omega)$ is given by

$$A^n_{f,E}(\omega) = A_{f,E}((n-1)\alpha + \omega) \times \cdots \times A_{f,E}(\omega). \tag{3}$$

We also write $A^n_f = A^n_{f,0}$ and $\gamma_f = \gamma_f(0)$.

The dynamics of these cocycles are crucial in the study of the spectral properties of the discrete quasi-periodic Schrödinger operator

$$[H_{\alpha,f}\psi](n) = \psi(n+1) + \psi(n-1) + f(n\alpha + \omega)\psi(n). \tag{4}$$

Given an energy E, one often considers the one-parameter family of cocycles, $A^n_{\lambda f,E}(\omega)$, where $\lambda \in (0,\infty)$. Many papers have been devoted to the problem of proving that $\gamma_{\lambda f}(E)$ is positive for λ sufficiently large, possibly along with quantitative estimates that show that the large λ behavior is of order $\log \lambda$; for example, [2, 5, 11, 12, 18]. It is especially desirable to prove such quantitative bounds uniformly in the energy E.

Our first result shows that such uniform bounds are extremely unstable:

Theorem 1. *Let $\alpha \in \mathbb{T}$ be irrational. For every countable set $\{\lambda_m\}_{m \in \mathbb{Z}_+} \subset (0,\infty)$, we have that for a residual set of f's in $C(\mathbb{T})$,*

$$\inf_{E \in \mathbb{R}} \gamma_{\lambda_m f}(E) = 0$$

for every $m \in \mathbb{Z}_+$.

Date: October 13, 2005.

If $A_{f,E}$ is replaced by an arbitrary continuous map $A : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$, we obtain a general quasi-periodic $\mathrm{SL}(2, \mathbb{R})$-valued cocycle. Motivated by Mañé [16, 17], Bochi has shown that there is a residual set $\mathcal{R} \subset C(\mathbb{T}, \mathrm{SL}(2, \mathbb{R}))$ such that for $A \in \mathcal{R}$, either A is uniformly hyperbolic or $\gamma(A) = 0$ [3]; also see the paper [9] by Fabbri and Johnson for a similar result that holds for a generic set of pairs (α, f). Here, a cocycle A^n is called uniformly hyperbolic if $\gamma(A) > 0$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)\| = \gamma(A)$$

uniformly in $\omega \in \mathbb{T}$. Our second result shows that the analogue of the Bochi-Mañé result holds for the specific case of Schrödinger cocycles:

Theorem 2. *Let $\alpha \in \mathbb{T}$ be irrational. The set*

$$\left\{ f \in C(\mathbb{T}) : A_f^n \text{ is uniformly hyperbolic or } \gamma_f = 0 \right\}$$

is residual.

Theorem 2 is a special case of a recent result of Bochi and Viana that is obtained by completely different methods [4]. We feel that it is worthwhile to present our alternate proof since it appears to be somewhat simpler and is closely related to our arguments that lead to Theorem 1.

We note that Fabbri [8] has worked out the Schrödinger cocycle analogue of the Fabbri-Johnson result, that is, she proves this dichotomy for generic pairs (α, f). Her approach uses rational approximation and does not seem suitable to prove a result like Theorem 2.

Since we want to keep the frequency α fixed, we cannot work with rational approximations. Rather, we will approximate continuous f's by discontinuous ones. The key ingredients in our approach are recent results for f's taking on finitely many values [6, 7], along with a description of the spectrum of the operator $H_{\alpha,f}$ in terms of zero exponents or non-uniform hyperbolicity [15].

We will recall the results from [6, 7, 15] used in the proofs of Theorems 1 and 2 in Section 2 and then prove the two theorems in Section 3.

2. Preliminaries

In this section we recount some known results that we will need in our proofs of Theorems 1 and 2. The relevant papers are [6, 7, 15]. We only state the results in the form we will need later in the paper. Each theorem in this section holds in greater generality. We refer the interested reader to the papers listed above for more information and more general statements.

Consider a Schrödinger operator $H_{\alpha,f}$ of the form given in (4) with $\alpha \in \mathbb{T}$ irrational, $\omega \in \mathbb{T}$ arbitrary and $f \in C(\mathbb{T})$. Since f is continuous, the spectrum of $H_{\alpha,f}$ is independent of ω. Recall the definition of the cocycle $A_{f,E}^n$ as given in (1) and (3) and the associated Lyapunov exponent $\gamma_f(E)$. Here, E is a real number, called the energy.

Consider an energy E with $\gamma_f(E) > 0$. The cocycle $A_{f,E}^n$ is called uniformly hyperbolic if

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_{f,E}^n(\omega)\| = \gamma_f(E) \text{ uniformly in } \omega \in \mathbb{T}, \tag{5}$$

and it is called non-uniformly hyperbolic otherwise. Lenz [15] (see also Johnson [13]) has shown the following:

Theorem 3 ([15]). *Suppose $\alpha \in \mathbb{T}$ is irrational. Then,*

$$\sigma\left(H_{\alpha,f}\right) = \{E : \gamma_f(E) = 0 \text{ or } A^n_{f,E} \text{ is non-uniformly hyperbolic}\}. \tag{6}$$

This shows in particular that $\sigma\left(H_{\alpha,f}\right) = \{E : \gamma_f(E) = 0\}$ if (5) holds for every $E \in \mathbb{R}$. The papers [6, 7, 14] are devoted to proving the latter property for certain classes of base dynamics and Schrödinger cocycles. We only recall one specific consequence; compare [6, Theorem 1] and [7, Theorem 10].

Theorem 4 ([6, 7]). *Suppose $\alpha \in \mathbb{T}$ is irrational and $f : \mathbb{T} \to \mathbb{R}$ is of the form*

$$f(\omega) = \sum_{m=1}^{M} f_m \chi_{[\beta_{m-1},\beta_m)}(\omega),$$

where $0 = \beta_0 < \beta_1 < \cdots < \beta_M = 1$ are rational numbers and $f_1, \dots, f_M$ are real. Then (5) holds for every $E \in \mathbb{R}$ and, consequently, $\sigma\left(H_{\alpha,f}\right) = \{E : \gamma_f(E) = 0\}$.

Remark. The characterization (6) of the spectrum does not require f to be continuous. Even if $\sigma\left(H_{\alpha,f}\right)$ is not ω-independent, it is by general principles always ω-independent on a full measure subset of $\mathbb{T}$ and (6) gives a description of this set. The functions f from Theorem 4, however, lead to operators $H_{\alpha,f}$ whose spectra are ω-independent. This can be shown using minimality of the base dynamics and semi-continuity of the spectrum with respect to strong approximation by translates.

3. Proofs

In this section we prove Theorems 1 and 2. We begin with a result that will quickly yield Theorem 1. Its proof, which is inspired by [1], will also suggest how to prove Theorem 2.

Proposition 1. *The set*

$$\left\{ f \in C(\mathbb{T}) : \inf_{E \in \mathbb{R}} \gamma_f(E) = 0 \right\}$$

is residual.

Proof. Denote

$$M_\delta = \{f \in C(\mathbb{T}) : \gamma_f(E) < \delta \text{ for some } E \in \mathbb{R}\}.$$

We will to show that M_δ is open and dense for every $\delta > 0$. It follows that

$$M_0 = \bigcap_{\delta > 0} M_\delta$$

is residual.

It follows from upper-semicontinuity of the Lyapunov exponent that M_δ is open. Indeed, if $f \in M_\delta$ and there exist $f_n \in C(\mathbb{T}) \setminus M_\delta$ with $\|f - f_n\|_\infty \to 0$, then pick $E \in \mathbb{R}$ with $\gamma_f(E) < \delta$. But

$$\gamma_f(E) \geq \limsup_{n\to\infty} \gamma_{f_n}(E) \geq \delta,$$

which is a contradiction.

To show that M_δ is dense, we will approximate a given $g \in C(\mathbb{T})$ by a step function s whose points of discontinuity are all rational. This will ensure that $\gamma_s(E) = 0$ for some energy E. Then we approximate s by a continuous function and use upper-semicontinuity again. This will yield $f \in C(\mathbb{T})$ for which we have $\|f - g\|_\infty$ as small as prescribed and $\gamma_f(E) < \delta$.

Fix some $\delta > 0$ and let $\varepsilon > 0$ be given. For $g \in C(\mathbb{T})$, we need to find $f \in M_\delta$ such that $\|f - g\|_\infty < \varepsilon$. Choose a step function s that has finitely many points of discontinuity, all of which are rational, such that $\|s - g\|_\infty < \frac{\varepsilon}{2}$ and the jumps of s are bounded by $\frac{\varepsilon}{2}$. By Theorem 4, γ_s vanishes on the spectrum of $-\Delta + s(n\alpha + \omega)$, which is a non-empty subset of $\mathbb{R}$. Thus, choose a value of E with $\gamma_s(E) = 0$. Approximate s in L^1-sense by continuous functions f_n that obey $\|s - f_n\|_\infty < \frac{\varepsilon}{2}$. By upper semi-continuity, we have that

$$0 = \gamma_s(E) \geq \limsup_{n \to \infty} \gamma_{f_n}(E),$$

and hence we can choose a value of n such that the function $f = f_n$ has the desired properties, $f \in M_\delta$ and $\|f - g\|_\infty < \varepsilon$. □

Proof of Theorem 1. A slight modification of the proof just given shows that, for each fixed $\lambda > 0$, the set

$$G_\lambda = \left\{ f \in C(\mathbb{T}) : \inf_{E \in \mathbb{R}} \gamma_{\lambda f}(E) = 0 \right\}$$

is residual. Taking countable intersections, the assertion of the theorem thus follows from the Baire Category Theorem. □

Proof of Theorem 2. The proof is a refinement of the argument from the proof of Proposition 1. We have to show that

$$\left\{ f \in C(\mathbb{T}) : A_f^n \text{ is non-uniformly hyperbolic} \right\}$$

is nowhere dense. This will follow once we prove that

$$M_\gamma = \left\{ f \in C(\mathbb{T}) : A_f^n \text{ is non-uniformly hyperbolic and } \gamma_f \geq \gamma \right\}$$

is nowhere dense for every $\gamma > 0$. By upper-semicontinuity, M_γ is closed. We therefore need to prove that M_γ does not contain an open set.

The assertion will follow once we can find, for any given $\gamma > 0$, $f \in M_\gamma$, and $\varepsilon > 0$, a function $g \in C(\mathbb{T})$ with $\|f - g\|_\infty < \varepsilon$ and $\gamma_g < \gamma$. Choose a sequence of step functions s_m subject to the following conditions:

- For every m, $\|f - s_m\|_\infty < \frac{\varepsilon}{4}$ and s_m has a finite number of points of discontinuity, all of which are rational, and the jumps of s_m are bounded by $\frac{\varepsilon}{2}$.
- $\|f - s_m\|_\infty \to 0$ as $m \to \infty$.

The second condition guarantees that, for each $\omega \in \mathbb{T}$, the operator $H_m = \Delta + s_m(\cdot\alpha + \omega)$ converges strongly to the operator $H = \Delta + f(\cdot\alpha + \omega)$. Since A_f^n is non-uniformly hyperbolic, we have that $0 \in \sigma(H)$. By strong convergence, there are $E_m \in \sigma(H_m)$ such that $E_m \to 0$. Choose m large enough so that $|E_m| < \frac{\varepsilon}{4}$. Then $s = s_m - E_m$ is a step function satisfying

- $\|f - s\|_\infty < \frac{\varepsilon}{2}$ and s has a finite number of points of discontinuity, all of which are rational, and the jumps of s are bounded by $\frac{\varepsilon}{2}$.
- 0 belongs to the spectrum of $\tilde{H} = \Delta + s(\cdot\alpha + \omega)$.

Consequently, $\gamma_s = 0$ by Theorem 4. Approximate s in L^1-sense by continuous functions g_k that obey $\|s - g_k\|_\infty < \frac{\varepsilon}{2}$. By upper semi-continuity, we have that

$$0 = \gamma_s \geq \limsup_{k \to \infty} \gamma_{g_k},$$

and hence we can choose a value of k such that the function $g = g_k$ has the desired properties, $\gamma_g < \gamma$ and $\|f - g\|_\infty < \varepsilon$. $\square$

References

[1] A. Avila and D. Damanik, Generic singular spectrum for ergodic Schrödinger operators, to appear in *Duke Math. J.*

[2] K. Bjerklöv, Positive Lyapunov exponent and minimality for a class of 1-d quasi-periodic Schrödinger equations, *Ergod. Th. & Dynam. Sys.* **25** (2005), 1015–1045

[3] J. Bochi, Genericity of zero Lyapunov exponents, *Ergod. Th. & Dynam. Sys.* **22** (2002), 1667–1696

[4] J. Bochi and M. Viana, The Lyapunov exponents of generic volume-preserving and symplectic maps, *Ann. of Math.* **161** (2005), 1–63

[5] J. Bourgain and M. Goldstein, On nonperturbative localization with quasi-periodic potential, *Ann. of Math.* **152** (2000), 835–879

[6] D. Damanik and D. Lenz, A condition of Boshernitzan and uniform convergence in the Multiplicative Ergodic Theorem, to appear in *Duke Math. J.*

[7] D. Damanik and D. Lenz, Zero-measure Cantor spectrum for Schrödinger operators with low-complexity potentials, to appear in *J. Math. Pures Appl.*

[8] R. Fabbri, On the Lyapunov exponent and exponential dichotomy for the quasi-periodic Schrödinger operator, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **5** (2002), 149–161

[9] R. Fabbri and R. Johnson, On the Lyapounov exponent of certain $\mathrm{SL}(2, \mathbb{R})$-valued cocycles, *Differential Equations Dynam. Systems* **7** (1999), 349–370

[10] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic ergodic systems, *Ann. Inst. Henri Poincaré Probab. Statist.* **33** (1997), 797–815

[11] M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions, *Ann. of Math.* **154** (2001), 155–203

[12] M. Herman, Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant the caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, *Comment. Math. Helv* **58** (1983), 4453–502

[13] R. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients, *J. Differential Equations* **61** (1986), 54–78

[14] D. Lenz, Uniform ergodic theorems on subshifts over a finite alphabet, *Ergod. Th. & Dynam. Sys.* **22** (2002), 245–255

[15] D. Lenz, Singular continuous spectrum of Lebesgue measure zero for one-dimensional quasicrystals, *Commun. Math. Phys.* **227** (2002), 119–130

[16] R. Mañé, Oseledec's theorem from the generic viewpoint, In *Proceedings of the ICM* (*Warsaw, 1983*), 1269–1276, PWN, Warsaw, 1984

[17] R. Mañé, The Lyapunov exponents of generic area preserving diffeomorphisms, In *International Conference on Dynamical Systems* (*Montevideo, 1995*), 110–119, Pitman Res. Notes Math. Ser. **362**, Longman, Harlow, 1996

[18] E. Sorets and T. Spencer, Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials, *Commun. Math. Phys.* **142** (1991), 543–566

Department of Mathematics, University of Toronto, Canada
E-mail address: `bjerklov@math.toronto.edu`

Mathematics 253–37, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: `damanik@caltech.edu`

Dipartimento di Sistemi e Informatica, Università di Firenze, 50139 Firenze, Italy
E-mail address: `johnson@dsi.unifi.it`