For a proof, see, for example, Browder [**160**, p. 236].

The motivating special case is $X = \partial\mathbb{D}$ and $\mathfrak{A}$ the $\|\cdot\|_\infty$ closure of the polynomials. Then $\widehat{\mathfrak{A}}$ is $\bar{\mathbb{D}}$ and each $\varphi \in \bar{\mathbb{D}}$ has a unique representing measure. If $\varphi = re^{i\psi} \in \mathbb{D}$, then $d\mu_\varphi(\theta) = P_r(\psi,\theta)\frac{d\theta}{2\pi}$ and (2.6.17) is (2.6.13)!

The classes of algebra where it is known that each $\varphi \in \widehat{\mathfrak{A}}$ has a unique representing measure are *Dirichlet algebras*, where $\{\operatorname{Re} f \mid f \in \mathfrak{A}\}$ is $\|\cdot\|_\infty$-dense in $C_{\mathbb{R}}(X)$, the real functions in $\mathfrak{A}$, and *logmodular algebras* where $\{\log|f| \mid f \in \mathfrak{A},\ f \text{ is invertible}\}$ is $\|\cdot\|_\infty$-dense in $C_{\mathbb{R}}(X)$. The condition of uniqueness is far from automatic. For the case $X = \mathbb{D}$ and $\mathfrak{A}$, the $\|\cdot\|_\infty$ closure of the polynomials, representations of, say, $f \mapsto f(0)$ are highly nonunique.

The history leading to Theorem 2.6.5 is complex. The earliest hints of such a result are in Arens-Singer [**54**]. Bochner [**134**] made the important observation that ideas of Helson-Lowdenslager [**513**] used to prove Szegő-type theorems for the polydisk were more widely applicable. This led to proofs by Wermer [**1105**] for Dirichlet algebras and by Hoffman [**550**] for logmodular algebras. In particular, Hoffman's paper is partly expository and extremely clear.

The realization that all that is really needed is uniqueness of representing measures is due to Lumer [**742**]. There have been extensions from $C(X)$ to general measure spaces; see Barbey-König [**79**] and references therein. Interestingly enough, the Function Algebra literature invariably refers to the "Szegő-Kolmogorov-Krein" theorem, ignoring Verblunsky and Geronimus (see the Notes to Section 2.3).

2.7. Szegő Asymptotics and Analysis of Difference Equations

As we have shown, if $\sum_{j=0}^\infty |\alpha_j| < \infty$, then it is easily seen (see (1.5.16)) that $\sup_{n,z\in\overline{\mathbb{D}}} |\Phi_n(z)| < \infty$, from which (1.5.11) implies $\Phi_n^*(z)$ converges uniformly on $\overline{\mathbb{D}}$. That one has convergence inside $\mathbb{D}$ if only $\sum_{j=0}^\infty |\alpha_j|^2 < \infty$ is one of the wonderful consequences of Szegő's theorem, but its proof in Section 2.4 is a little bit magical. In this section, our goal is to discuss some general subtle results in the theory of difference equations which provide a second proof that $\Phi_n^*(z)$ has a limit when $|z| < 1$ and $\sum_{j=0}^\infty |\alpha_j|^2 < \infty$. From this point of view, the existence of a limit for

$$\begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \cdots \begin{pmatrix} z & -\bar{\alpha}_0 \\ -\alpha_0 z & 1 \end{pmatrix}$$

will depend on the fact that the limit of individual matrices, namely, $\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$, has eigenvalues of distinct magnitude and that the ℓ^2 terms are off-diagonal.

The results we will focus on are discrete analogs of ordinary differential equations results of Levinson and of Hartman-Wintner. We start with Levinson's theorems. We look for solutions of

$$u(n+1) = B(n)u(n) \qquad n = 1, 2, \dots \tag{2.7.1}$$

where $u \in \mathbb{C}^d$ and $B(n)$ is a $d \times d$ matrix with

$$\det(B(n)) \neq 0 \qquad n = 1, 2, \dots \tag{2.7.2}$$

We will start out with the case

$$B(n) = B_0 + (\delta B)(n) \tag{2.7.3}$$

where

$$\sum_{n=1}^{\infty} \|\delta B(n)\| < \infty \tag{2.7.4}$$

THEOREM 2.7.1 (Discrete Levinson Theorem). *Suppose* (2.7.2)–(2.7.4) *hold where* B_0 *has distinct eigenvalues* $\lambda_1, \dots, \lambda_d$ *with eigenvectors* $e_1, \dots, e_d$*. Then there exist initial conditions* $u^{(j)}(1)$ *so that, as* $n \to \infty$*, the solutions,* $u^{(j)}$*, of* (2.7.1) *obey*

$$\lambda_j^{-n} u^{(j)}(n) \to e_j \tag{2.7.5}$$

Remarks. 1. If $(\delta B)(n) \equiv 0$, the solutions are $\lambda_j^n e_j$, so this is a perturbation theorem.

2. It is not hard to see that the $u^{(j)}(n)$ are linearly independent, and so a basis of solutions. For if $\sum_{j=0}^d a_j u^{(j)} \equiv 0$, order the λ's so

$$|\lambda_d| \geq |\lambda_{d-1}| \geq \cdots \geq |\lambda_1| \tag{2.7.6}$$

Then, in an inner product where $\{e_\ell\}_{\ell=1}^d$ are orthonormal, $\langle e_d, \lambda_d^{-n}(\sum_{j=1}^d a_j u^{(j)}(n))\rangle \to a_d$, so $a_d = 0$. In this way, one sees inductively that all $a_j = 0$.

3. This result is related to the Poincaré-Perron theorem (see Theorem 9.6.5). It has stronger hypotheses and a stronger conclusion.

4. The proof works if the λ_j's are not distinct so long as as B_0 is diagonalizable. Moreover, to construct $u^{(j)}$, we only need that $\{\lambda \mid |\lambda| = |\lambda_j|\}$ are all free of Jordan anomalies (i.e., their algebraic and geometric multiplicities are equal). There are some extensions to allow nontrivial Jordan blocks.

PROOF. We begin with some simplifying remarks. First, we can suppose that $\lambda_j = 1$ by replacing $B(n)$ and B_0 by $\lambda_j^{-1}B(n)$ and $\lambda_j^{-1}B_0$. Second, we will pick an inner product on $\mathbb{C}^d$, in which $\{e_\ell\}_{\ell=1}^d$ is orthonormal, and the associated norms. Thus, B_0 is normal. Third, if we find $u^{(j)}(n_0)$ so that $\lambda_j^{-n}B(n)\dots B(n_0)u^{(j)}(n_0) \to e_j$, then since $\{B(k)\}_{k=1}^{n_0-1}$ are invertible, we can find $u^{(j)}(1)$. Picking n_0 so $\sum_{n_0}^\infty \|\delta B(n)\| < 1$, we see that, without loss of generality, we can suppose (by renumbering)

$$\sum_{n=1}^\infty \|\delta B(n)\| < 1 \tag{2.7.7}$$

Finally, we not only suppose (2.7.6) holds but that $|\lambda_{j-1}| < |\lambda_j| = 1$.

It will be convenient to define P_+ as the orthogonal projection onto the span of eigenspaces for $\lambda_j, \lambda_{j+1}, \dots, \lambda_d$ and $P_- = 1 - P_+$. Thus (with $|\lambda_{j-1}| < 1$), we have

$$\|P_- B_0^k\| \le |\lambda_{j-1}|^k \qquad \|P_+ B_0^{-k}\| \le 1 \tag{2.7.8}$$

for $k = 0, 1, 2, \dots$ since B_0 is normal.

To motivate the key to the proof, we begin by rewriting (2.7.1) in "integral form" (i.e., the analog of going from ODE's to integral equations). By induction,

$$\begin{aligned} B(n+k)\dots B(n) - B_0^{k+1} = (\delta B)(n+k)B(n+k-1)\dots B(n) \\ + B_0(\delta B)(n+k-1)B(n+k-2)\dots B(n) \\ + \dots + B_0^k(\delta B)(n) \end{aligned}$$

Thus, if (2.7.1) holds, then

$$u(n+k) - B_0^k u(n) = \sum_{\ell=0}^{k-1} B_0^{(k-1-\ell)}(\delta B)(n+\ell)u(n+\ell) \tag{2.7.9}$$

which can be rewritten

$$u(n) = B_0^{-k}u(n+k) - \sum_{\ell=0}^{k-1} B_0^{-1-\ell}(\delta B)(n+\ell)u(n+\ell) \tag{2.7.10}$$

If all λ's are in $\overline{\mathbb{D}}$, then (2.7.9) is good, because then $B_0^{k-1-\ell}$ remains bounded. If all λ's are in $\mathbb{C}\backslash\mathbb{D}$, (2.7.10) is good, because then $B_0^{-1-\ell}$ remains bounded. This motivates using (2.7.9) on $\operatorname{ran} P_-$ and (2.7.10) on $\operatorname{ran} P_+$ and suggests we consider the equation

$$u(n) = e_j + (\mathcal{A}u)(n) \tag{2.7.11}$$

where

$$(\mathcal{A}x)(n) = \sum_{\ell=0}^{n-1} P_- B_0^{n-1-\ell}(\delta B)(\ell)x(\ell) - \sum_{\ell=n}^{\infty} P_+ B_0^{n-1-\ell}(\delta B)(\ell)x(\ell) \tag{2.7.12}$$

Let $\mathcal{L}$ be the space of sequences x with values in $\mathbb{C}^d$ and

$$\|x\|_\infty \equiv \sup_n |x(n)| \tag{2.7.13}$$

Then, by (2.7.8),

$$|(\mathcal{A}x)(n)| \leq \left[\sum_{\ell=0}^{n-1} |\lambda_{j-1}|^{n-1-\ell} \|\delta B(\ell)\| + \sum_{\ell=n}^{\infty} \|\delta B(\ell)\| \right] \|x\|_\infty \tag{2.7.14}$$

showing that the sum in (2.7.12) converges absolutely and defines $\mathcal{A}$ as an operator from $\mathcal{L}$ to $\mathcal{L}$ with

$$\|\mathcal{A}\| \leq \sum_{\ell=0}^{\infty} \|\delta B(\ell)\| < 1 \tag{2.7.15}$$

by (2.7.7). Thus, $1-\mathcal{A}$ is invertible, and if e_j is the function in $\mathcal{L}$ with constant value e_j, we have

$$(1-\mathcal{A})^{-1}e_j = \sum_{\ell=1}^{\infty} \mathcal{A}^\ell e_j \tag{2.7.16}$$

solves (2.7.11).

We want to show that this solution, u, of (2.7.11) solves (2.7.1) and obeys (2.7.5), that is, $u(n) \to e_j$ as $n \to \infty$. By the bound (2.7.14), $|(\mathcal{A}x)(n)| \to 0$ since $|\lambda_j| < 1$ and (2.7.14) says

$$|(\mathcal{A}x)(n)| \leq \left[\sum_{\ell=n/2}^{\infty} \|\delta B(\ell)\| + |\lambda_{j-1}|^{-n/2-1} \sum_{\ell=0}^{n/2} \|\delta B(\ell)\| \right] \|x\|_\infty$$

Since $\mathcal{A}x \to 0$, (2.7.11) says that $u(n) \to e_j$.

To see that the solution of (2.7.11) obeys (2.7.1), we note that from (2.7.12),

$$(\mathcal{A}x)(n+1) = B_0[(\mathcal{A}x)(n)] + (\delta B)(n)x(n) \tag{2.7.17}$$

Thus, if u solves (2.7.11), we have

$$\begin{aligned} u(n+1) &= e_j + (\mathcal{A}u)(n+1) \\ &= B_0 e_j + B_0(\mathcal{A}u)(n) + (\delta B)(n)u(n) \\ &= (B_0 + \delta B)(n)u(n) \end{aligned}$$

since $B_0 e_j = e_j$. Thus, u solves (2.7.1). $\square$

In the basis where B_0 is diagonal, it is only the off-diagonal elements that need to be ℓ^1 if we strengthen the condition on the eigenvalues and slightly modify the conclusion:

THEOREM 2.7.2. *Let $B(n)$ obey (2.7.2) and*

$$B(n) = B_0(n) + (\delta B)(n) \tag{2.7.18}$$

where (2.7.4) holds, each $B_0(n)$ is diagonal, and $B_0(n) \to B_\infty$, a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$. Suppose for all $i \neq j$, $|\lambda_i| \neq |\lambda_j|$. Then for each j, there is a solution $u^{(j)}(n)$ of (2.7.1) so that

$$\left[\prod_{\ell=1}^{n} b_{jj}(\ell)\right]^{-1} u^{(j)}(n) \to e_j \tag{2.7.19}$$

PROOF. Relabel so $|\lambda_d| > |\lambda_{d-1}| > \cdots > |\lambda_1|$. By replacing $B(n)$ by $b_{jj}(n)^{-1}B_n$, we can suppose $b_{jj}(n) = 1$. Since for $k < j$,

$$\limsup_{n\to\infty} |b_{kk}(n)| < 1 \tag{2.7.20}$$

and for $k > j$,

$$\limsup_{n\to\infty} |b_{kk}(n)|^{-1} < 1 \tag{2.7.21}$$

we see

$$\sup_{n,\ell,k<j} \left|\prod_{m=n}^{n+\ell} b_{kk}(m)\right| + \sup_{n,\ell,k>j} \left|\prod_{m=n}^{n+\ell} b_{kk}(m)\right|^{-1} < \infty \tag{2.7.22}$$

and for $k < j$,

$$\lim_{n\to\infty} \prod_{m=n/2}^{n} b_{kk}(m) = 0 \tag{2.7.23}$$

we can mimic the proof of Theorem 2.7.1. By (2.7.22), there is C so that

$$\|P_- B_0(n+\ell)\dots B_0(n)\| < C \qquad \|P_+ B_0(n)^{-1}\dots B_0(n+\ell)^{-1}\| < C$$

Thus, so long as $\sum_n \|(\delta B)(n)\| < C^{-1}$, $\|\mathcal{A}\| < 1$. (2.7.23) is used to prove $\|(\mathcal{A}x)(n)\| \to 0$ for all $x \in \ell^\infty$. $\square$

If $|\lambda_{j+1}| = |\lambda_j|$, it can happen that both $\sup_n \prod_{m=1}^n |b_{jj}(m)/b_{j+1\,j+1}(m)|$ and $\sup_n \prod_{m=1}^n |b_{j+1\,j+1}(m)/b_{jj}(m)|$ are infinite, so one cannot place e_{j+1} in either P_+ or P_- and get boundedness! That is why we assume $|\lambda_i| \neq |\lambda_j|$. That said, there are assumptions that allow $|\lambda_j| = |\lambda_{j+1}|$; see the discussion in the Notes. This dichotomy of $\limsup|\mu_k| < 1$ or $\liminf|\mu_k| > 1$ being good is seen also in the following, which we will need to handle off-diagonal terms that are only ℓ^2:

PROPOSITION 2.7.3. *Let $b_n \in \ell^t$ with $1 \le t < \infty$, and let $\mu_1, \mu_2, \dots$ be given so that either*

(i)
$$\limsup_{j\to\infty} |\mu_j| < 1 \tag{2.7.24}$$
or

(ii)
$$\liminf_{j\to\infty} |\mu_j| > 1 \tag{2.7.25}$$
Then there is a solution q_n of
$$q_{n+1} = \mu_n q_n + b_n \tag{2.7.26}$$
with $q_n \in \ell^t$.

PROOF. If (i) holds, define $q_1 = 0$ and
$$q_n = b_{n-1} + \mu_{n-1} b_{n-2} + \mu_{n-1}\mu_{n-2} b_{n-3} + \cdots + \mu_{n-1} \dots \mu_2 b_1 \tag{2.7.27}$$
By (i), there is $r < 1$ and $C > 1$ so that for all j and k,
$$\left| \prod_{\ell=j}^{j+k} \mu_\ell \right| \le C r^k \tag{2.7.28}$$
so (2.7.27) implies
$$\begin{aligned} |q_n| &\le C \sum_{1\le k\le n} r^{n-k} |b_k| \\ &= C \sum_{0\le k\le n-1} r^k |b_{n-k}| \end{aligned} \tag{2.7.29}$$
By the fact that $\|\cdot\|_t$ is a norm (i.e., Minkowski's inequality),
$$\begin{aligned} \|q\|_t &\le C \sum_{k=0}^{\infty} r^k \|b_{\cdot-k}\|_t \\ &= C(1-r)^{-1} \|b\|_t < \infty \end{aligned} \tag{2.7.30}$$
If (ii) holds, instead of (2.7.28), we have
$$\left| \prod_{\ell=j}^{j+k} \mu_\ell \right|^{-1} \le C r^k \tag{2.7.31}$$
and we can get to infinity to define the infinite sum
$$q_n = -\mu_n^{-1} b_n - \mu_n^{-1} \mu_{n+1}^{-1} b_{n+1} \dots - \mu_n^{-1} \dots \mu_{n+j}^{-1} b_{n+j} \dots \tag{2.7.32}$$
In place of (2.7.29), we have
$$|q_n| \le C \sum_{k\ge n} r^{k-n+1} |b_k| \tag{2.7.33}$$

which again implies $q \in \ell^t$. □

This leads to our final general asymptotic result:

THEOREM 2.7.4 (Discrete Hartman-Wintner Theorem). *Let B_0 be a diagonal $d \times d$ matrix whose diagonal elements $\lambda_1, \dots, \lambda_d$ obey $|\lambda_i| \neq |\lambda_j|$ for all $i \neq j$. Let $(\delta B)(n)$ obey:*

(i)
$$(\delta B)(n)_{kk} \to 0 \tag{2.7.34}$$
as $n \to \infty$ for each k.

(ii)
$$\sum_n |(\delta B)(n)_{kj}|^2 < \infty \tag{2.7.35}$$
for all $k \neq j$.

(iii)
$$\det(B(n)) \neq 0 \tag{2.7.36}$$

Then, for any j, there exists $u^{(j)}(1)$ so the solution $u^{(j)}(n)$ of (2.7.1) *obeys*
$$\left[\prod_{\ell=1}^{n} (\lambda_j + (\delta B(\ell))_{jj})\right]^{-1} u^{(j)}(n) \to e_j \tag{2.7.37}$$

PROOF. By considering only $n \geq N$, we can suppose for all j and n, $\lambda_j + (\delta B)(n)_{jj} \neq 0$. We begin with the Harris-Lutz transform. Suppose $Q(n)$ is a $d \times d$ matrix with $1 + Q(n)$ invertible for all n. Clearly, if
$$w(n) \equiv (1 + Q(n))u(n) \tag{2.7.38}$$
then u obeys (2.7.1) if and only if
$$w(n+1) = \tilde{B}(n)w(n) \tag{2.7.39}$$
where
$$\tilde{B}(n) = (1 + Q(n+1))B(n)(1 + Q(n))^{-1} \tag{2.7.40}$$
Moreover, if $\|Q(n)\| \to 0$, (2.7.37) holds if and only if $w^{(j)}(n)$ obeys the same asymptotic formula as $u^{(j)}(n)$.

Define $\tilde{B}_0(n)$ to be B_0 plus the diagonal part of $(\delta B)(n)$ and let $(\delta C)(n)$ be the off-diagonal part of $(\delta B)(n)$.

By (2.7.40),
$$\tilde{B}(n) = \tilde{B}_0(n) + \widetilde{\delta B}(n) \tag{2.7.41}$$
where, using $\tilde{B}_0(n) = \tilde{B}_0(n)(1 + Q(n))(1 + Q(n))^{-1}$, we have
$$\widetilde{\delta B}(n) = [Q(n+1)\tilde{B}_0(n) - \tilde{B}_0(n)Q(n) + (1+Q(n+1))(\delta C)(n)](1+Q(n))^{-1} \tag{2.7.42}$$

This suggests we try to pick $Q(n)$ to obey

$$Q(n+1)\tilde{B}_0(n) - \tilde{B}_0(n)Q(n) = -(\delta C)(n) \tag{2.7.43}$$

This equation is equivalent to

$$Q(n+1)_{jk}[\lambda_k + (\delta B)(n)_{kk}] = Q(n)_{jk}[\lambda_j + (\delta B)(n)_{jj}] - (\delta C)(n)_{jk} \tag{2.7.44}$$

Since $(\delta C)(n)_{kk} = 0$, we can take $Q(n)_{kk} \equiv 0$. For $j \neq k$, (2.7.44) precisely has the form of (2.7.26), and $|\lambda_i| \neq |\lambda_j|$ implies that

$$\lim_{n\to\infty} \frac{[\lambda_j + (\delta B)(n)_{jj}]}{[\lambda_k + (\delta B)(n)_{kk}]} = \frac{|\lambda_j|}{|\lambda_k|}$$

obeys either (2.7.24) or (2.7.25). It follows that (2.7.43) has a solution with

$$\sum \|Q(n)\|^2 < \infty \tag{2.7.45}$$

By (2.7.43), (2.7.42) becomes

$$(\tilde{\delta} B)(n) = Q(n+1)(\delta C)(n)(1 + Q(n))^{-1} \tag{2.7.46}$$

By (2.7.45) and (2.7.35), $\sum \|\tilde{\delta} B\|_1 < \infty$, so Theorem 2.7.2 implies this theorem. $\square$

We now apply this to OPUC using the recursion relation (1.5.33) in the form (2.7.1) where

$$u(n) = \begin{pmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{pmatrix} \qquad B(n) = \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \tag{2.7.47}$$

Clearly, we take

$$B_0 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \qquad (\delta B)(n) = \begin{pmatrix} 0 & -\bar{\alpha}_n \\ -\alpha_n z & 0 \end{pmatrix}$$

If $z \in \mathbb{D}$, the eigenvalues of B_0 obey $|z| \neq 1$, so if

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \tag{2.7.48}$$

Theorem 2.7.4 applies.

Theorem 2.7.5. *If the Verblunsky coefficients of a set of OPUC obey* (2.7.48), *then for all* $z \in \mathbb{D}$,

$$\lim_{n\to\infty} \Phi_n(z) = 0 \qquad \lim_{n\to\infty} \Phi_n^*(z) = \ell(z) \tag{2.7.49}$$

where ℓ is an everywhere nonvanishing analytic function.

Proof. By Theorem 2.7.4, there exist solutions $u^{(1)}(n), u^{(2)}(n)$ so

$$z^{-n}u^{(1)}(n) \to \begin{pmatrix}1\\0\end{pmatrix} \tag{2.7.50}$$

$$u^{(2)}(n) \to \begin{pmatrix}0\\1\end{pmatrix} \tag{2.7.51}$$

Since these span all solutions,

$$\begin{pmatrix}\Phi_n(z)\\ \Phi_n^*(z)\end{pmatrix} = k(z)u^{(1)}(n) + \ell(z)u^{(2)}(n) \tag{2.7.52}$$

Since $|z| < 1$, $u^{(1)}(n) \to 0$, and since (2.7.51) holds, the top components of $u^{(2)}(n) \to 0$, so $\Phi_n(z) \to 0$.

By (2.7.50)/(2.7.51), $\lim \Phi_n^*(z) = \ell(z)$ and the proofs show the convergence is uniform. $\ell(z)$ is nonvanishing because (2.2.91) shows $\lim \Phi_n^*(z)$ cannot be zero. □

We thus have a "direct" proof of Szegő asymptotics. Do not think that $\Phi_n(z) \sim u^{(1)}(n)$, that is, is $O(z^n)$. It can be bigger since the upper component of $u^{(1)}$ only goes to zero. Indeed, we will see (TK) that there x-ref? are many examples where, for some $z \in \mathbb{D}$, $\Phi_n(z)$ has asymptotics different from z^n.

Later, in Section 3.2 (see Proposition 3.2.8, Remark 2, and Theorem 3.2.11), we will construct solutions, $u(n)$, which go to zero as $n \to \infty$ and which in general are bounded by Cz^n. When (2.7.48) holds, they actually obey z^{-n} (solution) has a nonzero limit.

Remarks and Historical Notes. The techniques and results in this section were developed initially to study ordinary differential equations (ODE) of the form

$$u'(x) = B(x)u(x) \tag{2.7.53}$$

Levinson's theorem and the method of proof are due to Levinson [**717**]. The Hartman-Wintner theorem is due to them [**507**]. Their proof was more involved — the idea of making a transformation $w(x) = (1 + Q(x))u(x)$ to prove the result is due to Harris-Lutz [**505**]. Eastham's book [**332**] deals with these ODE methods and applications.

In the original papers, (2.7.11) is solved by the method of successive approximation and an ad hoc convergence argument that is equivalent to approximating $(1-A)^{-1}e_j$ by $\sum_{\ell=1}^{N} A^\ell e_j$. Later authors (e.g., [**570**]) noted the more elegant (but equivalent) operator theoretic way of solving the problem.

These results were carried over to difference equations of the form (2.7.1) originally by Coffman [**212**], with later contributions by Benzaid-Lutz [**108**] and Janas-Moszyński [**570**]. Applications of these ideas to Jost and Szegő asymptotics for OPRL were found by Damanik-Simon [**233**]; see the Notes to Section 13.10. The application here to Szegő asymptotics for OPUC seems to be new.

To make the argument in Theorem 2.7.2 work, one does not need $|\lambda_j| \neq |\lambda_k|$ or even that $|b_{jj}|$ has a limit, but only that for each $j \neq k$, either

$$\sup_{n\geq m} \prod_{\ell=m}^{n} \left| \frac{b_{jj}(\ell)}{b_{kk}(\ell)} \right| < \infty \tag{2.7.54}$$

or that

$$\sup_{n\geq m} \prod_{\ell=m}^{n} \left| \frac{b_{kk}(\ell)}{b_{jj}(\ell)} \right| < \infty \tag{2.7.55}$$

These conditions imply (2.7.22) for the rescaled B. To get (2.7.23), one uses a preliminary argument to show that if (2.7.54) holds but (2.7.55) fails, then, in fact,

$$\lim_{n\to\infty} \prod_{\ell=n/2}^{n} \left| \frac{b_{jj}(\ell)}{b_{kk}(\ell)} \right| = 0$$

For details, see Eastham's book [**332**].

2.8. Khrushchev's Proof of Szegő's Theorem

In this section, we present a direct "real variable" proof of Szegő's theorem due to Khrushchev [**641**]. Complex analysis only enters through the fact that for the functions we are looking at, $\operatorname{Re} f(0) = \frac{1}{2\pi}\int \operatorname{Re} f(e^{i\theta})\frac{d\theta}{2\pi}$. We will only prove (2.3.1), but the same method easily extends to prove (2.3.21). As an extra bonus, we will obtain a limit theorem for the functions $\log(\varphi_n^*(e^{i\theta}))$ on $\partial\mathbb{D}$.

We begin with what is essentially a restatement of (2.2.7) when $r = 0$:

Proposition 2.8.1. *Let $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}$ be a nontrivial probability measure on $\partial\mathbb{D}$. Then, for all n,*

$$\kappa_n^{-2} \geq \exp\left(\int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi}\right) \tag{2.8.1}$$

In particular,

$$\kappa_\infty^{-2} \geq \exp\left(\int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi}\right) \tag{2.8.2}$$