Since $|\varphi_{j+1}(0)|^2 = |\alpha_j|^2\kappa_{j+1}^2$ and $\kappa_{j+1} \leq \kappa_\infty$, this implies (2.4.25). In essence, he replaces the proof of Lemma 2.4.3 with $\kappa_{n+1}^2 - \kappa_n^2 = |\varphi_{n+1}(0)|^2$ so that $1 - \kappa_\infty^{-2}\kappa_n^2 = \kappa_\infty^{-2}\sum_{j=n}^\infty |\varphi_{j+1}(0)|^2$. Geronimus [**414**] also has results equivalent to Theorem 2.4.9; indeed, these bounds are a major theme of Chapters 2 and 3 of [**414**].

(2.4.29) and Proposition 2.4.7 are implicit in Simon [**995**]. L. Golinskii urged me to make them explicit and place them here.

In 1932, Smirnov [**1009**] proved that when the Szegő condition holds and $d\mu_{\rm s} = 0$, then the $L^2(d\mu)$-closure of polynomials is $\{D^{-1}f \mid f \in H^2\}$; see also Freud [**370**, Theorem V.3.4].

2.5. Pointwise Convergence on the Unit Circle

We have just seen that if μ obeys a Szegő condition, then $\varphi_n^*(e^{i\theta}) \to D_{\rm ac}(e^{i\theta})^{-1}$ in $L^2(\partial\mathbb{D}, d\mu)$ and $\varphi_n^*(z) \to D(z)^{-1}$ uniformly on compact subsets of $\mathbb{D}$. Here we will concentrate on when $\varphi_n^*(e^{i\theta})$ has a uniform (and so, also pointwise) limit on some interval $I \subset \partial\mathbb{D}$. Clearly, if this happens, $w(\theta) = |D(e^{i\theta})|^2$ is continuous on I and, if $|D(e^{i\theta})|$ is bounded above on I, then by $\int |\varphi_n^*(e^{i\theta})|^2\, d\mu_{\rm s} \to 0$, we see $\mu_{\rm s}(I) = 0$. Remarkably, we will not need much more than $\mu_{\rm s}(I) = 0$ and continuity of w. We note that in Section 5.2, we will show that, under global conditions, one can prove convergence of φ_n^* and all its derivatives on $\partial\mathbb{D}$. We will discuss convergence of derivatives under just local hypotheses below; see Examples 1.6.3 and 1.6.4 revisited at the end of this section.

Continuity of w alone does not suffice, for if $\varphi_n^*(e^{i\theta}) \to D(e^{i\theta})^{-1}$ uniformly for $e^{i\theta}$ in some interval I, then $D(e^{i\theta})$ is continuous and nonvanishing there. We will see (Example 2.5.5 below) that there exist gloablly continuous w's where D is not continuous.

The conditions on w concern its modulus of continuity,

$$\omega_I(\delta, w) = \sup\{|w(\theta) - w(\theta')| \mid \theta, \theta' \in I;\ |\theta - \theta'| < \delta\} \tag{2.5.1}$$

Notice $\omega_I(\delta, w)$ is monotone in δ and that uniform continuity on I is equivalent to $\omega_I(\delta, w) \to 0$ as $\delta \downarrow 0$. One condition we will look at is (the upper limit 1 could be anything)

$$\int_0^1 \frac{\omega_I(\delta, w)}{\delta}\, d\delta < \infty \tag{2.5.2}$$

By the monotonicity, ω_I has a limit as $\delta \downarrow 0$ and (2.5.2) fails unless this limit is zero. Thus, (2.5.2) implies that w is continuous on I. On the other hand, if w is Hölder continuous on I, that is, for some $C > 0$, $\alpha > 0$, $\theta, \theta' \in I$ imply

$$|w(\theta) - w(\theta')| \leq C|\theta - \theta'|^\alpha \tag{2.5.3}$$

then $\omega_I(\delta) \le C\delta^\alpha$ and (2.5.2) holds, so (2.5.2) is not much more than continuity. Here is the best result known for this problem of uniform convergence under local conditions:

THEOREM 2.5.1 (Badkov's Theorem [**67**]). *Suppose $d\mu$ has the form* (1.1.5) *where w obeys the Szegő condition* (2.4.1). *Let I be an open interval in $\partial\mathbb{D}$ so that*

$$\text{(i)} \qquad (2.5.2)\ \textit{holds}$$

$$\text{(ii)} \qquad \inf_{\theta\in I} w(\theta) > 0 \tag{2.5.4}$$

$$\text{(iii)} \qquad \mu_{\mathrm{s}}(I) = 0 \tag{2.5.5}$$

Then D is continuous and nonvanishing on I and $\varphi_n^*(e^{i\theta}) \to D^{-1}(e^{i\theta})$ uniformly on compact subsets of I.

We will instead prove a very slightly weaker result:

THEOREM 2.5.2. *The conclusion of Theorem 2.5.1 holds if* (i) *is replaced by*

$$\text{(i}'\text{)} \qquad w \text{ is Hölder continuous on } I$$

that is, for some $0 < \alpha < 1$ and $C > 0$, (2.5.3) *holds.*

Remark. As we will explain, our proof only needs

$$\text{(i}''\text{)} \qquad \int_0^1 \frac{\omega_I(\delta, w)}{\delta\log(\delta^{-1})}\,d\delta < \infty \tag{2.5.6}$$

Note that (2.5.2) holds if $\omega(\delta) \sim C(\log(\delta^{-1}))^{-\alpha}$ with $\alpha > 1$ while (2.5.6) requires $\alpha > 2$, but (i), (i′), (i″) are "basically" the same.

We need to begin with some preliminaries on moduli of continuity and on conjugate harmonic functions.

PROPOSITION 2.5.3. (i) *If f is bounded, then*

$$\omega_I(\delta, f) \le 2\|f\|_\infty \tag{2.5.7}$$

so that finiteness of integrals like (2.5.2) *or* (2.5.6) *is only a statement about behavior near $\delta = 0$.*

(ii)

$$\omega_I(2\delta, f) \le 2\omega_I(\delta, f) \tag{2.5.8}$$

(iii) *If F is C^1 on* $\operatorname{ran} f$, *then*

$$\omega_I(\delta, F \circ f) \le \|F'\|_\infty \omega_I(\delta, f) \tag{2.5.9}$$

Proof. (i) follows from $|f(\theta) - f(\theta')| \leq 2\|f\|_\infty$.

(ii) If θ'' is the midpoint of the interval (θ, θ') and $|\theta - \theta'| < 2\delta$, then $|\theta - \theta''| < \delta$, $|\theta'' - \theta'| < \delta$, and

$$|f(\theta) - f(\theta')| \leq |f(\theta) - f(\theta'')| + |f(\theta'') - f(\theta')| \tag{2.5.10}$$

(iii)

$$|F \circ f(\theta) - F \circ f(\theta')| \leq \|F'\|_\infty |f(\theta) - f(\theta')| \tag{2.5.11}$$

$\square$

Next, we turn to conjugate functions, aka the Hilbert transform. Given $f \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ and real-valued, the analytic function

$$F_f(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} f(e^{i\theta}) \frac{d\theta}{2\pi} \tag{2.5.12}$$

is a difference of Carathéodory functions (writing $f = f_+ - f_-$), so by (1.3.31), for Lebesgue a.e. θ,

$$\lim_{r \uparrow 1} \operatorname{Re} F_f(re^{i\theta}) = f(e^{i\theta}) \tag{2.5.13}$$

and, as noted in Subsection 5 of Section 1.3, for Lebesgue a.e. θ,

$$\lim_{r \uparrow 1} \operatorname{Im} F_f(re^{i\theta}) = Cf(e^{i\theta}) \tag{2.5.14}$$

exists and defines a function $Cf(e^{i\theta})$, the *conjugate function* of f.

The relevance to D is obvious: $D(re^{i\theta})$ has a.e. boundary values $D(e^{i\theta})$—we have $|D(e^{i\theta})| = w(\theta)^{1/2}$ and $\arg D(e^{i\theta}) = \frac{1}{2}h(e^{i\theta})$ where h is the conjugate function to $\log(w(\theta))$.

Proposition 2.5.4. *Let I be an open interval in $\partial\mathbb{D}$. Suppose (2.5.2) holds for w replaced by $f \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$. Then Cf is continuous on I, and for $e^{i\theta} \in I$,*

$$(Cf)(e^{i\theta}) = \int_{\varphi = \theta - \pi}^{\theta + \pi} K(\theta - \varphi)[f(e^{i\varphi}) - f(e^{i\theta})] \frac{d\varphi}{2\pi} \tag{2.5.15}$$

where

$$K(\theta - \varphi) = \frac{\sin(\theta - \varphi)}{1 - \cos(\theta - \varphi)} \tag{2.5.16}$$

$$= \frac{\sin(\theta - \varphi)}{2\sin^2(\frac{1}{2}(\theta - \varphi))} \tag{2.5.17}$$

In (2.5.15), the integral is absolutely convergent for $\varphi \in I$, uniformly on compact subsets of I.

Remark. By (2.5.17),

$$K(\theta - \varphi) \sim \frac{2}{\theta - \varphi} + O(\theta - \varphi) \tag{2.5.18}$$

so this is a kind of circular Hilbert transform.

Proof. For $r < 1$, let

$$Q_r(\theta, \varphi) = \operatorname{Im}\left[\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}}\right] \tag{2.5.19}$$

$$= \frac{2r\sin(\theta - \varphi)}{1 + r^2 - 2r\cos(\theta - \varphi)} \tag{2.5.20}$$

Clearly, for any $\theta \neq \varphi$,

$$\lim_{r\uparrow 1} Q_r(\theta, \varphi) = K(\theta, \varphi) \tag{2.5.21}$$

We claim that for any $a \in [-1, 1]$ and $r \in [0, 1]$,

$$1 + r^2 - 2ra \geq \tfrac{1}{2}(2 - 2a) \tag{2.5.22}$$

($2 - 2a$ is the value of $1 + r^2 - 2ra$ at $r = 1$) for if a is fixed, the minimum of the left side of (2.5.22) is at $r = a$, so (2.5.22) is implied by

$$1 - a^2 = (1 + a)(1 - a) \geq 1 - a \tag{2.5.23}$$

for $|a| < 1$.

Thus,

$$|Q_r(\theta, \varphi)| \leq 2|K(\theta - \varphi)| \tag{2.5.24}$$

Finally, as regards Q_r, we note that since $(e^{i\varphi} + z)(e^{i\varphi} - z)^{-1}$ is analytic in $\mathbb{D}$ and real at $z = 0$,

$$\int Q_r(\theta, \varphi)\,\frac{d\varphi}{2\pi} = 0 \tag{2.5.25}$$

As a final preliminary,

$$|K(\theta - \varphi)| \leq \frac{\pi^2}{2}\,\frac{1}{|\theta - \varphi|} \tag{2.5.26}$$

This follows from (2.5.17) and

$$x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \frac{2x}{\pi} \leq \sin x \leq x \tag{2.5.27}$$

By (2.5.26) and (2.5.2), we see that uniformly for θ in compact subsets, K, of I,

$$\sup_{\theta\in K} \int_{|\varphi-\theta|<\varepsilon} |K(\theta - \varphi)|\,|f(e^{i\varphi}) - f(e^{i\theta})|\,\frac{d\varphi}{2\pi} \to 0 \tag{2.5.28}$$

since once ε is so small that all φ with $|\theta-\varphi|<\varepsilon$ have $\varphi\in I$, we have that the integral is bounded by

$$\text{const}\int_0^\varepsilon \frac{\omega_I(\delta,f)}{\delta}\,d\delta \tag{2.5.29}$$

Thus, the right side of (2.5.15) is continuous as a uniform limit of continuous functions on each K.

By the definition of (2.5.12),

$$\operatorname{Im} F_f(re^{i\theta}) = \int Q_r(\theta,\varphi) f(e^{i\varphi})\,\frac{d\varphi}{2\pi} \tag{2.5.30}$$

$$= \int Q_r(\theta,\varphi)[f(e^{i\varphi})-f(e^{i\theta})]\,\frac{d\varphi}{2\pi} \tag{2.5.31}$$

by (2.5.25). By (2.5.24) and the dominated convergence theorem, we see for all $\theta\in I$,

$$\operatorname{Im} F_f(re^{i\theta}) \to \text{RHS of (2.5.15)}$$

By definition of C, we have (2.5.15). $\square$

Example 2.5.5. Let

$$f(e^{i\theta}) = \begin{cases} \frac{1}{\log(\theta^{-1})} & 0<\theta\le\frac{\pi}{4} \\ 0 & -\frac{\pi}{4}<\theta\le 0 \\ C^\infty \text{ interpolation} & |\theta|>\frac{\pi}{4} \end{cases} \tag{2.5.32}$$

Then f is C^1 on each interval $I=\{e^{i\theta}\mid |\theta|>\varepsilon\}$, so Cf is given by (2.5.15) for $\theta\neq 0$. In particular, by the monotone convergence theorem and $\int_0^{1/2}\frac{dx}{x\log(x^{-1})}=\infty$,

$$\lim_{\theta\uparrow 0}(Cf)(e^{i\theta}) = -\infty \tag{2.5.33}$$

even though f is globally continuous. If $w(\theta)=\exp(\frac{1}{2}f(e^{i\theta}))$, then $|D(e^{i\theta})|$ is continuous but $D(e^{i\theta})$ is discontinuous at $\theta=0$ with infinite oscillations there. Notice, of course, that if $0\in I$, $\int\frac{\omega_I(\delta,f)}{\delta}d\delta=\infty$. $\square$

Example 2.5.6. For $z\in\mathbb{D}$, let

$$F(z) = (z-1)\log(1-z) \tag{2.5.34}$$

which has continuous boundary values on $\partial\mathbb{D}$.

$$\operatorname{Re} F(e^{i\theta}) = (\cos\theta-1)\log(|2\sin(\tfrac{\theta}{2})|) - \sin\theta\arg(1-e^{i\theta}) \tag{2.5.35}$$

$$\operatorname{Im} F(e^{i\theta}) = \sin\theta\log(|2\sin(\tfrac{\theta}{2})|) + (\cos\theta-1)\arg(1-e^{i\theta}) \tag{2.5.36}$$

If

$$f(e^{i\theta}) = \operatorname{Re} F(e^{i\theta}) \qquad (Cf)(e^{i\theta}) = \operatorname{Im} F(e^{i\theta}) \tag{2.5.37}$$

then

$$|f(e^{i\theta}) - f(e^{i\varphi})| \leq C|\theta - \varphi| \tag{2.5.38}$$

but

$$\frac{|Cf(e^{i\theta}) - (Cf)(1)|}{|\theta|} \to \infty \tag{2.5.39}$$

□

We are heading towards a proof of

THEOREM 2.5.7 (Plemelj-Privalov Theorem). *If f is Hölder continuous for some $0 < \alpha < 1$ (i.e., f obeys (2.5.3)), then Cf is Hölder continuous of the same order α.*

Remark. By Example 2.5.6, this fails for $\alpha = 1$.

We will prove a more general result:

THEOREM 2.5.8. *If*

$$\int_0^{\pi/4} \frac{\omega(\delta, f)}{\delta}\, d\delta < \infty \tag{2.5.40}$$

then for $0 < \delta < \frac{\pi}{4}$,

$$\omega(\delta, Cf) \leq Q\left[\int_0^{\delta} \frac{\omega(y,f)}{y}\, dy + \delta + \int_0^{\pi/4} \omega(y,f)\, \frac{\delta}{y(y+\delta)}\, dy\right] \tag{2.5.41}$$

where Q is an f-dependent constant.

PROOF OF THEOREM 2.5.7 GIVEN THEOREM 2.5.8. First,

$$\int_0^{\delta} \frac{y^\alpha}{y}\, dy = \frac{\delta^\alpha}{\alpha} \tag{2.5.42}$$

Second, letting $y = x\delta$,

$$\begin{aligned} \int_0^{\pi/4} \frac{y^\alpha \delta}{y(y+\delta)}\, dy &= \delta^\alpha \int_0^{\pi/4\delta} \frac{x^\alpha}{x(x+1)}\, dx \\ &\leq \delta^\alpha \int_0^{\infty} \frac{x^\alpha}{x(x+1)}\, dx \end{aligned} \tag{2.5.43}$$

where the integral is finite since $0 < \alpha < 1$. Thus, by (2.5.41), if $\omega(\delta, f) \leq c_1 \delta^\alpha$, then

$$\omega(\delta, Cf) \leq c_2(\delta^\alpha + \delta) \tag{2.5.44}$$

proving Cf is Hölder continuous of order α. □

Corollary 2.5.9. *If*

$$\int_0^{\pi/4} \frac{\omega(\delta, f)}{\delta \log(\delta^{-1})}\, d\delta < \infty \tag{2.5.45}$$

then

$$\int_0^{\pi/4} \frac{\omega(\delta, Cf)}{\delta}\, d\delta < \infty \tag{2.5.46}$$

Proof. If (2.5.45) holds, then

$$\begin{aligned}\int_0^{\pi/4} \frac{d\delta}{\delta}\biggl(\int_0^\delta \frac{\omega(y,f)}{y}\, dy\biggr) &= \int_0^{\pi/4} \frac{1}{y}\,\omega(y,f)\biggl(\int_y^{\pi/4} \frac{d\delta}{\delta}\biggr) \\ &< \infty \end{aligned}\tag{2.5.47}$$

Similarly,

$$\int_0^{\pi/4} \frac{d\delta}{\delta} \int_0^{\pi/4} \frac{\omega(y,f)\delta}{y(y+\delta)} = \int_0^{\pi/4} \frac{dy}{y}\,\omega(y,f)\biggl(\int_0^{\pi/4} \frac{d\delta}{y+\delta}\biggr) \tag{2.5.48}$$

is finite by (2.5.45). Thus, (2.5.41) and (2.5.45) imply (2.5.46). □

Lemma 2.5.10. *For $\theta < \varphi < \pi$, we have*

$$|K(\varphi - \theta) - K(\varphi)| \le Q_0\, \frac{|\theta|}{|\varphi|\,|\theta - \varphi|} \tag{2.5.49}$$

for some constant Q_0.

Proof. We have

$$|K'(\eta)| \le C|\eta|^{-2} \tag{2.5.50}$$

for some C, so

$$\begin{aligned}|K(\varphi - \theta) - K(\varphi)| &\le \int_0^1 \biggl|\frac{d}{dt}\, K(\varphi - t\theta)\biggr|\, dt \\ &\le C|\theta| \int_0^1 \frac{dt}{|\varphi - t\theta|^2} \\ &= C\biggl[\frac{1}{|\varphi - \theta|} - \frac{1}{|\varphi|}\biggr] \\ &= C\, \frac{|\theta|}{|\varphi - \theta|\,|\varphi|}\end{aligned}$$

□

Proof of Theorem 2.5.8. It suffices to estimate $(Cf)(e^{i\theta}) - (Cf)(1)$ for $0 < \theta < \frac{\pi}{4}$. We can write

$$(Cf)(e^{i\theta}) - (Cf)(1) = C_1 + C_2 + C_3 + C_4 \tag{2.5.51}$$

where $C_1, \dots, C_4$ are defined as follows. Let

$$I(\varphi,\theta) = K(\theta-\varphi)[f(e^{i\varphi}) - f(e^{i\theta})] - K(-\varphi)[f(e^{i\varphi}) - f(1)] \tag{2.5.52}$$

$$C_1 = \int_0^\theta I(\varphi,\theta)\,\frac{d\varphi}{2\pi} \tag{2.5.53}$$

$$C_2 = \int_{-\pi}^{-\pi+\theta} I(\varphi,\theta)\,\frac{d\varphi}{2\pi} \tag{2.5.54}$$

$$C_3 = \int_{-\pi+\theta}^{0} [I(\varphi,\theta) + K(\theta-\varphi)(f(e^{i\theta}) - f(1))]\,\frac{d\varphi}{2\pi} \tag{2.5.55}$$

$$C_4 = \int_\theta^\pi [I(\varphi,\theta) - K(-\varphi)(f(1) - f(e^{i\theta}))]\,\frac{d\varphi}{2\pi} \tag{2.5.56}$$

In C_3, C_4, we have added terms which cancel since $f(e^{i\theta}) - f(1)$ is φ-independent and $(-\psi = \theta - \varphi)$

$$\begin{aligned}\int_{-\pi+\theta}^{0} K(\theta-\varphi)\,\frac{d\varphi}{2\pi} &= \int_{-\pi}^{-\theta} K(-\psi)\,\frac{d\psi}{2\pi} \\ &= -\int_\theta^\pi K(-\psi)\,\frac{d\psi}{2\pi}\end{aligned}$$

since $K(-\psi) = -K(\psi)$.

Thus,

$$|C_1| \le \int_0^\theta [|f(e^{i\varphi}) - f(1)|\,|K(-\varphi)| + |f(e^{i\varphi}) - f(e^{i\theta})|\,|K(\theta-\varphi)|]\,\frac{d\varphi}{2\pi} \tag{2.5.57}$$

and

$$|C_2| \le \int_{-\pi}^{-\pi+\theta} |f(e^{i\varphi}) - f(1)|\,|K(-\varphi)|\,\frac{d\varphi}{2\pi} + \int_{\pi-\theta}^{\pi} |f(e^{i\varphi}) - f(e^{i\theta})|\,|K(\varphi-\theta)|\,\frac{d\varphi}{2\pi} \tag{2.5.58}$$

Finally,

$$|C_3| \le \int_{-\pi+\theta}^{0} |f(e^{i\varphi}) - f(1)|\,|K(\theta-\varphi) - K(-\varphi)|\,\frac{d\varphi}{2\pi} \tag{2.5.59}$$

$$|C_4| \le \int_\theta^\pi |f(e^{i\varphi}) - f(e^{i\theta})|\,|K(\theta-\varphi) - K(-\varphi)|\,\frac{d\varphi}{2\pi} \tag{2.5.60}$$

By (2.5.26),

$$|C_1| \le \frac{2\pi^2}{2}\int_0^\theta \omega(\delta,f)\,\frac{d\delta}{\delta} \tag{2.5.61}$$

Clearly,

$$C_2 \le 4 \sup_{\frac{3\pi}{4} \le \varphi \le \pi} |K(\varphi)|\,\|f\|_\infty |\theta| \tag{2.5.62}$$

using the fact that K is bounded on $\partial\mathbb{D}$ away from $e^{i\varphi} = 1$.

Finally, by (2.5.49),

$$\begin{aligned} |C_3| + |C_4| &\le 2|\theta| \int_\theta^\pi \omega(\varphi - \theta, f)\, \frac{1}{|\varphi|\,|\theta - \varphi|}\, \frac{d\varphi}{2\pi} \\ &\le 2|\theta| \int_0^\pi \omega(y, f)\, \frac{1}{y(y+\theta)}\, \frac{dy}{2\pi} \end{aligned} \tag{2.5.63}$$

Since $|\omega(y,f)| \le 2\|f\|_\infty$, the integral is bounded by $\int_0^{\pi/4}$ plus an f-dependent (but θ-independent) constant. Thus, (2.5.51), (2.5.61), (2.5.62), and (2.5.63) imply (2.5.41). □

We can now relate this to the Szegő function. It will be useful to define Δ by (2.2.92), that is,

$$\Delta(z) = D(z)^{-1} \tag{2.5.64}$$

Proposition 2.5.11. *Let $w(\theta)$ obey a Szegő condition* (2.4.1). *Let $I \subset \partial\mathbb{D}$ be an open interval and suppose $\omega \restriction I$ is Hölder continuous of order $\alpha \in (0,1)$, that is,*

$$\omega_I(\delta, w) \le C\delta^\alpha \tag{2.5.65}$$

and

$$\inf_{\theta \in I} w(\theta) > 0 \tag{2.5.66}$$

Then

(i) *$D(z)$ and $\Delta(z)$ have continuous extension from $\mathbb{D}$ to $\mathbb{D} \cup I$.*
(ii) *$\Delta(e^{i\theta})$ is Hölder continuous on each compact subinterval of I of the same α.*
(iii) *For each compact interval $J \subset I$ and each $\theta \in J$,*

$$\varphi \mapsto \frac{\Delta(e^{i\varphi}) - \Delta(e^{i\theta})}{1 - e^{i(\theta-\varphi)}} = \tilde{\Delta}(e^{i\varphi}, e^{i\theta}) \tag{2.5.67}$$

lies in all $L^p(J, \frac{d\varphi}{2\pi})$ with $1 \le p < (1-\alpha)^{-1}$.

(iv) *Let*

$$\tilde{\Delta}_\varepsilon(e^{i\varphi}, e^{i\theta}) = \frac{\Delta(e^{i\varphi}) - \Delta(e^{i\theta})}{1 - (1+\varepsilon)e^{i(\theta-\varphi)}} \tag{2.5.68}$$

Then, for J, θ as in (iii),

$$\tilde{\Delta}_\varepsilon(\,\cdot\,, e^{i\theta}) \to \tilde{\Delta}(\,\cdot\,, e^{i\theta}) \tag{2.5.69}$$

in $L^p(J, \frac{d\varphi}{2\pi})$, $1 \le p < (1-\alpha)^{-1}$. The convergence is uniform for $\theta \in J$.

PROOF. (i),(ii) By (2.5.66) on each compact $J \subset I$, $u \to \log(u)$ is C^∞ on $\mathrm{ran}(w \restriction J)$, so by (2.5.9), $\log w \restriction J$ is Hölder continuous. Write

$$\int \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \log w(\varphi) \, \frac{d\varphi}{2\pi} \tag{2.5.70}$$

as an integral over I and over $\partial\mathbb{D}\backslash I$.

The integral over $\partial\mathbb{D}\backslash I$ is analytic across I and, by the Plemelj-Privalov theorem, the integral over I is Hölder continuous on any compact $J \subset I$. Since exp is C^∞, we see D and Δ are continuous up to I and Hölder continuous on I.

(iii) For $p < (1-\alpha)^{-1}$, $\int_{-\pi}^{\pi} (\frac{|\theta|^\alpha}{|\theta|})^p \, d\theta < \infty$, proving the L^p result.

(iv) This is an easy consequence of (iii) and the dominated convergence theorem. □

That completes the preliminaries we need concerning the conjugate harmonic function. We turn now to the issue of convergence of $\varphi_n^*(e^{i\theta})$ to $D(e^{i\theta})^{-1}$ uniformly on intervals I. We begin with combining (1.5.45) with the CD kernel.

THEOREM 2.5.12. *Let w obey a Szegő condition and let $I \subset \partial\mathbb{D}$ be an open interval with $\mu_{\mathrm{s}}(I) = 0$ and $w \restriction I$ Hölder continuous of some order $\alpha \in (0,1)$. Then*

(i) *For $z \in \mathbb{D}$,*

$$\frac{\kappa_n}{\kappa_\infty} \varphi_n^*(z) - \Delta(z) = \int K_n(\zeta, z)(\Delta_{\mathrm{ac}}(\zeta) - \Delta(z)) \, d\mu(\zeta) \tag{2.5.71}$$

(ii) *For $e^{i\theta} \in I$,*

$$\frac{\kappa_n}{\kappa_\infty} \varphi_n^*(e^{i\theta}) - \Delta(e^{i\theta}) = a_n \varphi_n^*(e^{i\theta}) + b_n \varphi_n(e^{i\theta}) \tag{2.5.72}$$

where

$$a_n = \int \frac{\Delta_{\mathrm{ac}}(\zeta) - \Delta(e^{i\theta})}{1 - e^{i\theta}\bar\zeta} \, \overline{\varphi_n^*(\zeta)} \, d\mu(\zeta) \tag{2.5.73}$$

$$b_n = -\int \frac{\Delta_{\mathrm{ac}}(\zeta) - \Delta(e^{i\theta})}{1 - e^{i\theta}\bar\zeta} \, e^{i\theta}\bar\zeta \, \overline{\varphi_n(\zeta)} \, d\mu(\zeta) \tag{2.5.74}$$

PROOF. (i) (1.5.45) can be rewritten

$$\kappa_n \varphi_n^*(z) = \sum_{j=0}^n \overline{\varphi_j(0)} \, \varphi_j(z) \tag{2.5.75}$$

By definition of κ_∞, $\kappa_n \to \kappa_\infty$ and, by (2.4.34), as functions in $L^2(\partial\mathbb{D}, d\mu)$, $\varphi_n^*(z) \to \Delta_{\mathrm{ac}}(z)$. Since φ_j are an orthonormal set in L^2, (2.5.75) implies

$$\kappa_\infty \Delta_{\rm ac}(\zeta) = \sum_{j=0}^\infty \overline{\varphi_j(0)}\, \varphi_j(\zeta) \tag{2.5.76}$$

as functions in $L^2(\partial\mathbb{D}, d\mu)$. Since K_n is the integral kernel of projection onto the span of $\{\varphi_j\}_{j=0}^n$, (2.5.76) and (2.5.75) imply

$$\int \kappa_\infty \Delta_{\rm ac}(\zeta) K_n(\zeta, z)\, d\mu(\zeta) = \kappa_n \varphi_n^*(z) \tag{2.5.77}$$

This plus

$$\int K_n(\zeta, z)\, d\mu(\zeta) = 1 \tag{2.5.78}$$

implies (2.5.71).

(ii) This follows when $e^{i\theta}$ is replaced by $z \in \mathbb{D}$ from (2.5.71) and (2.2.42). Since (2.5.26) implies L^1 convergence as $z = (1-\varepsilon)e^{i\theta} \to e^{i\theta}$ and $\varphi_n, \varphi_n^* \in L^\infty$, we get the result for $z = e^{i\theta}$ by taking limits. $\square$

The proof, equivalently (2.5.69), shows

$$a_n = \lim_{\varepsilon\downarrow 0} a_n(\varepsilon) \qquad b_n = \lim_{\varepsilon\downarrow 0} b_n(\varepsilon) \tag{2.5.79}$$

where $a_n(\varepsilon), b_n(\varepsilon)$ are given by (2.5.73)/(2.5.74) with $(1 - e^{i\theta}\bar\zeta)$ replaced by $(1-(1+\varepsilon)e^{i\theta}\bar\zeta)$. We need to show that $a_n \to 0$, $b_n \to 0$. As a start, we claim

LEMMA 2.5.13. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} a_n(\varepsilon) = 0 \tag{2.5.80}$$

$$\lim_{n\to\infty} b_n(\varepsilon) = 0 \tag{2.5.81}$$

Moreover, the limits are uniform over $e^{i\theta} \in K$, any compact subinterval of I.

Remark. a_n, b_n are, of course, θ-dependent also, but we suppress that in the notation.

PROOF. For $\varepsilon > 0$,

$$f_{\theta,\varepsilon}(\varphi) = -\frac{\Delta_{\rm ac}(e^{i\varphi}) - \Delta(e^{i\theta})}{1-(1+\varepsilon)e^{i(\theta-\varphi)}}\, e^{i(\theta-\varphi)} \tag{2.5.82}$$

lies in $L^2(d\mu)$ (by $w(\varphi) = |D(e^{i\varphi})|^2$ and (2.4.3)) and, as an L^2 function of φ, is continuous in θ for $\theta \in I$. Thus, by Bessel's inequality,

$$b_n(\varepsilon) = \langle \varphi_n, f_{\theta,\varepsilon}\rangle \to 0 \tag{2.5.83}$$

uniformly over compact sets of θ.

For $z \in \mathbb{D}$, let

$$g_{\theta,\varepsilon}(z) = \frac{z(\Delta(z) - \Delta(e^{i\theta}))}{z - (1+\varepsilon)e^{i\theta}}\, D(z) \tag{2.5.84}$$

Then $g_{\theta,\varepsilon}$ lies in $H^2(\mathbb{D})$ since $\Delta(z)D(z) = 1$ and $D \in H^2$ and $g_{\theta,\varepsilon}$ is continuous in θ for $\theta \in I$. Let

$$h_{\theta,\varepsilon} = \frac{\Delta_{\mathrm{ac}}(\zeta) - \Delta(e^{i\theta})}{1 - (1+\varepsilon)e^{i\theta}\bar\zeta} \tag{2.5.85}$$

which is $L^2(\partial\mathbb{D}, d\mu)$, and so $L^2(\partial\mathbb{D}, d\mu_{\mathrm{s}})$ uniformly in $\theta \in I$.

By

$$d\mu(e^{i\varphi}) = |D(e^{i\varphi})|^2\, \frac{d\varphi}{2\pi} + d\mu_s(e^{i\varphi}) \tag{2.5.86}$$

we have that

$$a_n(\varepsilon) = \langle D\varphi_n^*, g_{\theta,\varepsilon}(z)\rangle_{L^2(\mathbb{D}, d\varphi/2\pi)} + \langle \varphi_n^*, h_{\theta,\varepsilon}\rangle_{L^2(\mathbb{D}, d\mu_{\mathrm{s}})} \tag{2.5.87}$$

By (2.4.11),

$$|\langle \varphi_n^*, h_{\theta,\varepsilon}\rangle| \le C\|\varphi_n\|^*_{L^2(d\mu_{\mathrm{s}})} \to 0$$

uniformly in $\theta \in I$. By (2.4.8), $D\varphi_n^* \to 1$ in $L^2(\mathbb{D}, \frac{d\varphi}{2\pi})$, so uniformly in $\theta \in K$ compact in I,

$$a_n(\varepsilon) \to \int g_{\theta,\varepsilon}(e^{i\varphi})\, \frac{d\varphi}{2\pi} = g_{\theta,\varepsilon}(z=0) = 0 \qquad \square$$

Proposition 2.5.14. (i) *If* $|a_n(\varepsilon) - a_n| + |b_n(\varepsilon) - b_n| \to 0$ *as* $\varepsilon \downarrow 0$ *uniformly in* n*, then uniformly over* $e^{i\theta} \in K$ *compact in* I*,*

$$\lim_{n\to\infty} a_n = 0 \tag{2.5.88}$$

$$\lim_{n\to\infty} b_n = 0 \tag{2.5.89}$$

(ii) *If* (2.5.88)/(2.5.89) *hold, then uniformly over* $e^{i\theta} \in K$ *compact in* I*,*

$$\varphi_n^*(e^{i\theta}) \to D(e^{i\theta})^{-1} \tag{2.5.90}$$

Proof. (i) A standard approximation argument.

(ii) Since $\kappa_n \ge 1$, we have $\kappa_\infty/\kappa_n \le \kappa_\infty$, and so (2.5.72) implies

$$|\varphi_n^*(e^{i\theta})| \le (|a_n| + |b_n|)\kappa_\infty |\varphi_n(e^{i\theta})|^* + |\Delta(e^{i\theta})| \tag{2.5.91}$$

If n is so large that $(|a_n| + |b_n|)\kappa_\infty \le \frac{1}{2}$, this implies

$$|\varphi_n^*(e^{i\theta})| \le 2|\Delta(e^{i\theta})| \tag{2.5.92}$$

so, by (2.5.72) again,

$$\left| \frac{\kappa_n}{\kappa_\infty}\, \varphi_n^*(e^{i\theta}) - \Delta(e^{i\theta}) \right| \le 2(|a_n| + |b_n|)|\Delta(e^{i\theta})| \tag{2.5.93}$$

goes to zero uniformly in $e^{i\theta}$ in compacts. Since $\kappa_n/\kappa_\infty \to 1$, this proves (2.5.90). $\square$

THEOREM 2.5.15. (i) *If w obeys the hypotheses of Theorem 2.5.2 with $\frac{1}{2} < \alpha < 1$, then* (2.5.90) *holds uniformly over compacts $K \subset I$.*

(ii) *If w obeys the hypotheses of Theorem 2.5.2 and for all compact $K \subset I$,*

$$\sup_{e^{i\theta} \in K, n} |\varphi_n^*(e^{i\theta})| < \infty \tag{2.5.94}$$

then (2.5.90) *holds uniformly over compacts $K \subset I$.*

Remark. Of course, this is weaker than Theorem 2.5.2. The point is that we will use it to prove Theorem 2.5.2.

PROOF. Fix $K \subset J^{\mathrm{int}} \subset J \subset I$ with J compact and write $a_n, a_n(\varepsilon)$, etc. as

$$a_n = a_n^J + a_n^{\partial\mathbb{D}\setminus J} \tag{2.5.95}$$

where a_n^S is the integral over S. Since $\Delta_{\mathrm{ac}} \in L^2(\partial\mathbb{D}, d\mu)$, and for $e^{i\theta} \in K$, $e^{i\varphi} \in \partial\mathbb{D}\setminus J$, $(1-(1+\varepsilon)e^{i(\theta-\varphi)})^{-1}$ is uniformly bounded as $\varepsilon \downarrow 0$, we trivially have

$$a_n^{\partial\mathbb{D}\setminus J}(\varepsilon) \to a_n^{\partial\mathbb{D}\setminus J} \tag{2.5.96}$$

and similarly for b_n uniformly for $e^{i\theta} \in K$.

By $\mu_{\mathrm{s}}(I) = 0$, $\sup_I w(\theta) < \infty$, and Proposition 2.5.11, if $1 \le p < (1-\alpha)^{-1}$ and

$$\sup_n \biggl(\int_{e^{i\varphi} \in J} |\varphi_n(e^{i\varphi})|^q \, \frac{d\varphi}{2\pi} \biggr)^{1/2} < \infty \tag{2.5.97}$$

(with $q^{-1} = 1 - p^{-1}$), then

$$a_n^J(\varepsilon) \to a_n^J \tag{2.5.98}$$

uniformly for $e^{i\theta} \in K$. If $\alpha > \frac{1}{2}$, we can take $p = 2$ and $q = 2$ so (2.5.97) is immediate from $\inf_{\theta\in J} w(\theta) > 0$. This proves (i). If (2.5.98) holds, we can take $p = 1$, $q = \infty$ and (ii) is proven. $\square$

Thus, we are reduced to proving (2.5.94) when the hypotheses of Theorem 2.5.2 hold. We will do this by a two-step process due to Badkov. The key will be a useful comparison formula for a pair of measures, μ and ν:

PROPOSITION 2.5.16. *Let μ, ν be any pair of measures on $\partial\mathbb{D}$. Then for any c,*

$$\begin{aligned}\varphi_n(z; d\mu) &= \langle \varphi_n(\,\cdot\,; d\nu), \varphi_n(\,\cdot\,; d\mu)\rangle_{L^2(\partial\mathbb{D}, d\nu)}\varphi_n(z; d\nu) \\ &\qquad + \int_{\partial\mathbb{D}} K_{n-1}(e^{i\theta}, z; d\nu)\varphi_n(e^{i\theta}; d\mu)[d\nu(e^{i\theta}) - c\, d\mu(e^{i\theta})]\end{aligned} \tag{2.5.99}$$

PROOF. $\varphi_n(z; d\mu)$ is a linear combination of $\{\varphi_j(z; d\nu)\}_{j=0}^n$. Thus,

$$\varphi_n(z; d\mu) = \sum_{j=0}^{n} \langle \varphi_j(\,\cdot\,; d\nu), \varphi_n(\,\cdot\,; d\mu)\rangle \varphi_j(z; d\nu) \tag{2.5.100}$$

By definition of K_{n-1}, φ_n is thus the sum of the first term in (2.5.99) and the $d\nu$ integral. But since $\overline{K_{n-1}(e^{i\theta}, z)}$ is a polynomial of degree $n-1$ in $e^{i\theta}$ and $\varphi_n(e^{i\theta}; d\mu)$ is $d\mu$ orthogonal to any such polynomial, the second term in the integral in (2.5.99) is zero. $\square$

We first prove (2.5.94) for weights which are nice near the edges of I.

PROPOSITION 2.5.17. *Let $I = (a, b)$ be an open interval in $\partial\mathbb{D}$ with $\mu_{\rm s}(I) = 0$. Suppose that for some $\varepsilon > 0$, w is Hölder continuous of order $\alpha_0 > \frac{1}{2}$ on $J_1 = (a, a+\varepsilon)$ and $J_2 \equiv (b-\varepsilon, b)$ and w is Hölder continuous of some order $\alpha_1 > 0$ on all of I. Then*

$$\sup_{n, e^{i\theta} \in [a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}]} |\varphi_n(e^{i\theta})| < \infty \tag{2.5.101}$$

PROOF. Suppose not. Let $K = [a + \frac{3\varepsilon}{4}, b - \frac{3\varepsilon}{4}]$, $\tilde J_1 = [a + \frac{\varepsilon}{4}, a + \frac{3\varepsilon}{4}]$, $\tilde J_2 = [b - \frac{3\varepsilon}{4}, b - \frac{\varepsilon}{4}]$. By Theorem 2.5.15(i), $|\varphi_n|$ is uniformly bounded on $\tilde J_1$ and $\tilde J_2$. Let u_n be a point in $[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}]$ where $|\varphi_n|$ takes its maximum value, ρ_n, over that interval. Since $|\varphi_n(u_n)|$ is unbounded by hypothesis, but $|\varphi_n|$ is uniformly bounded on $\tilde J_1 \cup \tilde J_2$, we see that for large n, $e^{iu_n} \in K$.

Let $d\nu$ be a measure that agrees with $d\mu$ on $(\partial\mathbb{D}\backslash I) \cup J_1 \cup J_2$, has $\nu_{\rm s}(I) = 0$, and so that on I,

$$d\nu(\theta) = q(\theta)\, d\theta \tag{2.5.102}$$

with q Hölder continuous of order α_0 and $\inf_{\theta\in I} q(\theta) > 0$. We will use (2.5.99) with c n-dependent,

$$c_n = \frac{q(u_n)}{w(u_n)} \tag{2.5.103}$$

Since $d\mu = d\nu$ on $\partial\mathbb{D}\backslash K$ and μ, ν are absolutely continuous with weights bounded away from zero and continuous on K, for some constant c_0,

$$d\nu \leq c_0\, d\mu \tag{2.5.104}$$

Thus,

$$\begin{aligned} |\langle \varphi_n(\,\cdot\,; d\mu), \varphi_n(\,\cdot\,; d\nu)\rangle_{L^2(d\nu)}| &\leq \|\varphi_n(\,\cdot\,; d\mu)\|_{L^2(d\nu)} \\ &\leq c_0^{1/2} \|\varphi_n(\,\cdot\,; d\mu)\|_{L^2(d\mu)} = c_0^{1/2} \end{aligned} \tag{2.5.105}$$

Since $d\nu$ has a Hölder continuous weight of order α_0,

$$\sup_{z\in J_1\cup J_2\cup K, n} |\varphi_n(z; d\nu)| < \infty \tag{2.5.106}$$

by using Theorem 2.5.15(i) again. It follows that for a constant, c_1, and all n,

$$|\text{First term in (2.5.99) for } z = e^{iu_n}| \leq c_1 \tag{2.5.107}$$

Next we note, by the CD formula for K_{n-1},

$$|K_{n-1}(e^{i\theta}, e^{iu_n}; d\nu)| \leq 2|\varphi_n(e^{i\theta}; d\nu)|\, |\varphi_n(e^{iu_n}; d\nu)|\, |1 - e^{i(u_n-\theta)}|^{-1} \tag{2.5.108}$$

$$\leq c_2 |\varphi_n(e^{i\theta}; d\nu)|\, |u_n - \theta|^{-1} \tag{2.5.109}$$

by (2.5.106). Here the ambiguity in θ is picked so $|u_n - \theta| \leq \pi$.

Let $\eta < \varepsilon/2$ to be picked shortly and break the integral in (2.5.99) into the region $R_n(\theta) = \{\theta \mid |\theta - u_n| < \eta\}$ and its complement, $\partial\mathbb{D}\backslash R_n(\eta)$. Since $R_n(\eta) \subset J_1 \cup J_2 \cup K$ (since $\eta < \varepsilon/2$), by (2.5.106),

$$\left| \int_{R_n(\eta)} \cdots \right| \leq c_3 \rho_n \int_{R_n(\eta)} \left| q(\theta) - \frac{q(u_n)}{w(u_n)}\, w(\theta) \right| |\theta - u_n|^{-1}\, d\theta \tag{2.5.110}$$

Since $w(u_n) > \inf_I w(\theta)$ and q, w are Hölder continuous of order $\alpha_1 > 0$, uniformly in n, we have

$$\text{LHS of (2.5.110)} \leq c_4 \rho_n \eta^{\alpha_1} \tag{2.5.111}$$

So we fix η once and for all so that

$$\text{LHS of (2.5.110)} \leq \tfrac{1}{2}\, \rho_n \tag{2.5.112}$$

On $\partial\mathbb{D}\backslash R_n(\eta)$, $|u_n - \theta|^{-1} \leq \eta^{-1}$, so by (2.5.109),

$$\begin{aligned} \left| \int_{\partial\mathbb{D}\backslash R_n(\eta)} \cdots \right| &\leq c_2 \eta^{-1} \langle |\varphi_n(\,\cdot\,; d\nu)|, |\varphi_n(\,\cdot\,; d\mu)| \rangle_{L^2(d\nu + d\mu)} \\ &\leq c_5 \end{aligned} \tag{2.5.113}$$

by repeating the argument that led to (2.5.105).

Thus,

$$\rho_n \leq c_1 + c_5 + \tfrac{1}{2}\,\rho_n \tag{2.5.114}$$

proving that ρ_n is bounded. This is a contradiction, so the theorem is proven. $\square$

Proof of Theorem 2.5.2. Given a compact $K \subset I$, pick $\varepsilon > 0$ so that

$$\min(|\theta - \varphi| \mid e^{i\theta} \in K,\, e^{i\varphi} \notin I) \geq 2\varepsilon \tag{2.5.115}$$

Let $I = (a, b)$ and let $d\nu$ be a measure so that

$$\mu = \nu \quad \text{on} \quad \partial\mathbb{D}\backslash[(a, a+\varepsilon) \cup (b-\varepsilon, b)] \tag{2.5.116}$$

and so that on I, (2.5.104) holds, $\inf_{\theta\in I} q(\theta) > 0$, and so q is Hölder continuous of order $\alpha_0 > \frac{1}{2}$ on $J \equiv [a, a+\varepsilon] \cup [b-\varepsilon, b]$. Thus, by Proposition 2.5.17,

$$\sup_{\theta\in K, n} |\varphi_n(e^{i\theta}; d\nu)| = c_1 < \infty \tag{2.5.117}$$

Now use (2.5.99) with $z = e^{i\varphi} \in K$ and $c = 1$. As in the last proof, (2.5.104) holds, so (2.5.107) holds. In the second term in (2.5.99), $d\nu - d\mu$ has support on J, so $|\theta - \varphi| \geq \varepsilon$ (by (2.5.115)). Thus, by (2.5.117) and the CD formula,

$$|K_{n-1}(e^{i\theta}, e^{i\varphi}; d\nu)| \leq c_2\varepsilon^{-1}|\varphi_n(e^{i\theta}; d\nu)| \tag{2.5.118}$$

It follows that for $e^{i\varphi} \in K$,

$$|\varphi_n(e^{i\varphi}; d\mu)| \leq c_0 + c_2\varepsilon^{-1}\langle|\varphi_n(\,\cdot\,; d\nu)|, |\varphi_n(\,\cdot\,; d\mu)|\rangle_{L^2(d\mu+d\nu)} \tag{2.5.119}$$

is bounded as in the last proof. $\square$

If one goes through our proof, one sees that all we really need is that (2.5.2) holds and that

$$\int_0^1 \frac{\omega_I(\delta, D)}{\delta}\, d\delta < \infty \tag{2.5.120}$$

By Corollary 2.5.9, this only requires (2.5.6). We have thus noted that our proof only requires (2.5.6).

As we mentioned, we will prove in Section 5.2 that, under the global hypotheses $\mu_{\rm s}(\partial\mathbb{D}) = 0$, w is C^∞, and $\inf_{\partial\mathbb{D}} w(\theta) > 0$, then all derivatives of φ_n^* converge to the derivatives of $D_{\rm ac}^{-1}$. One might wonder if there is a local version of this, that is, if w is C^∞ on I, $\mu_{\rm s}(I) = 0$, (2.5.4) holds, and a Szegő condition holds, then on compact subsets of I, all derivatives of φ_n converge. The answer is no!

Example 1.6.3, revisited. One can take I to be any open subinterval of $\partial\mathbb{D}\setminus\{1\}$. By (1.6.6),

$$\begin{aligned}\Phi_n^*(z) &= 1 - \alpha_{n-1}(z + z^2 + \cdots + z^n) \\ &= 1 - \alpha_{n-1} z \left(\frac{1 - z^n}{1 - z}\right) \end{aligned} \tag{2.5.121}$$

so

$$(\Phi_n^*)'(z) = \alpha_{n-1}\,\frac{nz^n}{1-z} + \alpha_{n-1}\,\frac{1-z^n}{(1-z)^2} + \alpha_{n-1}\,\frac{1-z^n}{1-z} \tag{2.5.122}$$

The second term goes to 0 uniformly on compact subsets of $\mathbb{D}\setminus\{1\}$ since $\alpha_n = O(\frac{1}{n})$ by (1.6.7). But since $n\alpha_{n-1} \to \gamma$, the first term does not have a limit but, for any $z \neq 1$, oscillates. Thus, $(\varphi_n^*)'(z)$ does not converge at any point in $\partial\mathbb{D}$! Even the simplest singular spectrum destroys convergence of derivatives everywhere in $\partial\mathbb{D}$! □

Example 1.6.4, revisited. Look at the measure $(1 - \cos\theta)\frac{d\theta}{2\pi}$ (i.e., (1.6.8) with $a = 1$). $w(\theta)$ is C^∞, but vanishes at a single point. By (1.6.26) and straightforward algebra,

$$\Phi_n^*(z) = \frac{1}{1-z} - \frac{1}{n+1}\;\frac{z(1-z^{n+1})}{(1-z)^2} \tag{2.5.123}$$

As required by Theorem 2.5.2 on $\partial\mathbb{D}$ away from $z = 1$, $\Phi_n^*(z) \to D(0)D(z)^{-1} = (1-z)^{-1}$. The derivative of the first term in (2.5.123) converges to $D(0)\frac{d}{dz}D(z)^{-1}$, but the derivative of the second is $\frac{z^{n+1}}{(1-z)^2} + O(\frac{1}{n})$ which, as in the last example, does not have a limit but oscillates. Even the simplest isolated zero of w destroys convergence of derivatives at all points of $\partial\mathbb{D}$! □

Remarks and Historical Notes. The results of this section—and, in particular, Badkov's theorem—represent the combination of two historical trends. One is control of limits of $\varphi_n^*(e^{i\theta})$ on $\partial\mathbb{D}$ under global assumptions. Such results appear already in Szegő's book [**1046**] under strong regularity conditions on w even allowing a finite number of zeros of special form. Grenander-Szegő [**491**] only required $\inf_{\theta\in[0,2\pi)} w(\theta) > 0$ and (a Dini condition)

$$\omega_{\partial\mathbb{D}}(\delta, w) \leq C|\log(\delta^{-1})|^{-\lambda} \tag{2.5.124}$$

for some $\lambda > 1$ and all $\delta \in (0, \frac{1}{2})$. Geronimus-Golinski [**418**] then noted that this earlier proof only requires (2.5.2) for $I = \partial\mathbb{D}$.

The second trend involved only local convergence and only local hypotheses except for a global Szegő condition. In his book ([**414**, Thm. 4.9]), Geronimus proved an analog of Theorem 2.5.1 but with (i) replaced by the stronger

$$\omega_I(\delta, w) \le C\delta^{1/2}(\log(\delta^{-1}))^\lambda \tag{2.5.125}$$

for $\lambda > 1$. After various other results in work by Freud [**370**], Geronimus-Golinskii [**418**], Golinskii [**453, 454, 455, 456**], Badkov [**69**] proved Theorem 2.5.1.

Our approach to the slightly weaker Theorem 2.5.2 is a combination of ideas of Freud (who found and exploited Theorem 2.5.12 to get Theorem 2.5.15(i)) and a part of Badkov's argument (who found and exploited Proposition 2.5.16 in the precise two-step process we use to prove (2.5.116)). I found these arguments in discussions with Eric Ryckman. In this joint work, we also found the two counter-examples for derivative convergence that appear at the end of this section.

The Plemelj-Privalov theorem goes back to Plemelj [**888**] and Privalov [**901**].

Uniform (in n) boundedness of $\varphi_n(e^{i\theta}, d\mu)$ on I implies uniform convergence of φ_n Fourier expansions for functions whose Fourier coefficients obey $\sum_n |a_n| < \infty$. For discussion of applications to convergence of φ_n Fourier expansions, see Badkov [**67, 68**].

2.6. Szegő's Theorem Using the Poisson Kernel

In this section, we will give a third proof of Szegő's theorem that is more tuned to actually solving the Szegő minimum problem directly. It basically has three steps:

(1) Prove $\lambda_\infty(z, d\mu) = \lambda_\infty(z, w\frac{d\theta}{2\pi})$, that is, prove directly that the singular part of $d\mu$ does not matter.
(2) Prove that if $d\mu_s = 0$, then the minimum problem can be replaced by one with a general class of analytic functions.
(3) Use a function related to the Szegő function as the trial function in the expanded variational principle.

As a bonus, this proof allows consideration of L^p norms, so we define for $0 < p < \infty$,

$$\lambda_n(\zeta, d\mu; p) = \min\left(\int |\pi(e^{i\theta})|^p \, d\mu \;\middle|\; \deg \pi \le n,\ \pi(\zeta) = 1\right\} \tag{2.6.1}$$

and

$$\lambda_\infty(\zeta, d\mu; p) = \inf_n \lambda_n(\zeta, d\mu; p) \tag{2.6.2}$$

Since $\lambda_{n+1} \le \lambda_n$, the $\inf_n$ is also $\lim_{n\to\infty}$. We eventually show that λ_∞ is independent of p. Since the set of π's with $\deg(\pi) \le n$ and