of (13.1.34) and (13.1.37) but with $\varphi_n^\pm$ the solution of (13.1.27) with different boundary conditions than $\varphi_1 = 1$, $\varphi_0 = 0$. Presumably, the other solution has $\varphi_{-1} = 0$, $\varphi_0 = 1$, and the two choices of boundary condition at $+2$ and -2 yield the four inverses. The restriction on whether a $d\gamma$ lies in the range of the other maps is connected with whether this second solution is positive for $+2$ and sign alternating for -2.

13.3. Canonical Moments and the Geronimus Relations

Thus far, we have emphasized the role of Verblunsky coefficients as the recursion coefficients for the Φ_n. But, as we saw in Section 3.1, they also measure relative positions of c_n among all values consistent with $c_0, \dots, c_{n-1}$. For OPRL, the recursion coefficients and relative positions are very different. In this section, we obtain the relation between the Jacobi parameters and these relative positions, and also see that the relation between the relative position parameters for μ and $\mathrm{Sz}(\mu)$ is simple. This will lead to another illuminating proof of the direct Geronimus relations.

Throughout, we fix an interval $[a,b] \subset \mathbb{R}$ and we will consider moments of measures supported on $[a,b]$ (below $[a,b]$ will be either $[-2,2]$ or $[0,1]$). Given $\rho \in \mathcal{M}_{+,1}([a,b])$, let $c_j(\rho)$ be its moments given by

$$c_j(\rho) = \int_a^b x^n \, d\rho(x) \tag{13.3.1}$$

For each fixed ρ_0 and k, $\{\rho \in \mathcal{M}_{+,1}([a,b]) \mid c_j(\rho) = c_j(\rho_0),\ j = 0,1,\dots,k-1\}$ is a compact convex set, so the set of values of $c_k(\rho)$ as ρ runs through this set is a bounded closed interval $[c_k^-(\rho_0), c_k^+(\rho_0)]$. We will prove below that if ρ_0 is nontrivial, then $c_k^+ \neq c_k^-$. We thus define the *canonical moments*, $p_k(\rho_0)$, for $k = 1, 2, \dots$ by

$$p_k(\rho_0) = \frac{[c_k(\rho_0) - c_k^-(\rho_0)]}{[c_k^+(\rho) - c_k^-(\rho_0)]} \tag{13.3.2}$$

We will also prove that ρ_0 nontrivial implies $c_k(\rho_0) \in (c_k^-(\rho_0), c_k^+(\rho_0))$, so

$$0 < p_k(\rho_0) < 1 \tag{13.3.3}$$

Note. It is unfortunate that the standard symbol, p_k, used for the canonical moments is also used for the orthonormal OPRL. In this section, we will only use the monic polynomials, so p_k only stands for the canonical moments.

Notice the p_k depend not only on ρ but also on $[a,b]$. We will see below that any set of p_k's in $(0,1)$ can occur, that is, the p_k's are a map of the nontrivial measures supported on $[a,b]$ to $(0,1)$.

It will also be useful to define

$$q_k(\rho) = 1 - p_k(\rho) \qquad \zeta_1 = p_1,\ \zeta_k = q_{k-1}p_k \qquad k \geq 2 \tag{13.3.4}$$

The relation of the α's to p's is trivial!

THEOREM 13.3.1. *Let $\mu \in \mathcal{M}_{+,1}(\partial\mathbb{D})$ and $\gamma = \mathrm{Sz}(\mu)$ viewed as a measure on $[-2,2]$. Then*

$$p_k(\gamma) = \tfrac{1}{2}\,(1 + \alpha_{k-1}(d\mu)) \tag{13.3.5}$$

$$q_k(\gamma) = \tfrac{1}{2}\,(1 - \alpha_{k-1}(d\mu)) \tag{13.3.6}$$

$$\zeta_k(\gamma) = \tfrac{1}{4}\,(1 + \alpha_{k-1}(d\mu))(1 - \alpha_{k-2}(d\mu)) \tag{13.3.7}$$

where, as usual, $\alpha_{-1} = 1$.

PROOF. (13.3.6) and (13.3.7) follow from (13.3.5). To prove that, note that

$$\begin{aligned} c_k(\gamma) &= \int (z + z^{-1})^k \, d\mu(\theta) \\ &= c_k(d\mu) + f(c_0, \dots, c_{k-1}) \end{aligned}$$

so the map of α to p is the unique orientation-preserving affine map of $[-1,1]$ onto $[0,1]$, that is, $\frac{1}{2}(1+\alpha)$. □

Below, we will prove the following:

THEOREM 13.3.2. *Let γ be a measure on $[-2,2]$, p_k its canonical moments, and $\{a_n, b_n\}_{n=1}^\infty$ its Jacobi parameters. Then*

$$a_{n+1}^2 = 16\zeta_{2n+2}\zeta_{2n+1} \tag{13.3.8}$$

$$b_{n+1} = 4\zeta_{2n+1} + 4\zeta_{2n} - 2 \tag{13.3.9}$$

COROLLARY 13.3.3 (Third Proof of the Direct Geronimus Relations). *Let $\gamma = \mathrm{Sz}(\mu)$. Let $\{a_n, b_n\}_{n=1}^\infty$ be the Jacobi parameters of γ and α_n the Verblunsky coefficients of μ. Then*

$$a_{n+1}^2 = (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1}) \tag{13.3.10}$$

$$b_{n+1} = (1 - \alpha_{2n+1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2} \tag{13.3.11}$$

PROOF. This follows from (13.3.7), (13.3.8), and (13.3.9). For example,

$$\begin{aligned} 4\zeta_{2n+1} + 4\zeta_{2n} - 2 &= (1 + \alpha_{2n})(1 - \alpha_{2n-1}) + (1 + \alpha_{2n-1})(1 - \alpha_{2n-2}) - 2 \\ &= \alpha_{2n}(1 - \alpha_{2n-1}) - \alpha_{2n-2}(1 + \alpha_{2n-1}) \end{aligned}$$ □

It will turn out to be easiest to study canonical moments on $[0,1]$, so we need to use the affine map of $[-2,2] \to [0,1]$, that is, $x \mapsto \frac{1}{4}(x+2)$.

Proposition 13.3.4. *Let $\tilde\gamma$ be a measure on $[0,1]$ and γ on $[-2,2]$ related by*

$$\int_0^1 f(x)\,d\tilde\gamma(x) = \int_{-2}^2 f(\tfrac{1}{4}\,(x+2))\,d\gamma(x) \tag{13.3.12}$$

Let P_n (resp. $\tilde P_n$) and a_n, b_n (resp. $\tilde a_n, \tilde b_n$) be the monic OPRL and Jacobi parameters for γ (resp. $\tilde\gamma$). Then

$$P_n(x) = 4^n \tilde P_n(\tfrac{1}{4}\,(x+2)) \tag{13.3.13}$$

$$p_n(\gamma) = p_n(\tilde\gamma) \tag{13.3.14}$$

$$a_n^2 = 16\tilde a_n^2 \tag{13.3.15}$$

$$b_n = 4\tilde b_n - 2 \tag{13.3.16}$$

Proof. (13.3.12) shows the RHS of (13.3.13) is orthogonal to $\{x^j\}_{j=0}^{n-1}$ in $L^2([-2,2], d\gamma)$. The 4^n makes this side monic. This proves (13.3.13). Since $x \mapsto \frac{1}{4}(x+2)$ is affine, the map preserves the canonical moments, p_n, proving (13.3.14).

We start with

$$\tilde P_{n+1}(y) = (y - \tilde b_{n+1})\tilde P_n(y) - \tilde a_n^2 \tilde P_{n-1}(y)$$

Replace y by $\frac{1}{4}(x+2)$ and multiply by 4^{n+1} to get

$$P_{n-1}(x) = (x + 2 - 4\tilde b_{n+1})P_n(x) - 16\tilde a_n^2 P_{n-1}(x)$$

from which (13.3.15) and (13.3.16) are immediate. □

By the relations (13.3.14), (13.3.15), and (13.3.16), Theorem 13.3.2 is equivalent to

Theorem 13.3.5. *Let $\tilde\gamma$ be a measure on $[0,1]$, p_k its canonical moments, and $\{\tilde\alpha_n, \tilde b_n\}_{n=1}^\infty$ its Jacobi parameters. Then*

$$\tilde a_{n+1}^2 = \zeta_{2n+2}\zeta_{2n+1} \tag{13.3.17}$$

$$\tilde b_{n+1} = \zeta_{2n+1} + \zeta_{2n} \tag{13.3.18}$$

We prove this by exploiting the Hankel determinants given by (1.2.3) and the Heine formula (1.2.30). Since we will deal with moments of several putative measures, we will make the measures explicit and use

$$H_{ij}^{(n)}(d\rho) = \int x^{i+j-2}\,d\rho(x) \tag{13.3.19}$$

but we only use c_n for the moments of $d\gamma$ so, for example,

$$H_{ij}^{(n)}(x(1-x)\,d\gamma) = c_{i+j-1} - c_{i+j} \tag{13.3.20}$$

Define

$$W_{2n} = H^{(n+1)}(d\gamma) \qquad n = 0,1,2,\dots \tag{13.3.21}$$

$$W_{2n+1} = H^{(n+1)}(x\,d\gamma) \qquad n = 0,1,2,\dots \tag{13.3.22}$$

$$Y_{2n} = H^{(n)}(x(1-x)\,d\gamma) \qquad n = 1,2,\dots \tag{13.3.23}$$

$$Y_{2n+1} = H^{(n+1)}((1-x)\,d\gamma) \qquad n = 0,1,2,\dots \tag{13.3.24}$$

The index is defined so W_k and Y_k depend on $c_0,\dots,c_k$ and are affine in c_k. For example, the lower right corner of W_k has c_k and that of Y_k has $c_{k-1} - c_k$.

We will let

$$w_n = \det(W_n) \qquad y_n = \det(Y_n)$$

As in Section 3.1, Dodgson's equality (Proposition 3.1.5) will play a major role. In particular, it will imply (see Corollary 13.3.14)

$$w_n y_n = w_{n-1} y_{n+1} + w_{n+1} y_{n-1} \tag{13.3.25}$$

We are heading towards proving that $\{c_n\}_{n=0}^\infty$ are the moments of a measure on $[0,1]$ if and only if all w_n and all y_n are nonnegative.

PROPOSITION 13.3.6. *Let f be a continuous function on $[0,1]$ and define*

$$f^{[n]}(x) = \sum_{j=0}^n \binom{n}{j} x^j (1-x)^{n-j} f\left(\frac{j}{n}\right) \tag{13.3.26}$$

Then

(i) $\|f^{[n]} - f\|_\infty \to 0$ *as* $n \to \infty$.

(ii) *If f is a polynomial of degree ℓ, so is each $f^{[n]}$ and the coefficients converge to those of f as $n \to \infty$.*

Remark. $f^{[n]}$ are called the Bernstein polynomials for f. That they approximate f is an expression of the law of large numbers that for fixed x and large n, $\binom{n}{j} x^j (1-x)^{n-j}$ is concentrated near those j's with $j \sim xn$. The proof even uses this law of large numbers intuition in a somewhat disguised form. Notice that this proposition proves Weierstrass' theorem on the density of the polynomials.

PROOF. (i) Define

$$\mathbb{E}_x^{(n)}(h(j,x)) \equiv \sum_{j=0}^n \binom{n}{j} x^j (1-x)^{n-j} h(j,x) \tag{13.3.27}$$

Since

$$\sum_{j=0}^{n} \binom{n}{j} (x+a)^j (1-x)^{n-j} = (1+a)^n \tag{13.3.28}$$

we see, taking $(\frac{d}{da})^\ell \big|_{a=0}$, that

$$\mathbb{E}_x^{(n)}(j(j-1)\dots(j-\ell)) = x^{\ell+1} n(n-1)\dots(n-\ell) \tag{13.3.29}$$

from which

$$\begin{aligned} \mathbb{E}_x^{(n)}\left(\left(x - \frac{j}{n}\right)^2\right) &= x^2 - 2x^2 + x^2\left(1 - \frac{1}{n}\right) + \frac{x}{n} \\ &= \frac{1}{n}\, x(1-x) \le \frac{1}{4n} \end{aligned} \tag{13.3.30}$$

for $0 \le x \le 1$.

For any bounded function f,

$$|f(x) - f(y)| \le \sup_{|x-y|\le\delta} |f(x) - f(y)| + 2\,\frac{(x-y)^2}{\delta^2}\, \|f\|_\infty$$

Thus, by (13.3.30),

$$\begin{aligned} |f^{[n]}(x) - f(y)| &\le \mathbb{E}_x^{(n)}\left(\left| f(x) - f\left(\frac{j}{n}\right)\right|\right) \\ &\le \sup_{|x-y|\le\delta} |f(x) - f(y)| + \frac{1}{2\delta^2 n}\, \|f\|_\infty \end{aligned} \tag{13.3.31}$$

Taking δ so small that the sup is less than $\varepsilon/2$ (since f is continuous) and then $n \ge \|f\|_\infty/(\delta^2\varepsilon)$, we see, for such n, that $\|f^{[n]} - f\|_\infty \le \varepsilon$, proving (i).

(ii) Since $j^{\ell+1} = j(j-1)\dots(j-\ell) +$ polynomial in j of degree ℓ, (13.3.29) implies $\mathbb{E}_x^{(n)}((\frac{j}{n})^{\ell+1}) = x^{\ell+1} +$ polynomial in x of degree ℓ with coefficients of $O(1/n)$. $\square$

We can use this to discuss solubility of the Hausdorff moment problem (moment problem on $[0,1]$).

Lemma 13.3.7. *Let A be an $n \times n$ Hermitian matrix and for $k = 1, 2, \dots, n$, let d_k be the determinant of the $k \times k$ matrix $\{a_{ij}\}_{1\le i,j\le k}$. Then A is strictly positive definite if and only if each $d_k > 0$.*

Proof. If A is strictly positive, so is each $k \times k$ submatrix, and so each $d_k > 0$. We prove the converse by induction. If $n = 1$, $d_1 = a_{11} > 0$ if $d_1 > 0$. If the theorem is true for $(n-1) \times (n-1)$ matrices and each $d_k > 0$, by induction, $\tilde{A} = (a_{ij})_{1\le i,j\le n-1}$ is strictly positive, so its eigenvalues obey $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_{n-1}$. The eigenvalues $\{\mu_\ell\}_{\ell=1}^n$

of A and $\tilde{A}$ interlace, so $\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \mu_n$. Thus $\mu_2 \dots \mu_n > 0$, and since $d_n = \mu_1 \dots \mu_n > 0$, we conclude $\mu_1 > 0$, that is, A is strictly positive definite. □

THEOREM 13.3.8. *Let $c_0, c_1, c_2, \dots$ be a sequence of reals. Then the following are equivalent:*

(i) *There is a positive measure, ρ, on $[0,1]$ with (13.3.1).*

(ii) *For all j, m,*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} c_{k+j} \ge 0 \tag{13.3.32}$$

(iii) *All the matrices $\{W_k\}_{k=0}^{\infty}$, $\{Y_k\}_{k=1}^{\infty}$ are positive definite.*

Moreover, ρ is nontrivial if and only if all W_k and Y_k are strictly positive definite and that happens if and only if $w_0 > 0$ and if $w_k > 0$ and $y_k > 0$ for $k = 1, \dots$.

PROOF. We will prove (i) ⇒ (iii) ⇒ (ii) ⇒ (i). It will help to define a linear function, L, on polynomials by

$$L\biggl(\sum_{j=0}^{n} \beta_j x^j\biggr) = \sum_{j=0}^{n} c_j \beta_j \tag{13.3.33}$$

<u>(i) ⇒ (iii).</u> W_{2k} is positive definite if and only if $L(Q^*Q) \ge 0$ for each nonzero polynomial, Q, of degree k, W_{2k+1} for $L(xQ^*Q)$, Y_{2k+2} for $L(x(1-x)Q^*Q)$, and Y_{2k+1} for $L((1-x)Q^*Q)$. If (i) holds, $L(Q) = \int Q(x)\, d\rho(x)$ and the nonnegativity of these functionals is immediate.

<u>(iii) ⇒ (ii).</u> We have

$$\text{LHS of (13.3.32)} = L(x^j(1-x)^m) \tag{13.3.34}$$

Since $x^j(1-x)^m = Q^*Q$ if j, m are even, xQ^*Q if j is odd and m is even, etc., the positivity in (iii) implies (ii).

<u>(ii) ⇒ (i).</u> Let Q be a polynomial which is positive on $[0,1]$. By Proposition 13.3.6, (13.3.32) and (13.3.34) imply $L(Q) \ge 0$. For any real Q, we have $L(\|Q\|_\infty 1 \pm Q) \ge 0$ so $|L(Q)| \le c_0 \|Q\|_\infty$ and L extends to a positive functional on $C([0,1])$, and so is a measure.

Next, to see nontriviality is equivalent to strict positivity, we note first that if ρ is nontrivial, then $L(Q^*Q) > 0$ for any Q since the zeros of Q are finite in number, and similarly for $L(xQ^*Q)$, etc. Thus nontriviality implies strict positivity. For the converse, if ρ is trivial and Q has zeros at the support of ρ but is nonzero, then $L(Q^*Q) = 0$.

Finally, the equivalence to positivity of w_k and y_k follows from Lemma 13.3.7. □

Given $c_0, \dots, c_{k-1}$ so $w_0, \dots, w_{k-1} > 0$ and $y_0, \dots, y_{k-1} > 0$, define $\tilde{c}_k^{\pm}$ by

$$w_k(c_0, \dots, c_{k-1}, \tilde{c}_k^-) = 0 \qquad y_k(c_0, \dots, c_{k-1}, \tilde{c}_k^+) = 0 \tag{13.3.35}$$

(We will see eventually that if $c_j = c_j(\rho)$, then $\tilde{c}_k^{\pm}(c_0(\rho), \dots, c_{k-1}(\rho)) = c_k^{\pm}(\rho)$, but we do not know this yet.) Since c_k appears in W_k only as c_k in the lower corner, $w_k = c_k w_{k-2} +$ const, and thus, since $w_{k-2} > 0$,

$$w_k(c_0, \dots, c_{k-1}, c_k) > 0 \Leftrightarrow c_k > \tilde{c}_k^- \tag{13.3.36}$$

Similarly, since $y_k = (c_{k-1} - c_k) y_{k-2} +$ const,

$$y_k(c_0, \dots, c_{k-1}, c_k) > 0 \Leftrightarrow c_k < \tilde{c}_k^+ \tag{13.3.37}$$

We have:

Proposition 13.3.9.

$$\tilde{c}_k^+ - \tilde{c}_k^- = \frac{w_{k-1} y_{k-1}}{w_{k-2} y_{k-2}} \tag{13.3.38}$$

and, in particular, this quantity is strictly positive.

Proof. Let $w_1, \dots, w_{k-1}$ denote the determinants with $c_0, \dots, c_{k-1}$ and $\tilde{w}_k^{\pm}$ with $c_0, \dots, c_{k-1}, \tilde{c}_k^{\pm}$, and similarly for y. As noted above,

$$\tilde{w}_k^{\pm} = w_{k-2} c_k^{\pm} + f(c_0, \dots, c_{k-1}) \tag{13.3.39}$$

Since $w_k^- = 0$ (by (13.3.35)), we see from this that

$$c_k^+ - c_k^- = \frac{\tilde{w}_k^+}{w_{k-2}} \tag{13.3.40}$$

By (13.3.25) for $n = k-1$, using $\tilde{y}_k^+ = 0$ (by (13.3.35)), we have

$$w_{k-1} y_{k-1} = \tilde{w}_k^+ y_{k-2} \tag{13.3.41}$$

(13.3.40) and (13.3.41) imply (13.3.38). □

Theorem 13.3.10. *The map from nontrivial ρ's in $\mathcal{M}_{+,1}([0,1])$ is well-defined, one-one, and onto. Moreover,*

$$c_k^+ - c_k^- = \frac{w_{k-1} y_{k-1}}{w_{k-2} y_{k-2}} \tag{13.3.42}$$

$$p_k = \frac{w_k y_{k-2}}{w_{k-1} y_{k-1}} \tag{13.3.43}$$

$$q_k = \frac{w_{k-2} y_k}{w_{k-1} y_{k-1}} \tag{13.3.44}$$

$$\zeta_k = \frac{w_k w_{k-3}}{w_{k-1} w_{k-2}} \tag{13.3.45}$$

Proof. Given $c_1, \dots, c_{k-1}$ with $w_0, \dots, w_{k-1}, y_1, \dots, y_{k-1} > 0$, by the above, $\tilde{c}_k^+ > \tilde{c}_k^-$, so we can pick c_k in (c_k^-, c_k^+) with $w_k > 0$, $y_k > 0$. In this way, we can inductively complete $c_1, \dots, c_{k-1}$ to an infinite sequence with all w, y positive and so corresponding to a nontrivial measure by Theorem 13.3.8.

Thus, given ρ and $c_0(\rho), \dots, c_{k-1}(\rho)$, for any choice of c_k in (c_k^-, c_k^+), we can extend to a measure. Thus, $\tilde{c}_k^\pm = c_k^\pm$, and by (13.3.42), p_k is well-defined in $(0,1)$.

This shows that given any $p_1, p_2, \dots \in (0,1)$ and $c_0 = 1$, we can uniquely find $c_0, c_1, \dots$ with those ρ_j showing the map is one-one and onto. Moreover, (13.3.42) holds.

By (13.3.39),

$$c_k - c_k^- = \frac{w_k}{w_{k-2}} \tag{13.3.46}$$

so, by (13.3.2) and (13.3.4), (13.3.43) holds. Similarly,

$$c_k^+ - c_k = \frac{y_k}{y_{k-2}} \tag{13.3.47}$$

which leads to (13.3.44). Since $\zeta_k = q_{k-1}p_k$, we get (13.3.45) for $k \geq 3$.

This also holds for $k = 1, 2$ if we interpret $w_{-2} = w_{-1} = 1$. For $k = 1, 2$, then, say, $\zeta_1 = w_1 = c_1$ and $\zeta_2 = w_2/w_0 w_1$. $\zeta_1 = c_1$ is immediate and the formula ζ_2 holds since (13.3.42) is true for $k = 2$ if $y_0 = 1$. $\square$

We now turn to Heine's formulae by defining two polynomials of degree m, each an $m \times m$ matrix

$$W_{2m-1}(x) = \begin{vmatrix} c_0 & \dots & c_m \\ \vdots & & \vdots \\ c_{m-1} & \dots & c_{2m-1} \\ 1 & \dots & x^m \end{vmatrix} \qquad W_{2m}(x) = \begin{vmatrix} c_1 & \dots & c_{m+1} \\ \vdots & & \vdots \\ c_m & \dots & c_{2m} \\ 1 & \dots & x^m \end{vmatrix} \tag{13.3.48}$$

The index on W is the largest index on c. Let P_n be the monic OPRL for the measure γ with moments c_j, and Q_n the monic OPRL for $x\,d\gamma$. Then Heine's formula implies

Proposition 13.3.11.

$$P_n(x) = \frac{W_{2n-1}(x)}{w_{2n-2}} \qquad Q_n(x) = \frac{W_{2n}(x)}{w_{2n-1}} \tag{13.3.49}$$

The connection to the y's is

Proposition 13.3.12.

$$W_n(1) = y_n \tag{13.3.50}$$

PROOF. In $W_{2n-1}(1)$, subtract column 2 from column 1, then column 3 from column 2, …, finally column n from column $n-1$ to get

$$W_{2m-1}(1) = \begin{vmatrix} c_0 - c_1 & \dots & c_{m-1} - c_m & c_m \\ \vdots & & \vdots & \vdots \\ c_{m-1} - c_m & \dots & c_{2m-2} - c_{2m-1} & c_{2m-1} \\ 0 & \dots & 0 & 1 \end{vmatrix}$$

which now has a trivial last row, so the determinant is y_{2m-1}. The proof for $n = 2m$ is identical. □

PROPOSITION 13.3.13. *We have, for* $m \geq 1$,

$$w_{2m-1} W_{2m+1}(x) = x w_{2m} W_{2m}(x) - w_{2m+1} W_{2m-1}(x) \qquad (13.3.51)$$

and for $m \geq 0$,

$$w_{2m} W_{2m+2}(x) = w_{2m+1} W_{2m+1}(x) - w_{2m+2} W_{2m}(x) \qquad (13.3.52)$$

PROOF. If we take W_{2m+1} given by (13.3.48) and delete the last two rows and first and last column, we get w_{2m}. If we remove the last row and first column, we get w_{2m+1}, and the second-to-last row and last column, $W_{2m-1}(x)$. If we remove the last row and last column, we get w_{2m}, and the next-to-last row and first column, $xW_{2m}(x)$ (by factoring an x out of the remaining last row). Thus (13.3.51) is just Dodgson's equality.

To obtain (13.3.52), we first write for $m \geq 0$,

$$W_{2m+2}(x) = \begin{vmatrix} 1 & c_0 & \dots & c_{m+1} \\ 0 & c_1 & \dots & c_{m+2} \\ \vdots & \vdots & & \vdots \\ 0 & c_{m+1} & \dots & c_{2m+2} \\ 0 & 1 & \dots & x^{m+1} \end{vmatrix}$$

and use Dodgson's equality as before, deleting the last two rows and first and last column. □

COROLLARY 13.3.14.

$$w_n y_n = w_{n-1} y_{n+1} + w_{n+1} y_{n-1}$$

Remark. This is (13.3.25).

PROOF. Let $x = 1$ and use (13.3.50) in the last proposition to get

$$w_{2m-1} y_{2m+1} = w_{2m} y_{2m} - w_{2m+1} y_{2m-1}$$
$$w_{2m} y_{2m+2} = w_{2m+1} y_{2m+1} - w_{2m-2} y_{2m}$$

which is (13.3.25) for $n = 2m$ and $2m + 1$. □

PROPOSITION 13.3.15. *We have, for* $m \geq 0$,

$$P_{m+1}(x) = xQ_m(z) - \zeta_{2m+1}P_m(x) \tag{13.3.53}$$

$$Q_{m+1}(x) = P_{m+1}(x) - \zeta_{2m+2}Q_m(x) \tag{13.3.54}$$

PROOF. Divide (13.3.51) by $w_{2m-1}w_{2m}$ and get

$$P_{m+1}(x) = xQ_m(x) - \frac{w_{2m+1}w_{2m-2}}{w_{2m-1}w_{2m}}\,P_m(x)$$

by using (13.3.49). By (13.3.45), this is (13.3.53). Similarly, (13.3.52) leads to (13.3.42) by dividing by $w_{2m}w_{2m+1}$. □

We are now ready to prove the result that implies the the Geronimus relation:

PROOF OF THEOREM 13.3.5. Multiply (13.3.54) for $m = n-1$ by x and use (13.3.53) for $m = n$ and $m = n-1$ to replace xQ_n and xQ_{n-1}. The result is

$$P_{n+1} + \zeta_{2n+1}P_n = xP_n - \zeta_{2n}[P_n + \zeta_{2n-1}P_{n-1}]$$

which implies

$$\tilde{b}_{n+1} = \zeta_{2n+1} + \zeta_{2n}$$
$$\tilde{a}_n^2 = \zeta_{2n}\zeta_{2n-1}$$

which implies (13.3.18). □

Notes and Historical Remarks. The theory of canonical moments for OPRL was developed in the book of Dette-Studden [**279**] based in part on earlier work of Krein [**657**], Karlin-Studden [**614**], and Krein-Nudelman [**666**]. They discussed the analog of OPUC without apparently being aware of Verblunsky's work discussed in Section 3.1. They did not realize the Geronimus connection. That was done in a paper of Faybusovich-Gekhtman [**344**], who did not seem to know of Geronimus' earlier work!

13.4. Szegő's Theorem for OPRL: A First Look

In this section, we will use the Szegő mapping to carry over Szegő's theorem to OPRL. Of necessity, our real measures $d\gamma$ will obey $\text{supp}(d\gamma) = [-2,2]$. In Theorem 13.9.9 and Section 13.10, we will discuss extensions of the theory to some cases with $\text{ess supp}(d\mu) = [-2,2]$, which is why we call this a first look.

The main theorem is the following: