I prefer to state them as

$$\lim_{n\to\infty} |\alpha_n| = 0 \Leftrightarrow \lim_{n\to\infty} b_{n,\ell} = 0 \quad \text{for each } \ell = 1, 2, \dots$$
$$|\Omega| = 2\pi \Leftrightarrow \lim_{n\to\infty} b_{n,\ell} = 0 \quad \textit{uniformly} \text{ in } \ell = 1, 2, \dots$$

Rakhmanov's theorem has been extended to general Jacobi matrices by Denisov [**272**]; see Section 13.4 for the proof and further history.

9.2. Khrushchev's Proof of Rakhmanov's Theorem

In this section and the next, we discuss an approach to a number of issues connected with asymptotics of φ_n and properties of $d\mu$ that rely on a study of the Schur algorithm associated to $d\mu$ (see Section 1.3). The key will be an analysis of the Schur iterates, $f_n(z, d\mu)$, given by iterating (1.3.34)/(1.3.35), that is,

$$\alpha_n(d\mu) = f_n(0, d\mu) \tag{9.2.1}$$

$$f_{n+1}(z, d\mu) = \frac{1}{z} \frac{f_n(z, d\mu) - \alpha_n(d\mu)}{1 - \overline{\alpha_n(d\mu)}\, f_n(z, d\mu)} \tag{9.2.2}$$

with $f_0(z, d\mu) = f(z, d\mu)$, the Schur function given by

$$\frac{1 + zf(z)}{1 - zf(z)} = \int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \tag{9.2.3}$$

In (9.2.1), we use Geronimus' theorem that the Schur parameters and Verblunsky coefficients are the same (see Theorems 3.1.4, 3.2.7, 3.2.10, 3.4.7, and 4.5.9).

Khrushchev's approach to Rakhmanov's theorem depends on

THEOREM 9.2.1. *Let $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\rm s}$ and $\Omega = \{\theta \mid w(\theta) > 0\}$. Then $|\partial\mathbb{D}\backslash\Omega| = 0$ if and only if*

$$\lim_{n\to\infty} \int_0^{2\pi} |f_n(e^{i\theta}, d\mu)|^2\, \frac{d\theta}{2\pi} = 0 \tag{9.2.4}$$

Because $f_n(0) = \alpha_n$, we have

$$\alpha_n = \int_0^{2\pi} f_n(e^{i\theta}, d\mu)\, \frac{d\theta}{2\pi} \tag{9.2.5}$$

so

$$|\alpha_n| \leq \biggl(\int_0^{2\pi} |f_n(e^{i\theta}, d\mu)|^2\, \frac{d\theta}{2\pi}\biggr)^{1/2} \tag{9.2.6}$$

and thus (9.2.4) implies $\lim |\alpha_n| = 0$, providing a proof of Rakhmanov's theorem.

As in the last section, there are quantitative versions of the relation. We will establish

$$\limsup \int |f_n(e^{i\theta}, d\mu)|^2\, \frac{d\theta}{2\pi} \leq 2\biggl(1 - \frac{|\Omega|}{2\pi}\biggr)^{1/2} \tag{9.2.7}$$

while for all n,

$$\int |f_n e^{i\theta}, d\mu)|^2\, \frac{d\theta}{2\pi} \geq 1 - \frac{|\Omega|}{2\pi} \tag{9.2.8}$$

In particular, (9.2.7) and (9.2.6) imply that

$$\limsup_n |\alpha_n| \leq \sqrt{2}\biggl(1 - \frac{|\Omega|}{2\pi}\biggr)^{1/4} \tag{9.2.9}$$

which is better than (9.1.2) when $|\Omega|$ is small but much worse when $|\Omega|/2\pi$ is near 1; this bound is $O((2\pi - |\Omega|)^{1/4})$ and (9.1.2) is $O((2\pi - |\Omega|)^{1/2})$.

We begin with a proof of (9.2.8):

THEOREM 9.2.2. *For any n,*

$$\{\theta \mid |f_n(e^{i\theta}, d\mu)| < 1\} = \Omega \tag{9.2.10}$$

In particular, on $\partial\mathbb{D}\backslash\Omega$, $|f_n(e^{i\theta}, d\mu)| = 1$ and thus, (9.2.8) *holds.*

PROOF. By (9.2.3) and the fact that $w \mapsto (1+w)/(1-w)$ maps $\mathbb{D}$ to $\{z \mid \operatorname{Re} z > 0\}$, we see that

$$\{\theta \mid |f(e^{i\theta}, d\mu)| < 1\} = \{\theta \mid w(\theta) > 0\} \tag{9.2.11}$$

If $\mu^{(n)}$ is the measure with $\alpha_k(d\mu^{(n)}) = \alpha_{k+n}(d\mu)$ and $d\mu^{(n)} = w^{(n)}(\theta)\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}^{(n)}$, then (9.2.11) says that

$$\{\theta \mid |f_n(e^{i\theta}, d\mu)| < 1\} = \{\theta \mid w^{(n)}(\theta) > 0\} \tag{9.2.12}$$

Theorem 3.4.4 implies

$$\{\theta \mid w(\theta) > 0\} = \{\theta \mid w^{(n)}(\theta) > 0\} \tag{9.2.13}$$

(9.2.12) and (9.2.13) imply (9.2.10).

This in turn implies $|f_n| = 1$ on $\partial\mathbb{D}\backslash\Omega$ so

$$\begin{aligned}\int_0^{2\pi} |f_n(e^{i\theta})|^2 \,\frac{d\theta}{2\pi} &\geq \int_{|\partial\mathbb{D}\backslash\Omega|} |f_n(e^{i\theta})|^2 \,\frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi}\,|\partial\mathbb{D}\backslash\Omega| = 1 - \frac{|\Omega|}{2\pi}\end{aligned}$$ $\square$

A key role in what follows is played by the functions, called the *inverse Schur iterates*,

$$b_n(z, d\mu) = \frac{\varphi_n(z; d\mu)}{\varphi_n^*(z; d\mu)} \tag{9.2.14}$$

PROPOSITION 9.2.3. (i) *b_n is analytic in a neighborhood of $\bar{\mathbb{D}}$ and meromorphic in $\mathbb{C}$.*

(ii) *If $\{z_j\}_{j=1}^n$ are the zeros of φ_n, then b_n is the finite Blaschke product*

$$b_n(z) = \prod_{j=1}^n \left(\frac{z - z_j}{1 - \bar{z}_j z}\right) \tag{9.2.15}$$

(iii)

$$b_{n+1} = \frac{z b_n(z) - \bar{\alpha}_n}{1 - \alpha_n z b_n(z)} \tag{9.2.16}$$

(iv) *The Schur parameters of $b_n(z)$ are $(-\bar{\alpha}_{n-1}, -\bar{\alpha}_{n-2}, \dots, -\bar{\alpha}_0, 1)$.*

PROOF. (i) is immediate, given that φ_n^* is nonvanishing in $\bar{\mathbb{D}}$ (Theorem 1.7.1).

(ii) Since $\varphi_n(z) = \kappa_n \prod_{j=1}^n (z - z_j)$, we have

$$\varphi_n^*(z) = z_n\,\overline{\varphi_n(1/\bar{z})} = \kappa_n \prod_{j=1}^n (1 - \bar{z}_j z)$$

and (9.2.15) follows from (9.2.14).

(iii) This is immediate from the Szegő recursion; to make the calculation simpler, note that $b_n = \Phi_n/\Phi_n^*$ so

$$b_{n+1} = \frac{z\Phi_n - \bar{\alpha}_n \Phi_n^*}{\Phi_n^* - \alpha_n z\Phi_n}$$

and (9.2.16) follows dividing numerator and denominator by Φ_n^*.

(iv) (9.2.16) says that $b_n(0) = -\bar{\alpha}_{n-1}$ and that the first Schur iterate of b_n is b_{n-1} (by (1.3.39)). The result follows by induction until we reach $b_0(z) \equiv 1$. □

The key to much of Khrushchev's analysis is

THEOREM 9.2.4 (Khrushchev's Formula). *Let $d\mu = w\frac{d\theta}{2\pi} + d\mu_{\rm s}$ be a nontrivial probability measure on $\partial\mathbb{D}$. Then*

(i) *For a.e. $e^{i\theta} \in \partial\mathbb{D}$,*

$$|\varphi_n(e^{i\theta})|^2 w(\theta) = \frac{1 - |f_n(e^{i\theta})|^2}{|1 - e^{i\theta} b_n(e^{i\theta}) f_n(e^{i\theta})|^2} \tag{9.2.17}$$

(ii) *We have that for $z \in \mathbb{D}$,*

$$\int \frac{e^{i\theta} + z}{e^{i\theta} - z} |\varphi_n(e^{i\theta})|^2 \, d\mu(\theta) = \frac{1 + zf_n(z)b_n(z)}{1 - zf_n(z)b_n(z)} \tag{9.2.18}$$

PROOF. (i) By (1.3.32),

$$w(\theta) = \frac{1 - |f(e^{i\theta})|^2}{|1 - e^{i\theta} f(e^{i\theta})|^2} \tag{9.2.19}$$

and, by (1.3.79),

$$f(z) = \frac{A_{n-1}(z) + zB_{n-1}^*(z) f_n(z)}{B_{n-1}(z) + zA_{n-1}^*(z) f_n(z)} \tag{9.2.20}$$

If $\omega_n = \prod_{j=0}^n (1 - |\alpha_j|^2)$, then (1.3.83) says that

$$|B_n(e^{i\theta})|^2 - |A_n(e^{i\theta})|^2 = \omega_n \tag{9.2.21}$$

Straightforward arithmetic from (9.2.20) using (9.2.21) shows that

$$1 - |f(e^{i\theta})|^2 = \frac{(1 - |f_n(e^{i\theta})|^2)\omega_{n-1}}{|B_{n-1} + e^{i\theta} A_{n-1}^* f_n|^2} \tag{9.2.22}$$

and using Theorem 3.2.10 that

$$|1 - e^{i\theta} f(e^{i\theta})|^2 = \frac{|\Phi_n^*(e^{i\theta}) - e^{i\theta}\Phi_n(e^{i\theta}) f(e^{i\theta})|^2}{|B_{n-1} + e^{i\theta} A_{n-1}^* f_n|^2} \tag{9.2.23}$$

Since $\varphi_n = \omega_{n-1}^{-1/2} \Phi_n$, (9.2.19) implies

$$\begin{aligned} w(\theta) &= \frac{(1 - |f_n|^2)}{|\varphi_n^* - e^{i\theta} \varphi_n f_n|^2} \\ &= \frac{1 - |f_n|^2}{|\varphi_n|^2 |1 - e^{i\theta} b_n f_n|^2} \end{aligned}$$

since $|\varphi_n^*| = |\varphi_n|$ on $\partial\mathbb{D}$. This is (9.2.17).

(ii) We have that

$$\mathrm{Re}\biggl(\frac{1 + zb_n f_n}{1 - zb_n f_n}\biggr) = \frac{1 - |zb_n f_n|^2}{|1 - zb_n f_n|^2} \tag{9.2.24}$$

so if F_n is the left side of (9.2.18) and R_n is the right side, we have, by (9.2.17), that for a.e. θ,

$$\operatorname{Re} F_n(e^{i\theta}) = \operatorname{Re} R_n(e^{i\theta}) \tag{9.2.25}$$

Suppose $\ell \geq 1$ and $d\mu_\ell = \frac{d\theta}{2\pi}|\varphi_\ell(e^{i\theta})|^{-2}$ is a Bernstein-Szegő approximation for $d\mu$. In that case, $F_n(z, d\mu_\ell)$ is analytic in a neighborhood of $\bar{\mathbb{D}}$, and $f(z, d\mu_\ell)$, and so $f_n(z, d\mu_\ell)$ are analytic in a neighborhood of $\bar{\mathbb{D}}$ with $|f_n(e^{i\theta}, d\mu_\ell)| < 1$ for all θ. Thus both sides of (9.2.18) are analytic in a neighborhood of $\bar{\mathbb{D}}$, so (9.2.25) implies the two sides agree up to an imaginary constant. Since both sides are 1 at $z = 0$, this constant is zero. Thus (9.2.18) holds for μ_ℓ.

Since

$$\varphi_n(z; d\mu_\ell) = \varphi_n(z; d\mu) \tag{9.2.26}$$

if $\ell > n$, for $z \in \mathbb{D}$, the left side of (9.2.18) converges to the result for $d\mu$ as $\ell \to \infty$ by the Bernstein-Szegő approximation theorem (Theorem 1.7.8). By (9.2.26), $b_n(z, d\mu_\ell) = b_n(z, d\mu)$ if $\ell \geq n$. Since $f_n(z, d\mu_\ell)$ has Schur parameters $(\alpha_n, \alpha_{n+1}, \dots, \alpha_\ell, 0, \dots)$ and $f_n(z, d\mu)$ has Schur parameters $(\alpha_n, \alpha_{n+1}, \dots, \alpha_\ell, \alpha_{\ell+1}, \dots)$, for $z \in \mathbb{D}$, $f_n(z, d\mu_\ell) \to f_n(z, d\mu)$ by Theorem 1.5.6. Thus for $z \in \mathbb{D}$, the right side of (9.2.18) converges. It follows that (9.2.18) holds for $d\mu$. $\square$

Remarks. 1. (9.2.18) can be restated as

$$f(z, |\varphi_n|^2 d\mu) = f_n(z, d\mu) b_n(z, d\mu) \tag{9.2.27}$$

2. (4.4.7) provides another formula for the left side of (9.2.18). Reconciling the two provides an alternate proof of (9.2.18). Define u_k and u_k^* by

$$u_k = \psi_k + F(z)\varphi_k \qquad u_k^* = -\psi_k^* + F(z)\varphi_k^* \tag{9.2.28}$$

Thus

$$\frac{u_0^*}{u_0} = \frac{-1 + F}{1 + F} = zf \tag{9.2.29}$$

by (1.3.2). By Theorem 3.2.11, $\Xi_k = \binom{u_k}{u_k^*}$ is the unique solution of (3.2.7) which is in ℓ^2. It follows that

$$\begin{pmatrix} u_\ell(z; \alpha_k, \alpha_{k+1}, \dots) \\ u_\ell^*(z; \alpha_k, \alpha_{k+1}, \dots) \end{pmatrix} = c \begin{pmatrix} u_{k+\ell}(z; \alpha_0, \alpha_1, \dots) \\ u_{k+\ell}^*(z; \alpha_0, \alpha_1, \dots) \end{pmatrix}$$

for some c. In particular, taking $\ell = 0$, by (9.2.29),

$$\frac{u_k^*}{u_k} = zf_k \tag{9.2.30}$$

Next, note that by (9.2.28) and (3.2.21),

$$\varphi_k^* u_k - \varphi_k u_k^* = 2z^k \tag{9.2.31}$$

Since

$$\frac{e^{i\theta} + z}{e^{i\theta} - z} = \left(\frac{2z}{e^{i\theta} - z} + 1 \right)$$

we have that

$$\begin{aligned} \text{LHS of (9.2.18)} &= 2z((\text{LHS of (4.4.7)})) + 1 \\ &= \frac{2\varphi_k u_k^* + 2z^k}{2z^k} \qquad \text{(by (4.4.7))} \end{aligned}$$

$$= \frac{\varphi_k u_k^* + \varphi_k^* u_k}{\varphi_k^* u_k - \varphi_k u_k^*} = \frac{1 + \frac{\varphi_k}{\varphi_k^*} \frac{u_k^*}{u_k}}{1 - \frac{\varphi_k}{\varphi_k^*} \frac{u_k^*}{u_k}} = \frac{1 + zb_k f_k}{1 - zb_k f_k}$$

by (9.2.14), (9.2.30), and (9.2.31). Thus (4.4.7) is equivalent to (9.2.18); its proof provides another proof of Khrushchev's formula.

3. For a third proof of (9.2.18) using finite rank perturbations of unitaries, see Theorem 4.5.10.

Proposition 9.2.5. *Let*

$$g_n(\theta, d\mu) = \frac{2|\varphi_n(e^{i\theta}, d\mu)|^2 w(\theta)}{1 + |\varphi_n(e^{i\theta}, d\mu)|^2 w(\theta)} \tag{9.2.32}$$

Then

$$\int_0^{2\pi} |f_n(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le 2 \int_0^{2\pi} |1 - g_n(\theta)| \, \frac{d\theta}{2\pi} \tag{9.2.33}$$

$$\le 2\biggl(\int_0^{2\pi} |1 - g_n(\theta)|^2 \, \frac{d\theta}{2\pi}\biggr)^{1/2} \tag{9.2.34}$$

Proof. Since for $z = e^{i\theta} \in \partial\mathbb{D}$,

$$|1 - zb_n f_n|^2 = 1 + |f_n|^2 - 2\,\mathrm{Re}(zb_n f_n)$$

(since $|b_n| = 1$ on $\partial\mathbb{D}$), we have, by (9.2.17), that

$$1 - |f_n|^2 = [1 + |f_n|^2 - 2\,\mathrm{Re}(zb_n f_n)]\, |\varphi_n|^2 w$$

Solving for $|f_n|^2$,

$$|f_n|^2 = \frac{1 - |\varphi_n|^2 w}{1 + |\varphi_n|^2 w} + \mathrm{Re}(zb_n f_n)\biggl[1 + \frac{|\varphi_n|^2 w - 1}{1 + w|\varphi_n|^2}\biggr] \tag{9.2.35}$$

Since $h_n \equiv zb_n f_n$ is analytic in $\mathbb{D}$ and bounded there, the Cauchy formula for H^∞ functions implies that

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(zb_n f_n)\bigg|_{z=e^{i\theta}} \frac{d\theta}{2\pi} = \mathrm{Re}\, h_n(0) = 0$$

so using $|\mathrm{Re}(zb_n f_n)| \le 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le 2 \int \frac{|1 - |\varphi_n|^2 w|}{|1 + |\varphi_n|^2 w|} \, \frac{d\theta}{2\pi}$$

$$= 2 \int |1 - g_n(e^{i\theta})| \, \frac{d\theta}{2\pi}$$

(9.2.34) then follows from the Schwarz inequality. □

Proposition 9.2.6. (i)

$$\int_0^{2\pi} |g_n(e^{i\theta})|^2 \, \frac{d\theta}{2\pi} \le 1 - \int |\varphi_n(e^{i\theta})|^2 \, d\mu_{\mathrm{s}}(\theta) \tag{9.2.36}$$

(ii)

$$\frac{w}{g_n} \frac{d\theta}{2\pi} \to \frac{1}{2}\biggl(d\mu + w\frac{d\theta}{2\pi}\biggr) \tag{9.2.37}$$

weakly as measures.

(iii) *If* $\Omega = \{\theta \mid w(\theta) > 0\}$, *then*

$$\liminf \int_\Omega g_n \frac{d\theta}{2\pi} \geq \frac{|\Omega|}{2\pi} \tag{9.2.38}$$

PROOF. (i) We have for $y \geq 0$,

$$4y + (1-y)^2 = (1+y)^2$$

so

$$\frac{4y^2}{(1+y)^2} \leq y$$

It follows that

$$|g_n(e^{i\theta})|^2 \leq |\varphi_n(e^{i\theta})|^2 w(\theta) \tag{9.2.39}$$

so integrating,

$$\int |g_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq \int |\varphi_n(e^{i\theta})|^2 d\mu - \int |\varphi_n(e^{i\theta})|^2 \, d\mu_{\rm s}$$

which is (9.2.36).

(ii)

$$\frac{w}{g_n} = \frac{1}{2}\biggl(\frac{1}{|\varphi_n(e^{i\theta})|^2} + w\biggr)$$

so (9.2.37) follows from the Bernstein-Szegő approximation theorem (Theorem 1.7.8).

(iii) Let $f \in C(\partial\mathbb{D})$. Writing

$$f\sqrt{w} = f\sqrt{g_n}\,\sqrt{\frac{w}{g_n}}$$

and using the Schwarz inequality, we find that

$$\biggl(\int f(e^{i\theta})\sqrt{w(e^{i\theta})}\,\frac{d\theta}{2\pi}\biggr)^2 \leq \biggl(\int_\Omega g_n(e^{i\theta})\,\frac{d\theta}{2\pi}\biggr)\biggl(\int |f(e^{i\theta})|^2\,\frac{w(e^{i\theta})}{g_n(e^{i\theta})}\,\frac{d\theta}{2\pi}\biggr)$$

so taking n to infinity and using (9.2.37), we have

$$\biggl(\int f(e^{i\theta})\sqrt{w(e^{i\theta})}\,\frac{d\theta}{2\pi}\biggr)^2 \leq L\biggl\{\tfrac{1}{2}\int |f(e^{i\theta})|^2[d\mu(e^{i\theta}) + d\mu_{\rm ac}(e^{i\theta})]\biggr\} \tag{9.2.40}$$

where

$$L = \liminf \int_\Omega g_n(e^{i\theta})\,\frac{d\theta}{2\pi}$$

We now use an argument familiar from the last section and approximate any f in $L^2(d\mu)$ by f's in $C(\partial\mathbb{D})$ using the density of $C(\partial\mathbb{D})$ in $L^2(d\mu)$. The right side of (9.2.40) converges trivially and the left side does by writing $f\sqrt{w} = (f\sqrt{w}\,)(1)$ and using the Schwarz inequality. Thus (9.2.40) holds for all $f \in L^2(\partial\mathbb{D}, d\mu)$.

Pick a set $A \subset \partial\mathbb{D}$ with $|A| = 0$ and $\mu_{\rm s}(\partial\mathbb{D}\backslash A) = 0$ and let

$$f(e^{i\theta}) = \begin{cases} 0 & \text{if } e^{i\theta} \notin \Omega \text{ or } e^{i\theta} \in A \\ \frac{1}{\sqrt{w(e^{i\theta})}} & \text{if } e^{i\theta} \in \Omega\backslash A \end{cases}$$

which is in $L^2(\partial\mathbb{D}, d\mu)$. Then (9.2.40) becomes

$$\biggl(\frac{|\Omega|}{2\pi}\biggr)^2 \leq L\biggl(\frac{|\Omega|}{2\pi}\biggr)$$

which is (9.2.38). $\square$

We are prepared to prove (9.2.7):

THEOREM 9.2.7. *Let $d\mu = w\frac{d\theta}{2\pi} + d\mu_{\rm s}$ be a nontrivial probability measure on $\partial\mathbb{D}$. Let $\Omega = \{\theta \mid w(\theta) > 0\}$. Then*

$$\limsup\left[\left\{\int |f_n(e^{i\theta}, d\mu)|^2 \, \frac{d\theta}{2\pi}\right\}^2 + 4\int |\varphi_n(e^{i\theta}, d\mu)|^2 \, d\mu_{\rm s}(\theta)\right] \leq 8\left(1 - \frac{|\Omega|}{2\pi}\right) \tag{9.2.41}$$

PROOF. We have

$$\begin{aligned}
\text{LHS of (9.2.41)} &\leq \limsup\left[4\int |1-g_n|^2 \, \frac{d\theta}{2\pi} + 4\int |\varphi_n|^2 \, d\mu_{\rm s}\right] \quad \text{(by (9.2.34))} \\
&\leq \limsup\left\{4 + 4\left[\int_0^{2\pi} |g_n|^2 \, \frac{d\theta}{2\pi} + \int |\varphi_n|^2 \, d\mu_{\rm s}\right] - 8\int_0^{2\pi} |g_n| \, \frac{d\theta}{2\pi}\right\} \\
&\leq 8 - 8 \liminf \int_0^{2\pi} |g_n| \, \frac{d\theta}{2\pi} \quad \text{(by (9.2.36))} \\
&\leq 8 - \frac{8|\Omega|}{2\pi} \quad \text{(by (9.2.38))}
\end{aligned}$$
$\square$

PROOF OF THEOREM 9.2.1 AND (9.2.9). We proved (9.2.8) in Theorem 9.2.2 and (9.2.7) in Theorem 9.2.7. This implies Theorem 9.2.1. As noted in the introduction to this section, (9.2.9) then follows from (9.2.6) and (9.2.7). $\square$

THEOREM 9.2.8 (Second Proof of Rakhmanov's Theorem). *Let $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\rm s}$ be a nontrivial probability measure on $\partial\mathbb{D}$. If $w(\theta) > 0$ for $\frac{d\theta}{2\pi}$-a.e. θ, then*

$$\lim_{n\to\infty} |\alpha_n| = 0$$

PROOF. Immediate from (9.2.9). $\square$

We next want to find an analog of (9.1.27) in a setting where the Schur iterates are the fundamental objects:

THEOREM 9.2.9. *Let $d\mu = w\frac{d\theta}{2\pi} + d\mu_{\rm s}$ be a nontrivial probability measure on $\partial\mathbb{D}$ and $f_n(z)$ its Schur iterates. Then*

$$\int_0^{2\pi} \big|\, |\varphi_n(e^{i\theta})|^2 w(\theta) - 1 \big| \, \frac{d\theta}{2\pi} \leq 8\int_0^{2\pi} |f_n(e^{i\theta})| \, \frac{d\theta}{2\pi} \tag{9.2.42}$$

PROOF. Given $y \in \mathbb{R}$, let $y_\pm = \frac{1}{2}(|y| \pm y)$, so $y = y_+ - y_-$ and $|y| = y_+ + y_-$. If

$$q_n(e^{i\theta}) = |\varphi_n(e^{i\theta})|^2 w(\theta) - 1 \tag{9.2.43}$$

then

$$\begin{aligned}
\int |q_n| \, \frac{d\theta}{2\pi} &= 2\int (q_n)_- \, \frac{d\theta}{2\pi} + \int q_n \, \frac{d\theta}{2\pi} \\
&\leq 2\int (q_n)_- \, \frac{d\theta}{2\pi}
\end{aligned} \tag{9.2.44}$$

since $\int q_n \frac{d\theta}{2\pi} = \int |\varphi_n|^2 \, d\mu - 1 - \int |\varphi_n|^2 \, d\mu_{\rm s} = -\int |\varphi_n|^2 \, d\mu_{\rm s} \leq 0$.

By (9.2.39), $-q_n \leq 1 - g_n^2$ so

$$(q_n)_- \leq (1 - g_n^2)_+ \leq 2(1 - g_n)_+ \tag{9.2.45}$$

since $(1-g_n^2)_+ \neq 0 \Rightarrow 0 \leq g_n \leq 1 \Rightarrow (1-g_n^2) = (1+g_n)(1-g_n) \leq 2(1-g_n)$. But (9.2.35) says

$$1 - g_n = |f_n|^2 - \mathrm{Re}(zb_nf_n)g_n$$

so if $(1-g_n)_+ \geq 0$, we have $g_n \leq 1$ and thus

$$(1-g_n)_+ \leq |f_n|^2 + |zb_nf_n| \leq 2|f_n| \tag{9.2.46}$$

since $|z| = 1$, $|b_n| \leq 1$, and $|f_n| \leq 1$. It follows that

$$(q_n)_- \leq 4|f_n| \tag{9.2.47}$$

so

$$\int (q_n)_- \frac{d\theta}{2\pi} \leq 4 \int |f_n| \frac{d\theta}{2\pi} \tag{9.2.48}$$

(9.2.44) and (9.2.48) imply (9.2.42). $\square$

Complementing (9.2.42) is

Proposition 9.2.10. *Let $d\mu = w\frac{d\theta}{2\pi} + d\mu_s$ be a nontrivial probability measure on $\partial\mathbb{D}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq 2\int_0^{2\pi} |\,|\varphi_n(e^{i\theta})|^2 w(\theta) - 1|\frac{d\theta}{2\pi} \tag{9.2.49}$$

Proof. With g_n given by (9.2.6),

$$|1-g_n| = \frac{|1-|\varphi_n|^2w|}{|1+|\varphi_n|^2w|} \leq |\,|\varphi_n|^2w - 1|$$

so (9.2.33) implies (9.2.49). $\square$

Interestingly enough, these results allow a link to the ideas of Section 9.1:

Corollary 9.2.11. *In terms of quantity $b_{n,\ell}$ of* (9.1.1), *we have*

$$|\alpha_n|^2 \leq 2b_{n,\ell} \qquad \ell = 1,2,\dots \tag{9.2.50}$$

and

$$b_{n,1} \leq 8|\alpha_n| \tag{9.2.51}$$

Proof. Let $d\mu_k$ be the Bernstein-Szegő approximation $d\mu_k = d\theta/2\pi|\varphi_k(e^{i\theta})|^2$. Then $d\mu_k$ has Verblunsky coefficients $(\alpha_0,\alpha_1,\dots,\alpha_{k-1},0,\dots)$, so

$$f_m(0,d\mu_k) = \alpha_m \qquad m = 0,1,\dots,k-1 \tag{9.2.52}$$

and

$$f_{k-1}(z,d\mu_k) = \alpha_{k-1} \tag{9.2.53}$$

If $d\mu_k = w_k(\theta)\frac{d\theta}{2\pi}$, then since $\varphi_n(z;d\mu_k) = \varphi_n(z;d\mu)$ for $n \leq k$,

$$k > n \Rightarrow \int_0^{2\pi} |\,|\varphi_n(e^{i\theta};d\mu_k)|^2 w_k(\theta) - 1|\frac{d\theta}{2\pi} = b_{n,k-n} \tag{9.2.54}$$

Thus (9.2.49) and (9.2.52) imply for $\ell \geq 1$,

$$\begin{aligned}|\alpha_n|^2 &\leq \int |f_n(e^{i\theta}, d\mu_{n+\ell})|^2 \frac{d\theta}{2\pi} \\ &\leq 2b_{n,\ell}\end{aligned}$$

proving (9.2.50) and, by (9.2.42) and (9.2.53),

$$b_{n,1} \leq 8\int_0^{2\pi} |f_n(e^{i\theta}, d\mu_{n+1})|\frac{d\theta}{2\pi} = 8|\alpha_n|$$

proving (9.2.51). $\square$

Recall that a sequence of functions, f_n, are said to *converge in measure* to f if and only if for all $\varepsilon > 0$,

$$\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$$

as $n \to \infty$. Note that

PROPOSITION 9.2.12. *Let f_n, f be a sequence of functions on a probability measure space so that for some $p \in (1, \infty]$, $\sup_n \|f_n\|_p < \infty$ with $\|\cdot\|_p$ the L^p-norm. Then the following are equivalent:*
(i) *$f_n \to f$ in measure.*
(ii) *$\|f_n - f\|_q \to 0$ for one q in $[1, p)$.*
(iii) *$\|f_n - f\|_q \to 0$ for all q in $[1, p)$.*

PROOF. (iii) $\Rightarrow$ (ii) is obvious, so we will prove (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii). Let

$$S_{\varepsilon,n} = \{x \mid |f_n(x) - f(x)| > \varepsilon\} \tag{9.2.55}$$

(ii) $\Rightarrow$ (i) We have, by looking at the contribution to the integral when $x \in S_{\varepsilon,n}$,

$$\int |f_n - f|^q \, d\mu \geq \varepsilon^q \mu(S_{\varepsilon,n})$$

so

$$\mu(S_{\varepsilon,n}) \leq \varepsilon^{-q} \|f_n - f\|_q^q$$

(i) $\Rightarrow$ (iii) Given q, let $r = (\frac{1}{q} - \frac{1}{p})^{-1}$. Then, by Hölder's inequality,

$$\int_{S_\varepsilon} |f_n - f|^q \, d\mu \leq \|f - f_n\|_p^q \mu(S_{\varepsilon,n})^{q/r}$$

while, clearly,

$$\int_{\Omega \setminus S_\varepsilon} |f_n - f|^q \, d\mu \leq \varepsilon^q$$

so

$$\|f_n - f\|_q^q \leq \varepsilon^q + \big[2^q \sup_n \|f_n\|_p^q\big] \mu(S_{\varepsilon,n})^{q/r}$$

By (i), the $\limsup$ of the second term is 0, so for each ε, $\limsup \|f_n - f\|_q \leq \varepsilon$, that is, $f_n \to f$ in L^q. $\square$

We can summarize in the following:

THEOREM 9.2.13. *Let $d\mu = w\frac{d\theta}{2\pi} + d\mu_{\rm s}$ be a nontrivial probability measure on $\partial\mathbb{D}$. The following are equivalent:*
(i) *$w(\theta) > 0$ for $\frac{d\theta}{2\pi}$-a.e. θ.*
(ii) *The Schur iterates, f_n, restricted to $\partial\mathbb{D}$ converge to zero in measure.*
(iii) *For some $p < \infty$,*

$$\lim_{n\to\infty} \int |f_n(e^{i\theta})|^p \, \frac{d\theta}{2\pi} = 0 \tag{9.2.56}$$

(iv) *(9.2.56) holds for all $p < \infty$.*
(v) $\lim_{n\to\infty} \int_0^{2\pi} \big| |\varphi_n(e^{i\theta})|^2 w(\theta) - 1 \big| \frac{d\theta}{2\pi} = 0$
(vi) *For some $\alpha \in (0, 1)$, we have*

$$\lim_{n\to\infty} \int_0^{2\pi} [|\varphi_n(e^{i\theta})|^2 w(\theta)]^\alpha \, \frac{d\theta}{2\pi} = 1 \tag{9.2.57}$$

(vii) (9.2.57) *holds for all* $\alpha \in (0,1)$.

PROOF. (ii), (iii), (iv) are equivalent, given Proposition 9.2.12 and $\|f_n\|_\infty \le 1$. So we need only show (i) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) $\Rightarrow$ (i), (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (v).

(i) $\Rightarrow$ (iii) for $p=2$ by (9.2.7).

(iv) $\Rightarrow$ (v) by (9.2.42).

(v) $\Rightarrow$ (i) since if $\Omega = \{\theta \mid w(\theta) > 0\}$, then

$$\frac{1}{2\pi}\,|2\pi\backslash\Omega| \le \int_0^{2\pi} |\,|\varphi_n(e^{i\theta})|^2 w - 1|\,\frac{d\theta}{2\pi}$$

(v) $\Rightarrow$ (vi) In the proof of Theorem 9.1.14, we show (9.1.29) implies (9.1.32), so (v) implies that

$$\lim_{n\to\infty}\int_0^{2\pi} |(|\varphi_n|^2 w)^{1/2} - 1|^2\,\frac{d\theta}{2\pi} = 0$$

Thus (9.2.57) holds for $\alpha = \frac{1}{2}$.

(vi) $\Rightarrow$ (vii) $\beta_n(\alpha) = \log\{\int_0^{2\pi}[|\varphi_n(e^{i\theta})|^2 w(\theta)]^\alpha \frac{d\theta}{2\pi}\}$ is convex in α by Hölder's inequality and $\beta_n(0) = \beta_n(1) = 1$. It follows that if $\beta_n(\alpha_0) \to 1$, it holds for all $\alpha \in [0,1]$.

(vii) $\Rightarrow$ (v) 1 and $\sqrt{|\varphi_n|^2 w}$ are both vectors of length at most 1 in $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, so $\|\sqrt{|\varphi_n|^2 w} - 1\|_2^2 \le 2 - 2\langle 1, \sqrt{|\varphi_n|^2 w}\rangle$. Thus (vii) implies $\sqrt{|\varphi_n|^2 w} \to 1$ in L^2, so $|\varphi_n|^2 w \to 1$ in L^1, that is, (v) holds. $\square$

As a final result in this section, we want to use Khrushchev's formula, (9.2.17), to improve Theorem 2.6.4.

THEOREM 9.2.14. *Let* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}$ *be a nontrivial probability measure obeying the Szegő condition,* $\kappa_\infty < \infty$. *Then* (2.6.17) *holds in the sense of norm convergence, that is,*

$$\lim_{n\to\infty}\int |\log|\varphi_n(e^{i\theta})|^{-2} - \log(w(\theta))|\,\frac{d\theta}{2\pi} = 0 \tag{9.2.58}$$

PROOF. The integrand, $I_n(\theta)$, in (9.2.58) is $|-\log(w(\theta)|\varphi_n(e^{i\theta})|^2)|$, which, by (9.2.17), is bounded by

$$I_n(\theta) \le \log\frac{1}{[1-|f_n|^2]} + 2|\log|1 - e^{i\theta}b_n f_n|\,| \tag{9.2.59}$$

By Lemma 2.7.8,

$$\int \log|1 - e^{i\theta}b_n f_n|\,\frac{d\theta}{2\pi} = 0$$

since $zb_n(z)f_n(z)|_{z=0} = 0$.

Thus

$$\begin{aligned}\int |\log|1 - e^{i\theta}b_n f_n|\,|\,\frac{d\theta}{2\pi} &= 2\int (\log|1 - e^{i\theta}b_n f_n|)_+\,\frac{d\theta}{2\pi} \\ &\le 2\int |f_n(e^{i\theta})|\,\frac{d\theta}{2\pi} &(9.2.60)\\ &\le 2\biggl(\int \log\biggl(\frac{1}{1-|f_n|^2}\biggr)\frac{d\theta}{2\pi}\biggr)^{1/2} &(9.2.61)\end{aligned}$$

where (9.2.60) comes from $(\log|1+x|)_+ \le \log(1+|x|) \le |x|$ (since $\log(1+y) = \int_0^y \frac{du}{1+u} \le y$ for $y > 0$) and (9.2.61) comes from the Schwarz inequality and $y \le -\log(1-y) = y + \frac{y^2}{2} + \cdots$ for $y > 0$. By (9.2.59), we see that (9.2.58) is implied by

$$\lim_{n\to\infty} -\int \log(1-|f_n|^2)\,\frac{d\theta}{2\pi} = 0 \tag{9.2.62}$$

By (2.7.16) (a version of Szegő's theorem), the integral on the left of (9.2.62) is

$$-\sum_{j=n}^{\infty} \log(1-|\alpha_j|^2)$$

which goes to zero as $n \to \infty$ since $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$. □

Remarks and Historical Notes. This section is a rearrangement of part of Khrushchev [**625**], which we follow. The one innovation here is our use of (9.2.39). This occurs first in our proof of Proposition 9.2.6(i), which improves on Khrushchev's Lemma 6.2. It also allows an 8 in (9.2.42) where Khrushchev gets a 12. (9.2.22) appears earlier in Foias-Frazho [**358**]. As we showed, Khrushchev's formula, (9.2.18), is equivalent to (4.4.7). That formula in turn is essentially the same as equation (2.37) of Geronimo-Teplyaev [**399**] — so these authors had Khrushchev's formula five years before he did, but they did not realize its usefulness as Khrushchev did.

9.3. Further Aspects of Khrushchev's Theory

Two themes and their relation dominate this section. In the last two sections, we proved that if $w(\theta) > 0$ for a.e. θ, then $|\varphi_n(e^{i\theta})|^2\,d\mu \to \frac{d\theta}{2\pi}$ in norm. Here we will instead consider the question of when the following holds:

Definition. Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$. We say $d\mu$ obeys the *Rakhmanov condition* if and only if $|\varphi_n(e^{i\theta})|^2\,d\mu(\theta)$ converges weakly to $\frac{d\theta}{2\pi}$, that is, if and only if for all $f \in C(\partial\mathbb{D})$,

$$\lim_{n\to\infty} \int f(e^{i\theta})|\varphi_n(e^{i\theta})|^2\,d\mu(\theta) = \int f(e^{i\theta})\,\frac{d\theta}{2\pi} \tag{9.3.1}$$

Definition. A sequence of Verblunsky coefficients is said to obey the *Máté-Nevai* (MN) *condition* if and only if for each fixed $\ell = 1, 2, \dots$,

$$\lim_{n\to\infty} \alpha_n \alpha_{n+\ell} = 0 \tag{9.3.2}$$

One of the main results in this section is

Theorem 9.3.1 (Khrushchev [**625**]). *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$. $d\mu$ obeys the Rakhmanov condition if and only if its Verblunsky coefficients $\{\alpha_n(d\mu)\}_{n=0}^{\infty}$ obey the MN condition.*

The second theme involves the Schur approximates, $f^{[n]}(z) = A_n(z)/B_n(z)$, defined in Section 1.3. Recall (1.3.42) that says $f^{[n]}(z) \to f(z)$ uniformly on compact subset of $\mathbb{D}$. The issue is convergence on $\partial\mathbb{D}$.

Definition. We say $d\mu$ obeys the *Khrushchev condition* if and only if either
(i) $d\mu_{\rm ac} = 0$ (i.e., $d\mu$ is purely singular), or
(ii) $\alpha_n(d\mu) \to 0$ as $n \to \infty$
(or both).