proofs follow his, except that his derivation of (13.1.24) uses (13.1.14) and is more complicated than our proof using (13.1.22).

The relation (13.1.7) is also from Geronimus [**405**]; see also [**406**, Section 30]. Its usefulness was emphasized to me by Peherstorfer.

Damanik-Killip [**221**] have an interesting alternate discussion of the formulae of this section. They derive the direct relation using Schur functions. Their derivation of the inverse formula is an inductive argument close to Geronimus, but their formulae look different since they involve our $1/u_n$ and $1/v_n$, so that our $(v_n - u_n)/(v_n + u_n)$ is their $(u_n^{-1} - v_n^{-1})/(u_n^{-1} + v_n^{-1})$, and their formulae are for α_{2n} and α_{2n+1} in terms of u_n^{-1} and v_n^{-1} rather than Geronimus' α_{2n} and α_{2n-1}. Since $u_n^{-1} + v_n^{-1}$ can be used to write $u_{n+1} + v_{n+1}$, that shift is to be expected.

There are two continuum analogs of the Szegő mapping. One is via Krein systems. If $A = \bar{A}$, the transfer matrix for (1.1.32)/(1.1.33) becomes

$$D_A = \begin{pmatrix} 0 & \frac{d}{dx} - A \\ -\frac{d}{dx} - A & 0 \end{pmatrix}$$

and

$$D_A^* D_A = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix}$$

where $H^\pm = -\frac{d}{dx^2} + q^\pm$ and $q^\pm = A^2 \mp A'$, the analog of the direct Geronimus relation.

Damanik-Killip [**221**] introduce a different connection. If $-\frac{d^2}{dx^2} \pm V \geq 0$, they look at the function $\varphi^\pm$ obeying

$$-\varphi^\pm \pm V\varphi_\pm = 0$$

with $\varphi \pm (0) = 0$, $\varphi^{\pm\prime}(0) = 1$. If one defines $u = \varphi_+'/\varphi_+$, $v = \varphi_-'/\varphi_-$, their basic functions are

$$\Gamma_e(x) = \tfrac{1}{2}\,[u(x) - v(x)] \qquad \Gamma_o(x) = -\tfrac{1}{2}\,[u(x) + v(x)]$$

and their basic equations are

$$\Gamma_e'(x) = V(x) + 2\Gamma_e(x)\Gamma_o(x) \qquad \Gamma_o'(x) = \Gamma_o^2(x) + \Gamma_e^2(x)$$

and where they analyze $\alpha_{2n}, \alpha_{2n-1}$ to study certain Jacobi matrices, they analyze $\Gamma_o(x), \Gamma_e(x)$ to study certain Schrödinger operators.

13.2. CMV Matrices and the Geronimus Relations

Mathematicians tend to despise Dirac notation, because it can prevent them from making important distinctions, but physicists love it, because they are always forgetting such distinctions exist and the notation liberates them from having to remember . . .

— *N. D. Mermin*

The proof of the Geronimus relations (13.1.23)/(13.1.24) was not difficult, but we feel an alternate proof of Killip-Nenciu [**632**] based on the CMV matrix is very illuminating, so we present it in this section. We will also see that there are four natural maps of symmetric measures on $\partial\mathbb{D}$ to measures on $[-2, 2]$, each with its Geronimus relations. The second of these four is implicit in the Szegő construction and the other pair was discovered by Berriochoa, Cachafeiro, and García-Amor [**116, 117**], who use methods like those from Section 13.1 to establish the Geronimus relations for these cases.

Let $d\xi$ be a measure on $\partial\mathbb{D}$ symmetric under $z \to z^{-1} = \bar{z}$. There is a natural map $M : L^2(\partial\mathbb{D}, d\xi) \to L^2(\partial\mathbb{D}, d\xi)$ by

$$(Mf)(z) = f(z^{-1}) \tag{13.2.1}$$

Clearly,

$$MzM^{-1} = z^{-1} \tag{13.2.2}$$

so

$$M(z + z^{-1})M^{-1} = (z + z^{-1}) \tag{13.2.3}$$

Notice that $M^2 = \mathbf{1}$, so

$$L^2(\partial\mathbb{D}, d\xi) = L^2_e(\partial\mathbb{D}, d\xi) \oplus L^2_o(\partial\mathbb{D}, d\xi) \tag{13.2.4}$$

where $M = 1$ (resp. -1) on L^2_e (resp. L^2_o). (13.2.3) implies $z + z^{-1}$ leaves each of these subspaces fixed, so

$$z + z^{-1} = J_e \oplus J_o \tag{13.2.5}$$

If $\mathcal{H}_e^{(n)}$ (resp. $\mathcal{H}_o^{(n)}$) is the set of Laurent polynomials of degree at most $2n$ in L^2_e (resp. L^2_o), then $\mathcal{H}_e^{(n)}$ is of dimension $n+1$ (resp. n) and is spanned by $\{z^j + z^{-j}\}_{j=0}^n$ (resp. $\{z^j - z^{-j}\}_{j=1}^n$). Clearly, J_e (resp. J_o) maps $\mathcal{H}_e^{(n)}$ to $\mathcal{H}_e^{(n+1)}$ (resp. $\mathcal{H}_o^{(n)}$ to $\mathcal{H}_o^{(n+1)}$) and thus, since J_e and J_o are Hermitian, they define Jacobi matrices in the orthonormalization of the nested subspaces $\mathcal{H}_e^{(0)} \subset \mathcal{H}_e^{(1)} \subset \cdots$ (i.e., in some basis $\varphi_j \in \mathcal{H}_e^{(j)} \cap [\mathcal{H}_e^{(j-1)}]^\perp$).

This is, of course, an abstraction of the scheme we implemented in terms of polynomials in the last section. What does it have to do with the CMV matrix? Look at the relation (4.2.11) of the two CMV bases. When all Verblunsky coefficients are real, we have, by induction (using the Szegő recursion), that

$$\overline{\Phi_n(\bar{z})} = \Phi_n(z) \tag{13.2.6}$$

(which just says that all coefficients of Φ_n are real!) and this implies the same for χ_n and x_n. Thus, (4.2.11) becomes

$$x_n(z) = \chi_n(z^{-1}) \tag{13.2.7}$$

or equivalently,

$$M\chi_n = x_n \tag{13.2.8}$$

One can also see this by noting M takes the ordered set $1, z, z^{-1}, z^2, z^{-2}, \ldots$ to $1, z^{-1}, z, z^{-2}, z^2, \ldots$ and since M preserves orthogonality, it respects the Gram-Schmidt process leading to (13.2.8).

Since $M^2 = 1$, we have

$$Mx_n = \chi_n \tag{13.2.9}$$

In particular, in the χ basis, we have

$$\begin{aligned} M_{ij} &= \langle \chi_i, M\chi_j \rangle = \langle M\chi_i, \chi_j \rangle \\ &= \langle x_i, \chi_j \rangle \end{aligned}$$

so M is given by the matrix $\mathcal{M}$ (see (4.2.16)) of the CMV factorization!

Thus, the key to computing J_e and J_o is to diagonalize $\mathcal{M}$. $\mathcal{M}$ is, by (4.2.21), a sum of 2×2 matrices of the form Θ_j given by (4.2.20). Since α_j is real, these matrices have the form

$$\begin{pmatrix} -\cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \tilde{S}_\varphi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{S}_\varphi^{-1} \tag{13.2.10}$$

where

$$\tilde{S}_\varphi = \begin{pmatrix} \cos(\frac{\varphi}{2}) & \sin(\frac{\varphi}{2}) \\ -\sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix} \tag{13.2.11}$$

This follows from $\cos(\varphi) = \cos^2(\frac{\varphi}{2}) - \sin^2(\frac{\varphi}{2})$, $\sin(\varphi) = 2\cos(\frac{\varphi}{2})\sin(\frac{\varphi}{2})$ and can be understood geometrically by noting the left side is a reflection that takes $\binom{0}{1}$ to $\binom{\sin(\varphi)}{\cos(\varphi)}$ and so is a rotation by the half-angle conjugating the reflection $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see Figure 13.1). Alternatively, one can compute the eigenvectors of Θ.

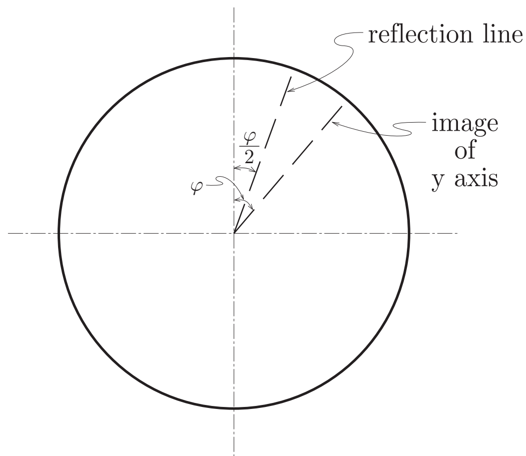

FIGURE 13.1. Why $\tilde{S}_\varphi$ has half angles

Since $\cos(\frac{\varphi}{2}) = \sqrt{(1+\cos(\varphi))/2}$ $\sin(\frac{\varphi}{2}) = \sqrt{(1-\cos(\varphi))/2}$, (13.2.10) can be rewritten for Θ_j,

$$S(\alpha)\Theta(\alpha)S(\alpha)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.2.12}$$

where (α real, $\rho = (1-|\alpha^2|)^{1/2}$)

$$\Theta(\alpha) = \begin{pmatrix} \alpha & \rho \\ \rho & -\alpha \end{pmatrix} \qquad S(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1-\alpha} & -\sqrt{1+\alpha} \\ \sqrt{1+\alpha} & \sqrt{1-\alpha} \end{pmatrix} \tag{13.2.13}$$

which can be checked directly by noting that $\sqrt{1-\alpha}\,\sqrt{1+\alpha} = \rho$ and $(\sqrt{1-\alpha})^2 - (\sqrt{1+\alpha})^2 = -2\alpha$.

Thus, we define

$$\mathcal{S} = \begin{pmatrix} 1 & & & \\ & S(\alpha_1) & & \\ & & S(\alpha_3) & \\ & & & \ddots \end{pmatrix} \tag{13.2.14}$$

It follows that $\mathcal{S}\mathcal{M}\mathcal{S}^{-1} = \mathcal{R}$ where

$$\mathcal{R} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{pmatrix} \tag{13.2.15}$$

and if $\mathcal{B} = \mathcal{S}\mathcal{L}\mathcal{S}^{-1}$, then

$$\mathcal{S}(\mathcal{C} + \mathcal{C}^{-1})\mathcal{S} = \mathcal{R}\mathcal{B} + \mathcal{B}\mathcal{R} \tag{13.2.16}$$

Now we see, since $\mathcal{R}^2 = \mathbf{1}$,

$$\begin{aligned} \mathcal{R}(\mathcal{R}\mathcal{B} + \mathcal{B}\mathcal{R}) &= \mathcal{R}\mathcal{B} + \mathcal{R}\mathcal{B}\mathcal{R} \\ &= (\mathcal{R}\mathcal{B} + \mathcal{B}\mathcal{R})\mathcal{R} \end{aligned}$$

(this is just (13.2.3)). Thus, $\mathcal{R}\mathcal{B} + \mathcal{B}\mathcal{R}$ vanishes on odd off-diagonals. Since $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{S}^{-1}$ are tridiagonal, $\mathcal{B}$ is 7-diagonal and thus, so are $\mathcal{R}\mathcal{B}$ and $\mathcal{B}\mathcal{R}$ and, a priori, $\mathcal{R}\mathcal{B} + \mathcal{R}\mathcal{B}$, but since the diagonals 1 and 3 from the main diagonal vanish, there are only three nonvanishing diagonals.

Let $\mathcal{H}_e$ (resp. $\mathcal{H}_o$) be the vectors, φ, in ℓ^2 with $R\varphi = \varphi$ (resp. $\mathcal{R}\varphi = -\varphi$), that is, with φ labelled $(\varphi_0, \varphi_1, \varphi_2, \dots)$ so $\mathcal{H}_e$ is the set of vectors $(u_1, 0, u_2, 0, u_3, \dots)$, that is, $\varphi_{2n} = u_{n-1}$ for $n = 0, 1, 2, \dots$ and $\varphi_{2n+1} = 0$. Similarly, $\mathcal{H}_o$ is these φ's with $\varphi_{2n-1} = u_n$ and $\varphi_{2n} = 0$. We thus have

$$\mathcal{R}\mathcal{B} + \mathcal{B}\mathcal{R} = \mathcal{S}(\mathcal{C} + \mathcal{C}^{-1})\mathcal{S} = \mathcal{J}_e \oplus \mathcal{J}_o$$

where $\mathcal{J}_e$ and $\mathcal{J}_o$ are tridiagonal.

With these wordy preliminaries out of the way, we are ready to prove

THEOREM 13.2.1. *We have for $y = e$ or o,*

$$\mathcal{J}_y = \begin{pmatrix} b^{(y)} & a_1^{(y)} & 0 & \dots \\ a_1^{(y)} & b_2^{(y)} & a_2^{(y)} & \dots \\ 0 & a_2^{(y)} & b_3^{(y)} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{13.2.17}$$

where

$$b_{n+1}^{(e)} = \alpha_{2n}(1 - \alpha_{2n-1}) - \alpha_{2n-2}(1 + \alpha_{2n-1}) \tag{13.2.18}$$

$$(a_{n+1}^{(e)})^2 = (1 + \alpha_{2n+1})(1 - \alpha_{2n}^2)(1 - \alpha_{2n-1}) \tag{13.2.19}$$

$$b_{n+1}^{(o)} = \alpha_{2n}(1 - \alpha_{2n+1}) - \alpha_{2n+2}(1 + \alpha_{2n+1}) \tag{13.2.20}$$

$$(a_{n+1}^{(o)})^2 = (1 - \alpha_{2n+3})(1 - \alpha_{2n+2}^2)(1 + \alpha_{2n+1}) \tag{13.2.21}$$

Moreover, $\mathcal{J}_e$ is the Jacobi matrix for the spectral measure $\gamma_e \equiv \mathrm{Sz}(\xi)$ given by (13.1.1)/(13.1.4) *and $\mathcal{J}_o$ is the Jacobi matrix for the spectral measure*

$$d\gamma_o \equiv c^2(4 - |x|^2)\, d(\mathrm{Sz}(\xi))(x) \tag{13.2.22}$$

with

$$c = [2(1 - |\alpha_0|^2)(1 - \alpha_1)]^{-1/2} \tag{13.2.23}$$

In particular, if $d\xi(\theta) = w(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta)$, then

$$f_e(x) = \pi^{-1}(4 - x^2)^{-1/2} w\biggl(\arccos\Bigl(\frac{x}{2}\Bigr)\biggr) \tag{13.2.24}$$

$$f_o(x) = c\pi^{-1}(4-x^2)^{1/2} w\biggl(\arccos\biggl(\frac{x}{2}\biggr)\biggr) \tag{13.2.25}$$

where $d\gamma_y(x) = f_y(x)\,dx + d\gamma_{y,\mathrm{s}}$.

Remarks. 1. (13.2.18)/(13.2.19) are, of course, just (13.1.23)/(13.1.24) and this is just the promised second proof.

2. By changing the signs of the δ_j, we can flip the sign of the off-diagonal terms, which is one reason we only list the a_{n+1}^2.

3. As usual, we interpret $\alpha_{-1} = -1$.

4. Notice that for $\alpha_n \equiv 0$, $d\gamma_o$ is the free Jacobi matrix measure with $a_n \equiv 1$, $b_n = 0$. This can also be seen by looking at the measure (13.2.22).

Proof. For φ, ψ both in $\mathcal{H}_e$ (resp. both in $\mathcal{H}_o$), $\langle \varphi, (\mathcal{R}\mathcal{B}+\mathcal{B}\mathcal{R})\psi\rangle = 2\langle\varphi,\mathcal{B}\psi\rangle$ (resp. $-2\langle\varphi,\mathcal{B}\psi\rangle$), so we need only compute matrix elements of $\mathcal{B}$. It will be convenient to use Dirac notation $|n\rangle$ for δ_n and so $\langle\delta_n,\mathcal{B}\delta_m\rangle = \langle n|\mathcal{B}|m\rangle$.

Since J_e is tridiagonal in $\{|2j\rangle\}_{j=0}^\infty$ basis, we need to compute

$$b_{n+1}^{(e)} = 2\langle 2n|\mathcal{B}|2n\rangle \tag{13.2.26}$$

$$[a_{n+1}^{(e)}]^2 = [2\langle 2n+2|\mathcal{B}|2n\rangle]^2 \tag{13.2.27}$$

We write $\mathcal{B} = \mathcal{S}\mathcal{L}\mathcal{S}^t$ since $\mathcal{S}^{-1} = \mathcal{S}^* = \mathcal{S}^t$ since $\mathcal{S}$ is unitary and real. $\mathcal{S}$ and $\mathcal{S}^t$ take $|2n\rangle$ to $|2n-1\rangle$ and $|2n-1\rangle$ to $|2n\rangle$. And $\mathcal{L}$ maps $|2n-1\rangle$ to $|2n-2\rangle$ and $|2n-2\rangle$ to $|2n-1\rangle$. We cannot go to $|2n-2\rangle$ and get back. It follows that

$$\langle 2n|\mathcal{B}|2n\rangle = |\langle 2n|\mathcal{S}|2n\rangle|^2\langle 2n|\mathcal{L}|2n\rangle + |\langle 2n-1|\mathcal{S}|2n\rangle|^2\langle 2n-1|\mathcal{L}|2n-1\rangle \tag{13.2.28}$$

and we use

$$\langle 2n|\mathcal{L}|2n\rangle = \Theta(\alpha_{2n})_{11} = \alpha_{2n} \tag{13.2.29}$$

$$\langle 2n-1|\mathcal{L}|2n-1\rangle = \Theta(\alpha_{2n-2})_{22} = -\alpha_{2n-2} \tag{13.2.30}$$

$$|\langle 2n|\mathcal{S}|2n\rangle|^2 = |S(\alpha_{2n-1})_{22}|^2 = \tfrac{1}{2}(1-\alpha_{2n-1}) \tag{13.2.31}$$

$$|\langle 2n-1|\mathcal{S}|2n\rangle|^2 = |S(\alpha_{2n-1})_{12}|^2 = \tfrac{1}{2}\,(1+\alpha_{2n-1}) \tag{13.2.32}$$

(13.2.23) is immediate from (13.2.26), (13.2.28), (13.2.29), (13.2.30), (13.2.31), and (13.2.32)

We will be more streamlined for the other calculations:

$$\begin{aligned}[a_{n+1}^{(e)}]^2 &= |2\langle 2n+2|\mathcal{B}|2n\rangle|^2 \\ &= |2\langle 2n+2|\mathcal{S}|2n+1\rangle\langle 2n+1|\mathcal{L}|2n\rangle\langle 2n|\mathcal{S}|2n\rangle|^2 \\ &= |2S(\alpha_{2n+1})_{21}\Theta(\alpha_{2n})_{21}S(\alpha_{2n-1})_{11}|^2 \\ &= (1+\alpha_{2n+1})\rho_{2n}^2(1-\alpha_{2n-1})\end{aligned}$$

proving (13.2.19).

Similarly,

$$\begin{aligned}b_{n+1}^{(o)} &= -2\langle 2n+1|\mathcal{B}|2n+1\rangle \\ &= -2[|\langle 2n+1|\mathcal{S}|2n+1\rangle|^2\langle 2n+1|\mathcal{L}|2n+1\rangle \\ &\qquad\qquad + |\langle 2n+2|\mathcal{S}|2n+1\rangle|^2\langle 2n+2|\mathcal{L}|2n+2\rangle] \\ &= -2[S(\alpha_{2n+1})_{11}^2\Theta(\alpha_{2n})_{22} + S(\alpha_{2n+1})_{12}^2\Theta(\alpha_{2n+2})_{11}] \\ &= \text{RHS of (13.2.20)}\end{aligned}$$

and

$$\begin{aligned}|a_{n+1}^{(o)}|^2 &= |2\langle 2n+3|\mathcal{B}|2n+1\rangle|^2 \\ &= 4|\langle 2n+3|\mathcal{S}|2n+3\rangle\langle 2n+3|\mathcal{L}|2n+2\rangle\langle 2n+2|\mathcal{S}^t|2n+1\rangle|^2 \\ &= 4|S(\alpha_{2n+3})_{11}\Theta(\alpha_{2n+2})_{21}S(\alpha_{2n+1})_{21}|^2 \\ &= \text{RHS of (13.2.21)}\end{aligned}$$

That provides the calculations of the connection formulae. To check the measures, we note that $|0\rangle$ corresponds to the function 1 in $L^2(\partial\mathbb{D}, d\xi(\theta))$ whose spectral measure for functions of $z+z^{-1}=x$ is given by

$$\int f(x)\, d\gamma_e(x) = \int f(2\cos\theta)\, d\xi(\theta)$$

that is, (13.1.2), so $\gamma_e = \mathrm{Sz}(\xi)$ as claimed.

To find γ_o, we note that $|1\rangle$ is an odd real, second-degree normalized Laurent polynomial so $c(z-z^{-1})$. Since (note $\int z^2\, d\xi = \int z^{-2}\, d\xi$)

$$\int |z-z^{-1}|\, d\xi = \int (2-2z^2)\, d\xi \tag{13.2.33}$$

$$= 2(1-c_2) \tag{13.2.34}$$

$$= 2(1-\alpha_0^2)(1-\alpha_1) \tag{13.2.35}$$

(by (1.3.52)), c is given by (13.2.23). Thus

$$\int f(x)\, d\gamma_0(x) = \int f(2\cos\theta)[4-(2\cos\theta)^2]c^2\, d\xi(\theta)$$

yielding (13.2.22). Here we used that $z = e^{i\theta}$ means

$$|z-z^{-1}|^2 = 4\sin^2\theta = 4-(2\cos\theta)^2 \qquad \square$$

The above depended on $\mathcal{M}^2=1$, but $\mathcal{L}^2=1$ also. Thus, if we look at the spaces with $\mathcal{L}=\pm 1$ (i.e., φ's with $z\varphi(z^{-1}) = \pm\varphi(z)$), $\mathcal{C}$ also is a direct sum of Jacobi matrices. Explicitly,

$$\mathcal{T} = \begin{pmatrix} S(\alpha_0) & & & \\ & S(\alpha_2) & & \\ & & S(\alpha_4) & \\ & & & \ddots \end{pmatrix} \tag{13.2.36}$$

and

$$\mathcal{T}\mathcal{L}\mathcal{T}^{-1} = -\mathcal{R} \tag{13.2.37}$$

with $\mathcal{R}$ given by (13.2.15). Thus, with $\tilde{\mathcal{B}} = \mathcal{T}\mathcal{M}\mathcal{T}^{-1}$, we have

$$\mathcal{T}(\mathcal{C}+\mathcal{C}^{-1})\mathcal{T}^{-1} = \mathcal{J}^+ \oplus \mathcal{J}^-$$

where $\mathcal{J}^+$ acts on $\mathcal{H}_e$ and $\mathcal{J}^-$ on $\mathcal{H}_o$.

Theorem 13.2.2. *$\mathcal{J}^\pm$ are given by* (13.2.18) *where*

$$b_{n+1}^{(\pm)} = \mp[\alpha_{2n+1}(1\pm\alpha_{2n}) - \alpha_{2n-1}(1\mp\alpha_{2n})] \tag{13.2.38}$$

$$(a_{n+1}^{(\pm)})^2 = (1\mp\alpha_{2n+2})(1-\alpha_{2n+1}^2)(1\pm\alpha_{2n}) \tag{13.2.39}$$

Moreover, the spectral measures $d\gamma_\pm$ for these Jacobi matrices are given by

$$d\gamma_\pm = c_\pm^2(2\mp x)\, d(\mathrm{Sz}(\xi))(x) \tag{13.2.40}$$

with

$$c_\pm = [2(1 \mp \alpha_0)]^{-1/2} \tag{13.2.41}$$

Remark. As usual, $\alpha_{-1} = 1$.

PROOF. As in the last theorem, noting that $-\mathcal{R}$ is -1 on $\mathcal{H}_e$ and $+1$ on $\mathcal{H}_o$,

$$\begin{aligned} b_{n+1}^{(+)} &= -2\langle 2n|\tilde{\mathcal{B}}|2n\rangle \\ &= -2[|\langle 2n|\mathcal{T}|2n\rangle|^2\langle 2n|\mathcal{M}|2n\rangle + |\langle 2n+1|\mathcal{T}|2n\rangle|^2\langle 2n+1|\mathcal{M}|2n+1\rangle] \\ &= -2|S(\alpha_{2n})_{11}|^2\Theta(\alpha_{2n-1})_{22} - 2|S(\alpha_{2n})_{12}|^2\Theta(\alpha_{2n+1})_{11} \\ &= \text{RHS of (13.2.38) for } (+) \end{aligned}$$

while

$$\begin{aligned} b_{n+1}^{(-)} &= 2\langle 2n+1|\tilde{\mathcal{B}}|2n+1\rangle \\ &= 2[|\langle 2n+1|\mathcal{T}|2n+1\rangle|^2\langle 2n+1|\mathcal{M}|2n+1\rangle + |\langle 2n+1|\mathcal{T}|2n\rangle|^2\langle 2n|\mathcal{M}|2n\rangle] \\ &= 2|S(\alpha_{2n})_{22}|^2\Theta(\alpha_{2n+1})_{11} + 2|S(\alpha_{2n})_{21}|^2\Theta(\alpha_{2n-1})_{22} \\ &= \text{RHS of (13.2.38) for } (-1) \end{aligned}$$

And we compute

$$\begin{aligned} [a_{n+1}^{(+)}]^2 &= 4|\langle 2n+2|\tilde{\mathcal{B}}|2n\rangle|^2 \\ &= 4|\langle 2n+2|\mathcal{T}|2n+2\rangle\langle 2n+2|\mathcal{M}|2n+1\rangle\langle 2n+1|\mathcal{T}^t|2n\rangle|^2 \\ &= 4|S(\alpha_{2n+2})_{11}\Theta(\alpha_{2n+1})_{21}S(\alpha_{2n})_{12}|^2 \\ &= \text{RHS of (13.2.39) for } (+) \end{aligned}$$

and

$$\begin{aligned} [a_{n+1}^{(-)}]^2 &= 4|\langle 2n+3|\tilde{\mathcal{B}}|2n+1\rangle|^2 \\ &= 4|\langle 2n+3|\mathcal{T}|2n+2\rangle\langle 2n+2|\mathcal{M}|2n+1\rangle\langle 2n+1|\mathcal{T}^t|2n+1\rangle|^2 \\ &= 4|S(\alpha_{2n+2})_{21}\Theta(\alpha_{2n+2})_{21}S(\alpha_{2n})_{22}|^2 \\ &= \text{RHS of (13.2.39) for } (-) \end{aligned}$$

This verifies the connection formulae.

To check the spectral measures, we note that $\mathcal{L} = \mathcal{C}\mathcal{M}$, since $\mathcal{M}^2 = 1$, and thus,

$$(\mathcal{L}\varphi)(z) = z\varphi(z^{-1})$$

Recalling that $\mathcal{L}|0\rangle = -|0\rangle$ and $\mathcal{L}|1\rangle = |1\rangle$ and that $|0\rangle$ and $|1\rangle$ are linear combinations of 1 and z, we see that

$$|0\rangle = c_+(1-z) \qquad |1\rangle = c_-(1+z)$$

with

$$1 = c_\pm^2\left(2 - 2\int z\,d\xi\right) = c_\pm^2(2 - 2\alpha_0)$$

so $c_\pm$ are given by (13.2.42). Since

$$|1 \mp z|^2 = 2 \mp (2\cos\theta)$$

we see the measures have the claimed form. $\square$

EXAMPLE 13.2.3. If $\alpha_n \equiv 0$, we have $a_n^{(\pm)} \equiv 1$, $b_1^{(\pm)} = \mp 1$, $b_n^{(\pm)} = 0$; $n \geq 2$. Just as the free Jacobi matrix (which is connected to the $\alpha_n^{(o)}$, $b_n^{(o)}$ for $\alpha_n \equiv 0$) is related to Chebyshev polynomials of the second kind and $\mathrm{Sz}(\frac{d\theta}{2\pi})$ (i.e., $a_n^{(e)}$, $b_n^{(e)}$ for $a_n \equiv 0$) to Chebyshev polynomials of the first kind, these examples are connected to Chebyshev polynomials of the third and fourth kind. These are defined by

$$W_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})} \tag{13.2.42}$$

$$V_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})} \tag{13.2.43}$$

By writing out $\sin(\theta)$ and $\cos(\theta)$ in terms of $e^{i\theta}$, we have $\sin(n+\frac{1}{2})\theta/\sin(\frac{\theta}{2}) = \sum_{j=-n}^{n} e^{ij\theta} = 1 + \sum_{j=1}^{n} 2\cos(j\theta)$, so

$$V_n(x) = T_0(x) + 2\sum_{j=1}^{n} T_j(x) \tag{13.2.44}$$

If we define

$$P_n^{(+)} = V_n\left(\frac{x}{2}\right) \qquad P_n^{(-)}(x) = W_n\left(\frac{x}{2}\right) \tag{13.2.45}$$

then

$$\begin{aligned} xP_n^{(\pm)}(x) &= P_{n+1}^{(\pm)} + P_{n-1}^{(\pm)} \qquad && n \geq 1 \\ xP_0^{\pm}(x) &= P_1^{\pm}(x) \mp P_0^{\pm}(x) && n = 0 \end{aligned} \tag{13.2.46}$$

which are the OPRL for these images of $\alpha_n \equiv 0$.

The orthogonality measure for V_n is $2\sin^2(\frac{\theta}{2})\frac{d\theta}{2\pi} = \frac{1}{\pi}\sin^2(\frac{\theta}{2})/\sin\theta\, d(\cos\theta)$, which leads directly to

$$d\gamma_+(x) = \frac{1}{2}\sqrt{\frac{2-x}{2+x}}\, dx \tag{13.2.47}$$

and similarly,

$$d\gamma_-(x) = \frac{1}{2}\sqrt{\frac{2+x}{2-x}}\, dx \tag{13.2.48}$$

□

Remarks and Historical Notes. The approach to the Geronimus relations for all four maps from α's to Jacobi matrices comes from Killip-Nenciu [**632**]. While it may well predate them, the earliest place I am aware of (13.2.20)/(13.2.21) appearing is Berriochoa, Cachafeiro, and García-Amor [**117**]. This paper introduced, from an orthogonal polynomial point of view, the mappings I call $d\xi \to d\gamma^{(\pm)}$ and [**116**] computed (13.2.38)/(13.2.39) by methods like those that appear in Section 13.1.

To compare their formulae and ours, we note the following dictionary: Their $\Phi_n(0)$ is our $-\bar{\alpha}_{n-1}$, and while their a's and ours agree, their b_n is our b_{n+1}.

Earlier, Peherstorfer [**846**] considered the measures $d\gamma^{(\pm)}$ and their relation of their OPs to the OPUC. For analogs of (13.1.13) for $d\gamma^{(o)}$ and $d\gamma^{(\pm)}$, see Berriochoa et al. [**117**] and Peherstorfer [**846**]. Peherstorfer [**846**] has a formula that sheds light on the relation of (13.2.18)/(13.2.19) to (13.2.20)/(13.2.21). Namely, he shows that if $d\tilde{\mu}$ is related to $d\mu$ by $\alpha_n \to -\alpha_n$ (i.e., $\tilde{F}(z) = 1/F(z)$), then the OPs for $d\tilde{\gamma}^{(o)}$ are the second kind polynomials for $d\gamma^{(e)}$. This means that one can get $a_n^{(o)}$ and

$b_n^{(o)}$ from $a_n^{(e)}$ and $b_n^{(e)}$ by changing the signs of the α's and taking $n \to n+1$ (i.e., α_j to α_{j+2}), which gives another — and illuminating — proof of (13.2.20)/(13.2.21)

When the α's are real, for any integer ℓ, $\mathcal{C}^\ell \mathcal{M} \equiv \mathcal{M}_\ell$ obeys $\mathcal{M}_\ell^2 = 1$ and $\mathcal{M}_\ell(\mathcal{C} + \mathcal{C}^t) = (\mathcal{C} + \mathcal{C}^t)\mathcal{M}_\ell$, but it is only for $\ell = 0, \pm 1$ ($\mathcal{M}_{-\ell} = \mathcal{M}_\ell^*$ and so $\mathcal{M}_{-1}$ and $\mathcal{M}_1$ are essentially equivalent) that $\mathcal{C} + \mathcal{C}^t \restriction (\mathcal{M}_\ell = \pm 1)$ is tridiagonal. [**117**] have a result related to this observation and show these four maps are the only ones within a potential class that map OPUC to OPRL.

We emphasize that $d\xi \to d\gamma^{(e)}$ is onto *all* measures supported on $[-2, 2]$ but that the other three maps are not surjective. For example,

$$\operatorname{ran}(d\xi \to d\gamma^{(o)}) = \left\{ \gamma \text{ supported in } [-2,2] \;\middle|\; \int (4 - x^2)^{-1}\, d\gamma < \infty \right\}$$

so, for example, the γ for the Chebyshev polynomials of the first kind (i.e., $\frac{1}{\pi}(4 - x^2)^{-1/2}\, dx$) is never a $d\gamma^{(o)}$.

It should be possible (and is hinted at in [**116**]) to write down analogs of the inverse Geronimus relations for the other three maps $d\xi \to d\gamma^{(0)}$, $d\xi \to d\gamma^{(\pm)}$ discussed here. These should have the form of (13.1.34) and (13.1.37) but with $\varphi_n^\pm$ the solution of (13.1.27) with different boundary conditions than $\varphi_1 = 1$, $\varphi_0 = 0$. Presumably, the other solution has $\varphi_{-1} = 0$, $\varphi_0 = 1$, and the two choices of boundary condition at $+2$ and -2 yield the four inverses. The restriction on whether a $d\gamma$ lies in the range of the other maps is connected with whether this second solution is positive for $+2$ and sign alternating for -2.

13.3. Szegő's Theorem for OPRL: A First Look

In this section, we will use the Szegő mapping to carry over Szegő's theorem to OPRL. Of necessity, our real measures $d\gamma$ will obey $\operatorname{supp}(d\gamma) = [-2, 2]$. In Theorem 13.8.9 and Section 13.9, we will discuss extensions of the theory to some cases with $\operatorname{ess\,supp}(d\mu) = [-2, 2]$, which is why we call this a first look.

The main theorem is the following:

THEOREM 13.3.1. *Let $d\gamma = f(x)\, dx + d\gamma_s$ be a measure on $[-2, 2]$. Let $a_n \equiv a_n(d\gamma)$ and $b_n \equiv b_n(d\gamma)$ be its Jacobi parameters, and let $\alpha_n = \alpha_n(\mathrm{Sz}^{-1}(d\gamma))$ be the Verblunsky coefficients of the measure $d\mu$ on $\partial\mathbb{D}$ with $d\gamma = \mathrm{Sz}(d\mu)$. Then the following are equivalent:*

(i)

$$\inf(a_n \dots a_1) > 0 \tag{13.3.1}$$

(ii)

$$\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \tag{13.3.2}$$

(iii) *All of the following*

(a) $$\sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty \tag{13.3.3}$$

(b) $\lim_{n\to\infty} a_n \dots a_1$ *exists and is nonzero and finite*

(c) $\lim_{n\to\infty} \sum_{j=1}^{n} b_j$ *exists*