CHAPTER 11

Periodic Verblunsky Coefficients

The splendid creations of complex function theory have excited the admiration of mathematicians mainly because they have enriched our science in an almost unparalleled way with an abundance of new ideas and opened up heretofore wholly unknown fields to research. The Cauchy integral formula, the Riemann mapping theorem and the Weierstrass power series calculus not only laid the groundwork for a new branch of mathematics but at the same time they furnished the first and until now the most fruitful example of the intimate connections between analysis and algebra. But it isn't just the wealth of novel ideas and discoveries which the new theory furnishes; of equal importance on the other hand are the boldness and profundity of the methods by which the greatest difficulties are overcome and the most recondite of truths, the mysteria functiorum, are exposed to the brightest light.

— *Richard Dedekind*

The study of one-dimensional periodic Schrödinger operators has been a mainstay of mathematical analysis for almost 125 years, with major developments starting with Hill's initial work (which have time as the dimension and looked at stability of the moon's orbit, not quantum theory!) and the follow up of Lyapunov and Floquet. In the early days of quantum theory, Bloch, Brillouin, and Wigner made significant discoveries codified by Gel'fand. As part of the KdV revolution, groups in the U.S. (including Flaschka, Kac, Lax, McKean, Moser, van Moerbeke) and Russia (including Dubrovin, Its, Marchenko, Matveev, and Novikov) found, in the period 1974–1976, a remarkable structure of abelian integrals, isospectral flows, and hyperelliptic functions.

On the OPUC side, there was early work of Geronimus on periodic Verblunsky coefficients and, more recently, important series of papers by Peherstorfer and collaborators and by Golinskii and collaborators (and related work on OPRL of which Akhiezer's work was especially important). But, surprisingly, the OPUC workers did not make much contact with the extensive work on Hill's equation and its discrete analog. The sole exception is a paper of Geronimo-Johnson [**398**], which limited itself to the almost periodic case but used the hyperelliptic function and abelian integral theory so extensively developed earlier.

Thus, the bulk of this chapter, which concerns an extremely beautiful theory, is new although it is very close to results in the Hill case.

Section 11.1 computes the spectrum by introducing the discriminant, a key object in the whole theory, and Section 11.2 discusses periodic CMV matrices and the related Floquet theory. Sections 11.3 and 11.4 introduce the inverse spectral theory problem, which will be a central focus of much of the chapter. Sections 11.5 and 11.6 describe two so far not totally successful attempts to prove the main results of the inverse theory without extensive machinery, while Sections 11.7 and 11.8 describe a successful attack using the full armory of the theory of meromorphic functions on hyperelliptic Riemann surfaces, and Section 11.10 a successful attack on one aspect of the inverse spectral theory. Section 11.11 is an introduction to the spectral flows that are lying in the background. Sections 11.9 and 11.12–11.14 deal with more specialized issues of interest.

11.1. The Discriminant

Consider a set of Verblunsky coefficients for which there exists a $p > 0$ so that for $j = 0, 1, 2, \dots$,

$$\alpha_{j+p} = \alpha_j \tag{11.1.1}$$

For a fixed p, $\{\alpha_j\}_{j=0}^\infty$ is determined by $\{\alpha_j\}_{j=0}^{p-1}$, so the set of such α's is $\mathbb{D}^p$.

If Ω is $\{0, \dots, p-1\}$, β is counting measure on Ω, $\mathcal{T}(k) = k + j \pmod p$, $f(j) = \alpha_j$, then $(\Omega, \beta, \mathcal{T}, f)$ is the foundation of a stochastic family of Verblunsky coefficients, and $\alpha_j(\omega = 0) = \alpha_j$ which occurs with positive probability, so the theory of stochastic Verblunsky coefficients is relevant.

In this case, since each point in Ω has positive measure, things normally only defined for a.e. ω are defined for all ω. In particular, $\mathcal{T}^{-1}$, and so α_{-1}, can be defined for all j. Essentially, $\{\alpha_j\}_{j=0}^\infty$ is extended to $\{\alpha_j\}_{j=-\infty}^\infty$ in the unique way that (11.1.1) holds.

The theory in this case is very rich, so we will do the following main highlights in this section. $d\mu$ is the measure for the α's and $d\nu$ the density of states.

(i) $\partial\mathbb{D}$ is decomposed naturally into $2p$ alternating sets: $G_1, B_1, G_2, \dots, B_p$ with each *gap*, G_j, open and each *band*, B_j, closed. Generically (i.e., for all α's in $\mathbb{D}^p$ except a closed set of measure zero; see Theorem 11.13.1), all G_j's and B_j's are nonempty and the B_j always have positive Lebesgue measure. But in nongeneric cases, some of the G_j can be empty (and then two B's overlap in a point). We then say the gap G_j is *closed*.

(ii) $d\mu$ is purely a.c. on $\cup_j B_j$ with $w(\theta) > 0$ on $\cup_j B_j^{\text{int}}$ and $d\mu$ has at most one pure point in any G_j and no other support there.

(iii) The density of zeros measure, $d\nu$, is the equilibrium measure for $\cup_{j=1}^p B_j$ and $\nu(B_j) = 1/p$.

(iv) The Lyapunov exponent is the equilibrium potential for $\cup_{j=1}^p B_j$.

For reasons that will be clear soon, the analysis is slightly easier if p is even, so we will assume that is so for the initial stages of the analysis. There are a number of ways to go from even p to odd p. First, if (11.1.1) holds for p, it holds for $2p$, so any set of Verblunsky coefficients of period p can be thought of as a set with period $2p$ where it turns out (at least) half the gaps are closed. Second, given $(\alpha_0, \alpha_1, \dots)$ of period p, we can look at Verblunsky coefficients $(0, \alpha_0, 0, \alpha_1, 0, \dots)$ which clearly has period $2p$. By Example 1.6.14, the original measure $d\mu$ and the new one $d\tilde\mu$ are related by $d\tilde\mu(\theta) = \frac{1}{2} d\mu(2\theta)$, so $\Phi_{2n}(z; d\tilde\mu) = \Phi_n(z^2; d\mu)$, and an analysis of $\tilde\mu$ yields results for μ. A third approach is to use $z^{1/2}$ instead of z as the basic variable in the function Δ below. This is essentially equivalent to the second approach. It is the second approach we will exploit below.

The basic function whose analysis will yield much information about periodic cases is

Definition. Let α_j obey (11.1.1) for an even integer p and let $T_n(z)$ be the transfer matrix (3.2.27) (or (10.5.2)). The *discriminant* is the function

$$\Delta(z) = z^{-p/2} \operatorname{Tr}(T_p(z)) \tag{11.1.2}$$

defined on $\mathbb{C}\backslash\{0\}$.

Here are the basic properties of Δ:

THEOREM 11.1.1. (i) *Δ is real on $\partial\mathbb{D}$, analytic in $\mathbb{C}\backslash\{0\}$, and*

$$\Delta(z) = \overline{\Delta(1/\bar{z})} \tag{11.1.3}$$

(ii) *The Lyapunov exponent is given by*

$$\gamma(z) = \frac{1}{2}\,\log|z| + \frac{1}{p}\,\log\left|\frac{\Delta}{2} + \sqrt{\tfrac{\Delta^2}{4} - 1}\,\right| \tag{11.1.4}$$

where the branch of the square root is taken that maximizes the right side of (11.1.4).

(iii) *For any $\omega \in \mathbb{C}$, $\Delta(z) = \omega$ has exactly p roots counting multiplicities and the product of these roots is* 1.

(iv) *If $\omega \in [-2, 2]$, all solutions of $\Delta(z) = \omega$ lie in $\partial\mathbb{D}$.*

(v) *If $\omega \in (-2, 2)$, all roots of $\Delta(z) = \omega$ are simple.*

(vi) *Starting with some convenient solution of $\Delta(z) = 2$, one can label these solutions (counting multiple roots multiple times) $z_1^+, z_2^+, \dots, z_p^+$, and the solutions of $\Delta(z) = -2$, $z_1^-, \dots, z_p^-$ so that if $a \dot{<} b \dot{<} c$ means a, b, c lie in $\partial\mathbb{D}$ with b between a and c on the arc going counterclockwise from a to c, then*

$$z_1^+ \dot{<} z_1^- \dot{\le} z_2^- \dot{<} z_2^+ \dot{\le} z_3^+ \dot{<} z_3^- \dot{\le} \cdots \tag{11.1.5}$$

with the arc from z_1^+ to z_2^+ to $\dots$ to z_p^+ to z_1^+ circling the origin once; see Figure 11.1.

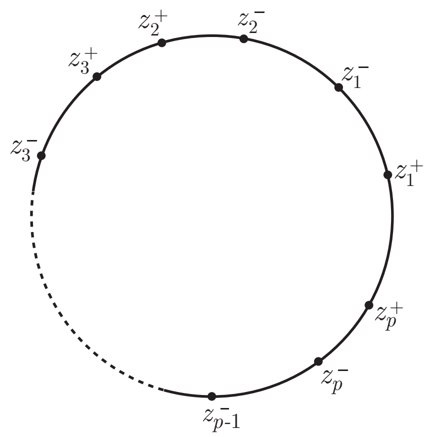

FIGURE 11.1. Ordering of solutions of $|\Delta(e^{i\theta})| = 2$

(vii) *$\Delta(z) \in (-2, 2)$ if and only if for some k, $z_{2k+1}^+ \dot{<} z \dot{<} z_{2k+1}^-$ or $z_{2k}^- \dot{<} z \dot{<} z_{2k}^+$, and on each band,*

$$B_k = \{z \mid z_k^{\sigma_k} \dot{\le} z \dot{\le} z_k^{-\sigma_k}\} \tag{11.1.6}$$

with $\sigma_k = (-1)^{k+1}$, $\Delta(z)$ is strictly monotone.

Remarks. 1. If we cut the circle at z_1^+, the graph of Δ along the circle is given in Figure 11.2.

2. Thus $\Delta(e^{i\theta})$ is a real-valued function on $\partial\mathbb{D}$, obeying $\frac{\partial\Delta}{\partial\theta} = 0 \Rightarrow |\Delta(e^{i\theta})| \ge 2$.

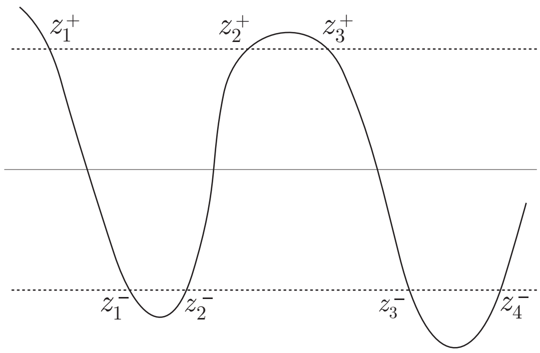

FIGURE 11.2. Graph of $\Delta(e^{i\theta})$

PROOF. (i) Since $\det(T_p(z)) = z^p$, if

$$Q(z) = z^{-p/2} T_p(z) \tag{11.1.7}$$

then

$$\det(Q(z)) = 1 \tag{11.1.8}$$

It follows that $Q(e^{i\theta})$ is in $\mathbb{SU}(1,1)$, so its trace is real by Corollary 10.4.2. Since p is even, $Q(z)$ is clearly analytic. (11.1.3) is the analytic continuation of reality of $\operatorname{Tr}(Q(e^{i\theta}))$.

(ii) Let $e_1(z)$ and $e_2(z)$ be the two eigenvalues of T_p. Then, by the spectral radius formula,

$$\lim_{n\to\infty} \|T_{np}\|^{1/np} = \lim_{n\to\infty} \|T_p^n\|^{1/np} = \max(|e_1|, |e_2|)^{1/p} \tag{11.1.9}$$

Moreover, since $\sup_k(\|A(\alpha_k, z)\|, \|A(\alpha_k, z)^{-1}\|)$ are bounded

$$\gamma(z) = \lim_{n\to\infty} \log \|T_{np}\|^{1/np} \tag{11.1.10}$$

If $\tilde{e}_1, \tilde{e}_2$ are the eigenvalues of Q, by (11.1.7),

$$e_j = z^{p/2} \tilde{e}_j \tag{11.1.11}$$

so, by (11.1.9) and (11.1.10),

$$\gamma(z) = \frac{1}{2} \log|z| + \frac{1}{p} \log\max(|\tilde{e}_1|, |\tilde{e}_2|) \tag{11.1.12}$$

Since $\tilde{e}_j$ are the roots of

$$x^2 - \Delta(z)x + 1 = 0 \tag{11.1.13}$$

they are

$$\frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} - 1}$$

so (11.1.12) is (11.1.4).

(iii) $\Delta(z) = \omega$ if and only if $\operatorname{Tr}(T_p(z)) - z^{p/2}\omega = 0$ which are roots of a polynomial equation of degree p. The degree is exactly p since $\rho_\infty^p z^{-p} T_p(z) \to \prod_{j=1}^p \left(\begin{smallmatrix} 1 & 0 \\ -\alpha_j & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -\alpha_p & 0 \end{smallmatrix}\right)$ with $\rho_\infty = (\prod_{j=0}^{p-1}(1 - |\alpha_j|^2)^{1/2})^{1/p}$, so $\rho_\infty^p \operatorname{Tr}(T_p(z))$ is monic, that is,

$$\rho_\infty^p z^{-p} \operatorname{Tr}(T_p(z)) \to 1 \tag{11.1.14}$$

as $|z| \to \infty$. Thus there are p roots counting multiplicity.

By (11.1.3), the $z^{p/2}$ and $z^{-p/2}$ coefficients of $\Delta(z)$ are complex conjugates, and so equal, since we have just seen that the $z^{p/2}$ coefficient is real. Thus, $\rho_\infty^p[\mathrm{Tr}[T_p(z) - z^{p/2}\omega]]$ is a monic polynomial with value 1 at $z=0$. Thus the product of its roots is 1.

(iv) By (11.1.4), if $\Delta(z) \in [-2,2]$, then $\left|\frac{\Delta}{2} + \sqrt{\frac{\Delta^2}{4} - 1}\,\right| = 1$ so $\gamma(z) = \frac{1}{2}\log|z|$. If $z \in \mathbb{D}$, by Theorem 10.5.11, $\gamma(z) \geq 0 > \frac{1}{2}\log|z|$, so we see $\Delta(z) \notin [-2,2]$. By (i), if $z \in \mathbb{C}\backslash\mathbb{D}$, $\Delta(z) = \overline{\Delta(1/\bar{z})} \notin \overline{[-2,2]} = [-2,2]$.

(v) In the neighborhood of an ℓ-th order zero, z_0, of an analytic function, there are 2ℓ smooth curves with end z_0 so that $f(z)$ is real and $|f(z)|$ small on the curves and the asymptotic phases are $e^{i(\theta_0 + j\pi/\ell)}$, $j = 0,1,2,\dots,2\ell-1$. Thus, if $\Delta(z) - \omega_0$ has a zero of order $\ell > 1$ at $z_0 \in \partial\mathbb{D}$ and $\omega_0 \in (-2,2)$, there are points near z_0 and not on $\partial\mathbb{D}$ with $\Delta(z) \in (-2,2)$. Since this is impossible, all zeros are simple.

For later purposes, we note that the same argument shows $\Delta(z) = \pm 2$ can only have single or double zeros, and if $\Delta(e^{i\theta_0}) = \pm 2$, $\frac{d}{d\theta}\Delta(e^{i\theta})\big|_{\theta=\theta_0} = 0$, then for all $(\theta - \theta_0)$ small and real, $\Delta(e^{i\theta}) \in \pm(2-\varepsilon, 2)$.

(vi) Pick any zero, ω, of $\Delta(z) = 2$. If $\Delta(e^{i\theta}\omega) \leq 2$ for $0 < \theta < \delta$, pick $z_1^+ = \omega$. Otherwise, by the last statement in the proof of (v), $\Delta(e^{i\theta}\omega) > 2$ for $0 < \theta < \delta$, in which case the next zero, call it z_1^+, has a simple zero at 2 and so obeys $\Delta(e^{i\theta}z_1^+) < 2$ for $0 < \theta < \delta$. By (v), $\Delta(e^{i\theta}z_1^+)$ is strictly monotone decreasing in θ until we reach a point z_1^- where $\Delta(z) = -2$. If z_1^- is a double zero, $z_2^- = z_1^-$. Otherwise, $\Delta(z) < -2$ just past z_1^-, but it must turn around, and then the next value in $[-2,2]$ is -2. In either event, $\Delta(z_2^- e^{i\theta}) > -2$ for $0 < \theta < \delta$. Repeating this argument shows the $+2, -2, -2, +2, +2, -2, -2, \dots$ alternation.

(vii) This is just the construction of (vi). $\square$

A major aim towards the later part of this chapter is to prove that a Laurent polynomial, $\Delta(z)$, of degree p even is the discriminant of a set of period p Verblunsky coefficients if and only if

(1) $\Delta(e^{i\theta})$ is real on $\partial\mathbb{D}$.
(2) All zeros of $\Delta(z)$ lie on $\partial\mathbb{D}$ and their product is 1.
(3)
$$\frac{\partial}{\partial\theta}\,\Delta(e^{i\theta}) = 0 \Rightarrow |\Delta(e^{i\theta})| \geq 2 \tag{11.1.15}$$

Notice we have proven above that these conditions are necessary for Δ to be a discriminant. The other direction is proven in Section 11.7 (as Theorem 11.4.4).

THEOREM 11.1.2. *Let $\{\alpha_j\}_{j=0}^\infty$ be a sequence of Verblunsky coefficients of some period p and let $d\mu$ be the associated measure. Then $\{e^{i\theta} \mid |\mathrm{Tr}(T_p(e^{i\theta}))| \leq 2\}$ is a closed set which is naturally the union of p closed intervals (which can only overlap in single points), $B_1, \dots, B_p$. $d\mu_{\mathrm{s}}[\cup B_j] = \emptyset$ and $\cup B_j$ is the essential support of the a.c. spectrum. In each disjoint open interval on $\partial\mathbb{D}\backslash \cup_{j=1}^n B_j$, μ has either no support or a single pure point.*

Remark. We will show later (see Theorem 11.3.2) that if $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}$, then w is continuous and $w(\theta) > 0$ on B^{int}.

PROOF. Suppose first that p is even. The condition that $\binom{1}{1}$ be an eigenfunction of $T_p(z)$ is that $\langle\binom{1}{-1}, T_p(z)\binom{1}{1}\rangle = 0$, which is a polynomial equation, and so it has at most p solutions. For each $z \in \partial\mathbb{D}\setminus \cup_{j=1}^{p} B_j \equiv G$, $|\mathrm{Tr}(T_p(z)| > 2$, so $T_p(z)$ is hyperbolic (see Theorem 10.4.3). Thus there is a subordinate solution, and if $\varphi_n(z)$ is that subordinate solution, $\binom{1}{1}$ must be an eigenvector of $T_p(z)$. Thus G has only singular spectrum and the singular spectrum on G is a finite set. It follows that the essential spectrum μ is contained in $\cup_{j=1}^{p} B_j$.

We claim that for any compact $K \subset B_j^{\mathrm{int}}$,

$$\sup_{z\in K}\sup_{n} \|T_n(z)\| < \infty \tag{11.1.16}$$

for on K, with ψ defined by $2\cos(\psi(z)) = \mathrm{Tr}(z^{-p/2}T_p(z))$,

$$T_p(z) = [e^{i\psi(z)}P_+(z) + e^{-i\psi(z)}(1-P_+(z))]z^{p/2}$$

where, by eigenvalue perturbation theory [**615**], $P_+(z)$ is analytic, and so continuous. Since

$$\|T_{mp}(z)\| \leq \|P_+(z)\| + \|(1-P_+(z)\|$$

we see

$$\sup_{z\in K}\sup_{m} \|T_{mp}(z)\| < \infty$$

Since

$$\sup_{z\in K}\sup_{j=0,1,2,\dots,p-1} \|T_j(z)\| < \infty$$

(11.1.16) holds by $\|T_{mp+j}\| \leq \|T_j\|\,\|T_{mp}\|$, since $T_{mp+j} = T_jT_{mp}$.

There are now many different ways to see that there is purely a.c. spectrum on $\cup_{j=1}^{p} B_j^{\mathrm{int}}$:

(1) By the Jitomirskaya-Last and Gilbert-Pearson ideas (see Corollary 10.8.4), $\mu_{\mathrm{s}}(B_j^{\mathrm{int}}) = 0$ and there is a.c. spectrum on B_j^{int}.
(2) By Carmona's criterion (Theorem 10.7.5), the spectrum is purely a.c. on B_j^{int}.
(3) By Kotani's theory (see Theorems 10.11.1 and 10.11.2), since (11.1.16) implies $\gamma(z) = 0$ on $\cup B_j^{\mathrm{int}}$ and B_j^{int} is open, the spectrum is purely a.c.
(4) Most directly, by computing $F(z)$, the Carathéodory function, and looking at the boundary values. We will do this in Section 11.3.

At the end points of B, $T_p(z)$ is parabolic, so $\lim_{n\to\infty} T_p(z)^n\binom{1}{1} \neq 0$, so there are not eigenvalues. Thus

$$\sigma_{\mathrm{ess}} = \bigcup_j B_j$$

and $\mu_{\mathrm{s}}(\cup_j B_j) = 0$.

By Theorem 10.16.3, there is at most one eigenvalue in each gap.

If p is odd, consider $(0, \alpha_0, 0, \alpha_1, 0, \dots)$ and read the results off this period $2p$ case. $\square$

Since we have γ in (11.1.4), we can compute the density of zeros, $d\nu$.

THEOREM 11.1.3. *Let $\{\alpha_j\}_{j=0}^{\infty}$ be a sequence of Verblunsky coefficients of some period p and let $d\mu$ be the associated measure. Let p be even. Then*

(1) *$d\nu$ is the equilibrium measure of $\cup_j B_j$, the essential spectrum for $d\mu$, and the equilibrium potential is $-[\gamma(z) + \log C_B]$. The capacity is*

$$C_B = \left[\prod_{j=0}^{p-1}(1-|\alpha_j|^2)\right]^{1/p} \tag{11.1.17}$$

(2) *The equilibrium measure, $d\nu$, is given in terms of the discriminant, Δ, by*

$$d\nu(\theta) = V(\theta)\,\frac{d\theta}{2\pi} \tag{11.1.18}$$

where

$$V(\theta) = \frac{1}{p}\,\frac{|\Delta'(e^{i\theta})|}{\sqrt{4-\Delta^2(e^{i\theta})}} \tag{11.1.19}$$

with

$$\Delta'(e^{i\theta}) = \frac{\partial}{\partial\theta}\,\Delta(e^{i\theta}) \tag{11.1.20}$$

(3)

$$\nu(B_j) \equiv \frac{1}{p} \tag{11.1.21}$$

for each j.

Remarks. 1. The results also hold for odd p. Δ is not well-defined, but it is defined up to a $\pm$ sign (associated to $z^{1/2}$). Since (11.1.19) involves Δ^2 and $|\Delta'|$, the sign drops out. The proof follows using sieving.

2. Equilibrium measures, capacities, etc. are discussed right after Theorem 8.1.12.

3. We will have a very different proof of (11.1.19) and (11.1.21) below (see Theorem 11.2.4 and (11.2.24)); in particular, the reason for (11.1.21), which here just follows from a calculation, will be transparent.

4. At an open gap edge, $\Delta^2 - 4$ has a simple zero, so $\Delta' \neq 0$ and thus, $V(\theta)$ has a square root divergence. At a closed gap, $\Delta' = 0$ and $4 - \Delta^2$ has a double zero, so $V(\theta)$ is regular.

Proof. (1) By the Thouless formula (10.5.21) (see Theorem 10.5.26),

$$\gamma(z) = -\log\rho_\infty + \int \log|z - e^{i\theta}|\,d\nu(e^{i\theta}) \tag{11.1.22}$$

where

$$\rho_\infty = \text{RHS of (11.1.17)} \tag{11.1.23}$$

By (11.1.4) and $\cup B_j = \{e^{i\theta} \mid |\Delta(e^{i\theta})| \leq 2\}$, $\gamma(z) = 0$ on $\cup_j B_j$. By the arguments in Section 8.1 (see the proof of Proposition 8.1.5), $d\nu$ is the equilibrium measure, $\mathcal{E}(d\nu) = -\log\rho_\infty$ (so $e^{-\mathcal{E}(d\nu)} = \rho_\infty$, which proves (11.1.17)), and $-[\gamma(z) + \log\rho_\infty]$ is the equilibrium potential.

(2) By (10.11.21) and (1.3.31), we have

$$V(\theta) = \lim_{r\uparrow 1} \operatorname{Re} F_\nu(re^{i\theta}) = 1 - \lim_{r\uparrow 1} \operatorname{Re} \left(2z\,\frac{\partial\Gamma}{\partial z}\right)\bigg|_{z=re^{i\theta}} \tag{11.1.24}$$

where Γ is any function analytic in $\mathbb{D}$ with

$$\operatorname{Re}\Gamma(z) = \gamma(z) \tag{11.1.25}$$

This determines Γ up to an imaginary constant which drops out of (11.1.24).

Define

$$H(z) = \frac{z^{p/2}\Delta(z)}{2} + \sqrt{\frac{z^p\Delta(z)^2}{4} - z^p} \tag{11.1.26}$$

where we take the branch of $\sqrt{}$ which is positive at $z=0$ where $z^{p/2}\Delta(z)\sim\rho_\infty^{-p}$. H is obviously nonvanishing near $z=0$ and $H(z)\sim\rho_\infty^{-p}$. For $z_0\neq 0$, we cannot have $H(z_0)=0$ since then

$$\left(\frac{z_0^{p/2}\Delta(z_0)}{2}\right)^2 = \left(-\sqrt{\frac{z_0^p\Delta(z_0)^2}{4}-z_0^p}\,\right)^2$$

and LHS − RHS $= z_0^p \neq 0$. Thus

$$\begin{aligned}\Gamma(z) &= \frac{1}{p}\,\log H(z)\\ &= \frac{1}{2}\,\log(z)+\frac{1}{p}\,\log\left[\frac{\Delta(z)}{2}+\sqrt{\frac{\Delta(z)^2}{4}-1}\,\right]\end{aligned} \tag{11.1.27}$$

is analytic in $\mathbb{D}$, and by (11.1.4), (11.1.24) holds.

Noting $\text{Re}(z\frac{\partial}{\partial z}\log(z))=1$ and that

$$\begin{aligned}\text{Re}\left(z\,\frac{\partial f}{\partial z}\right)_{z=re^{i\theta}} &= r\,\frac{\partial}{\partial r}\,\text{Re}\,f(re^{i\theta})\\ &= \frac{\partial}{\partial\theta}\,\text{Im}\,f(re^{i\theta})\end{aligned}$$

and taking limits as $r\uparrow 1$, we see that (11.1.19) is a direct consequence of (11.1.24) and (11.1.26).

(3) Noting that

$$V(\theta)=\frac{1}{p}\left|\frac{d}{d\theta}\,\arccos\left(\frac{1}{2}\,\Delta(e^{i\theta})\right)\right| \tag{11.1.28}$$

and that, since $\frac12\Delta$ runs from -1 to 1 monotonically over B_j, $\arccos(\frac12\Delta)$ runs from 0 to 2π, we conclude

$$\int_{B_j} V(\theta)\,\frac{d\theta}{2\pi}=\frac{1}{p}\,\frac{1}{2\pi}\,2\pi=\frac{1}{p} \qquad \square$$

Remark. Since $Q(z)\equiv\frac{\Delta(z)}{2}+\sqrt{\frac{\Delta(z)^2}{4}-1}$ has an analytic continuation to $\mathbb{C}\backslash\{\text{bands}\}$, it is tempting to use (11.1.27) to extend Γ to that set. But on a band, $Q(z)=e^{i\psi(z)}$ where $\cos(\psi(z))=\frac{\Delta(z)}{2}$, and under analytic continuity, $\psi(z)$ has opposite signs on the two sides of a band, so $\arg(Q(z))$ increases by 2π in going around one band. So $\log(Q(z))$ cannot be extended to a single-valued analytic function on $\partial\mathbb{C}\backslash\{\text{bands}\}$.

Example 11.1.4 (= Example 1.6.12 Revisited). Pick $\alpha\in\mathbb{D}$ and let $\alpha_j\equiv\alpha$. Ignoring for now the fact that $p=1$ is not even, if $z=e^{i\theta}$, then $\rho=(1-|\alpha|^2)^{1/2}$,

$$\Delta(e^{i\theta})=2\rho^{-1}\cos(\tfrac12\,\theta) \tag{11.1.29}$$

so $|\Delta|\leq 2$ if and only if $\cos(\frac12\theta)\leq\rho$ or $\sin(\frac12\theta)\geq|\alpha|$, that is, $\theta\geq\theta_{|\alpha|}=2\sin(|\alpha|)$. By (11.1.19),

$$\begin{aligned}V(\theta) &= \frac{|\sin(\theta/2)|}{2\rho\sqrt{1-\rho^{-1}\cos^2(\frac12\theta)}}\\ &= \frac{\sin(\theta/2)}{2\sqrt{\sin^2(\theta/2)-\sin^2(\theta_{|\alpha|}/2)}}\end{aligned}$$

which is (9.7.13).

To handle the fact that p is not even, we can sieve the problem and get a $p = 2$ problem with $\Delta(e^{i\theta}) = 2\rho^{-1}\cos(\theta)$ and then later map back to the $p = 1$ problem. Or we can use the fact that Δ is determined up to ± 1 and that the sign drops out of (11.1.29).

We can also see when there is a pure point in $(-\theta_{|\alpha|}, \theta_{|\alpha|})$. One needs $\binom{1}{1}$ to be an eigenvector of $T_1(z)$ with eigenvalue less than 1. The condition that $\binom{1}{1}$ be an eigenvector is that $z - \bar{\alpha} = -z\alpha + 1$, that is, that

$$z = \frac{1+\bar{\alpha}}{1+\alpha} \equiv z_\alpha \tag{11.1.30}$$

(i.e., (1.6.86)). The eigenvalue of T_1 is then $\rho(1+\alpha)^{-1}$, so the condition for a pure point is that $\rho^2 < |1+\alpha|^2$, that is, $2|\alpha|^2 + 2\operatorname{Re}(\alpha) > 0$ (in agreement with Theorem 1.6.13).

The eigenvalues of the transfer matrix also give one ratio asymptotics; this repeats the calculation that led to Theorem 9.6.9.

One can ask the structure of the set of periodic α's with Δ given by (11.1.29). It is those α's with $|\alpha| = \sqrt{1-\rho^2}$ and so, a circle. This circle can be understood as mapping to the gap by $\alpha \to z_\alpha$. The map is one-one at the endpoints and two-one on the interior of the gap. □

EXAMPLE 11.1.5. Let

$$\alpha_{2j} = \alpha \qquad \alpha_{2j+1} = \alpha' \tag{11.1.31}$$

be of period 2. Computing the trace of

$$\frac{1}{\rho\rho'}\begin{pmatrix} z & -\bar{\alpha}' \\ -\alpha' z & 1 \end{pmatrix}\begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix} \tag{11.1.32}$$

we see

$$\Delta(e^{i\theta}) = 2(\rho\rho')^{-1}[\cos(\theta) + 2\operatorname{Re}(\bar{\alpha}\alpha')] \tag{11.1.33}$$

Let $\theta_\pm$ solve

$$\cos(\theta_\pm) = -\operatorname{Re}(\bar{\alpha}\alpha') \pm \rho\rho'$$

with $\theta_\pm \in [0,\pi)$, so $0 \le \theta_+ < \theta_- \le \pi$. The Schwarz inequality implies $|\operatorname{Re}(\bar{\alpha}\alpha')| + \rho\rho' \le 1$ with equality only if $\alpha = \pm\alpha'$.

$|\Delta(e^{i\theta})| \le 2$ if and only if $\pm\theta \in [\theta_+, \theta_-]$ yields precisely the two bands in the essential support of $d\mu$.

A straightforward calculation yields

$$V(\theta) = \frac{|\sin\theta|}{2((\cos(\theta_+) - \cos(\theta))(\cos(\theta) - \cos(\theta_-)))^{1/2}} \tag{11.1.34}$$

To determine the structure of the set of pairs (α_0, α_1) to lead to a given Δ of the form (11.1.33), note the Δ fixes $A = \rho_0\rho_1$ and $B = \operatorname{Re}(\bar{\alpha}_0\alpha_1)$. Suppose first that $|B| + A = 1$. As noted above, this implies $\alpha_0 = \pm\alpha_1$, so B determines $|\alpha_0| = |\alpha_1|$. If $B = 0$, $A = 1$ and we have a single point $\alpha_0 = \alpha_1 = 0$. If $B > 0$, then $\alpha_0 = \alpha_1 = \sqrt{B}\, e^{i\theta}$ and we have a circle of values. If $B < 0$, then $\alpha_0 = -\alpha_1 = \sqrt{|B|}\, e^{i\theta}$ and we again get a circle.

Finally, if $|B| + A < 1$,

$$|\alpha_0\alpha_1| + \rho_0\rho_1 \le 1 \tag{11.1.35}$$

implies

$$|\operatorname{Im}(\alpha_0\alpha_1)| \le [(1-A)^2 - B^2]^{1/2} \equiv C \tag{11.1.36}$$

If $|\mathrm{Im}(\alpha_0\alpha_1)| = C$, then $|\alpha_0| = |\alpha_1|$ (by equality in the Schwarz inequality (11.1.35)) and $|\alpha_0|$ is uniquely determined as $\sqrt{1-A}$. If $|\mathrm{Im}(\alpha_0\alpha_1)| < C$, $|\alpha_0\alpha_1|^2$ is determined by $|\mathrm{Re}(\bar\alpha_0\alpha_1)|^2 + |\mathrm{Im}(\bar\alpha_0\alpha_1)|^2$ and $|\alpha_0|^2 + |\alpha_1|^2 = \frac{1}{2}(1 + |\alpha_0\alpha_1|^2 - A^2)$, so we get a quadratic equation for $|\alpha_0|$ with two solutions corresponding to interchanging $|\alpha_0|$ and $|\alpha_1|$. Thus, $\{(|\alpha_0|, |\alpha_1|)\}$ consistent with A and B is a circle (two copies of $[-C, C]$ glued at the ends). $|\alpha_0|$, $|\alpha_1|$, and $\mathrm{Re}(\bar\alpha_0\alpha_1)$ determine $\bar\alpha_0\alpha_1$, and then there is one circular degree of freedom from the phase of α_0 if $|\alpha_0| \neq 0$ and from the phase of α_1 if $|\alpha_0| = 0$.

Thus, if there are no gaps ($|\alpha_0| = |\alpha_1| = 0$), we get one solution; one gap ($\alpha_0 = \pm\alpha_1$), we get a circle; two gaps, we get a two-dimensional torus. This is a theme we return to in Section 11.4. □

Remarks and Historical Notes. As we have noted earlier (see Example 11.1.4, Sections 9.5, 9.6, 9.7, and Corollary 9.10.7), there has been extensive study of the Geronimus polynomials, $\alpha_j \equiv a$, and its perturbations, but much less about the more general periodic case.

In [**404**], Geronimus used continued fractions to compute the measure associated to a general family of periodic Verblunsky coefficients, in particular, obtaining the structure of purely a.c. spectrum on a collection of closed arcs, in general, equal in number to the minimal period. We will discuss this further in the Notes to Section 11.3. This calculation from another point of view, the one we use in Section 11.3, can be found in Peherstorfer-Steinbauer [**856**].

Calculations of the density of zeros as an equilibrium measure of zeros via potential theory can be found for a finite family of arcs in [**930**] and [**997**]. The two basic approaches in this section and the next, namely, $\Delta(z)$ as the trace of a transfer matrix and the use of the operator $\mathcal{E}_q(\beta)$, are new within OPUC, but, as we will explain shortly, are analogs of a well-known theory for ODEs, and so quite natural. Indeed, it is surprising that no previous workers on OPUC exploited this well-known theory.

The second-order ODE, $Hu = -u'' + Vu$ with V periodic, is called *Hill's equation*, after fundamental work of Hill [**510**]. He came to the equation by asking about stability of closed (hence periodic) orbits under small perturbations. The key notion of intervals of stability and instability (what we would call bands and gaps) is due to Lyapunov [**724**].

The use of discriminants in Hill's equation and their properties goes back to Lyapunov [**724**], Hamel [**490**], Haupt [**494**], and Kramers [**650**]. For monograph discussions of Hill's equation, see the delightful short book of Magnus and Winkler [**731**] and Eastham [**326**] and the briefer discussion in Reed-Simon [**899**].

The quantity $\Delta^2 - 4$ enters often. From one point of view, it is just the discriminant of the quadratic equation (11.1.13) (still Δ, not $4 - \Delta^2$, is called the discriminant in analog with the theory of Hill's equation!). Note that if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a 2×2 matrix with eigenvalues λ_1 and λ_2, then

$$\begin{aligned}(\lambda_1 - \lambda_2)^2 &= (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 \\ &= \mathrm{Tr}(A)^2 - 4\det(A) \\ &= (a-d)^2 + 4bc\end{aligned} \tag{11.1.37}$$

and for $z^{-p/2}T_p(z)$, (11.1.37) is $\Delta^2 - 4$.

By Theorem 11.1.1(iii), if $\{e^{i\theta_\ell}\}_{\ell=0}^{2p}$ are the roots of the polynomial $z^p(\Delta^2-4)$, then $\prod_{\ell=0}^{2p} e^{i\theta_\ell}=1$, and so

$$\begin{aligned}\Delta^2(e^{i\theta})-4 &= \rho_\infty^{-2p}e^{-ip\theta}\prod_{\ell=0}^{2p}(e^{i\theta}-e^{i\theta_\ell}) \\ &= \rho_\infty^{-2p}2^{2p}(-1)^p\prod_{\ell=0}^{2p}\sin\biggl(\frac{\theta-\theta_\ell}{2}\biggr)\end{aligned}$$

The function

$$2^{2p}\prod_{\ell=0}^{2p}\sin\biggl(\frac{\theta-\theta_\ell}{2}\biggr)=\mathcal{R}(e^{i\theta}) \tag{11.1.38}$$

is the central function of Peherstorfer-Steinbauer [**856**]. They discuss a polynomial, T, defined by $T^2-RU^2=L^2z^p$ which, for a special case, is a multiple of $z^{p/2}\Delta$.

11.2. Floquet Theory

We next present a totally different way of understanding the relation of Δ to $d\nu$, that is, of proving (11.1.19). It is a version of Floquet theory for OPUC that is quite close to the OPRL case, although various factors of $z^{p/2}$ will float in and out of view. We will let q be an integer so α_j is q periodic. In applications, q will be a large multiple of p so even if p is odd, we will take q to be an even multiple. We suppose $q\geq 6$.

Let the extended CMV matrix, $\mathcal{E}$, act on ℓ^∞. Since rows are finite and since the sum of the absolute values of any row is bounded by 4, $\mathcal{E}$ is bounded on ℓ^∞. If $(Mu)_m=u_{m+q}$, then

$$M\mathcal{E}=\mathcal{E}M \tag{11.2.1}$$

since q is even and α is periodic (q even is needed because of the mod 2 structure of $\mathcal{C}$ and $\mathcal{E}$). In particular, if $\beta\in\partial\mathbb{D}$ and

$$X_\beta=\{u\in\ell_\infty \mid Mu=\beta u\} \tag{11.2.2}$$

then $\mathcal{E}$ takes X_β to itself. Clearly, if $u\in X_\beta$, u is determined by $\{u_j\}_{j=0}^{q-1}$ and these coordinates are arbitrary, so

$$\dim(X_\beta)=q \tag{11.2.3}$$

We define

$$\mathcal{E}_q(\beta)=\mathcal{E}\restriction X_\beta \tag{11.2.4}$$

For $j=0,\dots,p-1$, let $\delta_j\in X_\beta$ be the vector with

$$(\delta_j)_m=\begin{cases}\beta^\ell & m=\ell q+j \\ 0 & m\not\equiv j \mod q\end{cases} \tag{11.2.5}$$

Then $\{\delta_j\}_{j=0}^{p-1}$ are a basis for X_β. In this basis, $\mathcal{E}_q(\beta)$ is a matrix with four nonzero elements on each row, obtained by taking the infinite $\mathcal{E}$, cutting out the $[0,q-1]\times[0,q-1]$ block, and adjusting the first two and last two rows, as follows: The top rows in $\mathcal{E}$ have one element each cut off in passing to $\mathcal{E}_q(\beta)$, shift that element right p places and multiply by β^{-1}. Similarly, in the bottom row, shift left by p places and multiply by β; see Figure 11.3.