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Tietze extension theorem

(Theorem)

 Let  $X$  be a [topological space](#). Then [the following are equivalent](#):

1.  $X$  is [normal](#).
2. If  $A$  is a [closed subset](#) in  $X$ , and  $f: A \rightarrow [-1, 1]$  is a [continuous function](#), then  $f$  has a continuous extension to all of  $X$ . (In other [words](#), there is a continuous function  $f^*: X \rightarrow [-1, 1]$  such that  $f$  and  $f^*$  coincide on  $A$ .)

*Remark:* If  $X$  and  $A$  are as above, and  $f: A \rightarrow (-1, 1)$  is a continuous function, then  $f$  has a continuous extension to all of  $X$ .

The present result can be found in [1].

## Bibliography

- 1 A. Mukherjea, K. Pothoven, *Real and Functional analysis*, Plenum press, 1978.

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### Attachments:

[proof of Tietze extension theorem](#) (Proof) by bbukh

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▲ proof of Tietze extension theorem

(Proof)

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To prove the [Tietze Extension Theorem](#), we first need a [lemma](#).

**Lemma 1** *If  $X$  is a normal topological space and  $A$  is closed in  $X$ , then for any continuous function  $f: A \rightarrow \mathbb{R}$  such that  $|f(x)| \leq 1$ ,*

*there is a continuous function  $g: X \rightarrow \mathbb{R}$  such that  $|g(x)| \leq \frac{1}{3}$  for*

*$x \in X$ , and  $|f(x) - g(x)| \leq \frac{2}{3}$  for  $x \in A$ .*

*Proof.* The sets  $f^{-1}((-\infty, -\frac{1}{3}])$  and  $f^{-1}([\frac{1}{3}, \infty))$  are disjoint and closed in  $A$ . Since  $A$  is closed, they are also closed in  $X$ . Since  $X$  is normal, then by [Urysohn's lemma](#) and the fact that  $[0, 1]$  is

[homeomorphic](#) to  $[-\frac{1}{3}, \frac{1}{3}]$ , there is a continuous function

$g: X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$  such that  $g\left(f^{-1}((-\infty, -\frac{1}{3}c])\right) = -\frac{1}{3}$  and

$g\left(f^{-1}([\frac{1}{3}, \infty))\right) = \frac{1}{3}$ . Thus  $|g(x)| \leq \frac{1}{3}$  for  $x \in X$ . Now if

$-\frac{1}{3} \leq f(x) \leq -\frac{1}{3}$ , then  $g(x) = -\frac{1}{3}$  and thus  $|f(x) - g(x)| \leq \frac{2}{3}$ .

Similarly if  $\frac{1}{3} \leq f(x) \leq 1$ , then  $g(x) = \frac{1}{3}$  and thus

$|f(x) - g(x)| \leq \frac{2}{3}$ . Finally, for  $|f(x)| \leq \frac{1}{3}$  we have that

$|g(x)| \leq \frac{1}{3}$ , and so  $|f(x) - g(x)| \leq \frac{2}{3}$ . Hence  $|f(x) - g(x)| \leq \frac{2}{3}$

holds for all  $x \in A$ .  $\square$

This puts us in a position to prove the main [theorem](#).

*Proof of the Tietze extension theorem.* First suppose that for any continuous function on a closed [subset](#) there is a continuous [extension](#). Let  $C$  and  $D$  be disjoint and closed in  $X$ . Define

$f: C \cup D \rightarrow \mathbb{R}$  by  $f(x) = 0$  for  $x \in C$  and  $f(x) = 1$  for  $x \in D$ . Now  $f$  is continuous and we can extend it to a continuous function  $F: X \rightarrow \mathbb{R}$ . By Urysohn's lemma,  $X$  is normal because  $F$  is a continuous function such that  $F(x) = 0$  for  $x \in C$  and  $F(x) = 1$  for  $x \in D$ .

Conversely, let  $X$  be normal and  $A$  be closed in  $X$ . By the lemma, there is a continuous function  $g_0: X \rightarrow \mathbb{R}$  such that  $|g_0(x)| \leq \frac{1}{3}$  for  $x \in X$  and  $|f(x) - g_0(x)| \leq \frac{2}{3}$  for  $x \in A$ . Since  $(f - g_0): A \rightarrow \mathbb{R}$  is continuous, the lemma tells us there is a continuous function  $g_1: X \rightarrow \mathbb{R}$  such that  $|g_1(x)| \leq \frac{1}{3}(\frac{2}{3})$  for  $x \in X$  and  $|f(x) - g_0(x) - g_1(x)| \leq \frac{2}{3}(\frac{2}{3})$  for  $x \in A$ . By repeated application of the lemma we can construct a [sequence](#) of continuous functions  $g_0, g_1, g_2, \dots$  such that  $|g_n(x)| \leq \frac{1}{3}(\frac{2}{3})^n$  for all  $x \in X$ , and  $|f(x) - g_0(x) - g_1(x) - g_2(x) - \dots| \leq (\frac{2}{3})^n$  for  $x \in A$ .

Define  $F(x) = \sum_{n=0}^{\infty} g_n(x)$ . Since  $|g_n(x)| \leq \frac{1}{3}(\frac{2}{3})^n$  and  $\sum_{n=0}^{\infty} \frac{1}{3}(\frac{2}{3})^n$  [converges](#) as a [geometric series](#), then  $\sum_{n=0}^{\infty} g_n(x)$  [converges absolutely](#) and uniformly, so  $F$  is a continuous function defined everywhere. Moreover  $\sum_{n=0}^{\infty} \frac{1}{3}(\frac{2}{3})^n = 1$  [implies](#) that  $|F(x)| \leq 1$ .

Now for  $x \in A$ , we have that  $|f(x) - \sum_{n=0}^k g_n(x)| \leq (\frac{2}{3})^{k+1}$  and as  $k$  goes to [infinity](#), the [right side](#) goes to zero and so the [sum](#) goes to  $F(x)$ . Thus  $|f(x) - F(x)| = 0$  Therefore  $F$  extends  $f$ .  $\square$

*Remarks:* If  $f$  was a [function](#) satisfying  $|f(x)| < 1$ , then the theorem can be strengthened as follows. Find an extension  $F$  of  $f$  as above. The set  $B = F^{-1}(\{-1\} \cup \{1\})$  is closed and disjoint from  $A$  because  $|F(x)| = |f(x)| < 1$  for  $x \in A$ . By Urysohn's lemma there is a continuous function  $\phi$  such that  $\phi(A) = \{1\}$  and  $\phi(B) = \{0\}$ . Hence  $F(x)\phi(x)$  is a continuous extension of  $f(x)$ , and has the [property](#) that  $|F(x)\phi(x)| < 1$ .

If  $f$  is [unbounded](#), then Tietze extension theorem holds as well. To see that consider  $t(x) = \tan^{-1}(x)/(\pi/2)$ . The function  $t \circ f$  has the property that  $(t \circ f)(x) < 1$  for  $x \in A$ , and so it can be extended to a continuous function  $h: X \rightarrow \mathbb{R}$  which has the property  $|h(x)| < 1$ . Hence  $t^{-1} \circ h$  is a continuous extension of  $f$ .

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