Example 9 ($(1/x)_+$ renormalized) Consider the function $(1/x)_+ = H(x)x^{-1}$. Since $\int_0^1 (1/x)\,dx = \infty$, $(1/x)_+$ does *not* define a distribution. Let $\mathcal{S}(\mathbb{R}\backslash 0) = \{f \in \mathcal{S} \mid \operatorname{supp} f \in \mathbb{R}\backslash 0\}$. If $f \in \mathcal{S}(\mathbb{R}\backslash 0)$, $\int (1/x)_+ f(x)\,dx$ makes sense. Thus $(1/x)_+$ does define a linear functional on $\mathcal{S}(\mathbb{R}\backslash 0)$ which is continous as we shall see. By the Hahn–Banach theorem, this functional on $\mathcal{S}(\mathbb{R}\backslash 0)$ has extensions to all of $\mathcal{S}(\mathbb{R})$ which we call "renormalizations of $(1/x)_+$." As explicit examples, consider

$$\left(\frac{1}{x}\right)_{+,M}(f) = \int_0^M \frac{f(x) - f(0)}{x}\,dx + \int_M^\infty \frac{f(x)}{x}\,dx$$

Since these maps are continuous on $\mathcal{S}$, $(1/x)_+$ is continuous on $\mathcal{S}(\mathbb{R}\backslash 0)$. How much arbitrariness is there in the renormalization? If T and S are two renormalizations of $(1/x)_+$, then $T - S$ vanishes on $\mathcal{S}(\mathbb{R}\backslash 0)$ and so has support $\{0\}$; thus $T - S = \sum_{|\alpha| \le m} c_\alpha D^\alpha \delta$. For example,

$$\left(\frac{1}{x}\right)_{+,M} - \left(\frac{1}{x}\right)_{+,N} = \cancel{-}\ln\left(\frac{M}{N}\right)\delta(x)$$

With this definition of renormalization, there are an infinity of free constants in the renormalization of $(1/x)_+$. However, one could argue that as long as $f \in \mathcal{S}$ and $f(0) = 0$, $\int_0^\infty [f(x)/x]\,dx < \infty$; so we really want to extend $(1/x)_+$ from $\{f \in \mathcal{S} \mid f(0) = 0\}$ to $\mathcal{S}$. If we adopt this requirement, the only real renormalizations of $(1/x)_+$ are the $(1/x)_{+,M}$ and there is only one free constant in the renormalization (see Problem 32 for a link between these definitions).

Bogoliubov and Hepp have treated the renormalization of x space Feynman graphs in the spirit of Example 9; the renormalization constants, for example, the renormalized mass and charge, enter as the free constants analogous to the $\ln(M/N)$ in Example 9. For more details, see the references in the Notes.

There is one final theorem about $\mathcal{S}$ and $\mathcal{S}'$ which is often useful. To appreciate its significance, let us first consider the case of L^p where the analogous theorem fails. Suppose $p^{-1} + q^{-1} = 1$, $p < \infty$, $q < \infty$, and let $F \in L^q(\mathbb{R}^2) = [L^p(\mathbb{R} \times \mathbb{R})]^*$. Let $f, g \in L^p(\mathbb{R})$; then $f(x)g(y) \in L^p(\mathbb{R}^2)$ so

$$F(f, g) = \int F(x, y) f(x) g(y)\,dx\,dy < \infty$$

Moreover

$$|F(f, g)| \le \|F\|_{L^q(\mathbb{R}^2)} \|f\|_p \|g\|_p$$

so F defines a continuous bilinear form on L^p. Not every bilinear form is of this type. For example, if $p = 2$, the bilinear form $(\bar{f}, g) = \int f(x) g(x)\,dx$ cannot be expressed as $(\bar{f}, g) = \int F(x, y) f(x) g(y)\,dx\,dy$ for some $F \in L^2(\mathbb{R}^2)$.

The situation for $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R} \times \mathbb{R})^* = \mathcal{S}'(\mathbb{R}^2)$ is very different:

Theorem V.12 (kernel or nuclear theorem) Let $B(f, g)$ be a separately continuous bilinear functional on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$. Then there is a unique tempered distribution $T \in \mathcal{S}'(\mathbb{R}^{n+m})$ with $B(f, g) = T(f \otimes g)$ where

$$(f \otimes g)(x_1, \ldots, x_{n+m}) = f(x_1, \ldots, x_n) g(x_{n+1}, \ldots, x_{n+m})$$

That separate continuity implies joint continuity is a consequence of the fact that $\mathcal{S}$ is Fréchet (Theorem V.9) and the corollary of Theorem V.7. We prove that jointly continuous functionals have the requisite form in the appendix to this section (Corollary 4 to Theorem V.14). Theorem V.12 can be extended to multilinear functionals (Problems 34 and 35).

Appendix to V.3 The N-representation for $\mathcal{S}$ and $\mathcal{S}'$

In this appendix, we will prove some of theorems about $\mathcal{S}$ and $\mathcal{S}'$. These proofs rely on the realization of $\mathcal{S}$ and therefore $\mathcal{S}'$ as sequence spaces (in fact as the space s of Section III.1). This realization depends in turn on two elements. The first element is topologizing $\mathcal{S}$ by an equivalent family of L^2 norms. To forcefully distinguish these norms from the $\|\cdot\|_{\alpha,\beta}$ norms we write

$$\|f\|_{\alpha,\beta,\infty} = \|x^\alpha D^\beta f\|_\infty$$

rather than merely $\|\cdot\|_{\alpha,\beta}$ and define

$$\|f\|_{\alpha,\beta,2} = \|x^\alpha D^\beta f\|_{L^2(\mathbb{R}^n)}$$

Then:

Lemma 1 The families of seminorms $\{\|\cdot\|_{\alpha,\beta,\infty}\}$ and $\{\|\cdot\|_{\alpha,\beta,2}\}$ on $\mathcal{S}(\mathbb{R}^n)$ are equivalent.

Proof We provide the proof in the case $n = 1$ for simplicity of notation. Since $(1 + x^2)^{-1} \in L^2$, $\|f\|_2 \le \|(1 + x^2)^{-1}\|_2 \|(1 + x^2) f\|_\infty$ so

$$\|f\|_{\alpha,\beta,2} \le C(\|f\|_{\alpha,\beta,\infty} + \|f\|_{\alpha+2,\beta,\infty})$$

On the other hand, $f(x) = \int_{-\infty}^{x} f'(x)\,dx$, so

$$\|f\|_\infty \le \|f'\|_1 \le \|(1+x^2)f'\|_2 \|(1+x^2)^{-1}\|_2$$

Since $(x^\alpha D^\beta f)' = \alpha x^{\alpha-1}D^\beta f + x^\alpha D^{\beta+1} f$ we have

$$\|f\|_{\alpha,\beta,\infty} \le C(\alpha\|f\|_{\alpha-1,\beta,2} + \|f\|_{\alpha,\beta+1,2} + \alpha\|f\|_{\alpha+1,\beta,2} + \|f\|_{\alpha+2,\beta+1,2}) \qquad ▮$$

The second element involves some special properties of Hermite functions (the eigenfunctions of the harmonic oscillator). Consider the maps $A: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $A^\dagger: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ given by

$$A = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \qquad A^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

and $N = A^\dagger A$. Let $\|f\|_n \equiv \|(N+1)^n f\|_2$ which is a seminorm on $\mathcal{S}$.

Lemma 2 The seminorms $\{\|\cdot\|_n\}$ are ~~a~~ directed family equivalent to the $\{\|\cdot\|_{\alpha,\beta,2}\}$ family of seminorms on $\mathcal{S}$.

Proof One need only use the inequality $\|A_1^\# \cdots A_m^\# f\|_2 \le \|(N+m)^{m/2} f\|_2$ where $A^\#$ stands for either A or $A^\dagger$. The details are left to the reader (Problem 36).

Now consider the function ϕ_0 defined by $A\phi_0 = 0$ and $\int_{-\infty}^{\infty} (\phi_0)^2\,dx = 1$, that is, $\phi_0(x) = \pi^{-1/4}e^{-\frac{1}{2}x^2}$ and let

$$\phi_n = (n!)^{-1/2}(A^\dagger)^n\phi_0 = (2^n n!)^{-1/2}(-1)^n \pi^{-1/4} e^{+\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

The $\{\phi_n\}_{n=0}^\infty$ are called the **Hermite functions** or the harmonic oscillator wave functions since

$$\left(-\frac{d^2}{dx^2} + x^2\right)\phi_n = (2n+1)\phi_n$$

One has:

Lemma 3 The set $\{\phi_n\}_{n=0}^\infty$ is an orthonormal basis for $L^2(\mathbb{R})$.

Proof See the following problems: Problems 40 or 41 of Chapter IX or Problem 20 of Chapter XIII.

Notice that $N\phi_n = n\phi_n$. Suppose $f \in \mathcal{S}$ and consider the L^2-convergent expansion $f = \sum_{n=0}^\infty a_n\phi_n$ where $a_n = (\phi_n, f) \equiv \int_{-\infty}^{\infty} \overline{\phi_n(x)} f(x)\,dx$. Since $N^m: \mathcal{S} \to \mathcal{S}$, $N^m f \in \mathcal{S}$ and thus in L^2. But $N^m f = \sum_{n=0}^\infty a_n n^m \phi_n$, so $\sum_{n=0}^\infty |a_n|^2 n^{2m} < \infty$. In particular $\sup_n |a_n| n^m < \infty$. We have thus proven the first part of:

Theorem V.13 (the N-representation theorem for $\mathcal{S}$) Let s_k be the set of multisequences $\{a_\alpha\}_{\alpha \in I_+{}^k}$ with the property

$$\sup_{\alpha \in I_+{}^k} |a_\alpha|\,|\alpha|^m < \infty$$

for each m. Topologize s_k with the seminorms

$$\|\{a_\alpha\}\|_\beta^2 = \sum_\alpha (\alpha+1)^{2\beta}|a_\alpha|^2$$

where $\beta \in I_+^k$ and $(\alpha+1)^{2\beta} = \prod_{i=1}^k (\alpha_i+1)^{2\beta_i}$. Let $f \in \mathcal{S}(\mathbb{R}^k)$. Then the sequence $\{a_\alpha\}$, $a_\alpha = (\phi_\alpha, f)$ with $\phi_\alpha(x) = \prod_{i=1}^k \phi_{\alpha_i}(x_i)$, is in s_k and the map $f \mapsto \{a_\alpha\}$ is a topological isomorphism. The **Hermite expansion** $f = \sum_\alpha a_\alpha\phi_\alpha$ converges in $\mathcal{S}$. The $\{a_\alpha\}$ are called **Hermite coefficients.**

Proof We give the details in the case $k = 1$. By our previous discussion, if $f \in \mathcal{S}$ and $a_n = (\phi_n, f)$, then $\{a_n\} \in s$. Moreover, $\|\{a_n\}\|_m = \|f\|_m$ in the notation of Lemma 2. Since the $\|\ \|_m$ are norms on $\mathcal{S}$, the map $f \mapsto \{a_n\}$ is injective. Now let $\{a_n\}_{n=0}^\infty \in s$ and let $f_N = \sum_{n=0}^N a_n\phi_n$. A simple computation shows that

$$\|f_N - f_M\|_m^2 = \sum_{n=N+1}^{M} |a_n|^2(n+1)^{2m} \to 0$$

as $N, M \to \infty$. Thus f_N is Cauchy in each of $\|\ \|_m$ and thus in $\mathcal{S}$ (by Lemmas 1 and 2). Since $\mathcal{S}$ is complete, $f_N \to f$ for some $f \in \mathcal{S}$. But then $f_N \to f$ in L^2 so $(\phi_n, f) = a_n$. Thus the image of our map of $\mathcal{S} \to s$ is all of s. The equivalence of the topologies follows from the equality of the norms $\|\cdot\|_m$ on $\mathcal{S}$ and s. ▮

We can now identify $\mathcal{S}'$ with a sequence space also:

Theorem V.14 (the N-representation theorem for $\mathcal{S}'$) Let $T \in \mathcal{S}'(\mathbb{R}^k)$. Let $b_\alpha = T(\phi_\alpha)$ for each $\alpha \in I_+^k$. Then for some $\beta \in I_+^k$, $|b_\alpha| \le C(\alpha+1)^\beta$ for all α. Conversely, if $|b_\alpha| \le C(\alpha+1)^\beta$ for all α, there is a unique $T \in \mathcal{S}'$ with $T(\phi_\alpha) = b_\alpha$. If $T \in \mathcal{S}'$ and $b_\alpha = T(\phi_\alpha)$ are its **Hermite coefficients,** then $\sum_\alpha b_\alpha\phi_\alpha$ converges in the $\sigma(\mathcal{S}', \mathcal{S})$ topology to T.

Proof Again, we consider only $k = 1$. Let $T \in \mathcal{S}'$. Then $|T(\phi)| \le C\|\phi\|_m$ for some m and C since $\{\|\cdot\|_m\}$ is a directed set. $\|\phi_n\|_m = (n+1)^m$, so

$|b_n| \le C(n+1)^m$. Conversely, suppose $|b_n| \le C(n+1)^m$. For $\{a_n\} \in s$ define $B(\{a_n\}) = \sum_{n=0}^{\infty} b_n a_n$. Then

$$\begin{aligned} |B(\{a_n\})| &\le \sum_{n=0}^{\infty} |b_n|\,|a_n| \le C \sum_{n=0}^{\infty} (n+1)^m |a_n| \\ &\le C\left(\sum_{n=0}^{\infty} (n+1)^{2m+2} |a_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} (n+1)^{-2}\right)^{1/2} \\ &\le \frac{\pi^2}{6} C \|\{a_n\}\|_{m+1} \end{aligned}$$

Thus B defines a continuous linear functional on s. Under the association of $\mathscr{S}$ and s, there is a $T \in \mathscr{S}'$ with $T(\sum_{n=0}^{\infty} a_n \phi_n) = \sum_{n=0}^{\infty} a_n b_n$; in particular, $T(\phi_n) = b_n$. The weak convergence of $\sum_n b_n \phi_n$ to T is easy. ▮

We can now easily prove many interesting theorems about $\mathscr{S}$ with this machinery which has *two* important simplifications: (1) Sequences are easier to deal with than functions. (2) The two conditions in $\mathscr{S}$, fall-off at ∞ and the C^∞ condition, are replaced by a *single* fall-off condition in s.

Corollary 1 $\mathscr{S}$ is dense in $\mathscr{S}'$ in the $\sigma(\mathscr{S}', \mathscr{S})$ topology.

Proof $\sum_{|\alpha| \le N} b_\alpha \phi_\alpha \in \mathscr{S}$ and converges weakly to $T \in \mathscr{S}'$ as $N \to \infty$ if $b_\alpha = T(\phi_\alpha)$. ▮

Corollary 2 $\mathscr{S}$ is separable in the Fréchet topology. $\mathscr{S}'$ is separable in the $\sigma(\mathscr{S}', \mathscr{S})$ topology (and also in the $\tau(\mathscr{S}', \mathscr{S})$ topology we introduce in Section V.7).

Corollary 3 The regularity theorem for distributions—Theorem V.10.

Proof Again we only consider the case $k = 1$. Since $\|f\|_\infty \le C\|(1+x^2)f'\|_2$ we conclude that $\|\phi_n\|_\infty \le C'(n+1)^{3/2}$, using A and $A^\dagger$ and the estimate in the proof of Lemma 2. (More detailed studies of the ϕ_n show $\|\phi_n\|_\infty \sim D(n+1)^{-1/12}$). Let $T \in \mathscr{S}'$ and let $\{b_n\}$ be its Hermite coefficients. Then $|b_n| \le E(n+1)^m$ for some m. Let $a_n = (n+1)^{-m-3} b_n$. Then $\sum |a_n|\, \|\phi_n\|_\infty \le E \sum (n+1)^{-3/2} < \infty$, so $\sum a_n \phi_n$ converges uniformly to some continuous function F on $\mathbb{R}$. F has Hermite coefficients (as an element of $\mathscr{S}'$), $\{a_n\}$. Extend $A^\dagger$, A and $N = \frac{1}{2}(-d^2/dx^2 + x^2 - 1)$ to $\mathscr{S}'$. Then

$$T = (N+1)^{m+3} F = \frac{1}{2^{m+3}} \left(-\frac{d^2}{dx^2} + x^2 + 1\right)^{m+3} F$$

Thus T can be written as a sum of polynomials times weak derivatives of polynomially bounded continuous functions. Simple manipulations (Problem 37) now complete the proof. ▮

Corollary 4 (nuclear theorem) Every jointly continuous bilinear functional $B(\cdot, \cdot)$ on $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^m)$ is of the form $B(f, g) = T(f \otimes g)$ for some $T \in \mathscr{S}'(\mathbb{R}^{n+m})$.

Proof Since B is jointly continuous, $|B(f, g)| \le C\|f\|_r \|g\|_s$ for some $r \in I_+^n$, $s \in I_+^m$. Then $|B(\phi_\alpha, \phi_\beta)| \le C(\alpha+1)^r(\beta+1)^s = C[\langle\alpha, \beta\rangle + 1]^{\langle r, s\rangle}$ where

$$\langle\alpha, \beta\rangle = \langle\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\rangle \in I_+^{n+m}$$

As a result $b_{\langle\alpha,\beta\rangle} \equiv B(\phi_\alpha, \phi_\beta)$ are the Hermite coefficients of a distribution $T \in \mathscr{S}'(\mathbb{R}^{n+m})$ with $T(\phi_{\langle\alpha,\beta\rangle}) \equiv T(\phi_\alpha \otimes \phi_\beta) = b_{\langle\alpha,\beta\rangle}$. Let $f = \sum a_\alpha \phi_\alpha$, $g = \sum c_\beta \phi_\beta$. Since these expansions converge in $\mathscr{S}$,

$$T(f \otimes g) = \sum_{\alpha,\beta} a_\alpha c_\beta T(\phi_\alpha \otimes \phi_\beta) = \sum_{\alpha,\beta} a_\alpha c_\beta b_{\langle\alpha,\beta\rangle} = B(f, g) \ ▮$$

V.4 Inductive limits: generalized functions and weak solutions of partial differential equations

In an intuitive sense, the distributions of the last section had the restriction of being polynomially bounded at infinity. We saw this in Theorem V.10 which told us any $T \in \mathscr{S}'$ is the derivative of a polynomially bounded function. The growth of a tempered distribution $T \in \mathscr{S}'$ is in some sense dual to the decrease restrictions imposed on functions $f \in \mathscr{S}$. This suggests we construct "distributions" without any growth restriction at ∞ as the dual of a space with the severest possible decrease conditions at ∞, that of vanishing outside of a compact set. That is, we want to topologize the C^∞ functions of compact support, $C_0^\infty(\mathbb{R}^n)$, so that it is a complete locally convex space. If K is a compact set in $\mathbb{R}^n$, the functions $C_0^\infty(K)$ which are C^∞ and have support in K have a natural topology given by $\|f\|_{\alpha,\infty} \equiv \sup_{\mathbb{R}^n} |D^\alpha f|$. $C_0^\infty(\mathbb{R}^n)$ is *not* complete when given the $\{\|\cdot\|_{\alpha,\infty}\}_{\alpha \in I_+^n}$ family of norms (see Problem 38), even though $C_0^\infty(K)$ is for each compact set K. In some sense, we want to think of $C_0^\infty(\mathbb{R}^n)$ as $\bigcup_m C_0^\infty(K_m)$ for some family of compact sets $\{K_m\}_{m=1}^{\infty}$ with $\bigcup_m K_m = \mathbb{R}^n$, and topologize it with a "limit" topology. To do this we describe a general construction.