Topology Q+A Board

Ask A Topologist 2003

Thread

- Topology by Alexci (April 20, 2003)
  - Re: Topology by mars (April 21, 2003)
  - normality of metric spaces and non-normal Hausdorff spaces. by Henno Brandsma (April 23, 2003)

From: Henno Brandsma
Date: April 23, 2003
Subject: normality of metric spaces and non-normal Hausdorff spaces.

In reply to "Topology", posted by Alexci on April 20, 2003:
>Q9) A topological space is called normal if any two disjoint closed
> sets can be separated by disjoint open sets.
> Prove that a metric space is normal. Construct (or find)
> a space which is Hausdorff, but not normal. Justify your answer.
>

A basic proof that metric spaces are normal:

Let A and B be closed and disjoint.

Then for each a \in A we have that there is an epsilon e(a)>0
such that B(a,e(a)) misses B. This is true because all a \in A are NOT in B
and B is closed.
Similarly, for each b \in B there is an e(b)>0 such that B(b,e(b)) misses A.
Now consider
U = Union {B(a,e(a)/2): a in A} and V = Union {B(b,e(b)/2): b in B}
These are open as unions of open balls, and a is in B(a,e(a)/2) for each a in A,
so A \subset U and also B \subset V.
Suppose that x is in U /\ V.
Then for some a \in A ,b \in B we have that x is in B(a,e(a)/2) /\ B(b,e(b)/2).
Suppose that e(a) <= e(b) (or reverse the role of a and b in the sequel).
Then d(a,b) <= d(a,x) + d(x,b) < e(a)/2 + e(b)/2 <= e(b)/2 + e(b)/2 = e(b).
So d(a,b) < e(b), but then a is in B(b,e(b)) which couldn't be because of the choice of e(b).

This shows that U and V are disjoint open neighbourhoods of A and B.

As to a Hausdorff but non-normal example:
Let X be R with the topology generated by the usual topology (called R-open sets, for now) and the subset Q.
So we take the smallest topology that is larger than the usual one in which the rationals are also open. So open sets are O (O R-open) or O /\ Q (with O R-open again).
Then X is Hausdorff, because it is already Hausdorff with the R-open sets, so all points can still be separated.
But the irrationals P = R\Q are now a closed set, disjoint from the point (or closed set) 0 (resp. {0}).
Suppose U and V are disjoint neighbourhoods of 0 and P respectively.
V cannot be of the form O /\ Q (because it contains P), so V is R-open.
It is easy to see that cl(V) = X, so that V intersects U.
Hence X is not normal (or regular).

Henno Brandsma
