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Ask A Topologist 2003

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From: Henno Brandsma

Date: April 23, 2003

Subject: normality of metric spaces and non-normal Hausdorff spaces.

In reply to "Topology", posted by Alexci on April 20, 2003:

>Q9) A topological space is called normal if any two disjoint closed
> sets can be separated by disjoint open sets.
> Prove that a metric space is normal. Construct (or find)
> a space which is Hausdorff, but not normal. Justify your answer.
>

A basic proof that metric spaces are normal:

Let A and B be closed and disjoint.

Then for each $a \in A$ we have that there is an $\epsilon(a) > 0$ such that $B(a, \epsilon(a))$ misses B . This is true because all $a \in A$ are NOT in B and B is closed.

Similarly, for each $b \in B$ there is an $\epsilon(b) > 0$ such that $B(b, \epsilon(b))$ misses A .

Now consider

$U = \text{Union } \{B(a, \epsilon(a)/2) : a \in A\}$ and $V = \text{Union } \{B(b, \epsilon(b)/2) : b \in B\}$

These are open as unions of open balls, and a is in $B(a, \epsilon(a)/2)$ for each a in A , so $A \subset U$ and also $B \subset V$.

Suppose that x is in $U \cap V$.

Then for some $a \in A, b \in B$ we have that x is in $B(a, \epsilon(a)/2) \cap B(b, \epsilon(b)/2)$.

Suppose that $\epsilon(a) \leq \epsilon(b)$ (or reverse the role of a and b in the sequel).

Then $d(a, b) \leq d(a, x) + d(x, b) < \epsilon(a)/2 + \epsilon(b)/2 \leq \epsilon(b)/2 + \epsilon(b)/2 = \epsilon(b)$.

So $d(a, b) < \epsilon(b)$, but then a is in $B(b, \epsilon(b))$ which couldn't be because of the choice of $\epsilon(b)$.

This shows that U and V are disjoint open neighbourhoods of A and B .

As to a Hausdorff but non-normal example:

Let X be \mathbb{R} with the topology generated by the usual topology (called \mathbb{R} -open sets, for now) and the subset Q .

So we take the smallest topology that is larger than the usual one in which the rationals are also open. So open sets are O (O \mathbb{R} -open) or $O \cap Q$ (with O \mathbb{R} -open again).

Then X is Hausdorff, because it is already Hausdorff with the \mathbb{R} -open sets, so all points can still be separated.

But the irrationals $P = \mathbb{R} \setminus Q$ are now a closed set, disjoint from the point (or closed set) 0 (resp. $\{0\}$).

Suppose U and V are disjoint neighbourhoods of 0 and P respectively.

V cannot be of the form $O \cap Q$ (because it contains P), so V is \mathbb{R} -open.

It is easy to see that $\text{cl}(V) = X$, so that V intersects U .

Hence X is not normal (or regular).

Henno Brandsma

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