L5.4 Lemma 5.3. *The Schur product of two positive definite matrices is positive definite.*

Proof. For any vector φ, let $Q^{(\varphi)}$ be the matrix

$$Q^{(\varphi)}_{mk} = \bar\varphi_m \varphi_k \tag{5.5}$$ 5.5x

which is clearly positive definite. Since any positive definite matrix has an orthonormal basis of eigenvectors with nonnegative eigenvalues, any positive definite X is a sum of $Q^{(\varphi)}$'s. Thus, it suffices to prove the result for $X = Q^{(\varphi)}$, $Y = Q^{(\psi)}$. Since

$$Q^{(\varphi)} \odot Q^{(\psi)} = Q^{(\varphi \odot \psi)}$$

where $(\varphi \odot \psi)_m = \varphi_m \psi_m$, the special case is obvious. □

As a second preliminary, it will be useful to have two approximation theorems: One is a simple extension of the Weierstrass approximation theorem. The other is specific to matrix monotone functions.

P5.5 Proposition 5.4. (a) *Let f be a continuous function on some bounded open interval (a,b). Then there exist polynomials P_m so that for any $[c,d] \subset (a,b)$,*

$$\sup_{c \le x \le d} |f(x) - P_m(x)| \to 0 \tag{5.6}$$ 5.4

as $m \to \infty$.

Moreover, if f is C^k, then for $\ell = 0, 1, \dots, k$,

$$\sup_{c \le x \le d} \left| \frac{d^\ell}{dx^\ell} f - \frac{d^\ell}{dx^\ell} P_m \right| \to 0 \tag{5.7}$$ 5.5

as $m \to \infty$.

(b) *Let (a,b) be a finite interval and $f \in \mathcal{M}_n(a,b)$ be continuous. Then there exist C^∞ functions f_m on $(a+\frac{1}{m}, b-\frac{1}{m})$ so that $f_m \in \mathcal{M}_n(a+\frac{1}{m}, b-\frac{1}{m})$ and (5.6) holds with P_m replaced by f_m. If f is C^k, (5.7) holds with P_m replaced by f_m. If f is only assumed monotone, then f_m exists in $\mathcal{M}_n(a+\frac{1}{m}, b-\frac{1}{m})$ so $\sup_{c \le x \le d; m} |f_m(x)| < \infty$ and $f_m(x) \to f(x)$ for almost all x.*

Proof. (a) If we prove the result for closed intervals, $[c,d]$, uniformly on $[c,d]$, it follows for open intervals (a,b) by a two-step approximation. By scaling, we can take $[c,d] = [0,1]$, so suppose f is continuous on $[0,1]$.

Define the Bernstein polynomials $B_m(x)$ by

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} x^j (1-x)^{m-j} f\left(\frac{j}{m}\right) \tag{5.8}$$ 5.6

Introduce the shorthand

$$\mathbb{E}_{m,x}(Q(j,x)) = \sum_{j=0}^{m} \binom{m}{j} x^j (1-x)^{m-j} Q(j,x) \qquad (5.9) \quad \boxed{\texttt{5.7}}$$

$\mathbb{E}$ stands for expectation since

$$\mathbb{E}_{m,x}(1) = 1$$

by the binomial theorem (see also the Notes). We have that

$$\frac{d^\ell}{da^\ell}(1+a)^m = \frac{d^\ell}{da^\ell}\left[\sum_{j=0}^{m}\binom{m}{j}(x+a)^j(1-x)^{m-j}\right]$$

so evaluating both sides at $a=0$, we get

$$\mathbb{E}(j(j-1)\dots(j-\ell+1)) = m(m-1)\dots(m-\ell+1)x^\ell \qquad (5.10) \quad \boxed{\texttt{5.8}}$$

In particular,

$$\mathbb{E}(j) = mx \qquad \mathbb{E}(j(j-1)+j) = m(m-1)x^2 + mx \qquad (5.11) \quad \boxed{\texttt{5.9}}$$

so

$$\begin{aligned}\mathbb{E}\left(\left(x-\frac{j}{m}\right)^2\right) &= x^2 + \left(1-\frac{1}{m}\right)x^2 + \frac{x}{m} - 2x^2 \\ &= \frac{x(1-x)}{m} \qquad (5.12) \quad \boxed{\texttt{5.10}}\end{aligned}$$

Thus,

$$\begin{aligned}|B_m(x)-f(x)| &= \left|\mathbb{E}_{m,x}\left(f\left(\frac{j}{m}\right)-f(x)\right)\right| \\ &\le 2\sup_x |f(x)|\,\mathbb{E}_{m,x}(\chi_{\{j\,|\,|x-\frac{j}{m}|>\delta\}}) + \sup_{|x-y|\le\delta}|f(x)-f(y)| \\ &\le 2\delta^{-2}\,\frac{x(1-x)}{m}\,\sup_x|f(x)| + \sup_{|x-y|\le\delta}|f(x)-f(y)|\end{aligned}$$

It follows that

$$\limsup_{m\to\infty}\sup_x|B_m(x)-f(x)| \le \sup_{|x-y|\le\delta}|f(x)-f(y)| \qquad (5.13) \quad \boxed{\texttt{5.11}}$$

Taking $\delta \downarrow 0$ and using uniform continuity of f, we see $B_m \to f$ uniformly.

Now suppose f is C^k. Since

$$\frac{d}{dx}\binom{m}{j}x^j(1-x)^{m-j} = m\left[\binom{m-1}{j-1}x^{j-1}(1-x)^{m-j} - \binom{m-1}{j}x^j(1-x)^{m-1-j}\right]$$